21-737 Probabilistic Combinatorics
Homework II: Hints
Follow-up solutions due Friday, February 24

1. The probability space Ω is the collection of all graphs on vertex set $[n]$. We consider the sequence of partitions $\mathcal{F}_2, \mathcal{F}_3, \ldots, \mathcal{F}_n$ of Ω in which two graph are in the same part of $\mathcal{F}_i$ if they agree on $[i]$. We 'reveal' the edges in the graph one vertex at a time and color greedily as we go. Note, for example, if the graphs G_1 and G_2 have $G_1[i] = G_2[i]$ (i.e. these two graphs are in the same part of $\mathcal{F}_i$) then the coloring we assign to these two graphs agrees on $[i]$. When we 'reveal' the edges between vertex $i+1$ and $[i]$ we color vertex $i+1$ in an arbitrary way that proper with respect to the coloring already assigned to the vertices in $[i]$ if possible. If at any stage we arrive at a graph and a coloring that cannot be extended to vertex $i+1$ then we simply stop the coloring process and say that we fail.

   Show that the probability that we fail at step i is at most $o(1/n)$. To do this condition on the number of times we have used each color so far and apply Jensen's Inequality (the randomness is in the edges between $i+1$ and $[i]$, which each appear with probability $1/2$).

2. Use the following:

   Chernoff Bound. If $X_1, X_2, \ldots, X_n$ are i.i.d. variables with $Pr(X_i = -1) = Pr(X_i = 1) = 1/2$ then

   $$Pr\left(\sum_{i=1}^{n} X_i > \lambda\sqrt{n}\right) \leq e^{-\lambda^2/2}.$$

   Note that the LLL provides a lower bound on the probability that none of the 'bad' events occur.

3. Prove

   $$Pr(A \subseteq Y \mid i \notin Y) \geq Pr(A \subseteq Y).$$

   Use a *coupling* to relate the space conditioned on $i \notin Y$ and the unconditioned probability space.

4. Let the bipartition of the vertex set be X, Y. Let a color for each $x \in X$ be chosen at random from its list. Use the Local Lemma (and the previous problem) to prove that there exists a way to properly color the vertices in Y.

5. Apply the Lovász Local Lemma. Each bad events should be defined by an edge *and* a color.

6. Note that we may assume that $\mathcal{A}$ and $\mathcal{B}$ are *maximal* with respect to the given property (i.e. we cannot add a set to either collection without violating the condition). Once we make this assumption, there is a very natural partition of $2^{[n]}$. Use Kleitman to relate the sizes of the parts in this partition to $|\mathcal{A}|$ and $|\mathcal{B}|$.