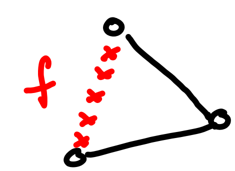
$\triangle$-free process

$G(0)$ = empty graph on $[n]$.

$G(i+1) = G(i) + e_{i+1}$ ← random edge

f is closed

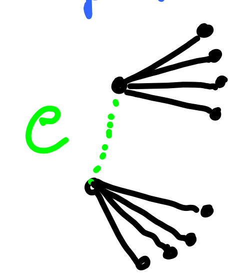


$E(i)$ = edge set of $G(i)$

$$C(i) = \left\{ f \in \binom{[n]}{2} \setminus E(i) : \triangle \subseteq G(i) + f \right\}$$

"closed pairs"

e is open



$$O(i) = \binom{[n]}{2} \setminus \left(E(i) \cup C(i) \right)$$

"open pairs"

e_{i+1} is chosen uniformly at random from $O(i)$.

Set M = # of steps in the process

note: M is a random variable.

Questions:

1. what is the likely value of M.
2. what is the likely independence # of $G(M)$?

Heuristic: $G(i)$ resembles $G_{n,i}$

Notation: $G_{n,m}$ has n vertices, m uniform random edges

$$\Pr(xy \in O(i)) \approx \Big(1 - \underbrace{\Big(\frac{2i}{n^2}\Big)}_{\text{edge probability}}{}^{2}\Big)^{n} \approx \exp\{-4i^2/n^3\}$$

$$E[|O(i)|] \approx \frac{n^2}{2} \exp\{-4i^2/n^3\}$$

So, the process should end when $i^2/n^3 = \Theta(\log n)$, which is when

$$i = \Theta\left(n^{3/2}\sqrt{\log n}\right).$$

Thm (B, 2009) With high probability

$$M = \Theta\left(n^{3/2}\sqrt{\log n}\right) \quad \text{and} \quad \alpha(G_M) = O\left(n^{1/2}\sqrt{\log n}\right)$$

conjectured by Spencer

note that proving this gives the Corollary as well as the upper bound on M (since each neighborhood is an independent set).

Corollary $R(3,t) = \Theta\left(t^2/\log t\right)$

<u>note:</u> (i) upper bound on $R(3,t)$ due to Ajtai, Komlós, Szemerédi: (improved by Shearer).

(ii) lower bound previously established by Jeong Han Kim 1995. This was an application of the Rödl nibble. (There are some similarities.)

(iii) See Also Erdős, Suen, Winkler and Bollobás, Riordan and Osthus, Taraz

Random variables

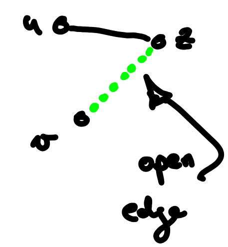
$Q(i) = \#$ of open edges in $G(i)$

This is the variable we care about most as $Q(M) = 0$

$$E[Q(i+1) - Q(i) \mid \mathcal{F}_i]$$
$$= \sum_{uv \in Q(i)} \frac{1}{|Q(i)|} \left[\left(\# \text{ of edges closed we add } uv = e_{i+1} \right) + 1 \right]$$

Say z is partial w.r.t. uv at step i if

- 1 of uz, vz is in $E(i)$
- 1 of uv, vz is in $O(i)$



z is partial w.r.t. uv.

$$Y_{uv}(i) = \left| \{ z : z \text{ is partial w.r.t } uv \text{ at step } i \} \right|$$

note: $E[Q(i+1) - Q(i) \mid \mathcal{F}_i] = \sum_{uv \in Q(i)} \frac{Y_{uv}(i)}{Q(i)}$

abuse of notation: Q, Y etc are sets and numbers

$$E\left[Y_{uv}(i+1) - Y_{uv}(i) \mid \mathcal{F}_i\right]$$

$$= -\sum_{z \in Y_{uv}} \frac{1}{Q}\left(Y_{zz^*}(i) + 1\right) + ?$$

closing zz^*

choosing zz^*

we can get new partial vertices

notational convention:

If z is partial w.r.t. uv then let $z^* \in \{u,v\}$ s.t. zz^* is open.

Say z is open w.r.t. uv in $G(i)$ if uz and vz are open

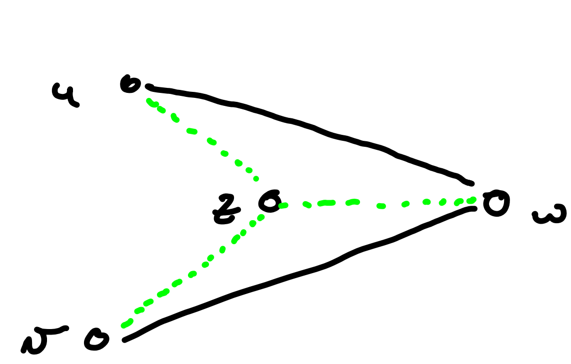
$$X_{uv}(i) = \left|\{z : z \text{ is open w.r.t. to } uv \text{ in } G(i)\}\right|$$

then $? = \dfrac{2X_{u,v}(i)}{Q}$

$$E\left[X_{u,v}(i+1) - X_{u,v}(i) \mid \mathcal{F}_i\right] = -\sum_{z \in X_{uv}} \frac{Y_{zu}(i) + Y_{zv}(i) + 1}{Q}$$

is this exactly correct?

what about this situation:



we "count" zw twice, but when we choose it only z leaves $X_{u,v}$.

<u>observations:</u> (i) uv is already closed. Do we need to know $Y_{uv}(i)$ for uv closed? for uv an edge?

(ii) These vertices w should be rare.

Say z is closed w.r.t. uv in $G(i)$ if uz and vz are edges.

$$W_{uv}(i) = |\{z : z \text{ is closed w.r.t. to } uv \text{ in } G(i)\}|$$

we will not track $W_{uv}(i)$. We will only <u>bound</u> it.

Trajectories

we set $t = i/n^{3/2}$. (this should seem natural, given the discussion above)

we would like to show

$$\left.\begin{aligned} Q(i) &\approx q(t)n^2 \\ Y_{uv}(i) &\approx y(t)n^{1/2} \\ X_{uv}(i) &\approx x(t)n \end{aligned}\right\}$$

note that these seem sensible. If we have a random graph with about $n^{3/2}$ edges and we add $\varepsilon n^{3/2}$ random edges then Q changes by about $\varepsilon n^{3/2} \cdot n^{1/2}$ etc.

We use the heuristic to "guess" values of q, y, x.

$$\Pr(uv \in O(i)) \approx \left(1 - \left(\frac{2i}{n^2}\right)^2\right)^n = \exp\left\{-4i^2/n^3\right\}.$$

Set $q(t) = \frac{1}{2}\exp\{-4t^2\}$

$$\Pr(uv \in E(i)) \approx \frac{2i}{n^2} = 2t\,n^{-1/2}$$

Set
$$y(t) = 4t\exp\{-4t^2\}$$
$$x(t) = \exp\{-8t^2\}$$

We now confirm that these make sense with respect to the expected changes.

$\vdots$

Q:

$$E[Q(i+1) - Q(i) \mid \mathcal{F}_i] = -\sum_{uv \in Q(i)} \frac{Y_{u,v}(i)+1}{Q(i)} \approx -y(t)\,n^{1/2}$$

So we should have

$$\boxed{q' = -y\,,\quad q(0) = 1/2}$$

note that the 1-step change in $q(t)n^2$ is about $q'(t)n^{1/2}$

$\underline{Y}$:

$$E\left[Y_{uv}(i+1) - Y_{uv}(i) \mid \mathcal{F}_i\right] = -\sum_{z \in Y_{uv}} \frac{Y_{z,z^*}(i)+1}{Q(i)} + \sum_{z \in X_{u,v}} \frac{2}{Q(i)}$$

$$\approx -\frac{\left(y(t)n^{1/2}\right)^2}{q(t)n^2} + \frac{2x(t)n}{q(t)n^2}$$

$$= -\frac{y^2(t)}{q(t)} \cdot \frac{1}{n} + \frac{2x(t)}{q(t)} \cdot \frac{1}{n}$$

So we should have

$$\boxed{y' = -y^2/q + 2x/q, \quad y(0) = 0}$$

$\underline{X}$:

$$E\left[X_{uv}(i+1) - X_{uv}(i) \,\middle|\, \mathcal{F}_i\right] \approx -\sum_{z \in X_{uv}(i)} \frac{Y_{zu}(i) + Y_{zv}(i) + 2}{Q(i)}$$

$$\approx -\frac{2x(t)y(t)\,n^{3/2}}{q(t)\,n^2}$$

So, we are interested in

$$\boxed{x' = \frac{-2xy}{q}, \quad x(0) = 1}$$

The system

$$\boxed{\begin{aligned} q(t) &= e^{-4t^2/2} \\ y(t) &= 4t\,e^{-4t^2} \\ x(t) &= e^{-8t^2} \end{aligned}}$$

satisfies all of the desired conditions.

Claim 1: $\exists$ a constant $\mu > 0$ s.t. w.h.p. we have

$$\left.\begin{aligned} Q(i) &= q(t)n^2 \pm f(t)n^{\alpha} & & \\ Y_{uv}(i) &= y(t)n^{1/2} \pm g(t)n^{\beta} & & \forall uv \notin E(i) \\ X_{uv}(i) &= x(t)n \pm h(t)n^{\rho} & & \forall uv \notin E(i) \\ W_{uv}(i) &< \log^2 n & & \forall uv \end{aligned}\right\} \text{ where } t = i/n^{3/2}$$

for $i = 1, \dots, \mu n^{3/2}\sqrt{\log n}$

as usual, $\alpha, \beta, \rho, f, g, h$ TBA

Proof:

Let the stopping time T be first step at which any of these are violated.

For each $uv \in \binom{[n]}{2}$ let $T_{uv} = T \wedge \{i : e_i = uv\}$

We stop the uv-variables at T_{uv}.

__note:__ this Claim says $G(i)$ indeed resembles $G_{n,i}$.

We treat the four statements in turn.

$\boxed{Q(i)}$ This has no appeal to Azuma-Hoeffding or probability theory.

If $i < \mu n^{3/2}\sqrt{\log n}$, T then

$$Q(i) = \binom{n}{2} - i - \sum_{j=1}^{i} Y_{e_j}(j-1)$$

$$= \frac{n^2}{2} - \sum_{j=0}^{i-1} y\left(j/n^{3/2}\right) n^{1/2} \pm g(t)n^{\beta} \pm O\left(n^{3/2}\sqrt{\log n}\right)$$

See $\boxed{\text{Ex}}$ below

$$= \frac{n^2}{2} - n^2\int_0^t y(s)\,ds \pm n^{\beta+3/2}\int_0^t g(s)\,ds \pm O\left(n^{3/2}\sqrt{\log n}\right)$$

this requires that $g(t) << n^{1-\beta}$

$$= n^2\left[\frac{1}{2} - \int_0^t 4te^{-4t^2}\,dt\right] \pm n^{\beta+3/2}\int_0^t g(s)\,ds \pm O\left(n^{3/2}\sqrt{\log n}\right)$$

$$= n^2 g(t) \pm n^{\beta+3/2} \int_0^t g(s)\,ds \pm O(n)$$

we need

$$\alpha \geq \beta + 3/2$$

if $\alpha = \beta + 3/2 \rightsquigarrow$ $f(t) > \int_0^t g(s)\,ds$

$\boxed{\text{EX}}$

$$\left| \int_{j/n^{3/2}}^{j/n^{3/2} + 1/n^{3/2}} y(s)\,ds - \frac{y(j/n^{3/2})}{n^{3/2}} \right| \leq \int_{j/n^{3/2}}^{j/n^{3/2} + 1/n^{3/2}} \frac{|y'(s)|}{n^{3/2}}\,ds$$

why?

$$= \left| \int_{j/n^{3/2}}^{j/n^{3/2} + 1/n^{3/2}} \underbrace{y(s) - y(j/n^{3/2})}_{\int_{j/n^{3/2}}^{s} y'(r)\,dr}\,ds \right|$$

so $\sum y(j/n^{3/2})$

$$= n^{3/2} \int y \pm \int_0^t |y'(s)|\,ds$$

$\boxed{X_{uv}(i)}$ We begin with the upper bound. Define

$$Z(i) = X_{uv}(i) - x(t)n - h(t)n^{\rho}$$

convention: if $e_i = uv$ then $Z(i) = Z(i-1)$

The supermartingale condition

$$E\left[Z(i+1) - Z(i) \mid \mathcal{F}_i\right]$$

assumes $h'' n^{\rho-1} = O(1)$ and uses the bound on $W(i)$

$$\leq -\sum_{z \in X_{uv}} \frac{Y_{uz} + Y_{vz}}{Q} - x'(t)n^{-1/2} - h'(t)n^{\rho-3/2} + O(n^{-1} \cdot \log^2 n)$$

$$\leq -2\frac{\left(x(t)n + h(t)n^{\rho}\right)\left(y(t)n^{1/2} + g(t)n^{\beta}\right)}{q(t)n^2 - f(t)n^{\alpha}} - x'(t)n^{-1/2} - h'(t)n^{\rho-3/2} + O(n^{-1} \cdot \log^2 n)$$

$$\leq -\frac{2xy}{q}n^{-1/2} + \frac{2hy}{q}n^{\rho-3/2} + \frac{2gx}{q}n^{\beta-1} + \frac{2hg}{q}n^{\rho+\beta-2} + \frac{4xyf}{q^2}n^{\alpha-5/2}$$

$$\left|\frac{A}{B\pm\varepsilon} - \frac{A}{B}\right| = \left|\frac{\mp\varepsilon A}{(B\pm\varepsilon)B}\right| < \frac{\frac{3}{2}\varepsilon A}{B^2}$$

assuming $\varepsilon < B/3$

(we assume that no error term exceeds $5/6$ of the main term

The orange terms cancel exactly.

$$-\underline{x'(t)n^{-1/2}} - h'(t)n^{\rho-3/2} + O(n^{-1}\cdot\log^2 n)$$

we need $$h' > \frac{2hy}{\xi} + \frac{2gx}{\xi}n^{\frac{1}{2}-\rho+\beta} + \frac{4xyf}{\xi^2}n^{-1-\rho+\alpha}$$

$$\boxed{\text{we set} \quad \rho = \alpha - 1 \quad \beta = \alpha - 3/2}$$

with foresight

Then we need $h' > \underline{\underline{16th}} + 8\xi g + 64t\xi f$

this will require $h > e^{8t^2}$

Since we stop at $t = \mu\sqrt{\log n}$ (μ small) such an h would be fine.

We apply the variation on Azuma-Hoeffding from the homework. Note

$$Z(i+1) \geq Z(i) - O(\sqrt{n})$$

$$Z(i+1) \leq Z(i) + O(1/\sqrt{n})$$

Each edge we close effects at most 1 vertex in $X_{u,v}(i)$.

$\therefore X_{u,v}(i+1) \geq X_{u,v}(i) - 1 - Y_{e_{i+1}}(i)$

we achieve this when $X_{u,v}(i+1) = X_{u,v}(i)$

Setting $N = O(\sqrt{n})$

$$\eta = O(1/\sqrt{n})$$

note: we have the condition $n^p < \eta \cdot (n^{3/2}\sqrt{\log n})$ That is needed for the Lemma.

we see that

$$\Pr\left(T < \mu n^{3/2}\sqrt{\log n} \text{ and } Z_{u,v}(T) \geq 0\right)$$

$$\leq \exp\left\{-\Omega\left(\frac{n^2 p}{n^{3/2}\sqrt{\log n}}\right)\right\}$$

note: we can multiply this by $\binom{n}{2}$ and it is still good.

We need $p > 3/4$

$\boxed{Y_{u,v}(i)}$ Lower bound.

We cannot do the usual thing here. If we consider

$$Z_{u,v}(i) := Y_{u,v}(i) - y(t)n^{1/2} - g(t)n^{\beta}$$

Then we could have $Z(i+1) > Z(i) + 1/2$ or $Z(i+1) < Z(i) - 1/2$.

So we would take $\eta, N > \Omega(1)$. The smallest variation we could keep would be about $n^{3/4}$. This is too big.

Idea: Both additions to and removals from $Y_{u,v}$ are rare. we should treat them seperately.

Note that we can write $y(t) = -a(t) + b(t)$ where

$$a(t) = \int_0^t \frac{y^2(s)}{q(s)}\,ds \qquad b(t) = \int_0^t \frac{2x(s)}{q(s)}\,ds$$

We track removals from $Y_{u,v}$ relative to $a(t)$ and additions to $Y_{u,v}$ relative to $b(t)$.

upper bound – removals

Set $A_{u,v}(i)$ be the # of vertices removed from $Y_{u,v}$ through i steps.

Define

$$Z(i) = A_{u,v}(i) - a(t)n^{1/2} - \frac{g(t)n^{\beta}}{2}$$

we split the allowed error evenly between additions and removals.

$\boxed{\begin{array}{l}\underline{\text{convention:}}\ \text{if } e_i = uv \\ \text{then } Z(i) = Z(i-1)\end{array}}$

supermartingale

$$E\left[Z(i+1) - Z(i) \mid \mathcal{F}_i\right]$$

$$\leq \sum_{z \in Y_{uv}(i)} \frac{1 + Y_{z,z^*}(i) - 1}{Q(i)} - a'(t)n^{-1} - \frac{g'(t)}{2} n^{\beta - 3/2} + O(n^{-2})$$

The 1 is for choosing zz^*

The -1 is because we don't change Z if $e_{i+1} = uv$ (note that $uv \in Y_{zz^*}$)

$$\leq \frac{\left(y(t)n^{1/2} + g(t)n^{\beta}\right)^2}{\xi(t)n^2 - f(t)n^{\alpha}} - \frac{y^2(t)}{\xi(t)} n^{-1} - \frac{g'(t)}{2} n^{\beta - 3/2} + O\left(n^{-3/2}\right)$$

cancels the main term exactly

$$\leq \underline{\frac{2gy}{\xi} n^{\beta - 3/2}} + \underline{\frac{2fy^2}{\xi^2} n^{\alpha - 3}} + \frac{g^2}{\xi} n^{2\beta - 2} \underline{- \frac{g'(t)}{2} n^{\beta - 3/2}} + O\left(n^{-3/2}\right)$$

These are the main terms.

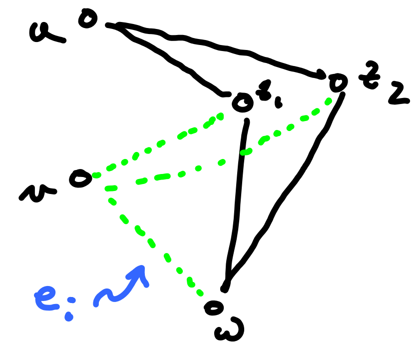
Since we already set $\underline{\alpha = \beta + 3/2}$ we need

$$g' > \underline{16t}\, g + 256 t^2 f$$

again we see that we are getting error $= e^{\Theta(t^2)}$

Now we apply the modified Azuma Hoeffding with

$$N = O(\log^2 n)$$
$$\eta = O(1/n)$$



All the z_i's are in $W_{u,w}$

we see that

$$\Pr\left(T < \mu n^{3/2}\sqrt{\log n} \wedge A_{uv}(T) > 0\right)$$

$$\leq \exp\left\{-\Omega\left(\frac{n^{2\beta}}{n^{1/2}\log^{5/2} n}\right)\right\}$$

note:
$n^{\beta} < n^{3/2}\sqrt{\log n} \cdot \eta$

So, we need $\beta > 1/4$

note: if we "allowed" uv to be added in this calculation then we could get $n^{1/2}$ things removed from Y_{uv} in 1 step.

lower bound – additions

Set $B_{u,v}(i)$ be the # of vertices added to $Y_{u,v}$ through i steps.

Define

$$Z(i) = B_{u,v}(i) - b(t)n^{1/2} + \frac{g(t)n^{\beta}}{2}$$

submartingale

$$E\left[Z(i+1) - Z(i) \mid \mathcal{F}_i\right]$$

$$\geq \sum_{z \in X_{u,v}} \frac{2}{Q(i)} - b'(t)n^{-1} + \frac{g'(t)n^{\beta - 3/2}}{2} + O(n^{-2})$$

cancels the main term exactly

$$\geq \frac{2\left(x(t)n - h(t)n^{\beta}\right)}{g(t)n^2 + f(t)n^{\alpha}} - \frac{2x(t)}{g(t)}n^{-1} + \frac{g'(t)n^{\beta-3/2}}{2} + O(n^{-2})$$

$$\geq -\frac{2h}{\rho}n^{\rho-2} - \frac{4f\alpha}{\rho^2}n^{\alpha-3} + \frac{g'}{2}n^{\beta-3/2} + O(n^{-2})$$

Since we are setting $\underline{\rho = \beta + 1/2}$, $\underline{\alpha = \beta + 3/2}$

we need.

$$g' > \frac{8h}{\rho} + 32f$$

Now we apply the modified Azuma-Hoeffding with

$N = O(1)$ and $\eta = O(1/n)$

we see that

$$\Pr\left(T < \mu n^{3/2}\sqrt{\log n} \text{ and } B_{u,v}(T) < 0\right)$$

$$\leq \exp\left\{-\Omega\left(\frac{n^{2\beta}}{n^{1/2}\sqrt{\log n}}\right)\right\}$$

So, we need $\beta > 1/4$

again, $\eta \cdot n^{3/2}\sqrt{\log n} > n^{\beta}$ and we can multiply by $\binom{n}{2}$ with no problem

reconciliation

variable	condition
Q:	$f' > g$
X:	$h' > 16th + 8\frac{g}{f}g + 64t\frac{g}{f}f$
Y	$\begin{cases} g' > 16tg + 256t^2 f \\ g' > \frac{8h}{g} + 32f \end{cases}$

first guess $\begin{cases} f = e^{At^2} \\ g = e^{Bt^2} \\ h = e^{Ct^2} \end{cases}$

looking at the exponents:

$\left.\begin{array}{l} A \geq B \\ C \geq B-4, A-4 \\ B \geq A, C+4 \end{array}\right\}$ we should take

$B = A, \; C = A-4$

second guess
$$\begin{cases} f = u(t)e^{At^2} \\ g = v(t)e^{At^2} \\ h = w(t)e^{(A-4)t^2} \end{cases}$$

we will choose A large

<u>then we need</u>

$$2Atu + u' > v$$

$$2(A-4)tw + w' > \underline{16tw} + 4v + 32tu$$

$$2Atv + v' > \underline{16tv} + 256t^2u$$

$$2Atv + v' > 16w + 16u$$

<u>note:</u> (i) we can ignore the underlined terms by taking A large.

(ii) when t is large we will likely want $v \geq tu$ and $w \geq u$

third guess
$$\begin{cases} f = u(t)e^{At^2} \\ g = u(t)\cdot(t+1)\,e^{At^2} \\ h = u(t)e^{(A-4)t^2} \end{cases}$$

$$2Atu + u' > (t+1)u$$

$$2(A-4)tu + u' > \underline{16tu} + 4(t+1)u + \underline{32tu}$$

$$2At(t+1)u + u + (t+1)u' > \underline{16t(t+1)u} + \underline{256t^2u}$$

$$2At(t+1)u + u + (t+1)u' > 16u + 16u$$

<u>note:</u> we can ignore these terms by taking A large

It seems that we are left with

$$u' > u$$

$$u' > 4u \qquad \text{So we take } u = e^{32t}$$

$$u' > 31u$$

The following error functions suffice.

$$f = e^{At^2 + 32t}$$

$$g = (t+1)e^{At^2 + 32t}$$

$$h = e^{(A-4)t^2 + 32t}$$

$\boxed{W_{u,v}(i)}$ We apply the first moment method. If $i<T$

$$\Pr\left(W_{u,v}(i+1) = W_{u,v}(i)+1\right) = \frac{Y_{u,v}(i)}{Q(i)} < 2\,\frac{4t\cdot e^{-4t^2}\cdot n^{1/2}}{\left(e^{-4t^2}/2\right)\cdot n^2} = 16t\,n^{-3/2}$$

So long as t is not very close to 0.

$$\Pr\left(T<\mu n^{3/2}\sqrt{\log n} \text{ and } \exists\, u,v \text{ s.t. } W_{u,v}(T)>\log^2 n\right)$$

$$< \binom{n}{2}\binom{\mu n^{3/2}\sqrt{\log n}}{\log^2 n}\left(16\mu\sqrt{\log n}\; n^{-3/2}\right)^{\log^2 n}$$

$$\le \frac{n^2}{2}\left[\frac{\mu n^{3/2}\sqrt{\log n}\cdot e}{\log^2 n}\cdot\frac{16\mu\sqrt{\log n}}{n^{3/2}}\right]^{\log^2 n}$$

$\boxed{\text{Ex}}$ this estimate holds even for t near 0

$\square$ Claim 1 is proved.

Note: Claim 1 establishes the lower bound of the main theorem, as it shows that $Q(i)>0$ for $i=\mu n^{3/2}\sqrt{\log n}$.