The giant component via dynamic concentration

Def: $C_1 = C_1(G)$ is the largest component in graph G.

Theorem (Erdös, Renyi 1960)

Let $p = c/n$ where c is a constant

(i) If $c < 1$ then $|C_1(G_{n,p})| = O(\lg n)$

(ii) If $c > 1$ then $|C_1(G_{n,p})| = \Omega(n)$

with high probability

Random Graph Process

Let $u_1, v_1, u_2, v_2, u_3, v_3, \ldots$ be chosen independently and u.a.r. from $[n]$.

Set $e_i = \{u_i, v_i\}$ for $i = 1, 2, \ldots$

$G(i)$ = graph on $[n]$ with edge set $\{e_1, \ldots, e_i\}$.

Suppose $G(i)$ has connected components $C_1, C_2, C_3, \ldots$

note:

1. Loops and multiple edges can occur.
2. $G(i)$ is very similar to $G_{n,p}$ with $i = n^2p/2$

$X(i) = \sum |C_k|^2$ is the <u>susceptibility</u> of $G(i)$

$$E\left[X(i+1) - X(i) \mid G(i)\right] = \sum_{k \neq j} \frac{|C_k|}{n} \frac{|C_j|}{n} 2 |C_k| |C_j|$$

$$= \frac{2}{n^2}\left(\sum_k |C_k|^2\right)^2 - \underbrace{\frac{2}{n^2}\sum_k |C_k|^4}_{\text{should be small, call this } (*)}$$

$$= \frac{2}{n^2} X^2$$

So, we should have

$$X(i) \simeq n\,x(i/n) \quad \text{where} \quad \underbrace{x(0) = 1 \qquad \frac{dx}{dt} = 2x^2}_{\Rightarrow\; x(t) = \frac{1}{1-2t}}$$

Theorem (B-Kravitz Spencer-Wormald).

Let $\varepsilon > 0$.

With high probability

$$X(i) = \frac{n}{1-2t}\left(1 + o(1)\right),$$

uniformly, for $i = 0, 1, \dots, \left(\tfrac{1}{2} - \varepsilon\right)n$.

(where $t = i/n$).

first try at a proof (upper bound)

Set $Z(i) = Z^+(i) = X(i) - n\,x(t) - e(t)$

a slowly growing function, will also depend on n

note: $Z(0) = -e(0)$

$t = i/n$ this is our time-scaling

Outline:

1. Show $Z(i)$ is a supermartingale.
2. Apply Azuma-Hoeffding to conclude

$$\Pr(\exists\, i \text{ s.t. } Z(i) > 0) = o(1)$$

Convention: Let the stopping time T be the smallest index i s.t.

$$X(i) \in n\,x(t) \pm e(t) \qquad (1)$$

fails to hold. Set $i \wedge T = \min\{i, T\}$. If $Z(i)$ is a supermartingale then $Z(i \wedge T)$ is also a supermartingale.

We work with $Z(i \wedge T)$, and therefore we can assume that (1) holds.

<u>Convention:</u> Any condition that must be satisfied by a choice of a parameter made later is stated in red.

$\boxed{\text{Step 1.}}$

<u>Notation:</u> $\Delta Z = Z(i+1) - Z(i)$

$$E[\Delta Z] \leq \underbrace{\frac{2X^2(i)}{n^2}}_{\text{contribution from } X(i)} - 2x^2(t) - \frac{e'(t)}{n}$$

here we use $X^4 > 0$ and $e^4 > 0$.

$$\le \frac{2}{n^2}\left(nx(t)+e(t)\right)^2 - 2x^2(t) - \frac{e'(t)}{n} \quad \text{by (1)}$$

$$\le \frac{4x(t)e(t)}{n} + \frac{e^2(t)}{n^2} - \frac{e'(t)}{n}$$

to get the supermartingale condition it suffices to choose $e(t)$ so that

$$e' > \frac{4e}{1-2t} \quad \text{and} \quad e = o(nx)$$

We set $e = \dfrac{S_e}{(1-2t)^3}$

S_e, the scaling of the error term, will be a function of n

$\boxed{\text{Step 2}}$

To apply Hoeffding-Azuma we need a bound on $|\Delta z|$

e.g. if we assume $|c_k| < B$ (i.e. $B = \log^2 n$ seems sensible)

then $\Delta z = O(B^2)$ and

$$\Pr(z_i - z_0 > A) \leq \exp\left\{\frac{-A^2}{3 \cdot i \cdot B^4}\right\}.$$

Thus if we take

$$e(0) = S_e = \omega\left(B^2 \sqrt{n} \log n\right)$$

this $\log n$ can be any $\omega(1)$ function.

then

$$\Pr(z_i \geq 0) = \Pr(z_i - z_0 > \log n \, B\sqrt{n})$$

$$< \exp\{-\Omega(\log^2 n)\}$$

So (i) we need a bound $|C_k|$ that holds throughout the process.

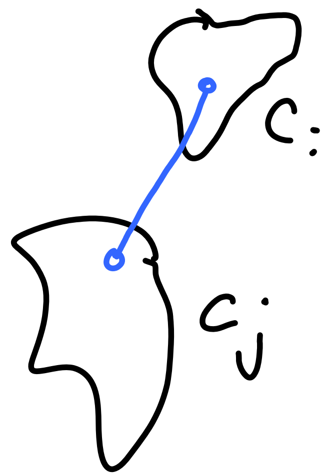
(ii) we also need to set-up a submartingale for the lower bound. And here we need the bound on $|C_k|$ to deal with (*).

Idea: ignore edges that touch components on n^α or more vertices.

If we do this then

(i) $\Delta X_i \le O(n^{2\alpha})$ and so long as $\alpha < 1/4$ the application of Azuma-Hoeffding works.

(ii) There will be an impact on $E[\Delta X]$.



$$\sum_{C_i \text{ bad},\, C_j} \frac{|C_i|}{n} \frac{|C_j|}{n} 2|C_i||C_j|$$

recall: $X = \sum |C_e|^2$

$$= \sum_{C_i \text{ bad}} \frac{|C_i|^2}{n} \cdot \frac{2 \sum_{C_j} |C_j|^2}{n}$$

if $|C_i|$ or $|C_j|$ too large add this as a blue edge.

$$\le \frac{1}{n} \cdot (\#\text{large}) (2n^{\alpha})^2 \cdot \underbrace{\frac{2X}{n}}_{\text{constant}}$$

In order to maintain the supermartingale condition on $Z(i)$ we need

$\#$ large components $= o\left(S_e \, n^{-2\alpha}\right)$

__Next idea:__ introduce another random variable to keep control on the # of large components.

$$Y(i) = \sum_k |C_k|^3$$

__note:__ (i) $Y = O(n) \Rightarrow$ # large components $\leq n^{1-3\alpha}$

Setting $S_e = n^{1-\alpha/2}$ for a sufficiently small will satisfy the conditions above.

(ii) $Y = O(n) \Rightarrow (*) \leq n^{\alpha-1}$. We need this for the lower bound on $X(i)$.

It remains to show that $Y(i)$ is __at most__ linear for $i \leq (\frac{1}{2} - \varepsilon)n$.

$$E[\Delta Y] = \sum_{i,j} \frac{|C_i|}{n} \frac{|C_j|}{n} \left(3|C_i|^2 |C_j| + 3|C_i||C_j|^2 \right)$$

$$\leq \frac{6}{n^2} \left[\sum_i |C_i|^2 \right] \left[\sum_j |C_j|^3 \right]$$

$$= \frac{6XY}{n^2}$$

<u>If</u> we want to show

$Y(i) \lesssim n\,y(i/n)$

we take the solution of

$$y' = 6xy \qquad y(0) = 1$$

so

$$y = \frac{1}{(1-2t)^3}$$

<u>note:</u> as we need only an upper bound we can ignore

(i) the diagonal term

(ii) the edges we fail to add.

The application of Azuma-Hoeffding is very similar to the one above.

$\blacksquare$