21-484 Graph Theory
Assignment # 7
Due: Friday, April 11

1. Let G be a 2-connected planar graph with n vertices and degree sequence $(d_i)_{i=1}^n$. Show that
$$\sum_{i:d_i\le 6}(6-d_i) \ge \sum_{i=1}^{n}(6-d_i) \ge 12.$$
Deduce that if $\delta(G) \ge 5$ then G has at least 12 vertices of degree 5, and if $\delta(G) \ge 4$ tehn G has at least 6 vertices of degree at most 5.

2. A graph is **outerplanar** if it can be embedded in the plane so that there is face F with the property that every vertex of the graph is on the boundary of F.
   (a) Use Euler's formula to prove that K_4 and $K_{2,3}$ are not outerplanar.
   (b) Prove that a graph G is outerplanar if and only if G contains neither K_4 nor $K_{2,3}$ as a minor.

3. Show that a 2-connected plane graph G is bipartite if and only if the boundary of every face of G is an even cycle.

Let $G = (V, E)$ be a 2-connected plane graph, and assume that the intersection of the boundaries of any pair of faces contains at most one edge. Let $\mathcal{F}$ be set of faces G. We define the dual of G to be the graph G^* which has vertex set $\mathcal{F}$ and an edge joining $F_1, F_2 \in \mathcal{F}$ if and only if the boundaries of F_1 and F_2 meet in an edge. For each edge $e \in E$ let e^* be the edge in G^* that joins the two faces that have e on their boundary. Note that the map from E to $E(G^*)$ that takes e to e^* is a natural bijection between E and $E(G^*)$.

4. Let G be a 2-connected plane graph, and assume that the intersection of the boundaries of any pair of faces contains at most one edge. Show that $\chi(G^*) = 2$ if and only if the degree of every vertex in G is even.

   Hint: Recall the definition of an Eulerian graph, and find a decomposition of the G into edge-disjoint cycles.

5. Let $G = (V, E)$ be a 2-connected plane graph, and assume that the intersection of the boundaries of any pair of faces contains at most one edge. Let T be a spanning tree of G. Show that the graph on vertex set $\mathcal{F}$ with edge set
$$\{e^* : e \in E \setminus E(T)\}$$
is a spanning tree of G^*.