21-484 Graph Theory
Assignment # 6 Due: Friday, March 28

1. Given a graph G and $k \in \mathbb{N}$ let $P_G(k)$ be the number of k-colorings of G. Show that P_G is a polynomial in k of degree $n = |V(G)|$. Show that the coefficent of k^n is 1 and the coefficient of k^{n-1} is $-|E(G)|$. The polynomial P_G is known as the *chromatic polynomial* of G.

   Hint. Go by induction on $|E(G)|$, appealing to $G - e$ and G/e.

2. Consider the chromatic polynomial P_G, which is defined in the previous problem. Determine the class of graphs for which $P_G(k) = k(k-1)^{n-1}$.

3. Consider the odd cycle C_{2k+1}. We assign each vertex a list C_v of allowable colors such that $|C_v| = 2$. Show that if there are vertices u, v such that $C_u \neq C_v$ then there is a proper coloring f of the vertices of C_{2k+1} such that $f(v) \in C_v$ for all v.

4. A multi-graph is a graph in which we allow multiple edges between any pair of vertices.

   (a) Let d be even. Give an example of a d-regular multigraph G such that
   $$\chi'(G) = \frac{3d}{2}$$

   (b) Prove that if G is a multigraph then we have
   $$\chi'(G) \leq \frac{3\Delta(G)}{2}.$$

   Hint: Go by induction on $|E(G)|$ and consider the colors that appear at x, y and a carefully chosen neighbor z of x in an edge-coloring of $G - xy$.

Let $G = (V, E)$ be a graph. We define a **fractional vertex packing** of a graph G to be a map $f : V \to \mathbb{R}^+$ such that
$$\sum_{x \in X} f(x) \leq 1$$
for all cliques X in G. The **fractional vertex packing number** of G is
$$\alpha^*(G) = \max_f \sum_{v \in V} f(v),$$
where the maximum is taken over all fractional vertex packings f.

5. (a) Show that for any graph G we have $\alpha(G) \leq \alpha^*(G)$.

   (b) Show $\alpha^*(C_{2k+1}) = \frac{2k+1}{2}$ and $\alpha^*(\overline{C}_{2k+1}) = \frac{2k+1}{k}$ for $k \geq 2$.

6. Let G_1 and G_2 be graphs. The strong (or Shannon) product of G_1 and G_2 is the graph $G_1 \times G_2$ with vertex set $V(G_1) \times V(G_2)$ and an edge joining distinct vertices (x_1, x_2) and (y_1, y_2) if and only if $x_i = y_i$ or $x_i y_i \in E(G_i)$ for $i = 1, 2$.

   (a) Prove $\alpha^*(G \times H) = \alpha^*(G)\alpha^*(H)$.

   (b) Prove $c(G) \leq \alpha^*(G)$, where $c(G)$ is the Shannon capacity of G.