21-484 Graph Theory
Review sheet for the final exam

Note. The exam is cumulative. So all material covered on the previous review sheets can be tested on the final. However, the material on this review sheet will be over-weighted on the final as this material has not yet been tested.

Definitions. The test will assume that you know the following definitions as well as the definitions that appear on the previous review sheets.

- density $d(X, Y)$ of pair of sets of vertices (where the sets X, Y are either equal are disjoint).
- ϵ-regular pair.
- ϵ-regular partition.

Theorems. The test assumes knowledge of the following theorems

- **Szemerédi Regularity Lemma.** For every $\epsilon > 0$ there exists an integer M such that every graph G has an ϵ-regular partition with at most M parts.
- **Triangle Counting Lemma.** Let $G = (V, E)$ be a graph such that $V = X \dot\cup Y \dot\cup Z$ and
  - (X, Y), (X, Z) and (Y, Z) are ϵ-regular, and
  - $d(X, Y) = \alpha$, $d(X, Y) = \beta$, and $d(X, Y) = \gamma$.

  Let T be the number of copies of K_3 in G that have exactly on vertex in each of X, Y and Z. If $\alpha, \beta, \gamma > 2\epsilon$ then
  $$|T| \geq (1 - 2\epsilon)(\alpha - \epsilon)(\beta - \epsilon)(\gamma - \epsilon)|X||Y||Z|.$$
- **Triangle Removal Lemma.** For every $\epsilon > 0$ there exists a $\delta > 0$ such that any graph on n vertices that has at most δn^3 triangles can be made triangle free by the removal of at most ϵn^2 edges.

Note. You will be asked to prove a Theorem (or Theorems) proved in class on the final exam. Possibilities include the following:

- the Handle Theorem (regarding 2-connected graphs)
- Hall's Theorem
- the fact that a graph G has an induced subgraph H such that $\delta(H) \geq \chi(G) - 1$.
- Euler's Formula
- Turán's Theorem
- Kővari, Sós, Turán
- the triangle counting lemma (in the context of the ϵ-regular partitions)

Review Problems. Doing these problems should help in preparation for the final exam.

1. Show that an ϵ-regular partition of a graph G is also an epsilon regular partition for $\overline{G}$.
2. Show that for all $\epsilon_1 > \epsilon_0 > 0$ there is an $\eta > 0$ such that for any sets of vertices X_0, X_1, Y_0, Y_1 such that
   - $X_0 \subset X_1$ and $Y_0 \subset Y_1$,
   - Y_1 and X_1 are disjoint
   - (X_0, Y_0) is ϵ_0-regular
   - $|X_1| < (1 + \eta)|X_0|$ and $|Y_1| < (1 + \eta)|Y_0|$

   the pair (X_1, Y_1) is ϵ_1-regular.
3. Prove that for each $n \geq 1$ the number of graphs on vertex set $\{1, 2, \dots, n\}$ will all degrees even is $2^{\binom{n-1}{2}}$.
4. Let G be a connected graph with n vertices. Prove that G contains a path of length at least $\min\{n - 1, 2\delta(G)\}$.
5. Let G be a bipartite graph with bipartition $V(G) = A\dot{\cup}B$ and let X be the set of vertices of G that have maximum degree. Prove that there is a matching M in G such that
$$X \cap A \subset \bigcup_{e \in M} e.$$
6. Let G be a connected graph with an even number of edges. Use Tutte's Theorem to prove that the set of edges of G can be partitioned into pairwise disjoint paths of length two.

7. Let $G = (V, E)$ be a connected graph with the property that for every pair of vertices u, v there is a path from u to v of length at least 3 (i.e. contains at least 3 edge). Let the G^2 be the graph on a vertex set V that has edge set E together with all pairs xy with the property that x and y have a common neighbor in G. Prove that G^2 is 2-connected.

8. A soccer ball is made from (not necessarily regular) pentagons and hexagons, sewn together so that their seams form a (connected) 3-regular graph. How many pentagons were used? Prove your answer.

9. Prove that if n is sufficiently large and the complete graph K_n is embedded in the plane then there is a constant $c > 0$ such that there are at least cn^4 pairs of edges that cross (i.e. have interior intersection).

10. (a) Show that for every odd $n \geq 5$ there is a triangle-free non-bipartite graph with $\frac{1}{4}(n-1)^2 + 1$ edges.

    (b) Let G be a graph that contains an odd cycle and has more then $\frac{1}{4}(n-1)^2 + 1$ edges. Show that G contains a triangle.

    Hint. Go by induction on n.