21-484 Graph Theory
Review sheet for test 3

Definitions. The test will assume that you know the following definitions as well as the definitions that appear on the previous review sheets.

- k list colorable, list chromatic number (aka choice number),
- list chromatic index
- perfect graph
- plane graph, planar graph, embedding, face, outer face
- edge contraction, H is an MX, branch set, minor of a graph G
- subdivision, H is a TX, branch vertex, topological minor of a graph G
- Turán number $ex(n, G)$.
- Turán graph $T_{n,r}$.

Theorems. The test assumes knowledge of the following theorems

- **Weak Perfect Graph Theorem.** If G is a graph then G is perfect if and only if $\overline{G}$ is perfect.
- **Strong Perfect Graph Theorem.** A graph G is perfect if and only if G contains neither an odd cycle on at least 5 vertices nor the complement of an odd cycle on at least 5 vertices as an induced subgraph.
- **Euler's Formula.** If G is a connected plane graph with n vertices, m edges and f faces then
$$n - m + f = 2$$
- **Theorem.** If H is a graph such that the maximum degree of H is at most 3 then a graph G contains H as a minor if and only if G contains H as a topological minor.
- **Kuratowski's Theorem.** Let $G = (V, E)$ be a graph. The following are equivalent
  1. G is planar
  2. G contains neither K_5 nor $K_{3,3}$ as minor
  3. G contains neither K_5 nor $K_{3,3}$ as a topological minor
- **Theorem.** If G is a planar graph then the list-chromatic number of G is at most 5. In other words, every planar graph is 5-choosable.
- **Mantel's Theorem.** If $G = (V, E)$ is a graph that does not contain K_3 as a subgraph then
$$|E| \leq \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil .$$
- **Turán's Theorem.** If $G = (V, E)$ is a graph that does not contain K_{r+1} as a subgraph then $|E| \leq |E(T_{n,r})|$. In other words
$$ex(n, K_{r+1}) = |E(T_{n,r})|.$$
- **Theorem (Erdős, Stone, Simonovits).** If G is a graph then
$$\lim_{n\to\infty} \frac{ex(n, G)}{\binom{n}{2}} = 1 - \frac{1}{\chi(G) - 1}.$$
- **Theorem (Kövari, Sós, Turán).** For any fixed $2 \leq s \leq t$ there exists a constant C such that
$$ex(K_{s,t}) \leq Cn^{2-\frac{1}{s}}.$$
- **Theorem.** $ex(n, K_{2,2}) = \left(\frac{1}{2} - o(1)\right) n^{3/2}$.
- **Theorem.** For any fixed $2 \leq s \leq t$ there exists a constant C' such that
$$ex(K_{s,t}) \geq C' n^{2-\frac{s+t-2}{st-1}}.$$

Review Problems. Doing these problems should help in preparation for the third test.

1. Show that for every k there is a graph G such that $\chi(G) = 2$ and $\chi_\ell(G) = ch(G) \geq k$.

2. Prove that a graph G is perfect if and only if for every subset X of the vertex set of G there is a set $Y \subseteq X$ such that Y is an independet set in $G[X]$ and $\omega(G[X \setminus Y]) < \omega(G[X])$.

3. A graph G is a threshold graph if it is possible to assign real-valued weights $f(v)$ to each vertex $v \in V$ and specify a threshold $T \in \mathbb{R}$ such that $uv \in E(G)$ if and only if $f(u) + f(v) \geq T$. Prove from the definitions that threshold graphs are perfect.

4. A graph G is face-regular if all of its faces are cycles with the same number of edges.
   (a) Characterize all 2-connected plane graphs that are both regular and face regular.
   (b) Show that exactly five of these graphs are 3-connected. (These are the platonic solids.)

5. Recall that a graph is cubic if it is 3-regular.
   (a) Show that there is no bipartite cubic planar graph on 10 vertices.
   (b) Show that for every $n \geq 4$ such that $n \neq 5$ there is a connected planar bipartite cubic graph on $2n$ vertices.

   Hint. Find an explicit construction, and start with n even.

6. Show that K_5 is a minor of the Petersen graph.

7. Is the following statement True or False. Explain your answer

   If the graph $G = (V, E)$ does not contain K_4 as a topological minor then G is planar.

8. Prove that if $n \geq 5$ then every graph on n vertices with at least $\lfloor n^2/4 \rfloor + 2$ edges contains two copies of K_3 that share a vertex.

9. Suppose that n, m and t are the number of vertices, edges, and triangles (respectively) in a graph G. Show that

$$3t \geq 4m^2/n - mn.$$

   Note that this implies $ex(n, K_3) \leq n^2/4$.

   Hint. Consider $\sum_{xy \in E(G)} d(x) + d(y)$.

10. (a) Let P_3 be the path with three edges (and four vertices). Determine $ex(n, P_3)$.
    (b) Determine $ex(n, K_{1,r})$.

11. Show that there is some constant c such that for any set X of n distinct points in the plane at most $cn^{3/2}$ of the pairs in $\binom{X}{2}$ are pairs of points which are a unit distance apart.

12. Suppose that $v_1, \ldots, v_n$ are distinct unit vectors in $\mathbb{R}^3$. Prove that there are at most $4n^{5/3}$ pairs $\{i, j\}$ such that v_i and v_j are orthogonal.