

21-301 Combinatorics

Course Review 1

This document contains a list of the important theorems that have been covered thus far in the course. It is *not* a complete listing of what has happened in lecture. Following the list you will find a collection of review exercises and sample test questions.

Note. Some of the Theorems appear in boxes. You could be asked to prove these Theorems, or closely related statements, on the test.

2. MATCHINGS:

Definition 1. A **matching** in a graph $G = (V, E)$ is a collection of edges $X \subseteq E$ such that

$$e_1, e_2 \in X \Rightarrow e_1 \cap e_2 = \emptyset.$$

In words, the edges in X are pairwise disjoint. The matching X is a **perfect matching** if $|X| = |V|/2$ (i.e. if every vertex is in an edge in the matching).

Definition 2. A graph $G = (V, E)$ is **bipartite** if there is a partition of V into parts V_1, V_2 (called the **bipartition** of the graph) such that

$$\binom{V_1}{2} \cap E = \emptyset, \quad \binom{V_2}{2} \cap E = \emptyset.$$

In words, every edge in G has one vertex in each part of the partition.

Definition 3. Let $G = (V, E)$ be a graph and $X \subseteq V$. The **neighborhood** of X is

$$N(X) = \{v \in V \setminus X : \exists u \in X \text{ such that } \{u, v\} \in E\}.$$

In words, $N(X)$ is the set of vertices outside of X that are joined by edges to vertices in X .

Note 4. If $G = (V, E)$ is a bipartite graph with bipartition V_1, V_2 and $X \subseteq V_1$ then $N(X) \subseteq V_2$.

Theorem 5 (Marriage theorem). A bipartite graph with bipartition V_1, V_2 has a perfect matching if and only if $|V_1| = |V_2|$ and

$$|N(X)| \geq |X| \quad \forall X \subseteq V_1. \tag{1}$$

Note 6. Condition (1) is called **Hall's condition**.

Definition 7. A graph $G = (V, E)$ is **regular** (or **d-regular**) if there is an integer d such that $d(x) = d$ for all $x \in V$ (recall that $d(x)$ denotes the degree of vertex x).

Corollary 8. If $G = (V, E)$ is a regular bipartite graph then G has a perfect matching.

Definition 9. Let $G = (V, E)$ be a bipartite graph with bipartition A, B . A matching $X \subseteq E$ is **A-perfect** if $|X| = |A|$ (i.e. if every vertex in A is in an edge in the matching).

Theorem 10 (Hall's theorem). Let $G = (V, E)$ be a bipartite graph with bipartition A, B . G has an A -perfect matching if and only if

$$|N(X)| \geq |X| \quad \forall X \subseteq A.$$

Definition 11. Let $G = (V, E)$ be a graph and let $M \subseteq E$ be a matching.

- (a) A vertex $v \in V$ is **M -saturated** if there exists $e \in M$ such that $v \in e$.
- (b) A path $P = v_1, v_2, \dots, v_k$ in G is an **M -alternating path** if the edges in P alternate between edges in M and edges not in M .
- (c) An M -alternating path $P = v_1, v_2, \dots, v_k$ is an **M -augmenting path** if v_1 and v_k are not M -saturated.

Theorem 12. Let $G = (V, E)$ be a graph and $M \subseteq E$ be a matching. M is a maximum matching iff there does not exist an M -augmenting path.

3. PARTIALLY ORDERED SETS.

Definition 13. A **partially ordered set** (or **poset**) is a set P together with a binary relation $\leq$ on P satisfying the following three conditions.

- (Reflexivity) For all $x \in P$, $x \leq x$.
- (Antisymmetry) For all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.
- (Transitivity) For all $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

We write $(P, \leq)$ when we want to record the order relation explicitly, and simply P when $\leq$ is understood.

Notation 14. We write $x < y$ as shorthand for “ $x \leq y$ and $x \neq y$ ”.

Definition 15. Two elements x, y of a poset P are **comparable** if $x \leq y$ or $y \leq x$; otherwise they are **incomparable**. The word “partial” in “partial order” refers exactly to the fact that incomparable pairs are allowed.

Example 16 (The subset lattice). Let X be a set and let $P = 2^X = \{A : A \subseteq X\}$ be its power set, ordered by

$$A \leq B \iff A \subseteq B.$$

Reflexivity, antisymmetry and transitivity are the familiar properties of set inclusion. For example, with $X = \{1, 2, 3, 4, 5\}$ we have $\{2, 3\} \leq \{1, 2, 3, 5\}$, while $\{2, 3\} \not\leq \{1, 3, 4\}$. This poset is called the **subset lattice**, or **Boolean lattice**, of X .

Example 17 (Total orders). The usual order $\leq$ on $\mathbb{Z}$, on $\mathbb{Q}$, or on $\mathbb{R}$ is a partial order. These examples satisfy an extra condition:

$$\text{for all } x, y \in P : \quad x \leq y \quad \text{or} \quad y \leq x.$$

That is, any two elements are comparable. A partial order satisfying this condition is called a **linear order** or **total order**.

Definition 18. Let P be a poset and $Q \subseteq P$. Restricting $\leq$ to Q gives a partial order on Q ; we call Q with this order an **induced subposet** of P . All three axioms are inherited immediately.

Definition 19. Let P be a poset and $x, y \in P$. We say y **covers** x , written $x \triangleleft y$, if

- (a) $x < y$ (that is, $x \leq y$ and $x \neq y$), and
- (b) there is no $z \in P$ with $x < z < y$.

Thus y covers x if y is strictly above x with nothing strictly in between.

Definition 20. Let P be a poset. The **cover graph** of P is the graph with vertex set P having an edge joining x and y whenever $x \triangleleft y$ or $y \triangleleft x$.

The **Hasse diagram** of P is a depiction of the cover graph in the plane in which y is drawn strictly higher than x whenever $x \triangleleft y$. Edges are drawn as straight line segments and no arrowheads are used: the vertical direction encodes the order.

Theorem 21. Let P be a finite poset and let $x, y \in P$. Then $x < y$ if and only if there exist $k \geq 1$ and elements

$$x = x_0, x_1, \dots, x_k = y$$

of P such that $x_{i-1} \triangleleft x_i$ for $i = 1, \dots, k$.

Corollary 22. A finite poset is determined by its Hasse diagram: for $x \neq y$ we have $x < y$ if and only if the diagram contains an upward path of edges from x to y . Equivalently, $\leq$ is the reflexive–transitive closure of $\triangleleft$.

Definition 23. Let P be a poset and $m \in P$.

- (a) m is **minimal** if there is no $x \in P$ with $x < m$.
- (b) m is **maximal** if there is no $x \in P$ with $m < x$.
- (c) m is a **minimum** (or **least**) element if $m \leq x$ for all $x \in P$.
- (d) m is a **maximum** (or **greatest**) element if $x \leq m$ for all $x \in P$.

Theorem 24 (Existence of linear extensions). Let P be a finite poset with order $\leq$. Then there exists a linear order $\leq'$ on P such that

$$x \leq y \implies x \leq' y \quad \text{for all } x, y \in P.$$

Such an order is called a **linear extension** of P .

Definition 25. Let $(P, \leq)$ and $(P', \leq')$ be posets. A map $f : P \rightarrow P'$ is an **embedding** of P into P' if

- (a) f is injective, and
- (b) for all $a, b \in P$: $a \leq b \iff f(a) \leq' f(b)$.

If in addition f is surjective, f is an **isomorphism** of posets, and we say P and P' are isomorphic.

Theorem 26. Let P be a poset. Then there is an embedding of P into 2^P , the power set of P ordered by inclusion.

Definition 27. Let P be a poset.

- (a) A **chain** in P is a set $C \subseteq P$ such that every pair $x, y \in C$ is comparable.
- (b) The **height** of P , denoted $h(P)$, is the maximum size of a chain in P .
- (c) An **antichain** in P is a set $A \subseteq P$ such that no pair of distinct elements $x, y \in A$ is comparable.
- (d) The **width** of P , denoted $w(P)$, is the maximum size of an antichain in P .

Note 28.

- (a) If a poset P of height h is partitioned into antichains, then at least h antichains are required.
- (b) If a poset P of width w is partitioned into chains, then at least w chains are required.

Theorem 29 (Dilworth). A finite poset of width w can be partitioned into w chains.

Theorem 30 (Mirsky). A finite poset P can be partitioned into $h(P)$ antichains.

Corollary 31. For every finite poset P we have $h(P) \cdot w(P) \geq |P|$.

Theorem 32 (Sperner). Let X be a set with $|X| = n$. Then

$$w(2^X) = \binom{n}{\lfloor n/2 \rfloor}.$$

Corollary 33 (LYM inequality). If $A \subseteq 2^X$ is an antichain and $A_k = \{Y \in A : |Y| = k\}$, then

$$\sum_{k=0}^n \frac{|A_k|}{\binom{n}{k}} \leq 1.$$

REVIEW EXERCISES: Working the following problems should help in preparation for the test. These are more difficult than questions that might appear on the test (roughly speaking).

1. Let X be a set of n symbols. A *Latin square* is an arrangement of the symbols in X in the positions of an $n \times n$ matrix such that every symbol appears exactly once in every row and column.
Suppose that we have an assignment of the symbols in X to the positions in the first k columns of an $n \times n$ matrix such that each symbol appears once in every column and at most once in every row (i.e. we have a partially filled Latin square). Show that the remaining positions in the matrix can be filled with the symbols from X to form a full Latin square.
2. Let $G = (V, E)$ be a bipartite graph with bipartition A, B such that $|A| = |B| = n$. Show that if the degree of every vertex in A is at least $3n/4$ and the degree of every vertex in B is at least $n/4$ then G has a perfect matching.

3. An $n \times n$ matrix $A = (a_{i,j})$ over $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ is **doubly stochastic** if its row sums and column sums are all 1. A $n \times n$ matrix M with entries in $\{0, 1\}$ is a **permutation matrix** if M has exactly one 1 in every row and column.

Let A be an $n \times n$ doubly stochastic matrix. Use the marriage theorem to show that there exist permutation matrices $M_1, \dots, M_k$ and positive real numbers $\alpha_1, \dots, \alpha_k$ such that

$$A = \sum_{i=1}^k \alpha_i M_i.$$

4. Consider a sequence $a_1, a_2, \dots, a_{n^2+1}$ of $n^2 + 1$ distinct positive integers. Show that this sequence contains either an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $n + 1$.

Hint: Define a poset on the set $P = \{(i, a_i) : i = 1, \dots, n^2 + 1\}$ and apply Dilworth.

5. Let $a_1, \dots, a_n$ be real numbers with $a_i \geq 1$ for every i . For a sign vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, +1\}^n$ write

$$S(\varepsilon) = \sum_{i=1}^n \varepsilon_i a_i.$$

Prove that for any open interval $I \subseteq \mathbb{R}$ of length 2, the number of $\varepsilon \in \{-1, +1\}^n$ with $S(\varepsilon) \in I$ is at most $\binom{n}{\lfloor n/2 \rfloor}$.

Hint: Associate the sign vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, +1\}^n$ to the set $X(\varepsilon) = \{i : \varepsilon_i = +1\}$.

SAMPLE QUESTIONS:

- Let P be a finite poset of height h . For $i = 1, \dots, h$ let Y_i be the set of $x \in P$ for which the largest chain with greatest element x has exactly i elements. Show that the sets $Y_1, \dots, Y_h$ forms a partition of P into antichains.
- Let $\mathcal{F}$ be a collection of subsets of $\{1, 2, \dots, n\}$. We say that $\mathcal{F}$ is **semi-independent** if it contains no three distinct sets A, B, C such that $A \subset B \subset C$.

Adapt the proof of Sperner's Theorem given in class to prove that if $\mathcal{F}$ is a semi-independent collection of subsets of an n -element set then

$$|\mathcal{F}| \leq 2 \binom{n}{\lfloor n/2 \rfloor}.$$

The test will include some questions in the following format.

Are the following statement true or false? For each true statement give a brief explanation of why it is true. For each false statement provide a counterexample.

- If $G = (V, E)$ be a bipartite graph with bipartition A, B and there is an integer k such that

$$x \in A \Rightarrow d(x) \geq k \quad \text{and} \quad x \in B \Rightarrow d(x) \leq k$$

then G has a perfect matching.

4. If $G = (V, E)$ be a d -regular bipartite graph and $d \geq 1$ then there is a partition of E into sets $M_1, \dots, M_d$ such that M_i is a perfect matching for $i = 1, \dots, d$.
5. If P is a finite poset and $f : P \rightarrow 2^P$ is the map which takes x to

$$D(x) = \{w \in P : w \leq x\}$$

then $x \triangleleft y$ implies $|f(y)| = |f(x)| + 1$.