

21-301 Combinatorics
Assignment # 5
Due: Friday, October 23

1. (a) Let H be a graph. Prove that if $n \geq 2$ then

$$\frac{\text{ex}(n+1, H)}{\binom{n+1}{2}} \leq \frac{\text{ex}(n, H)}{\binom{n}{2}}.$$

- (b) Conclude that

$$\pi(H) := \lim_{n \rightarrow \infty} \frac{\text{ex}(n, H)}{\binom{n}{2}}$$

exists.

2. Prove that a graph $G = (V, E)$ on n vertices with m edges contains at least $\frac{m}{3n}(4m - n^2)$ copies of K_3 .

Hint. Start by finding a lower bound on the number of copies of K_3 that contain some edge $\{x, y\}$.

3. Let $1 < a \leq b$ be integers. Prove that there exists some constant C such that

$$\text{ex}(n, K_{a,b}) \leq Cn^{2-\frac{1}{a}}$$

Hint. Follow the outline of the proof of the upper bound on $\text{ex}(n, K_{2,2})$ given in class.

4. Let p be a positive integer, let $r = p+1$ and let $n = p^2 + p + 1$. Let $X = [n] = \{1, 2, \dots, n\}$ and suppose $A_1, \dots, A_m$ are sets in $\binom{X}{r}$ with the property

$$i \neq j \Rightarrow |A_i \cap A_j| \leq 1.$$

Prove $m \leq n$.

5. Let $G = (V, E)$ be a bipartite graph with bipartition A, B that does not contain C_4 as a subgraph. Let $a = |A|$ and $b = |B|$. Prove that there is some constant C such that

$$|E| \leq C(b\sqrt{a} + a).$$

Note. This is a stronger bound than the one proved in lecture when b is significantly smaller than a .

6. Let L be a set of n distinct lines in the plane and let P be a set of n distinct points in the plane. Define the set of incidences to be the set of ordered pairs

$$I = \{(p, \ell) : p \in P, \ell \in L \text{ and } p \in \ell\}.$$

Prove that there is a constant c such that $|I| < cn^{3/2}$.