

21-301 Combinatorics
Assignment # 4
Due: Friday, October 2

Throughout this assignment we set $X = [n] = \{1, 2, \dots, n\}$.

1. If $\mathcal{F} \subseteq \binom{X}{k}$ and $k \geq 1$ we define the *shadow* of $\mathcal{F}$ to be the set

$$\partial\mathcal{F} = \left\{ A \in \binom{X}{k-1} : \exists B \in \mathcal{F} \text{ such that } A \subset B \right\}.$$

Use double counting to show

$$\frac{|\partial\mathcal{F}|}{\binom{n}{k-1}} \geq \frac{|\mathcal{F}|}{\binom{n}{k}}.$$

Note. This can be viewed as a 'local' version of the LYM inequality.

2. Use the previous problem to prove the LYM inequality: If $\mathcal{A} \subset 2^X$ is an antichain and we define $\mathcal{A}_k := \{B \in \mathcal{A} : |B| = k\}$ then we have

$$\sum_{k=0}^n \frac{|\mathcal{A}_k|}{\binom{n}{k}} \leq 1.$$

3. On Homework 3 we established that if $\mathcal{F} \subseteq 2^X$ is intersecting then $|\mathcal{F}| \leq 2^{n-1}$. For $t \geq 2$ we say that $\mathcal{F} \subseteq 2^X$ is t -intersecting if

$$A, B \in \mathcal{F} \Rightarrow |A \cap B| \geq t.$$

Is the following statement true or false?

$$\text{If } \mathcal{F} \subseteq 2^X \text{ is } t\text{-intersecting then } |\mathcal{F}| \leq 2^{n-t}.$$

4. Let and set $M = \binom{n}{\lfloor n/2 \rfloor}$. Show that there is a partition of 2^X into chains $C_1, C_2, \dots, C_M$ such that each chain C_i has the following properties:

$$\exists A \in C_i \text{ such that } |A| = \lfloor n/2 \rfloor$$

and

$$A, B \in C_i \text{ and } |B| \geq |A| + 2 \Rightarrow \exists D \in C_i \text{ such that } |D| = |A| + 1.$$

In other words, each chain C_i has the property that the list of cardinalities of sets in C_i form a contiguous interval of integers.

5. Let $X = [n] = \{1, 2, \dots, n\}$ and $1 < k < n/2$. Use the previous problem to prove the following: If $\mathcal{F} \subset 2^X$ is an intersecting antichain and $|A| \leq k$ for all $A \in \mathcal{F}$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

6. Suppose $n = 2k$.

(a) Show that if $\mathcal{F} \subset \binom{X}{k}$ is intersecting then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

(b) Determine the number of intersecting families $\mathcal{F} \subset \binom{X}{k}$ that have maximum size.

Hint: These families are not always all sets of size k that contain some fixed element.