

21-301 Combinatorics
Assignment # 3
Due: Friday, September 18

1. Let $\leq_1, \leq_2, \dots, \leq_k$ be partial orders on the set X . We define a relation $\leq$ on X by setting

$$a \leq b \iff a \leq_i b \text{ for all } i.$$

- (a) Show that $(X, \leq)$ is a partially ordered set. Note that we can view this poset as the intersection of the partial orders $\leq_1, \dots, \leq_k$.
- (b) Prove that any finite poset P can be expressed as the intersection of **linear** orders.
2. Let k, ℓ be positive integers. Prove that there is a poset P on $n = k\ell$ elements with the property that every chain in P has at most k elements and every antichain in P has at most ℓ elements.
3. A poset P is an *interval order* if there is a collection of intervals $([a_x, b_x] : x \in P)$ in the real line such that $x < y$ if and only if $b_x < a_y$ in $\mathbb{R}$.

- (a) Show that $2^{\{u,v\}}$ (i.e. the 2-dimensional Boolean lattice) is an interval order.
- (b) Define $2C_2$ to be the 4-element poset given by 2 disjoint 2-element chains with no relations between them. Prove that a finite poset P is an interval order if and only if there is not embedding of $2C_2$ into P .

Hint: Go by induction and consider an element x for which the down set $D(x) = \{y \in P : y \leq x\}$ has maximum size.

4. Use Dilworth's Theorem to prove Hall's Theorem.
5. On the previous homework we proved the following Theorem using Hall's Theorem:

König's Theorem. If G is a bipartite graph then the matching number of G is equal to the vertex cover number of G . (In symbols, $\nu(G) = \tau(G)$.)

Use König's Theorem to prove Dilworth's Theorem.

Hint. Given a poset P form a bipartite graph using two copies of P .

Note. Taken together, these three exercises give Hall $\Rightarrow$ König $\Rightarrow$ Dilworth $\Rightarrow$ Hall.

6. Let $n \geq 1$ and let $\mathcal{F}$ be a collection of subsets of $\{1, \dots, n\}$. We say that $\mathcal{F}$ is **intersecting** if every pair of sets in $\mathcal{F}$ intersect; to be precise, the collection $\mathcal{F}$ is intersecting if

$$A, B \in \mathcal{F} \implies A \cap B \neq \emptyset.$$

- (a) Prove that if $\mathcal{F}$ is an intersecting collection of subsets of $\{1, \dots, n\}$ then $|\mathcal{F}| \leq 2^{n-1}$.
- (b) Give an example of an intersecting collection $\mathcal{F}$ of subsets of $\{1, \dots, n\}$ such that $|\mathcal{F}| = 2^{n-1}$.