

21-301 Combinatorics
Assignment # 2
Due: Friday, September 11

1. Let $G = (V, E)$ be a bipartite graph with bipartition A, B such that $|A| = |B|$. Show that if $|N(X)| \geq |X| - 1$ for all $X \subseteq A$ then G has a matching of size $|A| - 1$ (i.e. an almost perfect matching).
2. Let G be a graph that has $2n$ vertices and in which every vertex has degree at least n . Show that G has a perfect matching.
3. Let G be a bipartite graph with maximum degree Δ (so every vertex is in at most Δ edges). Prove that G has a matching M that saturates every vertex of degree Δ in G .
4. Let $G = (V, E)$ be a graph and let M be a matching of G . Let X be the set of M -saturated vertices. Prove that there is a maximum matching M' such that every vertex in X is M' -saturated.
5. Consider the following game. The board is a graph $G = (V, E)$. The players alternately choose vertices in V , producing a sequence $u_1, v_1, u_2, v_2, \dots$ of vertices (so $u_1, u_2, \dots$ are the moves of player I and $v_1, v_2, \dots$ are the moves of Player II). The players must obey the following rules:
 - (i) No vertex is chosen more than once (so $u_1, v_1, \dots$ is a sequence of distinct vertices),
 - (ii) Each vertex chosen must be connected by an edge to the last vertex chosen.Note that $u_1, v_1, u_2, v_2, \dots$ is a path in G . The first player who cannot make a move is the loser.
 - (a) Show that if G has a perfect matching then the second player has a winning strategy.
 - (b) Show that if G does not have a perfect matching then the first player has a winning strategy.
6. The matching number of a graph G , denoted $\nu(G)$ is the maximum number of edges in a matching in G . The vertex cover number of G , denoted $\tau(G)$ is the minimum number of vertices in a set of vertices that meets every edge of G (such a set is called a vertex cover). Use Hall's Theorem to prove the following: If G is a bipartite graph then $\nu(G) = \tau(G)$.