

21-301 Combinatorics
Homework 1
Due: Friday, September 4

1. Use a **combinatorial** argument to prove the following: for all positive integers m_1, m_2, n we have

$$\sum_{k=0}^n \binom{m_1}{k} \binom{m_2}{n-k} = \binom{m_1 + m_2}{n}.$$

2. Consider the collection of $m \times n$ matrices with the property that each entry is either 1 or -1 . How many of these matrices have the property that the product of the entries in each row is 1 and the product of the entries in each column is 1?

Hint: Show that some of the entries can be filled in arbitrarily and then we can get the desired condition by assigning the remaining entries carefully.

3. A function $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ is **monotone** if $i < j$ implies $f(i) \leq f(j)$. How many monotone functions $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ are there?
4. Recall that a graph $G = (V, E)$ is a tree if G is connected and has no cycle. Suppose G is a tree on at least two vertices that has no vertex of degree 2. Show that G has more vertices of degree 1 than vertices of degree greater than 2. (Recall that the degree of a vertex v in a graph $G = (V, E)$ is the number of edges in E that contain v .)
5. Suppose $G = (V, E)$ is a graph on n vertices with the property that if $u, v \in V$ are not adjacent then $d(u) + d(v) \geq n - 1$. Prove that G is connected.
6. (a) Let m be a fixed integer. Show that the product of primes p such that $m < p \leq 2m$ is at most $\binom{2m}{m}$.
- (b) Use (a) to prove $\pi(n) = O(n/\ln n)$, where $\pi(n)$ is the number of primes less than or equal to n .

Hint: Choose a suitably large constant and go by induction.