21-228 Discrete Mathematics
Assignment # 7: Solutions

1. (a)

1	4	7	10
8	11	2	5
3	6	9	12

   (b) Assume for the sake of contradiction that there exists a Hamilton cycle in $G_{4,4}$. Let X be the set of edges in this cycle. For each vertex $v \in \{1,2,3,4\} \times \{1,2,3,4\}$ we have

$$|\{e \in X : v \in e\}| = 2.$$

   Since the degree of the vertex $(1,1)$ in $G_{4,4}$ is 2, both of the edges that contain $(1,1)$ must be in X; that is,

$$\{(1,1),(2,3)\}, \{(1,1),(3,2)\} \in X.$$

   Applying the same reasoning to the vertex $(4,4)$ we have

$$\{(2,3),(4,4)\}, \{(3,2),(4,4)\} \in X.$$

   The vertices

$$(1,1),(2,3),(3,2),(4,4)$$

   form a 4-cycle within our Hamilton cycle. This is a contradiction.

2. Suppose $G = (V,E)$ is a graph with $|V| = n$ and

$$|E| \geq \frac{(n-1)(n-2)}{2} + 1.$$

   Let $\overline{G}$ be the complement of G; that is, the graph $\overline{G}$ has vertex set V and edge set $\binom{[n]}{2} \setminus E$. Note that the edge set of $\overline{G}$ has at most

$$\binom{n}{2} - \frac{(n-1)(n-2)}{2} - 1 = n-2$$

   edges. As trees are minimally connected graphs and a tree on n vertices has $n-1$ edges, it follows that $\overline{G}$ is not connected.

   Now recall that we showed in last homework that for any graph G we have G connected or $\overline{G}$ connected. As we have shown that $\overline{G}$ is not connected, it follows that G is connected.

   For the example, consider the graph with vertex set V with $|V| = n$, $u \in V$ and edge set $\binom{V\setminus\{u\}}{2}$.

3. Let $G = (V, E)$ be a tree.

   Assume for the sake of contradiction that $x_0, x_1, \ldots, x_k$ and $y_0, y_1, \ldots, y_k$ are vertex-disjoint maximum paths in G.

   Since G is a tree, it is connected, and there is a path $x_0 = z_0, z_1, z_2, \ldots, z_\ell = y_0$ from x_0 to y_0. Let i and j be indices such that

   - $i < j$,
   - $z_i \in \{x_0, x_1, \ldots, x_k\}$,
   - $z_j \in \{y_0, y_1, \ldots, y_k\}$, and
   - $z_{i+1}, z_{i+1}, \ldots z_{j-1} \notin \{x_0, x_1, \ldots, x_k, y_0, y_1, \ldots, y_k\}$.

   (Why must such a pair of indices exist?) Let $z_i = x_a$ and $z_j = y_b$. We assume without loss of generality that $a, b \geq k/2$ (Why can we make this assumption?) The path

   $$x_0, x_1, \ldots, x_a = z_i, z_{i+1}, \ldots, z_{j-1}, z_j = y_b, y_{b-1}, \ldots y_0.$$

   is a path in G. The number of edges in this path is

   $$a + (j - i) + b \geq \frac{k}{2} + 1 + \frac{k}{2} = k + 1.$$

   The existence of this path contradicts the assumption that our original paths are maximum paths.

   We show that the claim does not hold for maximal paths by exhibiting a counterexample. Let $G = (V, E)$ be the tree defined by

   $$V = \{v_1, v_2, \ldots, v_6\} \qquad E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_5\}, \{v_4, v_5\}, \{v_5, v_6\}\}.$$

   In this tree the paths v_1, v_2, v_3 and v_4, v_5, v_6 are maximal but they do not intersect.

4. Let the vertices of the tree $T = (V, E)$ be $v_1, \ldots, v_n$.

   Applying a theorem proved in lecture (and a fact about the sums of the degrees in any graph proved earlier this semester) we have

   $$\sum_{i=1}^{n} d(v_i) = 2|E| = 2(|V| - 1) = 2n - 2.$$

   Subtracting $2n$ from both sides of this equation gives

   $$\sum_{i=1}^{n} (d(v_i) - 2) = \left(\sum_{i=1}^{n} d(v_i) \right) - 2n = 2n - 2 - 2n = -2.$$

   Multiplying by -1 gives

   $$\sum_{i=1}^{n} (2 - d(v_i)) = 2.$$

   We can express the summation as a double sum as follows:

   $$2 = \sum_{i=1}^{n} (2 - d(v_i)) = \sum_{j=1}^{n-1} \sum_{i:d(v_i)=j} (2 - j) = \sum_{j=1}^{n-1} (2 - j)|\{i : d(v_i) = j\}|$$
   $$= \sum_{j=1}^{n-1} (2 - j)p_j = p_1 - p_3 - 2p_4 - \cdots - (n - 3)p_{n-1}.$$

5. We modify the proof of inclusion/exclusion given in class, using the following fact.

Claim. Let $G = (V, E)$ be a tree. If $X \subseteq V$ then the number of edges in E that fall within X is at most $|X| - 1$. Formally, we write

$$\left| E \cap \binom{X}{2} \right| \leq |X| - 1.$$

Proof. Consider the graph with vertex set X and edge set $E \cap \binom{X}{2}$. Since H is a subgraph of a tree, the graph H has no cycles. We proved in lecture that a graph on vertex set X that contains no cycle is contained in a tree on vertex set X. We also showed that a tree on vertex set X has $|X| - 1$ edges. It follows that H has at most $|X| - 1$ edges. □

For each $x \in \Omega$ let N_x be the set of indices $i \in \{1, 2, \dots, n\}$ such that $x \in A_i$. We use the following facts

$$\begin{aligned} x \in A_i \quad &\Leftrightarrow \quad i \in N_x \\ x \in A_u \cap A_v \quad &\Leftrightarrow \quad u, v \in N_x \\ x \in \bigcup_{i=1}^{n} A_i \quad &\Leftrightarrow \quad N_x \neq \emptyset \end{aligned}$$

We have

$$\begin{aligned} \sum_{i=1}^{n} |A_i| - \sum_{\{u,v\} \in E} |A_i \cap A_j| &= \sum_{i=1}^{n} \sum_{x \in A_i} 1 - \sum_{\{u,v\} \in E} \sum_{x \in A_u \cap A_v} 1 \\ &= \sum_{x \in \Omega} \sum_{i \in V : x \in A_i} 1 - \sum_{x \in \Omega} \sum_{\{u,v\} \in E : x \in A_u \cap A_v} 1 \\ &= \sum_{x \in \Omega} |N_x| - \sum_{x \in \Omega} \left| E \cap \binom{|N_x|}{2} \right| \\ &\geq \sum_{x \in \Omega : N_x \neq \emptyset} |N_x| - (|N_x| - 1) \\ &= \sum_{x \in \Omega : N_x \neq \emptyset} 1 \\ &= |\{x \in \Omega : N_x \neq \emptyset\}| \\ &= \left| \bigcup_{v \in V} A_v \right| \end{aligned}$$