This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is not a complete listing of what has happened in lecture. The sections from the book that correspond with each topic are also given.

Following the list of important definitions and theorems you will find a collection of review exercises.

8. Graph Theory continued: Chapter 9.

**Definition 1.** Suppose $G = (V, E)$ is a connected graph and $f : E \rightarrow \mathbb{R}^+$ is an assignment of weights to the edges of $G$. Let $T$ be the collection of subsets $S$ of $E$ with the property that $(V, S)$ is a tree. A **minimum weight spanning tree** of $G$ is a tree $T \in T$ such that

$$\sum_{e \in T} w(e) = \min_{S \in T} \sum_{e \in S} w(e).$$

**Theorem 2.** Let $G = (V, E)$ be a connected graph and let $f : E \rightarrow \mathbb{R}^+$ be an assignment of weights to the edges of $G$. The following greedy algorithms produce minimum weight spanning trees.

**Kruskal’s Algorithm**

Set $T = \emptyset$

While $|T| \leq |V| - 2$

Let $X$ be the set of edges $e$ in $E \setminus T$

such that $(V, T) + e$ does not contain a cycle

Choose $e \in X$ of minimum weight

Add $e$ to $T$

**Prim’s Algorithm**

Let $v \in V$

Set $Y = \{v\}$ and $T = \emptyset$

While $|T| \leq |V| - 2$

Let $X$ be the set of edges $e \in E$ such that $|e \cap Y| = 1$

Choose $e \in X$ of minimum weight

Add $e$ to $T$ and set $Y = Y \cup \{e\}$
Review Exercises: Working the following problems should help in preparation for the test. Some (but not all) of these are more difficult than questions that might appear on the quiz.

Note: the final is cumulative. This list does represent the full scope of questions that could appear on the final. See the review sheets for the tests for a more complete list of review exercises.

1. Let \( X = \{1, 2, \ldots, n\} \). Let \( \mathcal{F} \) be a collection of 3-element subsets of \( \mathcal{F} \) (formally, \( \mathcal{F} \subseteq \binom{X}{3} \)). For each \( i \in X \) define \( d(i) = |\{A \in \mathcal{F} : i \in A\}| \). In words, \( d(i) \) is the number of sets in \( \mathcal{F} \) that contain \( i \). We define

\[
A = \{i \in X : d(i) \equiv 1 \mod 3\} \quad \text{and} \quad B = \{i \in X : d(i) \equiv 2 \mod 3\}.
\]

Prove that \(|A| \equiv |B| \mod 3\).

2. What is the number of strings consisting of \( p \) 0’s and \( q \) 1’s in which each pair of 1’s is separated by at least two 0’s? Explain your answer.

3. Prove that for \( n = 1, 2, \ldots \) we have

\[
2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.
\]

4. (a) Show that

\[
(1 - 4x)^{-1/2} = \sum_{n \geq 0} \binom{2n}{n} x^n.
\]

(b) Simplify

\[
\sum_{n \geq 0} \binom{2n - 1}{n} x^n
\]

5. (a) Determine the generating function for the sequence \( a_0, a_1, \ldots \) defined by the recurrence

\[
a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for} \quad n \geq 2
\]

and \( a_0 = a_1 = 1 \).

(b) Give a closed form expression for \( a_n \).

6. Let \( a_n \) be the number of ordered triples \((i, j, k)\) of integer numbers such that \( i \geq 0, j \geq 1, k \geq 1 \) and \( i + 3j + 3k = n \). Find the generating function for the sequence \( a_0, a_1, a_2, \ldots \).

7. Let \( k \) be a positive integer. Let \( G = (V, E) \) be a graph such that \(|V| = 6k\), half of the vertices in \( G \) have degree \( 2k \) and the other half of the vertices in \( G \) have degree \( 4k \). Show that \( G \) is Hamiltonian.

8. Ann and Bob play a game in which they alternately toss a fair die, with Ann rolling first. The one who is first to roll a 6 wins the game.

(a) Set up a probability space that describes this experiment.

(b) Determine the probability that Ann wins on her second roll.

(c) Determine the probability that Ann wins the game.
9. A coloring of a collection of $m$ indistinguishable balls with $n$ (distinguishable) colors is chosen uniformly at random from the set of all such colorings.

(i) Describe the probability space for this experiment.
(ii) Let the random variable $X$ count the number of colors that are used exactly twice. Determine $E[X]$.

10. Let $n$ be a positive integer. Let $X$ be the set of strings $x_1, x_2, \ldots, x_{2n}$ consisting of $n$ 1’s and $n$ −1’s such that

$$x_1 + x_2 + \cdots + x_k \geq 0 \quad \text{for } k = 1, 2, \ldots, 2n.$$

Show that

$$|X| = \frac{1}{n+1} \binom{2n}{n}$$

11. Let $G = (V, E)$ be a connected graph. Prove that there is a graph $H = (V, F)$ such that $F \subseteq E$ and $H$ is a tree.

12. Consider the following algorithm for the minimum weight spanning tree algorithm. Let $G = (V, E)$ be a connected graph and let $w : E \rightarrow \mathbb{R}^+$. Label the edges $e_1, e_2, \ldots, e_m$ in such a way that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$. Set $E_0 = E$ and

$$E_i = \begin{cases} E_{i-1} \setminus \{e_i\} & \text{if the graph } G - e_i \text{ is connected} \\ E_{i-1} & \text{otherwise.} \end{cases}$$

Prove that $(V, E_m)$ is a minimum weight spanning tree of $G$.

13. Let $G$ be a connected graph with a weight function $w$ on the edges, and assume that $w$ is an injective function (i.e. every edge gets a different weight). Prove that there is only one minimum weight spanning tree.

From the text: 9.2.4, 9.2.5