21-228 Discrete Mathematics
Course Review 4

This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is *not* a complete listing of what has happened in lecture. The sections from the book *Invitation to Discrete Mathematics, second edition* by Matousek and Nesetril that correspond with each topic are also given.

Following the list of important definitions and theorems you will find a collection of review exercises.

7. GRAPH THEORY CONTINUED: note: Hamilton cycles are not covered in Matousek and Nesetril.

   Definition 94. *A* **Hamilton cycle** *in a graph $G = (V, E)$ is a cycle that contains every vertex in V. The graph G is* **Hamiltonian** *if it has a Hamilton cycle.*

   Definition 95. *Let $G = (V, E)$ be a graph and let $e \in \binom{V}{2} \setminus E$. We define the graph $G + e$ to be the graph with vertex set V and edge set $E \cup \{e\}$.*

   Lemma 96. *Let $G = (V, E)$ and $|V| = n$. If $x, y \in V$, $\{x, y\} \notin E$ and $d_G(x)+d_G(y) \geq n$ then*

   $$G + e \text{ is Hamiltonian} \quad \Leftrightarrow \quad G \text{ is Hamiltonian.}$$

   Definition 97. *Let $G = (V, E)$ be a graph. The* **minimum degree** *of G is*

   $$\delta(G) = \min\{d(x) : x \in G\}.$$

   Corollary 98 (Dirac's Theorem)**.** *If $G = (V, E)$ is a graph with $|V| = n$ and $\delta(G) \geq n/2$ then G is Hamiltonian.*

8. OPTIMIZATION: Section 5.4 and 5.5

   Definition 99. *Suppose $G = (V, E)$ is a connected graph and $f : E \to \mathbb{R}^+$ is an assignment of weights to the edges of G. Let $\mathcal{T}$ be the collection of subsets S of E with the property that (V, S) is a tree. A* **minimum weight spanning tree** *of G is a tree $T \in \mathcal{T}$ such that*

   $$\sum_{e \in T} w(e) = \min_{S \in \mathcal{T}} \sum_{e \in S} w(e).$$

   Theorem 100. *Let $G = (V, E)$ be a connected graph and let $f : E \to \mathbb{R}^+$ be an assignment of weights to the edges of G The following greedy algorithm produces a minimum weight spanning tree.*

   Kruskal's Algorithm

   Set $T = \emptyset$
   While $|T| \leq |V| - 2$
       Let X be the set of edges e in $E \setminus T$
         such that $(V, T) + e$ does not contain a cycle
       Choose $e \in X$ of minimum weight
       Add e to T

Definition 101. *Let $G = (V, E)$. A* **connected component** *of G is a maximal connected subgraph of G. To be precise, a connected component of G is a graph*

$$\left(X, \binom{X}{2} \cap E \right)$$

such that

- $X \subseteq V$,
- $(X, \binom{X}{2} \cap E)$ *is connected, and*
- *If $X \subset Y \subseteq V$ then $(Y, \binom{Y}{2} \cap E)$ is not connected.*

Theorem 102. *Let $G = (V, E)$ be a connected graph and let $f : E \to \mathbb{R}^+$ be an assignment of* distinct *weights to the edges of G The following greedy algorithm produces a minimum weight spanning tree.*

Bubbles Algorithm

`Set` $E_0 = \emptyset$ `and` $t = 0$

`While` $|E_t| \leq |V| - 2$

    `Let` $X_1, X_2, \ldots, X_\ell$ `be the vertex sets of the`

        `connected components of` (V, E_t)

    `For` $i = 1, \ldots, \ell$ `let` $e_i \in E$ `be an edge of minimum weight`

        `such that` $|e_i \cap X_i| = 1$

    `set` $E_{t+1} = E_t \cup \{e \in E : \exists i \text{ such that } e = e_i\}$ `and increment` t

Review Exercises: Working the following problems should help in preparation for the test. Some (but not all) of these are more difficult than questions that might appear on the quiz.

Note: the final is cummulative. This list does represent the full scope of questions that could appear on the final. See the review sheets for the tests for a more complete list of review exercises.

1. Let $X = \{1, 2, \dots n\}$. Let $\mathcal{F}$ be a collection of 3-element subsets of $\mathcal{F}$ (formally, $\mathcal{F} \subseteq \binom{X}{3}$). For each $i \in X$ define $d(i) = |\{A \in \mathcal{F} : i \in A\}|$. In words, $d(i)$ is the number of sets in $\mathcal{F}$ that contain i. We define

$$A = \{i \in X : d(i) \equiv 1 \bmod 3\} \qquad B = \{i \in X : d(i) \equiv 2 \bmod 3\}.$$

   Prove that $|A| \equiv |B| \bmod 3$.

2. What is the number of strings consisting of p 0's and q 1's in which each pair of 1's is separated by at least two 0's? Explain your answer.

3. Prove that for $n = 1, 2, \dots$ we have

$$2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.$$

4. (a) Show that

$$(1-4x)^{-1/2} = \sum_{n \geq 0} \binom{2n}{n} x^n.$$

   (b) Simplify

$$\sum_{n \geq 0} \binom{2n-1}{n} x^n$$

5. (a) Determine the generating function for the sequence $a_0, a_1, \dots$ defined by the recurrence

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2$$

   and $a_0 = a_1 = 1$.

   (b) Give a closed form expression for a_n.

6. Let a_n be the number of ordered triples (i, j, k) of integer numbers such that $i \geq 0, j \geq 1, k \geq 1$ and $i + 3j + 3k = n$. Find the generating function for the sequence $a_0, a_1, a_2, \dots$.

7. Let k be a positive integer. Let $G = (V, E)$ be a graph such that $|V| = 6k$, half of the vertices in G have degree $2k$ and the other half of the vertices in G have degree $4k$. Show that G is Hamiltonian.

8. Ann and Bob play a game in which they alternately toss a fair die, with Ann rolling first. The one who is first to roll a 6 wins the game.

   (a) Set up a probability space that describes this experiment.

   (b) Determine the probability that Ann wins on her second roll.

   (c) Determine the probability that Ann wins the game.

9. A coloring of a collection of m indistinguishable balls with n (distinguishable) colors is chosen uniformly at random from the set of all such colorings.

   (i) Describe the probability space for this experiment.

   (ii) Let the random variable X count the number of colors that are used exactly twice. Determine $E[X]$.

10. Let n be a positive integer. Let X be the set of strings $x_1, x_2, \ldots, x_{2n}$ consisting of n 1's and n -1's such that

$$x_1 + x_2 + \cdots + x_k \geq 0 \qquad \text{for } k = 1, 2, \ldots, 2n.$$

   Show that

$$|X| = \frac{1}{n+1}\binom{2n}{n}$$

11. Let $G = (V, E)$ be a connected graph. Prove that there is a graph $H = (V, F)$ such that $F \subseteq E$ and H is a tree.

12. Consider the following algorithm for the minimum weight spanning tree algorithm. Let $G = (V, E)$ be a connected graph and let $w : E \to \mathbb{R}^+$. Label the edges $e_1, e_2, \ldots, e_m$ in such a way that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$. Set $E_0 = E$ and

$$E_i = \begin{cases} E_{i-1} \setminus \{e_i\} & \text{if the graph } G - e_i \text{ is connected} \\ E_{i-1} & \text{otherwise.} \end{cases}$$

   Prove that (V, E_m) is a minimum weight spanning tree of G.

13. Let G be a connected graph with a weight function w on the edges, and assume that w is an injective function (i.e. every edge gets a different weight). Prove that there is only one mimimum weight spanning tree.

14. Let G be a connected graph that is not a path. Prove that there are at least three vertices with the property that the removal of any one of them does not disconnect the graph.

15. Prove that a planar bipartite graph with $n \geq 3$ vertices has at most $2n - 4$ edges.

16. Prove that a planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges.

17. A **forest** is a graph that contains no cycle. Prove that a forest on n vertices with k connected components has $n - k$ edges.

18. Suppose that we run the Bubbles algorithm on a graph with n vertices. Show that the algorithm terminates in at most $\log_2 n$ steps.

From Lovász, Pelikán and Vesztergombi: 9.2.4, 9.2.5,