This document contains a list of the important definitions and theorems that have been covered in the course but do not appear on the previous review sheets. It is not a complete listing of what has happened in lecture. The sections from the book that correspond with each topic are also given.

Note: The final exam is cumulative. You should consult the previous review sheets for review materials that cover the material that was tested on exams 1-3.

Following the list of important definitions and theorems you will find a collection of review exercises.

8. Minimum Weight Spanning Tree:

Matoušek and Nešetřil. Sections 4.3 - 4.5.
Lovász, Pelikán and Vestergombi. Section 9.1

Definition 1. Suppose $G = (V, E)$ is a connected graph and $f : E \to \mathbb{R}^+$ is an assignment of weights to the edges of $G$. Let $T$ be the collection of subsets $S$ of $E$ with the property that $(V, S)$ is a tree. A minimum weight spanning tree of $G$ is a tree $T \in T$ such that

$$\sum_{e \in T} w(e) = \min_{S \in T} \sum_{e \in S} w(e).$$

Theorem 2. Let $G = (V, E)$ be a connected graph and let $w : E \to \mathbb{R}^+$ be an assignment of weights to the edges of $G$. The following greedy algorithm produces a minimum weight spanning tree.

Kruskal’s Algorithm

Set $T = \emptyset$
While $|T| \leq |V| - 2$

Let $X$ be the set of edges $e$ in $E \setminus T$

such that $(V, T) + e$ does not contain a cycle

Choose $e \in X$ of minimum weight

Add $e$ to $T$
Theorem 3. Let $G = (V, E)$ be a connected graph and let $w : E \rightarrow \mathbb{R}^+$ be an assignment of distinct weights to the edges of $G$ (i.e. $f$ is an injective map). The following greedy algorithm produces a minimum weight spanning tree

**Bubbles Algorithm**

Set $E_0 = \emptyset$

For $i = 0, \ldots, |V| - 2$

Let $X_1, \ldots, X_{|V| - i}$ be the connected components of $(V, E_i)$

Let $e_{i+1}$ be the edge of minimum weight among edges of $E$ that intersect two of the sets $X_1, \ldots, X_{|V| - i}$.

Set $E_{i+1} = E_i \cup \{e_{i+1}\}$.

**Review Exercises:** Working the following problems should help in preparation for the test. Some (but not all) of these are more difficult than questions that might appear on the quiz.

**Note:** the final is cumulative. This list does represent the full scope of questions that could appear on the final. See the review sheets for the tests for a more complete list of review exercises.

1. Let $X = \{1, 2, \ldots, n\}$. Let $\mathcal{F}$ be a collection of 3-element subsets of $X$ (formally, $\mathcal{F} \subseteq \binom{X}{3}$). For each $i \in X$ define $d(i) = |\{A \in \mathcal{F} : i \in A\}|$. In words, $d(i)$ is the number of sets in $\mathcal{F}$ that contain $i$. We define

   $$A = \{i \in X : d(i) \equiv 1 \mod 3\} \quad B = \{i \in X : d(i) \equiv 2 \mod 3\}.$$

   Prove that $|A| \equiv |B| \mod 3$.

2. What is the number of strings consisting of $p$ 0’s and $q$ 1’s in which each pair of 1’s is separated by at least two 0’s? Explain your answer.

3. Prove that for $n = 1, 2, \ldots$ we have

   $$2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.$$

4. (a) Show that

   $$(1 - 4x)^{-1/2} = \sum_{n \geq 0} \binom{2n}{n} x^n.$$

   (b) Simplify

   $$\sum_{n \geq 0} \binom{2n - 1}{n} x^n$$

5. (a) Determine the generating function for the sequence $a_0, a_1, \ldots$ defined by the recurrence

   $$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2$$

   and $a_0 = a_1 = 1$.

   (b) Give a closed form expression for $a_n$. 
6. Let \( a_n \) be the number of ordered triples \((i, j, k)\) of integer numbers such that \( i \geq 0, j \geq 1, k \geq 1 \) and \( i + 3j + 3k = n \). Find the generating function for the sequence \( a_0, a_1, a_2, \ldots \).

7. Let \( k \) be a positive integer. Let \( G = (V,E) \) be a graph such that \( |V| = 6k \), half of the vertices in \( G \) have degree \( 2k \) and the other half of the vertices in \( G \) have degree \( 4k \). Show that \( G \) is Hamiltonian.

8. Ann and Bob play a game in which they alternately toss a fair die, with Ann rolling first. The one who is first to roll a 6 wins the game.
   (a) Set up a probability space that describes this experiment.
   (b) Determine the probability that Ann wins on her second roll.
   (c) Determine the probability that Ann wins the game.

9. A coloring of a collection of \( m \) indistinguishable balls with \( n \) (distinguishable) colors is chosen uniformly at random from the set of all such colorings.
   (i) Describe the probability space for this experiment.
   (ii) Let the random variable \( X \) count the number of colors that are used exactly twice. Determine \( E[X] \).

10. Let \( n \) be a positive integer. Let \( X \) be the set of strings \( x_1, x_2, \ldots, x_{2n} \) consisting of \( n \) 1’s and \( n - 1 \)’s such that
    \[
    x_1 + x_2 + \cdots + x_k \geq 0 \quad \text{for} \quad k = 1, 2, \ldots, 2n.
    \]
    Show that
    \[
    |X| = \frac{1}{n+1} \binom{2n}{n}
    \]

11. Let \( G = (V,E) \) be a connected graph. Prove that there is a graph \( H = (V,F) \) such that \( F \subseteq E \) and \( H \) is a tree.

12. Let \( G \) be a connected graph with a weight function \( w \) on the edges, and assume that \( w \) is an injective function (i.e. every edge gets a different weight). Prove that there is only one minimum weight spanning tree.

13. Suppose we have a plane graph \( G = (V,E) \) with \( n \) vertices and we assign weights to the edges by setting the weight of edge \( \{x,y\} \) to be the distance between \( x \) and \( y \). Let \( T \) be a minimum weight spanning tree. Show \( T \) has no vertex of degree 7 or more.

14. Let \( G = (V,E) \) be a connected graph and let \( w : E \to \mathbb{R}^+ \) be a weight function for the edges. Two players, Optimist and Pessimist collaborate in forming a spanning tree in \( G \). They proceed in a series of rounds. At the start we set \( A = E \), where \( A \) is the set of available edges, and we set \( C = \emptyset \), where \( C \) is the set of chosen edges. In each round a coin is flipped. If the outcome is a Head then the Optimist chooses an edge \( e \) of \( A \) of minimum weight with the property that \( (V,C) + e \) does not form a cycle, and the edge is added to \( C \) and removed from \( A \). If the outcome is a Tail then the Pessimist selects an edge \( f \) of maximum weight among the edges of \( A \) with the property that \( (V,A \cup C) - f \) is connected. The edge \( f \) is removed from \( A \). This process continues until neither player can make a move. Show that this process results in a minimum weight spanning tree in \( G \) regardless of the outcome of the coin flips.

From LPV: 9.2.4.