21-228 Combinatorics
Course Review 4

This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is *not* a complete listing of what has happened in lecture. The sections from the book that correspond with each topic are also given.

Following the list of important definitions and theorems you will find a collection of review exercises and sample test questions.

11. Recurrence relations and generating functions cont.:

Notation 1. *If $f(x)$ is a polynomial in x then $[x^n]f(x)$ is the coefficient of x^n in f. For example, if $f(x) = 1 + 22x^2 + 3x^3 + 5x^4$ then $[x^2]f(x) = 22$.*

Note 2. *If $f_1(x), f_2(x), \ldots, f_k(x)$ are polynomials in x then*

$$[x^n]\prod_{i=1}^{k} f_i(x) = \sum_{\substack{i_1,\ldots,i_k \in \mathbb{N} \\ i_1+i_2+\ldots i_k = n}} [x^{i_1}]f_1(x)\cdot[x^{i_2}]f_2(x)\cdot\cdots\cdot[x^{i_k}]f_k(x).$$

Definition 3. *Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. The* **generating function** *for this sequence is*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Note 4. *A generating function can be viewed as either*

(i) a function of x (when we have convergence), or

(ii) a formal object with addition and multiplication.

Note 5.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

This can be viewed as either

(i) A fact for formal power series that follows from noting that

$$(1-x)(1+x+x^2+\cdots) = 1, \text{ or}$$

(ii) the power series for the function $1/(1-x)$, which converges for $|x| < 1$.

Definition 6. *The* **Catalan sequence** *is the sequence $c_0, c_1, \ldots$ where*

$$c_n = \frac{1}{n+1}\binom{2n}{n} \qquad \text{for } n = 0, 1, 2, \ldots$$

The number c_n is called the n^{th} **Catalan number**.

Theorem 7. *The number of triangulations of an n-gon (ways to draw line segments connecting the vertices of the n-gon in such a way that the line segments do not cross and all the regions formed are triangles) is* $c_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$.

12. Finite Geometries. Sections 14.1 and 14.2.

Definition 8. *A* **combinatorial projective plane** *is an ordered pair (X, L) where X is a finite set of* **points**, *$L \subseteq 2^X$ is a collection of* **lines** *and the following 4 conditions hold:*

1. *Each pair of points defines a line: If $\{u, , v\} \in \binom{X}{2}$ then there exists a unique $\ell \in L$ such that $u, v \in \ell$.*
2. *Any two lines intersect in exactly one point: If $\ell_1, \ell_2 \in L$ are distinct then $|\ell_1 \cap \ell_2| = 1$.*
3. *Every line contains at least 3 points.*
4. *$L \neq \{X\}$.*

Definition 9. *Let (X, L) be a projective plane. The* **incidence matrix** *of this plane is the $|L| \times |X|$, $0-1$ matrix with rows indexed by L, columns indexed by X and a 1 in the (ℓ, x) entry iff $x \in \ell$.*

Claim 10. *If ℓ_1, ℓ_2 are lines in a projective plane (X, L) then $|\ell_1| = |\ell_2|$.*

Definition 11. *The* **order** *of a projective plane (X, L) is n if every line in the plane consists of $n + 1$ points.*

Claim 12. *If x, y are points in a projective plane (X, L) then*

$$|\{\ell \in L : x \in \ell\}| = |\{\ell \in L : y \in \ell\}|$$

Claim 13. *If x is a point in a projective plane (X, L) of order n then*

$$|\{\ell \in L : x \in \ell\}| = n + 1$$

Claim 14. *If (X, L) is a projective plane of order n then $|X| = n^2 + n + 1$.*

Example 15. *Here we give a construction of a projective plane of order 3. First, we let $\mathbb{Z}_3 = \{0, 1, 2\}$. Arithmetic on this set is performed modulo 3 (e.g. 2+2 = 1). We introduce 2 sets*

$$A = \{(x, y) : x, y \in \mathbb{Z}_3\} \qquad\qquad B = \{x_0, x_1, x_2, x_\infty\}.$$

Let K be the set of lines in A; that is, the collection K consists of the following sets

$$\begin{array}{ll} \{(x,y) \in A : x + y = \gamma\} & \text{for} \quad \gamma = 0,1,2 \\ \{(x,y) \in A : x + 2y = \gamma\} & \text{for} \quad \gamma = 0,1,2 \\ \{(x,y) \in A : x = \gamma\} & \text{for} \quad \gamma = 0,1,2 \\ \{(x,y) \in A : y = \gamma\} & \text{for} \quad \gamma = 0,1,2 \end{array}$$

Note that any two lines in K either are in the same group or they intersect in exactly one point. Also note that there is exactly one line through any pair of points. We extend *the pair (A, K) to a projective plane (X, L) by setting $X = A \cup B$ and letting L be the following set of lines:*

$$\begin{array}{ll}
\{(x,y) \in A : x + y = \gamma\} \cup \{x_1\} & \text{for} \quad \gamma = 0,1,2 \\
\{(x,y) \in A : x + 2y = \gamma\} \cup \{x_2\} & \text{for} \quad \gamma = 0,1,2 \\
\{(x,y) \in A : x = \gamma\} \cup \{x_0\} & \text{for} \quad \gamma = 0,1,2 \\
\{(x,y) \in A : y = \gamma\} \cup \{x_\infty\} & \text{for} \quad \gamma = 0,1,2 \\
\{x_0, x_1, x_2, x_\infty\}. &
\end{array}$$

Fact 16. *If q is a prime power then there exists a projective plane of order q.*

Theorem 17. *Let q be a prime power and let $m = q^2 + q + 1$. For A an $m \times m$ matrix whose entries are either 0 or 1, define $\|A\|_1$ to be the number of 1's in A.*

(a) If A has no 2×2 submatrix of all 1's then the $\|A\|_1 \le m^{3/2} + m/2$.

(b) The incidence matrix of a projective plane of order q is an $m \times m$, $0 - 1$ matrix with no 2×2 submatrix of 1's. There are

$$(q^2 + q + 1)(q + 1) > m^{3/2}.$$

1's in this matrix.

REVIEW EXERCISES: Working the following problems should help in preparation for the test. They are not necessarily 'sample' test questions.

1. A word over the alphabet $\{a, b, c, \ldots, z\}$ is called *increasing* if its letters, apart from repetitions, appear in alphabetical order. For example *aabcc* is increasing but *abzc* is not. How many increasing words are there of length n?
2. Let $h_0, h_1, h_2, \ldots$ be the sequence defined by $h_n = \binom{n}{3}$. Determine the generating function for this sequence.
3. Suppose (X, L) is a projective plane of order n. Prove that $|L| = n^2 + n + 1$.

From the text: 14.1.2, 14.1.3, 14.1.4.