21-228 Discrete Mathematics
Course Review 3

This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is *not* a complete listing of what has happened in lecture. The sections from the book that correspond with each topic are also given.

Following the list of important definitions and theorems you will find a collection of review exercises.

6. Discrete Probability continued:

   Matoušek and Nešetřil. Sections 9.3 and 9.4.

   Lovász, Pelikán and Vesztergombi. Sections 5.2 and 5.3

Definition 53. *Let A be an event in a probability space defined on the set Ω. The* **probability of $\omega \in \Omega$ conditioned on A** *is*

$$Pr(\omega|A) = \begin{cases} \frac{Pr(\omega)}{Pr(A)} & \text{if } \omega \in A \\ 0 & \text{otherwise.} \end{cases}$$

If $B \subseteq \Omega$ the **probability of B conditioned on A** *is*

$$Pr(B|A) = \frac{Pr(A \cap B)}{Pr(A)}.$$

Theorem 54 (Law of Total Probability)**.** *Let a probability space be defined on set Ω and let $B_1, \ldots, B_n$ be a partition of Ω. For $A \subseteq \Omega$ we have*

$$Pr(A) = \sum_{i=1}^{n} Pr(A|B_i)Pr(B_i).$$

Definition 55. *Two events $A, B \subseteq \Omega$ are* **independent** *if*

$$Pr(A \cap B) = Pr(A)Pr(B).$$

Equivalently:

$$Pr(A|B) = Pr(A).$$

Definition 56. *A collection $A_1, \ldots, A_n$ of events is* **mutually independent** *if for every subset I of $\{1, 2, \ldots, n\}$ we have*

$$Pr\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} Pr(A_i)$$

Definition 57. *Let a probability space of set Ω be given. A function $f : \Omega \to \mathbb{R}$ is called a* **random variable**.

Definition 58. *If Z is a random variable on a probability space defined on the set Ω then*

$$E(Z) = \sum_{\omega \in \Omega} Z(\omega)\mathbb{P}(\omega)$$

is the **expected value** *of Z.*

Claim 59 (Linearity of Expectations)**.** *If $X_1, \dots, X_n$ are random variables defined on a common probability space and $c_1, c_2, \dots, c_n \in \mathbb{R}$ then*

$$E[c_1X_1 + c_2X_2 + \cdots + c_nX_n] = c_1E[X_1] + c_2E[X_2] + \cdots + c_nE[X_n].$$

Claim 60. *If X is a random variable then*

$$\mathbb{P}(X \geq E[X]) > 0 \qquad \text{and} \qquad \mathbb{P}(X \leq E[X]) > 0$$

Definition 61. *Let $n \in \{1, 2, \dots\}$ and $0 \leq p \leq 1$. A random variable X that takes values in $\{0, \dots, n\}$ is distributed as a* **Binomial random variable with parameters** *n, p if*

$$\mathbb{P}(X = k) = p^k(1-p)^{n-k}\binom{n}{k} \qquad \text{for } k = 0, \dots n.$$

We write X is distributed as $Bi(n, p)$.

Note 62. *Let $\Omega = \{H, T\}^n$ and consider the probability space on Ω in which*

$$\mathbb{P}(\omega) = p^k(1-p)^{n-k}$$

where k is the number of H's in ω (note that this probability space is given by a sequence of mutually independent flips of a coin that comes up H with probability p). The random variable $X : \Omega \to \{0, 1, \dots, n\}$ where $X(\omega)$ is defined to be the number of H's in ω is distributed as binomial random variable with parameters n, p.

Claim 63. *If X is distributed as $Bin(n, p)$ then $E[X] = np$.*

Definition 64. *Let $\lambda > 0$. A random variable Y that takes values in $\{0, 1, 2, \dots\}$ is distributed as a* **Poisson random variable** *with parameter λ if*

$$Pr(Y = k) = e^{-}\frac{\lambda^k}{k!} \qquad \text{for } k = 0, 1, 2, \dots.$$

We say Y is distributed as $Po(\lambda)$.

Claim 65. *If $\lambda > 0$ and Y is distributed as $Po(\lambda)$ then $E[Y] = \lambda$.*

Note 66. *Let $\lambda > 0$ be a fixed constant. Consider a sequence of random variables $X_2, X_3, \dots$ where X_n is distributed as $Bi(n, \lambda/n)$. For a fixed positive integer k we have*

$$Pr(X_n = k) = \binom{n}{k}p^k(1-p)^{n-k} \sim \frac{n^k}{k!}\left(\frac{\lambda}{n}\right)^k\left(1 - \frac{\lambda}{n}\right)^{n-k}$$
$$\sim e^{-\lambda}\frac{\lambda^k}{k!} = Pr(Y = k),$$

where Y is distributed as $Po(\lambda)$.

Claim 67 (Markov's Inequality)**.** *If X is a random variable that takes only non-negative values then for any positive t*

$$\mathbb{P}(X \geq t) \leq \frac{E[X]}{t}$$

Definition 68. *Let X be a random variable with $\mu = E[X]$. The* **variance** *of X is defined to be*

$$Var[X] = E[(X - \mu)^2].$$

Note 69. *If X is distributed as $Bi(n,p)$ then $Var(X) = np(1-p)$.*

Claim 70 (Chebyshev's inequality)**.** *Let Z be a random variable such that $E[Z] = \mu$ and $Var[Z] = \sigma^2 > 0$. For any $t > 0$ we have*

$$\mathbb{P}(|Z - \mu| \geq t\sigma) \leq \frac{1}{t^2}.$$

Definition 71. *Let X and Y be random variables defined on a common probability space. Define*

$$A = \{\alpha \in \mathbb{R} :| Pr(X = \alpha) > 0\} \qquad B = \{\beta \in \mathbb{R} \mid Pr(Y = \beta) > 0\}$$

We say that X and Y are **independent random variables** *if the events $\{\omega \in \Omega : X(\omega) = \alpha\}$ and $\{\omega \in \Omega : Y(\omega) = \beta\}$ are independent events for all $\alpha \in A, \beta \in B$.*

Definition 72. *A collection $X_1, X_2, \ldots, X_n$ of random variables defined on a common probability space is* **mutually independent** *if every collection of events of the form*

$$\{\{X_i = \alpha_i\} : i \in I\},$$

where $I \subseteq \{1, 2, \ldots, n\}$ and $\alpha_i \in \mathbb{R}$ for all $i \in I$, is mutually independent.

Theorem 73 (Law of Large Numbers)**.** *Let $X_1, X_2, \ldots$ be a sequence of pairwise independent random variabless with commone (finite) mean μ and common (finite) variance $\sigma^2 > 0$ then for any $\epsilon > 0$ we have*

$$\lim_{n\to\infty} Pr\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \epsilon\right) = 0.$$

7. Graph Theory:

Matoušek and Nešetřil. Sections 3.1-3.4, Section 4.1, and Sections 5.1 - 5.3.

Loász, Pelikán and Vesztergombi. Chapter 7, Sections 8.1-8.2 and Chapter 12.

Definition 74. *A* **graph** *is an ordered pair $G = (V, E)$ where V is a finite set and $E \subseteq \binom{V}{2}$. The set V is the vertex set of G, and E is the edge set of G.*

Definition 75. *If $G = (V, E)$ is a graph then the* **degree** *of a vertex x is the number of edges of G that contain x. Written formally,*

$$d_G(x) = d(x) = |\{e \in E : x \in E\}|.$$

Theorem 76 (Handshaking Lemma)**.** *If $G = (V, E)$ is a graph then*

$$\sum_{v\in V} d(v) = 2|E|.$$

Definition 77. *Let $G = (V, E)$ be a graph.*

- *A sequence of vertices $x_0, x_1, x_2, \ldots, x_k$ is a* **walk** *in G if $\{x_i, x_{i+1}\} \in E$ for $i = 0, \ldots, k-1$.*
- *A sequence of distinct vertices $x_0, x_1, x_2, \ldots, x_k$ such that $\{x_i, x_{i+1}\} \in E$ for $i = 0, \ldots, k-1$ is a* **path** *in G.*
- *A sequence of distinct vertices $x_0, x_1, x_2, \ldots, x_k$ such that $k \geq 2$ and $\{x_i, x_{i+1}\} \in E$ for $i = 0, \ldots, k-1$ and $\{x_0, x_k\} \in E$ is a* **cycle** *in G.*

Definition 78. *A graph $G = (V, E)$ is* **connected** *if for every pair of vertices $x, y \in V$ there is a path in G joining x and y.*

Definition 79. *A walk in a graph $G = (V, E)$ is an* **Eulerian walk** *if it traverses every edge in E exactly once.*

Theorem 80. *Let $G = (V, E)$ be a graph with at least 2 vertices. There is an Eulerian walk in G if and only if G is connected and the number of vertices in G of odd degree is either 0 or 2.*

Definition 81. *A* **Hamilton cycle** *in a graph $G = (V, E)$ is a cycle that contains every vertex in V. The graph G is* **Hamiltonian** *if it has a Hamilton cycle.*

Definition 82. *Let $G = (V, E)$ be a graph and let $e \in \binom{V}{2} \setminus E$. We define the graph $G + e$ to be the graph with vertex set V and edge set $E \cup \{e\}$.*

Lemma 83. *Let $G = (V, E)$ and $|V| = n$. If $x, y \in V$, $\{x, y\} \notin E$ and $d_G(x) + d_G(y) \geq n$ then*

$$G + e \text{ is Hamiltonian} \quad \Leftrightarrow \quad G \text{ is Hamiltonian.}$$

Definition 84. *Let $G = (V, E)$ be a graph. The* **minimum degree** *of G is*

$$\delta(G) = \min\{d(x) : x \in G\}.$$

The **maximum degree** *of G is*

$$\Delta(G) = \max\{d(x) : x \in G\}.$$

Theorem 85 (Dirac)**.** *If $G = (V, E)$ is a graph with $|V| = n$ and $\delta(G) \geq n/2$ then G is Hamiltonian.*

Definition 86. *A graph $G = (V, E)$ is a* **tree** *if it is connected and contains no cycles.*

Theorem 87. *(i) A graph G is a tree if it is connected but the deletion of any edge of G results in a disconnected graph*

(ii) A graph G is a tree if it contains no cycles but the addition of any edge to G results in a graph that contains a cycle.

Theorem 88. *If $G = (V, E)$ is a tree then $|E| = |V| - 1$.*

Corollary 89. *If $G = (V, E)$ is a tree then G has at least 2 vertices of degree 1.*

Definition 90. *For $x, y \in \mathbb{R}^2$ let $\ell(x, y)$ be the line segment in $\mathbb{R}^2$ joining x and y.*

Definition 91. *A graph $G = (V, E)$ is* **planar** *if there is an injective map $\varphi : V \to \mathbb{R}^2$ with the property that*

$$\{v_1, v_2\}, \{u_1, u_2\} \in E \quad \Rightarrow \quad \ell(\varphi(v_1), \varphi(v_2)) \text{ and } \ell(\varphi(u_1), \varphi(u_2)) \text{ do not have interior intersection.}$$

A planar graph G embedded in this way is called a **plane graph**. *When we discuss plane graphs we identify v and $\varphi(v)$.*

Definition 92. *If $G = (V, E)$ is a plane graph then the* **faces** *of G are the connected components of*

$$\mathbb{R}^2 \setminus \bigcup_{\{x,y\}\in E} \ell(x, y).$$

Theorem 93 (Euler's formula)**.** *If $G = (V, E)$ is a connected plane graph with f faces then*

$$|V| - |E| + f = 2.$$

REVIEW EXERCISES: Working the following problems should help in preparation for the test. Some (but not all) of these are more difficult than questions that might appear on the test.

1. An urn contains x Red balls, y Green balls and z Yellow ball. Balls are drawn from the urn one at a time and uniformly at random until a Yellow ball is drawn. When a Red ball is drawn it is returned to the urn. When a Green ball is drawn it is thrown away (and not returned to the urn). What is the probability that Yellow appears for the first time on the third draw?

2. A bag contains r red and b black balls. A ball is drawn uniformly at random from the bag and then returned to the bag. This is repeated until a black ball is drawn. What is the expected number of drawings in this process?

3. For simplicity assume that the probability of the birth of a boy and the birth of girl are the same (which is not quite so in reality). For a certain family, we know that they have exactly two children and that at least on of them is a boy. What is the probability that they have two boys?

4. Prove that for all integers n and $p \in [0, 1]$ we have

$$R(k, l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell} (1 - p)^{\binom{\ell}{2}}.$$

   Hint: Consider a random coloring of edge set of K_n in which we color each edge independently at random. Then remove some vertices from the graph in order to get a Ramsey coloring.

5. Let $0 < p < 1$. A $n \times n$, 0-1 matrix is chosen at random by independently flipping a biased coin (a coin that comes up heads with probability p and tails with probability $1 - p$) for each position. If the coin for a given position comes up heads then that position gets a 1 (if the coin comes up tails then the position gets a 0). Let X be the number of 2×2 submatrices of the random matrix that are all 1's.

(a) Determine $E[X]$.

(b) Let $p = 1/n$. Use Boole's inequality to show that $P(X = 0) \geq 3/4$.

(c) Let $p = 1/n$. Use (a) to show that $P(X = 0) \geq 3/4$.

(d) Let $p = 2/n$ and A be the event that the first row of the matrix is all 0's. Determine

$$\lim_{n\to\infty} P(A).$$

6. The *4-cycle* is the graph with vertex set $\{a, b, c, d\}$ and edge set

$$\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}.$$

This graph describes a network, and in this network each edge fails (independently) with probability p. Let G be the graph given by the edges in this network that *do not* fail. Let A be the event that there is a path in G from b to d. Let B be the event that there is a path in G from a to c. Compute $P(A)$ and $P(A|B)$. Are A and B independent events?

7. Let X be a distributed as a Poisson random variable with parameter λ_1 and let Y be a Poisson random variable with parameter λ_2. Let these be independent random variables; that is, for every $i, j \in \mathbb{N}$ the events $X = i$ and $Y = j$ are independent. What is $P(X + Y = n)$?

8. A group of n students are taking an exam and they are allowed to take their phones into the test. Every student has exactly one phone and each student gives the TA her or his phone as they enter the exam. After the exam the professor randomly returns the phones to students so that each student gets exactly one phone.

(a) Set up a probability space that describes this experiment.

(b) Let the random variable X be the number of students who get their own phone back. Determine $E[X]$ and $E[X^2]$.

(c) Use Markov's inequality to bound $Pr(X \geq 10)$.

(d) Use Chebyshev's inequality to bound $Pr(X \geq 10)$.

9. Let $G = (V, E)$ be a connected graph. Prove that there is a graph $H = (V, F)$ such that $F \subseteq E$ and H is a tree.

10. The *k-cube* is the graph with vertex set $\{(x_1, \ldots, x_k) : x_1, \ldots, x_k \in \{0, 1\}\}$ and an edge between two vertices if and only if they differ in exactly 1 coordinate. Show that the k-cube has a Hamilton cycle for every $k \geq 2$.

Hint: go by induction on k.

11. The *complete bipartite graph* $K_{m,n}$ has vertex set V with bipartition A, B where $|A| = n$ and $|B| = m$ and edge set

$$\{\{x, y\} : x \in A \text{ and } y \in B\}.$$

Which complete bipartite graphs have Hamilton cycles?

12. A connected graph $G = (V, E)$ has $|V| = 2k + 1$ vertices and exactly $k + 1$ vertices of degree 2, no two of which are adjacent. Show that G is not Hamiltonian.

13. Let G be a connected graph in which any two distinct vertices have either 0 or 7 common neighbors. Prove that G is a regular graph.

14. Prove that a **connected** planar bipartite graph with n nodes that contains a cycle has at most $2n-4$ edges.

15. For a graph $G=(V,E)$, Let $\mathcal{L}(G)$ denote the **line graph** of G, which we define as

$$\mathcal{L}(G)=\left(E,\left\{\{e,f\}\in\binom{E}{2}:e\cap f\neq\emptyset\right\}\right).$$

In words, the vertices of the line graph are the edge of G and we connect two vertices in the line graph if the corresponding edges intersect. Are the following statements true or false?

(a) G is connected if and only if $\mathcal{L}(G)$ is connected.

(b) G is Eulerian if and only if $\mathcal{L}(G)$ Hamiltonian.

16. Let T be a tree with n vertices, $n\geq 2$. For each positive integer i let p_i be the number of vertices of T of degree i. Prove

$$p_1-p_3-2p_4-\cdots-(n-3)p_{n-1}=2.$$

17. Is the Claim below true or false? If it is false find a counterexample. Is the proof correct? If not, why not?

Claim. The number of trees on vertex set $\{v_1,v_2,\ldots,v_n\}$ is $(n-1)!$.

Proof. We go by induction on n.

Base case: $n=2$. A connected graph on a vertex set of cardinality 2 must contain the edge connecting the two vertices. Therefore, there is only one such connected graph and only one such tree.

Inductive assumption: There are $(n-2)!$ trees on vertex set $\{v_1,v_2,\ldots,v_{n-1}\}$.
Let $n\geq 3$. Let $\mathcal{A}$ be the collection of all trees on vertex set $\{v_1,\ldots,v_{n-1}\}$.
Let $\mathcal{B}$ be the collection of trees on vertex set $\{v_1,\ldots,v_n\}$.
Since

(i) we get a tree on n vertices by attaching a leaf to a tree on $n-1$ vertices and

(ii) there are $n-1$ vertices in a tree in $\mathcal{A}$ to which we can attach the vertex v_n to get a tree in $\mathcal{B}$,

each tree in $\mathcal{A}$ corresponds to $n-1$ trees in $\mathcal{B}$. Therefore, $|\mathcal{B}|=(n-1)|\mathcal{A}|$.
Applying the inductive assumption we have

$$|\mathcal{B}|=(n-1)|\mathcal{A}|=(n-1)\cdot(n-2)!=(n-1)!$$

□

From *Lovász, Pelikán and Vesztergombi*: 5.4.5, 7.3.2, 7.3.10, 7.3.11, 7.3.13, 8.1.3, 8.5.3, 8.5.4, 8.5.8, 12.2.1, 12.2.2, 12.3.1, 12.3.6 *(For question 12.3.6: You may assume that the graph is 3 regular. Follow-up question: why can we assume this?).*