21-228 Discrete Mathematics
Course Review 3

This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is *not* a complete listing of what has happened in lecture. The sections from the book that correspond with each topic are also given.

Following the list of important definitions and theorems you will find a collection of review exercises.

9. Paths and Cycles: Chapter 7.

Theorem 1 (handshaking Lemma)**.** *If $G = (V, E)$ is a graph then*

$$\sum_{v \in V} d(v) = 2|E|.$$

Definition 2. *Let $G = (V, E)$ be a graph.*

- *A sequence of vertices $x_0, x_1, x_2, \ldots, x_k$ is a* **walk** *in G if $\{x_i, x_{i+1}\} \in E$ for $i = 0, \ldots, k-1$.*
- *A sequence of distinct vertices $x_0, x_1, x_2, \ldots, x_k$ such that $\{x_i, x_{i+1}\} \in E$ for $i = 0, \ldots, k-1$ is a* **path** *in G.*
- *A sequence of distinct vertices $x_0, x_1, x_2, \ldots, x_k$ such that $\{x_i, x_{i+1}\} \in E$ for $i = 0, \ldots, k-1$ and $\{x_0, x_k\} \in E$ is a* **cycle** *in G.*

Definition 3. *A graph $G = (V, E)$ is* **connected** *if for every pair of vertices $x, y \in V$ there is a path in G joining x and y.*

Definition 4. *A walk in a graph $G = (V, E)$ is an* **Eulerian walk** *if it traverses every edge in E exactly once.*

Theorem 5. *Let $G = (V, E)$ be a graph with no isolated vertices. There is an Eulerian walk in G if and only if G is connected and the number of vertces in G of odd degree is either 0 or 2.*

Definition 6. *A* **Hamilton cycle** *in a graph $G = (V, E)$ is a cycle that contains every vertex in V. A* **Hamilton path** *in G ia a path that contains every vertex in V.*

Definition 7. *Let $G = (V, E)$ be a graph and let $e \in \binom{V}{2} \setminus E$. We define the graph $G + e$ to be the graph with vertex set V and edge set $E \cup \{e\}$.*

Lemma 8. *Let $G = (V, E)$ and $|V| = n$. If $x, y \in V$, $\{x, y\} \notin E$ and $d_G(x) + d_G(y) \geq n$ then*

$$G + e \text{ is Hamiltonian} \quad \Leftrightarrow \quad G \text{ is Hamiltonian.}$$

Definition 9. *Let $G = (V, E)$ be a graph. The* **minimum degree** *of G is*

$$\delta(G) = \min\{d(x) : x \in G\}.$$

Corollary 10. *If $G = (V, E)$ is a graph with $|V| = n$ and $\delta(G) \geq n/2$ then G is Hamiltonian.*

10. TREES: Section 8.1, Section 9.1, Chapter 12.

Definition 11. *A graph $G = (V, E)$ is a* **tree** *if it is connected and contains no cycles.*

Theorem 12. *(i) A graph G is a tree if it is connected but the deletion of any edge of G results in a disconnected graph*

(ii) A graph G is a tree if it contains no cycles but the addition of any edge to G results in a graph that contains a cycle.

Theorem 13. *If $G = (V, E)$ is a tree then $|E| = |V| - 1$.*

Corollary 14. *If $G = (V, E)$ is a tree then G has at least 2 vertices of degree 1.*

Definition 15. *For $x, y \in \mathbb{R}^2$ let $\ell(x, y)$ be the line segment in $\mathbb{R}^2$ joining x and y.*

Definition 16. *A graph $G = (V, E)$ is* **planar** *if there is an embedding $\varphi : V \to \mathbb{R}^2$ with the property that*

$$\{v_1, v_2\}, \{u_1, u_2\} \in E \quad \Rightarrow \quad \ell(\varphi(v_1), \varphi(v_2)) \text{ and } \ell(\varphi(u_1), \varphi(u_2)) \text{ do not have interior intersection.}$$

A planar graph G embedded in this way is called a **plane graph**. *When we discuss plane graphs we identify v and $\varphi(v)$.*

Definition 17. *If $G = (V, E)$ is a plane graph then the* **faces** *of G are the connected components of*

$$\mathbb{R}^2 \setminus \bigcup_{\{x,y\} \in E} \ell(x, y).$$

Theorem 18 (Euler's formula)**.** *If $G = (V, E)$ is a connected plane graph with f faces then*

$$|V| - |E| + f = 2.$$

Definition 19. *Suppose $G = (V, E)$ is a connected graph and $f : E \to \mathbb{R}^+$ is an assignment of weights to the edges of G. Let $\mathcal{T}$ be the collection of subsets S of E with the property that (V, S) is a tree. A* **minimum weight spanning tree** *of G is a tree $T \in \mathcal{T}$ such that*

$$\sum_{e \in T} w(e) = \min_{S \in \mathcal{T}} \sum_{e \in S} w(e).$$

Definition 20. *Let $G = (V, E)$ be a connected graph and $\varphi : E \to \mathbb{R}^+$ be a set of edge costs. The following is an example of a greedy algorithm for producing a tree of small total cost. We begin with the $F = \emptyset$. While $|F| < |V| - 1$ we choose an edge $e \in E \setminus F$ such that*

(a) $(V, F) + e$ does not contains a cycle, and

(b) the cost of e (i.e. $\varphi(e)$) is smallest among the edges $e \in E \setminus F$ that satisfy condition (a).

and set $F = F \cup \{e\}$.

Theorem 21. *Let $G = (V, E)$ be a connected graph and $\varphi : E \to \mathbb{R}$ be a set of edge costs. The greedy algorithm given in the previous definition produces a minimum weight spanning tree on G.*

11. Recurrence Relations and Generating Function: Chapter 4 gives a nice example of a homogeneous linear recurrence relation.

Definition 22. *Let $a_0, a_1, a_2, \ldots$ be an infinite sequence. A* **recurrence relation** *for this sequence is a function $f : \mathbb{R}^{k+1} \to \mathbb{R}$ such that*

$$a_n = f(n, a_{n-1}, a_{n-2}, \ldots, a_{n-k})$$

for $n = k, k+1, \ldots$.

Definition 23. *Let $a_0, a_1, \ldots$ be an infinite sequence.*

(a) This sequence satisfies a **homogeneous linear recurrence relation of order** *k if there exist constants $c_1, c_2, \ldots, c_k$ such that*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$$

for $n = k, k+1, \ldots$.

(b) The **characteristic polynomial** *(or* **characteristic equation***) of this homogeneous linear recurrence relation is the polynomial*

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_{k-1} x - c_k.$$

(c) The roots of the characteristic polynomial are called the **characteristic roots** *of the homogeneous linear recurrence relation.*

Theorem 24. *If $q_1, \ldots, q_k$ are the distinct roots of the characteristic polynomial of a homogeneous linear recurrence relation of order k then*

$$a_n = \alpha_1 q_1^n + \alpha_2 q_2^n + \cdots + \alpha_k q_k^n,$$

where $\alpha_1, \ldots, \alpha_k$ are constants, is the general solution of the recurrence.

Definition 25. *Let $a_0, a_1, \ldots$ be an infinite sequence. The* **generating function** *for this sequence is*

$$\sum_{n=0}^{\infty} a_n x^n.$$

Review Exercises: Working the following problems should help in preparation for the test. Some (but not all) of these are more difficult than questions that might appear on the quiz.

1. Solve the recurrence relation $b_n = 3b_{n-2} - 2b_{n-3}$ for $n \geq 3$ with initial values $b_0 = 1$, $b_1 = 0$, $b_2 = 0$.

2. Solve the recurrence relation $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$ for $n \geq 3$ with initial values $h_0 = 0$, $h_1 = 1$, $h_2 = 2$. Is h_n positive for all n?

3. Let $a_n = n^2$ for $n = 0, 1, 2, \ldots$. Determine the generating function for this sequence.

4. Let $G = (V, E)$ be a connected graph. Prove that there is a graph $H = (V, F)$ such that $F \subseteq E$ and H is a tree.

5. The *k-cube* is the graph with vertex set $\{(x_1, \ldots, x_k) : x_1, \ldots, x_k \in \{0, 1\}\}$ and an edge between two vertices if and only if they differ in exactly 1 coordinate. Show that the k-cube has a Hamilton cycle for every $k \geq 2$.

   Hint: go by induction on k.

6. The *complete bipartite graph* $K_{m,n}$ has vertex set V with bipartition A, B where $|A| = n$ and $|B| = m$ and edge set

$$\{\{x, y\} : x \in A \text{ and } y \in B\}.$$

   Which complete bipartite graphs have Hamilton cycles?

7. Prove the following:

   If $G = (V, E)$ is a graph with $|V| = n$ and the property

$$\{x, y\} \notin E \quad \Rightarrow \quad d(x) + d(y) \geq n$$

   then G is Hamiltonian.

From the text: 4.3.7, 4.3.12, 7.3.2, 7.3.10, 7.3.13, 8.5.3, 8.5.4, 8.5.8, 8.5.10, 9.2.3, 9.2.4, 9.2.6, 12.2.1, 12.2.2, 12.3.1, 12.3.4 (*hint: is there a lower bound on the number of edges on the boundary of each face?*).