This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is not a complete listing of what has happened in lecture. The sections from the book that correspond with each topic are also given.

Following the list of important definitions and theorems you will find a collection of review exercises.

7. DISCRETE PROBABILITY CONTINUED: Chapter 5, Section 2.5. Note that part of what we cover here is not in the book.

Definition 1. Let a probability space of set $\Omega$ be given. A function $f: \Omega \rightarrow \mathbb{R}$ is called a random variable.

Definition 2. If $Z$ is a random variable on a probability space defined on the set $\Omega$ then

$$E(Z) = \sum_{\omega \in \Omega} Z(\omega)\mathbb{P}(\omega)$$

is the expected value of $Z$.

Claim 3. If $X_1, \ldots, X_n$ are random variables defined on a common probability space then

$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n].$$

Claim 4. If $X$ is a random variable then

$$\mathbb{P}(X \geq E[X]) > 0 \quad \text{and} \quad \mathbb{P}(X \leq E[X]) > 0$$

Theorem 5. For any positive integer $n$ we have

$$R(k, k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$$

Corollary 6.

$$R(k, k) > \frac{k-2}{e} \cdot 2^{k/2}.$$ 

Definition 7. Let $n \in \{1, 2, \ldots\}$ and $0 \leq p \leq 1$. A random variable $X$ that takes values in $\{0, \ldots, n\}$ is distributed as a Binomial random variable with parameters $n, p$ if

$$\mathbb{P}(X = k) = p^k(1-p)^{n-k}\binom{n}{k} \quad \text{for} \quad k = 0, \ldots n.$$ 

We write $X \sim Bi(n, p)$.

Note 8. Let $\Omega = \{H,T\}^n$ and consider the probability space on $\Omega$ in which

$$\mathbb{P}(\omega) = p^k(1-p)^{n-k}$$

where $k$ is the number of $H$’s in $\omega$ (note that this probability space is given by a sequence of mutually independent flips of a coin that comes up $H$ with probability $p$). The random variable $X : \Omega \rightarrow \{0, 1, \ldots, n\}$ where $X(\omega)$ is defined to be the number of $H$’s in $\omega$ is distributed as binomial random variable with parameters $n, p$. 
Claim 9. If $X \sim \text{Bin}(n, p)$ then $E[X] = np$.

Claim 10 (Markov's Inequality). If $X$ is a random variable that takes only non-negative values then for any positive $t$
$$
P(X \geq t) \leq \frac{E[X]}{t}$$

Definition 11. Let $X$ be a random variable with $\mu = E[X]$. The variance of $X$ is defined to be
$$
\text{Var}[X] = E[(X - \mu)^2].
$$

Note 12. If $X \sim \text{Bi}(n, p)$ then $\text{Var}(X) = np(1-p)$.

Claim 13 (Chebyshev's Inequality). Let $Z$ be a random variable such that $E[Z] = \mu$ and $\text{Var}[Z] = \sigma^2$. For any $t \geq 1$ we have
$$
P(|Z - \mu| \geq t\sigma) \leq \frac{1}{t^2}.$$

Definition 14. Let $X$ and $Y$ be random variables defined on a common probability space. Define
$$
A = \{\alpha \in \mathbb{R} \mid \text{Pr}(X = \alpha) > 0\} \quad B = \{\beta \in \mathbb{R} \mid \text{Pr}(Y = \beta) > 0\}
$$
We say that $X$ and $Y$ are independent random variables if the events $\{\omega \in \Omega : X(\omega) = \alpha\}$ and $\{\omega \in \Omega : Y(\omega) = \beta\}$ are independent events for all $\alpha \in A, \beta \in B$.

Definition 15. A collection $X_1, X_2, \ldots, X_n$ of random variables defined on a common probability space is independent if every collection of events of the form
$$
\{\{X_i \geq \alpha_i\} : i \in I\},
$$
where $I \subseteq \{1, 2, \ldots, n\}$ and $\alpha_i \in \mathbb{R}$ for all $i \in I$, is independent.

Theorem 16 (Law of Large Numbers). Let $X_1, X_2, \ldots$ be a sequence of independent random variables that all have the same distribution (i.e. $\text{Pr}(X_i = \alpha) = \text{Pr}(X_j = \alpha)$ for all $i, j$ and all $\alpha$). Suppose $\mu = E[X_i]$ and $\text{Var}[X_i]$ are finite. For any $\epsilon > 0$ we have
$$
\lim_{n \to \infty} \text{Pr}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \epsilon\right) = 0.
$$

8. Graph Theory: Chapter 7, Section 8.1, Chapter 12.

Definition 17. A graph is an ordered pair $G = (V, E)$ where $V$ is a finite set and $E \subseteq \binom{V}{2}$. The set $V$ is the vertex set of $G$, and $E$ is the edge set of $G$.

Definition 18. If $G = (V, E)$ is a graph then the degree of a vertex $x$ is the number of edges of $G$ that contain $x$. Written formally,
$$
d_G(x) = d(x) = |\{e \in E : x \in E\}|.
$$

Theorem 19 (Handshaking Lemma). If $G = (V, E)$ is a graph then
$$
\sum_{v \in V} d(v) = 2|E|.
$$
Definition 20. Let $G = (V, E)$ be a graph.

- A sequence of vertices $x_0, x_1, x_2, \ldots, x_k$ is a walk in $G$ if $\{x_i, x_{i+1}\} \in E$ for $i = 0, \ldots, k - 1$.
- A sequence of distinct vertices $x_0, x_1, x_2, \ldots, x_k$ such that $\{x_i, x_{i+1}\} \in E$ for $i = 0, \ldots, k - 1$ is a path in $G$.
- A sequence of distinct vertices $x_0, x_1, x_2, \ldots, x_k$ such that $k \geq 2$ and $\{x_i, x_{i+1}\} \in E$ for $i = 0, \ldots, k - 1$ and $\{x_0, x_k\} \in E$ is a cycle in $G$.

Definition 21. A graph $G = (V, E)$ is connected if for every pair of vertices $x, y \in V$ there is a path in $G$ joining $x$ and $y$.

Definition 22. A walk in a graph $G = (V, E)$ is an Eulerian walk if it traverses every edge in $E$ exactly once.

Theorem 23. Let $G = (V, E)$ be a graph with no isolated vertices. There is an Eulerian walk in $G$ if and only if $G$ is connected and the number of vertices in $G$ of odd degree is either 0 or 2.

Definition 24. A Hamilton cycle in a graph $G = (V, E)$ is a cycle that contains every vertex in $V$. The graph $G$ is Hamiltonian if it has a Hamilton cycle.

Definition 25. Let $G = (V, E)$ be a graph and let $e \in \binom{V}{2} \setminus E$. We define the graph $G + e$ to be the graph with vertex set $V$ and edge set $E \cup \{e\}$.

Lemma 26. Let $G = (V, E)$ and $|V| = n$. If $x, y \in V$, $\{x, y\} \notin E$ and $d_G(x) + d_G(y) \geq n$ then $G + e$ is Hamiltonian $\iff$ $G$ is Hamiltonian.

Definition 27. Let $G = (V, E)$ be a graph. The minimum degree of $G$ is

$$\delta(G) = \min\{d(x) : x \in G\}.$$ 

Corollary 28. If $G = (V, E)$ is a graph with $|V| = n$ and $\delta(G) \geq n/2$ then $G$ is Hamiltonian.

Definition 29. A graph $G = (V, E)$ is a tree if it is connected and contains no cycles.

Theorem 30. (i) A graph $G$ is a tree if it is connected but the deletion of any edge of $G$ results in a disconnected graph

(ii) A graph $G$ is a tree if it contains no cycles but the addition of any edge to $G$ results in a graph that contains a cycle.

Theorem 31. If $G = (V, E)$ is a tree then $|E| = |V| - 1$.

Corollary 32. If $G = (V, E)$ is a tree then $G$ has at least 2 vertices of degree 1.

Definition 33. For $x, y \in \mathbb{R}^2$ let $\ell(x, y)$ be the line segment in $\mathbb{R}^2$ joining $x$ and $y$.

Definition 34. A graph $G = (V, E)$ is planar if there is an injective map $\varphi : V \to \mathbb{R}^2$ with the property that

$$\{v_1, v_2\}, \{u_1, u_2\} \in E \implies$$

$$\ell(\varphi(v_1), \varphi(v_2)) \text{ and } \ell(\varphi(u_1), \varphi(u_2)) \text{ do not have interior intersection.}$$

A planar graph $G$ embedded in this way is called a plane graph. When we discuss plane graphs we identify $v$ and $\varphi(v)$. 

Definition 35. If $G = (V, E)$ is a plane graph then the faces of $G$ are the connected components of 
\[ \mathbb{R}^2 \setminus \bigcup_{\{x,y\} \in E} \ell(x,y). \]

Theorem 36 (Euler’s formula). If $G = (V, E)$ is a connected plane graph with $f$ faces then 
\[ |V| - |E| + f = 2. \]

**Review Exercises:** Working the following problems should help in preparation for the test. Some (but not all) of these are more difficult than questions that might appear on the quiz.

1. Suppose two dice are loaded with the same probabilities (i.e. for $k = 1, \ldots, 6$ the probability that the first die equals $k$ is equal to the probability that the second die equals $k$).
   (a) Set up a probability space that describes the experiment given by rolling the two loaded dice.
   (b) Show that the probability of doubles is always at least $1/6$.

2. We toss a sequence of $n$ fair, independent coins. (I.e. we consider $\Omega = \{H, T\}^n$ with the uniform distribution.) Let the random variable $X$ be the number of runs, where a run is a maximal sequence of consecutive tosses that all have the same result. (E.g. $X(HHHTHTTTTH) = 5$.) What is $E[X]$?

3. Let $X$ and $Y$ be arbitrary random variables defined on a common probability space. Which of the following statements are true?
   (a) $E[X^2] = E[X]^2$.
   (c) $E[3X + 5Y] = 3E[X] + 5E[Y]$.

4. Let $0 < p < 1$. A $n \times n$, 0-1 matrix is chosen at random by independently flipping a biased coin (a coin that comes up heads with probability $p$ and tails with probability $1 - p$) for each position. If the coin for a given position comes up heads then that position gets a 1 (if the coin comes up tails then the position gets a 0). Let $X$ be the number of $2 \times 2$ submatrices of the random matrix that are all 1’s.
   (a) Determine $E[X]$.
   (b) Let $p = 1/n$. Use Boole’s inequality to show that $P(X = 0) \geq 3/4$.
   (c) Let $p = 1/n$. Use (a) to show that $P(X = 0) \geq 3/4$.
   (d) Let $p = 2/n$ and $A$ be the event that the first row of the matrix is all 0’s. Show that 
   \[ \lim_{n \to \infty} P(A) = e^{-2}. \]

5. Let $X$ be a random variable defined a probability space on the set $\Omega$. Prove 
\[ \mathbb{P}(X \geq E[X]) > 0. \]
6. Consider the probability space given by $\Omega = \{0, 1\}^n$ with the uniform distribution. Let $X$ be the random variable that counts the number of entries $i$ in $\omega = (x_1, \ldots, x_n)$ such that $x_i = x_{i+1} = 0$. Determine $E(X)$.

7. A bag contains $r$ red and $b$ black balls. A ball is drawn uniformly at random from the bag and then returned to the bag. This is repeated until a black ball is drawn. What is the expected number of drawings in this process?

8. A group of $n$ students are taking an exam and they are not allowed to take their phones into the test. Every student has exactly one phone and each student gives the TA her or his phone as they enter the exam. After the exam the professor randomly returns the phones to students so that each student gets exactly one phone.
   (a) Set up a probability space that describes this experiment.
   (b) Let the random variable $X$ be the number of students who get their own phone back. Determine $E[X]$ and $E[X^2]$.
   (c) Use Markov’s inequality to bound $Pr(X \geq 10)$.
   (d) Use Chebyshev’s inequality to bound $Pr(X \geq 10)$.

9. The $k$-cube is the graph with vertex set $\{(x_1, \ldots, x_k) : x_1, \ldots, x_k \in \{0, 1\}\}$ and an edge between two vertices if and only if they differ in exactly 1 coordinate. Show that the $k$-cube has a Hamilton cycle for every $k \geq 2$.
   *Hint: go by induction on $k$.*

10. The complete bipartite graph $K_{m,n}$ has vertex set $V$ with bipartition $A, B$ where $|A| = n$ and $|B| = m$ and edge set

$$\{\{x, y\} : x \in A \text{ and } y \in B\}.$$ 

Which complete bipartite graphs have Hamilton cycles?

11. A connected graph $G = (V, E)$ has $|V| = 2k + 1$ vertices and exactly $k + 1$ vertices of degree 2, no two of which are adjacent. Show that $G$ is not Hamiltonian.

12. Let $G$ be a connected graph in which any two distinct vertices have either 0 or 7 common neighbors. Prove that $G$ is a regular graph.

13. Prove that a planar bipartite graph with $n$ nodes that contains a cycle has at most $2n - 4$ edges.

14. Let $G = (V, E)$ be a connected graph. Prove that there is a graph $H = (V, F)$ such that $F \subseteq E$ and $H$ is a tree.

15. For $n, m \geq 1$ let $G_{n,m}$ be the graph with vertex set

$$\{(x, y) : 1 \leq x \leq n \text{ and } 1 \leq y \leq m\}$$

and an edge connecting vertices $(x, y)$ and $(u, v)$ if

$$|x - u| = 1 \text{ and } |y - v| = 2 \quad \text{or} \quad (|x - u| = 2 \text{ and } |y - v| = 1).$$

In words, the vertices in $G_{n,m}$ are the squares on an $n \times m$ chessboard and we connect two squares with an edge if there is a ‘knight’s move’ that takes one to the other.
(a) Exhibit a Hamiltonian path in $G_{3,4}$.
(b) Show that $G_{4,4}$ does not have a Hamiltonian cycle.

*Hint: Which squares come before and after the corner squares in a Hamiltonian cycle?*

16. For a graph $G = (V, E)$, Let $\mathcal{L}(G)$ denote the **line graph** of $G$, which we define as

$$\mathcal{L}(G) = \left( E, \left\{ \{e, f\} \in \binom{E}{2} : e \cap f \neq \emptyset \right\} \right).$$

In words, the vertices of the line graph are the edge of $G$ and we connect two vertices in the line graph if the corresponding edges intersect. Are the following statements true or false?

(a) $G$ is connected if and only if $\mathcal{L}(G)$ is connected.
(b) $G$ is Eulerian if and only if $\mathcal{L}(G)$ is Hamiltonian.

17. Is the Claim below true or false? If it is false find a counterexample. Is the proof correct? If not, why not?

**Claim.** The number of trees on vertex set $\{v_1, v_2, \ldots, v_n\}$ is $(n-1)!$.

**Proof.** We go by induction on $n$.

**Base case:** $n = 2$. A connected graph on a vertex set of cardinality 2 must contain the edge connecting the two vertices. Therefore, there is only one such connected graph and only one such tree.

**Inductive assumption:** There are $(n-2)!$ trees on vertex set $\{v_1, v_2, \ldots, v_{n-1}\}$.

Let $n \geq 3$. Let $\mathcal{A}$ be the collection of all trees on vertex set $\{v_1, \ldots, v_{n-1}\}$.

Let $\mathcal{B}$ be the collection of trees on vertex set $\{v_1, \ldots, v_n\}$.

Since

(i) we get a tree on $n$ vertices by attaching a leaf to a tree on $n-1$ vertices and

(ii) there are $n-1$ vertices in a tree in $\mathcal{A}$ to which we can attach the vertex $v_n$ to get a tree in $\mathcal{B}$,

each tree in $\mathcal{A}$ corresponds to $n-1$ trees in $\mathcal{B}$. Therefore, $|\mathcal{B}| = (n-1)|\mathcal{A}|$.

Applying the inductive assumption we have

$$|\mathcal{B}| = (n-1)|\mathcal{A}| = (n-1) \cdot (n-2)! = (n-1)!$$

$\square$

18. We form a graph $G_n$ on vertex set $\{1, 2, \ldots, n\}$ by including each edge independently with probability $p = \frac{\log n}{2n}$. Let the random variable $Y$ be the number of isolated vertices (i.e. the number of vertices of degree 0).

(a) Determine $E[Y]$. 
(b) Use Chebyshev’s inequality to conclude that

\[ \lim_{n \to \infty} \mathbb{P}(Y = 0) = 0. \]

**Hint:** Note that one only needs an upper bound on \( \text{Var}(Y) \). You may also use the estimate \( \lim_{n \to \infty} x^2 n = 0 \) implies \( (1 - x)^n \sim e^{-xn} \).

From the text: 7.3.2, 7.3.10, 7.3.11, 7.3.12, 7.3.13, 8.1.3, 8.5.3, 8.5.4, 8.5.8, 12.2.1, 12.2.2, 12.3.1, 12.3.4, 12.3.6 (For question 12.3.6: You may assume that the graph is 3 regular. Follow-up question: why can we assume this?), 12.3.7.