This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is not a complete listing of what has happened in lecture. The sections from the book that correspond with each topic are also given.

Following the list of important definitions and theorems you will find a collection of review exercises.

5. **Generating Functions**: Chapter 4 gives a nice example of a sequence that is determined by a recurrence relation.

**Definition 31.** Let \(a_0, a_1, \ldots\) be an infinite sequence. The *generating function* for this sequence is

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n.
\]

**Note 32.** A generating function can be viewed as either

(i) A function of \(x\) (when we have convergence).

(ii) A formal object with addition and multiplication.

E.g.,

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.
\]

This can be viewed as either

(i) A fact for formal power series that follows from noting that

\[(1-x)(1+x+x^2+\cdots) = 1,\]

or

(ii) the power series for the function \(1/(1-x)\), which converges for \(|x| < 1\).

**Notation 33.** If \(f(x)\) is a polynomial in \(x\) or a generating function then we define \([x^n]f(x)\) to be the coefficient of \(x^n\) in \(f\). For example, if \(f(x) = 1 + 22x^2 + 3x^3 + 5x^4\) then \([x^2]f(x) = 22\).

**Note 34.** If \(f(x), g(x)\) are polynomials in \(x\) or generating functions then

\[
[x^n]f(x)g(x) = \sum_{i=0}^{n} [x^i]f(x) \cdot [x^{n-i}]g(x).
\]

**Definition 35.** Let \(a_0, a_1, \ldots\) be an infinite sequence. This satisfies a recurrence relation if there is a function that gives \(a_n\) in terms of \(a_1, \ldots, a_{n-1}\) and \(n\).

**Note 36.** In lecture we saw three main methods for finding and manipulating generating functions:

(i) Direct application of a recurrence,
(ii) use of the combinatorial implication of the multiplication of generating functions, and

(iii) differentiation (or integration).

**Definition 37.** The Catalan sequence is the sequence $c_0, c_1, \ldots$ where

$$c_n = \frac{1}{n+1} \binom{2n}{n} \quad \text{for } n = 0, 1, 2, \ldots$$

The number $c_n$ is called the $n^{th}$ Catalan number.

**Theorem 38.** The number of triangulations of an $n$-gon (ways to draw line segments connecting the vertices of the $n$-gon in such a way that the line segments do not cross and all the regions formed are triangles) is $c_{n-2} = \frac{1}{n-1} \binom{2n-4}{n-2}$.


**Definition 39.** A graph $G$ is an ordered pair $G = (V, E)$ where $V$ is the vertex set of $G$ and $E \subseteq \binom{V}{2}$ is the edge set of $G$.

**Definition 40.** The complete graph $K_n$ is the graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \binom{V}{2}$.

**Theorem 41 (Ramsey’s Theorem (two color, graph)).** Let $k, l \geq 2$ be integers. There is an integer $n$ such that every coloring of the edge set of $K_n$ with the colors Blue and Green has a Blue $K_k$ or a Green $K_l$.

**Note 42.** The smallest integer $n$ that satisfies the condition in Ramsey’s Theorem is called the Ramsey number $R(k, l)$.

**Claim 43.** $R(3, 3) = 6$ and $R(4, 3) = 9$.

**Note 44.** It follows from the proof of Ramsey’s theorem that we have

(a) If $k, \ell \geq 3$ then $R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$

(b) $R(k, l) \leq \binom{k+l-2}{l-1}$

**Corollary 45.**

$$R(k, k) \leq \binom{2k-2}{k-1} \leq \frac{1}{4} \cdot 4^k$$

**Claim 46.**

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1 \quad \Rightarrow \quad R(k, k) > n.$$
7. Discrete Probability: Sections 5.1, 5.2, 2.5. Note that part of what we cover here is not in the book.

**Definition 48.** A probability space is a finite or countable set (a set is countable if it can be indexed with the integers) $\Omega$ and a function

$$Pr : \Omega \rightarrow \{x \in \mathbb{R} : x \geq 0\}$$

such that

$$\sum_{\omega \in \Omega} Pr(\omega) = 1.$$

$Pr(\omega)$ is the probability of $\omega$.

**Definition 49.** The uniform distribution on a finite set $\Omega$ is the probability space in which

$$Pr(\omega) = \frac{1}{|\Omega|}$$

for all $\omega \in \Omega$.

**Definition 50.** An event in a probability space is a set $A \subseteq \Omega$. We set

$$Pr(A) = \sum_{\omega \in A} Pr(\omega).$$

**Theorem 51** (Union Bound (a.k.a Boole’s Inequality)). If $A_1, \ldots, A_n$ are events in a probability space then

$$Pr \left( \bigcup_{i=1}^{n} A_i \right) \leq \sum_{i=1}^{n} Pr(A_i)$$

**Example 52.** Suppose we throw $m$ balls into $n$ boxes uniformly at random. (Boxes and balls are assumed to distinguishable.) Let $A$ be the event that the first box is empty. If $c > 0$ is some real number and $m = \lfloor cn \rfloor$ then

$$\lim_{n \to \infty} Pr(A) = e^{-c}.$$ 

Note. We consider a sequence of probability spaces in order to make sense of the limiting statement.

**Definition 53.** Let $A$ be an event in a probability space defined on the set $\Omega$. The probability of $\omega \in \Omega$ conditioned on $A$ is

$$Pr(\omega | A) = \begin{cases} \frac{Pr(\omega)}{Pr(A)} & \text{if } \omega \in A \\ 0 & \text{otherwise.} \end{cases}$$

If $B \subseteq \Omega$ the probability of $B$ conditioned on $A$ is

$$Pr(B | A) = \frac{Pr(A \cap B)}{Pr(A)}.$$

**Theorem 54** (law of total probability). Let a probability space be defined on set $\Omega$ and $B_1, \ldots, B_n$ are a partition of $\Omega$. If $A \subseteq \Omega$ then

$$Pr(A) = \sum_{i=1}^{n} Pr(A | B_i) Pr(B_i).$$
Definition 55. Two events $A, B \subseteq \Omega$ are independent if

\[ \Pr(A \cap B) = \Pr(A)\Pr(B). \]

Equivalently:

\[ \Pr(A|B) = \Pr(A). \]

Definition 56. A collection $A_1, \ldots, A_n$ of events is independent if for every subset $I$ of $\{1, 2, \ldots, n\}$ we have

\[ \Pr \left( \bigcap_{i \in I} A_i \right) = \prod_{i \in I} \Pr(A_i) \]

Review Exercises: Working the following problems should help in preparation for the test. Some (but not all) of these are more difficult than questions that might appear on the quiz.

1. Solve the recurrence relation $b_n = 3b_{n-2} - 2b_{n-3}$ with initial values $b_0 = 1$, $b_1 = 0$, $b_2 = 0$.

2. Determine the coefficient of $x^4$ in $(2 + 3x)^5 \sqrt{1 - x}$.

3. Find the generating function for the sequence 1, 2, 1, 4, 1, 8, \ldots.

4. How many ways are there to distribute 20 identical bars of gold to 2 nuns and 2 pirates so that each nun gets at least 2 bars and each pirate gets at least 1 bar. Express your answer as a coefficient of a power of $x$ in some product of polynomials.

5. Let $a_n$ be number of ordered triples $(i, j, k)$ of non-negative integers such that $i + 2j + 2k = n$. Find the generating function for the sequence $a_0, a_1, \ldots$ and use this generating function to determine a formula for $a_n$.

6. Let $A_n$ denote the set of strings of elements of the set $\{1, 2\}$ of length $n$. Let $B_n$ be those strings in $A_n$ which do not contain 1, 2, 2 as a sub-string (in consecutive positions). Let $b_n = |B_n|$.

(a) What are $b_0, b_1, b_2, b_3, b_4$?

(b) Prove that for $n \geq 4$ we have

\[ b_n = b_{n-1} + b_{n-2} + 1. \]

(c) Determine the generating function for this sequence.

7. Prove $R(3, 5) \geq 11$.


9. Prove the following statement: For every positive integer $k$ there exists an $n$ such that any coloring of the edges of $K_n$ with $k$ colors contains a monochromatic triangle (i.e. a monochromatic copy of $K_3$).

10. Let $n$ and $k$ be integers such that $n > k \geq 2$. Use Boole’s inequality to show that if

\[ 3^{1-\binom{k}{2}} \binom{n}{k} < 1 \]

then there exists a coloring of the edges of $K_n$ with the colors red, blue and green with the property that there is no monochromatic $K_k$.
11. A particle starts at the origin in the plane. Each minute the particle makes a random move of length 1 in one of the following directions: Up, Down, Left, Right. In $n$ minutes all sequences of possible moves are equally likely. Set up the probability space for this experiment and determine the probability that the particle is back in the starting position after $n$ minutes.

12. Suppose two dice are loaded with the same probabilities (i.e. for $k = 1, \ldots, 6$ the probability that the first die equals $k$ is equal to the probability that the second die equals $k$).

(a) Set up a probability space that describes the experiment given by rolling the two loaded dice.

(b) Show that the probability of doubles is always at least 1/6.

13. $m$ indistinguishable balls are randomly colored using $n$ (distinguishable) colors. What is the probability that every color is used at least twice?

14. What is the probability that the top and bottom cards of a randomly shuffled deck are both aces?

15. An urn contains $x$ Red balls, $y$ Green balls and $z$ Yellow ball. Balls are drawn from the urn one at a time and uniformly at random until a Yellow ball is drawn. When a Red ball is drawn it is returned to the urn. When a Green ball is drawn it is thrown away (and not returned to the urn). What is the probability that Yellow appears for the first time on the third draw?

16. Suppose we have a probability space on a set $\Omega$ with the uniform distribution. Prove that if $|\Omega|$ is prime and $A, B \subset \Omega$ are proper, nontrivial events then $A$ and $B$ are not independent.

17. (a) Let $A, B$ and $C$ be sets chosen uniformly and independently at random from the collection of all subsets of $\{1, 2, \ldots, n\}$. Set up a probability space that describes this experiment and prove that

$$P(A \cap B \subseteq C) = \left(\frac{7}{8}\right)^n.$$  

Hint: Think of $A, B$ and $C$ as strings of 0’s and 1’s.

(b) Let $m < \left(\frac{2}{7}\right)^{n/3}$. Use (a) and Boole’s inequality to show that there exist sets $A_1, \ldots, A_m \subseteq \{1, \ldots, n\}$ such that for all distinct $i, j, k$ we have

$$A_i \cap A_j \not\subseteq A_k.$$  

Hint: Choose $A_1, \ldots, A_k$ at random.

18. The $4$-cycle is the graph with vertex set $\{a, b, c, d\}$ and edge set

$$\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}.$$  

This graph describes a network, and in this network each edge fails (independently) with probability $p$. Let $G$ be the graph given by the edges in this network that do not fail. Let $A$ be the event that there is a path in $G$ from $b$ to $d$. Let $B$ be the event that there is a path in $G$ from $a$ to $c$. Compute $P(A)$ and $P(A|B)$. Are $A$ and $B$ independent events?

From the text: 5.4.2, 5.4.4, 5.4.5