21-228 Discrete Mathematics
Course Review 2

This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is *not* a complete listing of what has happened in lecture. The sections from the book that correspond with each topic are also given.

Following the list of important definitions and theorems you will find a collection of review exercises.

4. Generating Functions:

Matoušek and Nešetřil: Section 10.1-10.4

Lovász, Pelikán, and Vesztergombi: Chapter 4 gives a nice example of a sequence that is determined by a recurrence relation.

Definition 31. *Let $a_0, a_1, \ldots$ be an infinite sequence. The* **generating function** *for this sequence is*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Note 32. *A generating function can be viewed as either*

(i) A function of x (when we have convergence).

(ii) A formal object with addition and multiplication.

E.g..

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

This can be viewed as either

(i) A fact for formal power series that follows from noting that

$$(1-x)(1+x+x^2+\cdots) = 1, \text{ or}$$

(ii) the power series for the function $1/(1-x)$, which converges for $|x| < 1$.

Notation 33. *If $f(x)$ is a polynomial in x or a generating function then we define $[x^n]f(x)$ to be the coefficient of x^n in f. For example, if $f(x) = 1 + 22x^2 + 3x^3 + 5x^4$ then $[x^2]f(x) = 22$.*

Note 34. *If $f(x), g(x)$ are polynomials in x or generating functions then*

$$[x^n]f(x)g(x) = \sum_{i=0}^{n} [x^i]f(x) \cdot [x^{n-i}]g(x).$$

Definition 35. *Let $a_0, a_1, \ldots$ be an infinite sequence. This satisfies a* recurrence relation *if there is a function that gives a_n in terms of $a_1, \ldots, a_{n-1}$ and n.*

Note 36. *In lecture we saw three main methods for finding and manipulating generating functions:*

(i) Direct application of a recurrence,
(ii) use of the combinatorial implication of the multiplication of generating functions, and
(iii) differentiation (or integration).

Definition 37. *The* **Catalan sequence** *is the sequence $c_0, c_1, \ldots$ where*

$$c_n = \frac{1}{n+1}\binom{2n}{n} \qquad \text{for } \; n = 0, 1, 2, \ldots$$

The number c_n is called the n^{th} **Catalan number**.

Theorem 38. *The number of triangulations of an n-gon (ways to draw line segments connecting the vertices of the n-gon in such a way that the line segments do not cross and all the regions formed are triangles) is $c_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$.*

5. Introduction to Ramsey Theory: Not covered in the books.

Definition 39. *A* **graph** *G is an ordered pair $G = (V, E)$ where V is the* **vertex set** *of G and $E \subseteq \binom{V}{2}$ is the* **edge set** *of G.*

Definition 40. *The* **complete graph** *K_n is the graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \binom{V}{2}$.*

Theorem 41 (Ramsey's Theorem (two color, graph))**.** *Let $k, \ell \geq 2$ be integers. There is an integer n such that every partition $\binom{V}{2} = R \cup B$ of the edge set of a complete graph on n vertices has the property that (V, R) contains a complete subgraph on k vertices or (V, B) contains a complete graph on l vertices.*

Note 42. *The smallest integer n that satisfies the condition in Ramsey's Theorem is called the* **Ramsey number** *$R(k, \ell)$.*

Claim 43. *$R(3, 3) = 6$ and $R(4, 3) = 9$.*

Note 44. *It follows from the proof of Ramsey's theorem that we have*

(a) If $k, \ell \geq 3$ then $R(k, \ell) \leq R(k-1, \ell) + R(k, \ell-1)$
(b) $R(k, \ell) \leq \binom{k+\ell-2}{\ell-1}$

Corollary 45.

$$R(k, k) \leq \binom{2k-2}{k-1} \leq \frac{1}{4} \cdot 4^k$$

Claim 46.

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1 \quad \Rightarrow \quad R(k, k) > n.$$

Corollary 47.

$$R(k, k) \geq \frac{1}{2\sqrt{2}e} k \cdot \left(\sqrt{2}\right)^k.$$

6. DISCRETE PROBABILITY:

Matoušek and Nešetřil: Sections 9.1, 9.2
Lovász, Pelikán, and Vesztergombi: Sections 5.1, 5.2, 2.5.

Definition 48. *A* **probability space** *is a finite or countable set (a set is countable if it can be indexed with the integers)* Ω *and a function*

$$\mathbb{P} : \Omega \to \{x \in \mathbb{R} : x \geq 0\}$$

such that

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$$

$\mathbb{P}(\omega)$ *is the* **probability** *of* ω.

Definition 49. *The* **uniform distribution** *on a finite set* Ω *is the probability space in which*

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

for all $\omega \in \Omega$.

Definition 50. *An* **event** *in a probability space is a set* $A \subseteq \Omega$. *We set*

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega).$$

Theorem 51 (Union Bound (a.k.a Boole's Inequality))**.** *If* $A_1, \ldots, A_n$ *are events in a probability space then*

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$$

Example 52. *Suppose we throw* m *balls into* n *boxes uniformly at random. (Boxes and balls are assumed to distinguishable.) Let* A *be the event that the first box is empty. If* $c \in \mathbb{N}^+$ *is a positive integer and* $m = cn$ *then*

$$\lim_{n \to \infty} \mathbb{P}(A) = e^{-c}.$$

Note. We consider a sequence of probability spaces in order to make sense of the limiting statement.

REVIEW EXERCISES: Working the following problems should help in preparation for the test. Some (but not all) of these are more difficult than questions that might appear on the quiz.

1. Solve the recurrence relation $b_n = 3b_{n-2} - 2b_{n-3}$ with initial values $b_0 = 1$, $b_1 = 0$, $b_2 = 0$.
2. Determine the coefficient of x^4 in $(2 + 3x)^5\sqrt{1 - x}$.
3. Find the generating function for the sequence $1, 2, 1, 4, 1, 8, \ldots$.

4. How many ways are there to distribute 20 identicle bars of gold to 2 nuns and 2 pirates so that each nun gets at least 2 bars and each pirate gets at least 1 bar. Express your answer as a coefficient of a power of x in some product of polynomials.

5. Let a_n be number of ordered triples (i, j, k) of non-negative integers such that $i + 2j + 2k = n$. Find the generating function for the sequence $a_0, a_1, \ldots$ and use this generating function to determine a formula for a_n.

6. Prove $R(3, 5) \geq 11$.

7. Prove $R(4, 4) > 12$.

8. Prove the following statement: For every positive integer k there exists an n such that any coloring of the edges of K_n with k colors contains a monochromatic triangle (i.e. a monochromatic copy of K_3).

9. Let n and k be integers such that $n > k \geq 2$. Use Boole's inequality to show that if

$$3^{1-\binom{k}{2}} \binom{n}{k} < 1 \qquad (1)$$

then there exists a coloring of the edges of K_n with the colors red, blue and green with the property that there is no monochromatic K_k.

10. Consider a poker hand: a set of five cards drawn at random from a standard deck. What is the probability space for this random experiment? Determine the probabilities of the following events:

    (a) flush: all 5 cards have the same suit.

    (b) 3 of kind: 3 card of the same value, which is different from the values of the other two cards, which are distinct. e.g. Q, Q, Q, 7 ,5.

    (c) full house: 3 card of the same value, which is different from the value of the other two cards, which are the same. e.g. J,J,J,7,7.

    (d) 4 of a kind: 4 cards of the same value. e.g. 9,9,9,9,J.

11. A particle starts at the origin in the plane. Each minute the particle makes a random move of length 1 in one of the following directions: Up, Down, Left, Right. In n minutes all sequences of possible moves are equally likely. Set up the probability space for this experiment and determine the probability that the particle is back in the starting position after n minutes.

12. Suppose two dice are loaded with the same probabilities (i.e. for $k = 1, \ldots, 6$ the probability that the first die equals k is equal to the probability that the second die equals k).

    (a) Set up a probability space that describes the experiment given by rolling the two loaded dice.

    (b) Show that the probability of doubles is always at least 1/6.

13. What is the probability that the top and bottom cards of a randomly shuffled deck are both aces?

14. (a) Let A, B and C be sets chosen uniformly and independently at random from the collection of all subsets of $\{1, 2, \ldots, n\}$. Set up a probability space that describes this experiment and prove that

$$P(A \cap B \subseteq C) = \left(\frac{7}{8}\right)^n.$$

Hint: Think of A, B and C as strings of 0's and 1's.

(b) Let $m < (\frac{8}{7})^{n/3}$. Use (a) and Boole's inequality to show that there exist sets $A_1, \ldots, A_m \subseteq \{1, \ldots, n\}$ such that for all distinct i, j, k we have

$$A_i \cap A_j \not\subseteq A_k.$$

Hint: Choose $A_1, \ldots, A_k$ at random.

15. We perform the following two stage experiment:

Stage I: A fair coin is repeatedly flipped until the outcome is a Head.

Stage II: If the first Head appeared on the i^{th} flip then we roll i fair dice.

(A) Set up a probability space that describes this experiment

(B) Determine the probability that *all* dice rolled in Stage II land on the same number.

From the text: 5.4.4, 5.4.5