21-228 Discrete Mathematics
Course Review 2

This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is *not* a complete listing of what has happened in lecture. The sections from the book that correspond with each topic are also given.

Following the list of important definitions and theorems you will find a collection of review exercises.

6. Discrete Probability: Sections 5.1, 5.2, 2.5. Note that a large part of what we cover here is not in the book.

Definition 1. *A* **probability space** *is a finite or countable set (a set is countable if it can be indexed with the integers)* Ω *and a function*

$$P : \Omega \to \{x \in \mathbb{R} : x \geq 0\}$$

such that

$$\sum_{\omega \in \Omega} P(\omega) = 1.$$

$P(\omega)$ *is the* **probability** *of* ω.

Definition 2. *The* **uniform distribution** *on a finite set* Ω *is the probability space in which*

$$P(\omega) = \frac{1}{|\Omega|}$$

for all $\omega \in \Omega$.

Definition 3. *An* **event** *in a probability space is a set* $A \subseteq \Omega$. *We set*

$$P(A) = \sum_{\omega \in A} P(\omega).$$

Theorem 4 (Boole's Inequality). *If* $A_1, \dots, A_n$ *are events in a probability space then*

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Theorem 5.

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1 \;\Rightarrow\; R(k,k) > n.$$

Corollary 6.

$$R(k,k) > \frac{1}{e\sqrt{2}} k 2^{k/2}.$$

Definition 7. *Let* A *be an event in a probability space defined on the set* Ω. *The* **probability of** $\omega \in \Omega$ **conditioned on** A *is*

$$P(\omega | A) = \begin{cases} \frac{P(\omega)}{P(A)} & \text{if } \omega \in A \\ 0 & \text{otherwise.} \end{cases}$$

If $B \subseteq \Omega$ *the* **probability of** B **conditioned on** A *is*

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Theorem 8 (law of total probability)**.** *Let a probability space be defined on set* Ω *and* $B_1, \ldots, B_n$ *are a partition of* Ω*. If* $A \subseteq \Omega$ *then*

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$

Definition 9. *Two events* $A, B \subseteq \Omega$ *are* **independent** *if*

$$P(A \cap B) = P(A)P(B).$$

Equivalently:

$$P(A|B) = P(A).$$

Definition 10. *A collection* $A_1, \ldots, A_n$ *of events is* **independent** *if for every subset* I *of* $\{1, 2, \ldots, n\}$ *we have*

$$Pr\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} Pr(A_i)$$

Definition 11. *Let a probability space of set* Ω *be given. A function* $f : \Omega \to \mathbb{R}$ *is called a* **random variable***.*

Definition 12. *If* Z *is a random variable on a probability space defined on the set* Ω *then*

$$E(Z) = \sum_{\omega \in \Omega} Z(\omega)P(\omega)$$

is the **expected value** *of* Z*.*

Definition 13. *Let* $n \in \mathbb{P}$ *and* $0 \leq p \leq 1$*. Let* $\Omega = \{H, T\}^n$ *and consider the probability space on* Ω *in which*

$$P(\omega) = p^k(1-p)^{n-k}$$

where k *is the number of* H*'s in* ω *(note that this probability space is given by a sequence of independent flips of a coin that comes up* H *with probability* p*). The function* $B_{n,p} : \Omega \to \mathbb{N}$ *in which* $B_{n,p}(\omega)$ *is the number of* H*'s in* ω *is called the* **binomial random variable** *with parameters* n, p*. Note that*

$$P(B_{n,p} = k) = \binom{n}{k} p^k(1-p)^{n-k}.$$

Claim 14. $E[B_{n,p}] = np$.

Definition 15. *Let* $\lambda > 0$*. The* **Poisson random variable** *with parameter* λ*, denoted* Po_λ*, takes values in* $\mathbb{N}$ *according to the following:*

$$P(Po_\lambda = k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

Claim 16. $E[Po_\lambda] = \lambda$.

Note 17. *Let $\lambda > 0$ be a fixed constant and $p = \lambda/n$ where $n \to \infty$. For a fixed positive integer k we have*

$$P(B_{n,p} = k) = \binom{n}{k} p^k (1-p)^{n-k} \sim \frac{n^k}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \sim e^{-\lambda} \frac{\lambda^k}{k!} = Pr(Po_\lambda = k).$$

Claim 18. *If $X_1, \ldots, X_n$ are random variables defined on a common probability space then*

$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n].$$

Claim 19. *If X is a random variable then*

$$Pr\left(X \geq E[X]\right) > 0 \qquad \text{and} \qquad Pr\left(X \leq E[X]\right) > 0$$

Theorem 20. *For any positive integer n we have*

$$R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$$

Corollary 21.

$$R(k,k) > \frac{k-2}{e} \cdot 2^{k/2}.$$

7. Notes on asymptotics: Section 2.5.

Claim 22. *If $x > 0$ then*

$$\frac{x-1}{x} \leq \ln x \leq x - 1$$

Claim 23. *If $0 \leq x \leq 3/5$ then $e^{-x-x^2} \leq 1 - x \leq e^{-x}$.*

Review Exercises: Working the following problems should help in preparation for the test. Some (but not all) of these are more difficult than questions that might appear on the quiz.

1. Consider a poker hand: a set of five cards drawn at random from a standard deck. What is the probability space for this random experiment? Determine the probabilities of the following events:
   (a) flush: all 5 cards have the same suit.
   (b) 3 of kind: 3 card of the same value, which is different from the values of the other two cards, which are distinct. e.g. Q, Q, Q, 7 ,5.
   (c) full house: 3 card of the same value, which is different from the value of the other two cards, which are the same. e.g. J,J,J,7,7.
   (d) 4 of a kind: 4 cards of the same value. e.g. 9,9,9,9,J.

2. A particle starts at the origin in the plane. Each minute the particle makes a random move of length 1 in one of the following directions: Up, Down, Left, Right. In n minutes all sequences of possible moves are equally likely. Set up the probability space for this experiment and determine the probability that the particle is back in the starting position after n minutes.

3. Suppose two dice are loaded with the same probabilities (i.e. for $k = 1, \ldots, 6$ the probability that the first die equals k is equal to the probability that the second die equals k).

   (a) Set up a probability space that describes the experiment given by rolling the two loaded dice.

   (b) Show that the probability of doubles is always at least 1/6.

4. m indistinguishable balls are randomly colored using n (distinguishable) colors. What is the probability that every color is used at least twice?

5. What is the probability that the top and bottom cards of a randomly shuffled deck are both aces?

6. An urn contains x Red balls, y Green balls and z Yellow ball. Balls are drawn from the urn one at a time and uniformly at random until a Yellow ball is drawn. When a Red ball is drawn it is returned to the urn. When a Green ball is drawn it is thrown away (and not returned to the urn). What is the probability that Yellow appears for the first time on the third draw?

7. A bag contains r red and b black balls. A ball is drawn uniformly at random from the bag and then returned to the bag. This is repeated until a black ball is drawn. What is the expected number of drawings in this process?

8. Let A and B be independent events in a probability space defined on set Ω. Prove that $\overline{A} = \Omega \setminus A$ and $\overline{B} = \Omega \setminus B$ are independent events.

9. (a) Let A, B and C be sets chosen uniformly and independently at random from the collection of all subsets of $\{1, 2, \ldots, n\}$. Set up a probability space that describes this experiment and prove that

$$P(A \cap B \subseteq C) = \left(\frac{7}{8}\right)^n.$$

   Hint: Think of A, B and C as strings of 0's and 1's.

   (b) Let $m < (\frac{8}{7})^{n/3}$. Use (a) and Boole's inequality to show that there exist sets $A_1, \ldots, A_m \subseteq \{1, \ldots, n\}$ such that for all distinct i, j, k we have

$$A_i \cap A_j \not\subseteq A_k.$$

   Hint: Choose $A_1, \ldots, A_k$ at random.

10. Prove that for all integers n and $p \in [0, 1]$ we have

$$R(k, l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell} (1-p)^{\binom{\ell}{2}}.$$

Hint: Consider a random coloring of edge set of K_n in which we color each edge independently at random. Then remove some vertices from the graph in order to get a Ramsey coloring.

11. Let $0 < p < 1$. A $n \times n$, 0-1 matrix is chosen at random by independently flipping a biased coin (a coin that comes up heads with probability p and tails with probability $1 - p$) for each position. If the coin for a given position comes up heads then that position gets a 1 (if the coin comes up tails then the position gets a 0). Let X be the number of 2×2 submatrices of the random matrix that are all 1's.

   (a) Determine $E[X]$.

   (b) Let $p = 1/n$. Use Boole's inequality to show that $P(X = 0) \geq 3/4$.

   (c) Let $p = 1/n$. Use (a) to show that $P(X = 0) \geq 3/4$.

   (d) Let $p = 2/n$ and A be the event that the first row of the matrix is all 0's. Show that

   $$\lim_{n \to \infty} P(A) = e^{-2}.$$

12. The complete graph on n vertices (denoted K_n) has vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \binom{V}{2}$ (K_n is 'complete' as it has all possible edges).

   (a) Suppose we color the edge set of K_n with 2 colors uniformly at random (so, we have $|\Omega| = 2^{\binom{n}{2}}$ and the uniform distribution). Let X be the number of monochromatic triangles in the random coloring. Determine $E(X)$.

   (b) Conclude that there exists a coloring of the edge set of K_n that has at most $\frac{1}{4}\binom{n}{3}$ monochromatic triangles.