21-228 Discrete Mathematics
Course Review 1

This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is *not* a complete listing of what has happened in lecture. The sections from the book that correspond with each topic are also given. Following the list of important definitions and theorems you will find a collection of review exercises.

1. COUNTING SETS AND FUNCTIONS: Chapter 1, Section 3.2-3.4.

Definition 1. *Let S be a set. A* **partition** *of S is a collection of sets $A_1, \ldots, A_k$ such that*

$$S = \bigcup_{i=1}^{k} A_i, \quad \text{and}$$
$$A_i \cap A_j = \emptyset \quad \text{for all } i \neq j.$$

Recall that the equivalence classes of an equivalence relation on S form a partition of S.

Note 2. *If S is a finite set and $A_1, \ldots, A_k$ form a partition of S then*

$$|S| = \sum_{i=1}^{k} |A_i|.$$

Definition 3. *If X is a finite set then 2^X is the collection of all subsets of X; to be precise,*

$$2^X = \{A : A \subseteq X\}.$$

Claim 4. *If X is a finite set then $|2^X| = 2^{|X|}$.*

Definition 5. *Let $X = \{1, \ldots, n\}$. The* **characteristic vector** *of a set $A \subseteq X$ is $1_A = (y_1, \ldots, y_n)$ where*

$$y_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A. \end{cases}$$

Definition 6. *Let X be a finite set. For $k \in \mathbb{N}$ we define $\binom{X}{k}$ to be the collection of all k-element subsets of X; that is,*

$$\binom{X}{k} = \{A \subseteq X : |A| = k\}.$$

Claim 7. *If X is a finite set and $k \in \mathbb{N}$ then*

$$\left|\binom{X}{k}\right| = \frac{n!}{k!(n-k)!} =: \binom{n}{k}.$$

Claim 8. *If $n \in \mathbb{N}$ then*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Claim 9. *If* $0 \le k \le n$ *then*
$$\binom{n}{k} = \binom{n}{n-k}.$$

Claim 10. *If* $0 \le k \le n$ *then*
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Definition 11. *Let* X *and* Y *be sets.* X^Y *is the collection of functions from* Y *to* X*; that is,*
$$X^Y = \{\text{functions } f : Y \to X\}.$$
There is a correspondence between the functions in X^Y *and the strings of elements of* X *indexed by* Y*. If* $|Y| = \{1, \ldots, s\}$ *then the string* $(x_1, \ldots, x_s)$ *corresponds to the function* f *such that* $f(i) = x_i$ *for* $i = 1, \ldots, s$*.*

Definition 12. *For* $m, k \in \mathbb{N}^+$*,* $S(m,k)$ *is defined to be the number of partitions of a set of size* m *into* k *nonempty parts. These are called the* **Stirling numbers of the second kind.**

Claim 13. *Let* X *and* Y *be finite sets such that* $|X| = n$ *and* $|Y| = s$*. We have*
$$\begin{aligned} |X^Y| &= |X|^{|Y|} = n^s \\ \big|\{f \in X^Y : f \text{ is an injection}\}\big| &= n(n-1)\cdots(n-s+1) \\ \big|\{f \in X^Y : f \text{ is a surjection}\}\big| &= S(s,n)n!. \end{aligned}$$
Note that if $n < s$ *then the number of injections is 0. Furthermore, if* $s < n$ *then the number of surjections is 0.*

Claim 14. *Let* Z *be a collection of* k **indistinguishable** *objects. Let* X *be a set of* n **distinguishable** *labels. We consider labellings of the objects of* Z *with the elements of* X*. We consider two such labellings distinct if there is a label in* X *that appears a different number of times in the two labellings. So, there is correspondence between the set of distinct labeling and the set*
$$\{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{N} \quad \text{and} \quad x_1 + \cdots + x_n = k\}.$$
(In this correspondence x_i *is the number of times label* i *is used.) We have*
$$\begin{aligned} \text{total number of labellings} &= \tbinom{n+k-1}{n-1} \\ \text{number of labellings in which every label is different} &= \tbinom{n}{k} \\ \text{number of labellings in which every label appears} &= \tbinom{k-1}{n-1} \end{aligned}$$

2. APPROXIMATIONS: Sections 2.2, 2.5, 3.1, and 3.5-3.8.

Definition 15. *If $f(n)$ and $g(n)$ are functions from $\mathbb{N}$ to $\mathbb{R}$ then write $f \sim g$ for*

$$\lim_{n\to\infty} f/g = 1.$$

Theorem 16 (Stirling's Formula)**.**

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Claim 17. *For any $n \in \mathbb{N}^+$ we have*

$$e\left(\frac{n}{e}\right)^n \leq n! \leq \frac{e^2}{4}(n+1)\left(\frac{n}{e}\right)^n.$$

Theorem 18 (Binomial Theorem)**.** *For any $n \in \mathbb{N}^+$ we have*

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

(Note: this holds over any field.)

Definition 19. *For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}^+$ we set*

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

For $\alpha \in \mathbb{R}$ we set $\binom{\alpha}{0} = 1$.

Theorem 20 (Newton's Binomial Theorem)**.** *If $\alpha, x \in \mathbb{R}$ and $|x| < 1$ then*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

Claim 21. *If $0 \leq k < n$ are integers then*

$$\binom{n}{k}(n-k) = \binom{n}{k+1}(k+1)$$

Note 22. *Every row of Pascal's triangle is symmetric and has its maximum in the middle. In particular,*

$$\binom{n}{0} < \binom{n}{1} < \binom{n}{2} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-2} > \binom{n}{n-1} > \binom{n}{n}.$$

Claim 23. *If $0 \leq t < m$ are integers then*

$$e^{-\frac{t^2}{m-t+1}} \leq \frac{\binom{2m}{m-t}}{\binom{2m}{m}} \leq e^{-\frac{t^2}{m+t}}.$$

Lemma 24. *For every $x > 0$*

$$\frac{x-1}{x} \leq \log x \leq x-1$$

(Where log *denotes the natural logarithm.)*

3. INCLUSION/EXCLUSION: See Section 2.3 for a brief introduction.

Theorem 25 (Principle of inclusion and exclusion)**.** *Let $A_1, \dots, A_n$ be subsets of a finite set Ω. For $S \subseteq \{1, \dots, n\}$ set*

$$A_S = \begin{cases} \cap_{i \in S} A_i & \text{if } S \neq \emptyset \\ \Omega & \text{if } S = \emptyset. \end{cases}$$

We have

$$\left| \Omega \setminus \left(\bigcup_{i=1}^{n} A_i \right) \right| = \sum_{S \subseteq \{1,\dots,n\}} (-1)^{|S|} |A_S|$$

Lemma 26. *If $n \geq 1$ is an integer then*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Claim 27. *Let X be a finite set. A bijection $\sigma : X \to X$ is a* **derangement** *if $\sigma(x) \neq x$ for all $x \in X$. The number of derangements of a set of cardinality n is*

$$n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

Claim 28.

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.$$

Theorem 29 (Bonferroni Inequalities)**.** *Let $A_1, \dots, A_n$ be subsets of a finite set Ω. For $S \subseteq \{1, \dots, n\}$ set*

$$A_S = \begin{cases} \cap_{i \in S} A_i & \text{if } S \neq \emptyset \\ \Omega & \text{if } S = \emptyset. \end{cases}$$

If k is even then we have

$$\left| \left(\bigcup_{i=1}^{n} A_i \right) \right| \geq \sum_{j=1}^{k} \sum_{S \in \binom{\{1,2,\dots,n\}}{j}} (-1)^{j+1} |A_S|$$

If k is odd then we have

$$\left| \left(\bigcup_{i=1}^{n} A_i \right) \right| \leq \sum_{j=1}^{k} \sum_{S \in \binom{\{1,2,\dots,n\}}{j}} (-1)^{j+1} |A_S|$$

Lemma 30. *Let $1 \leq k \leq n$ be integers. If k is even then*

$$\sum_{j=1}^{k} (-1)^{j+1} \binom{n}{j} \leq 1.$$

If k is odd then

$$\sum_{j=1}^{k} (-1)^{j+1} \binom{n}{j} \geq 1.$$

REVIEW EXERCISES: Working the following problems should help in preparation for the test. They are not necessarily 'sample' test questions.

1. Give a combinatorial proof of the following identity:

$$k^n = \sum_{i=1}^{n} \binom{k}{i} i! S(n,i).$$

If you use a partition, state explicitly the equivalence relation that gives the partition.

2. Recall that $D(n)$ denotes the number of derangements of an n-element set. Give a **combinatorial** proof of the following identity.

$$n! = D(n) + \binom{n}{1} D(n-1) + \cdots + \binom{n}{i} D(n-i) + \cdots + \binom{n}{n-2} D(2) + \binom{n}{n-1} D(1) + 1.$$

3. Compute the sum

$$\sum_{i=0}^{n} (-1)^n \binom{n}{i} \binom{n}{n-i}$$

4. Use the binomial theorem to prove the following:

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}.$$

5. Determine the number of strings of p 0's and q 1's where each pair of 1's is separated by at least 2 0's.

6. A *monotone path* in the plane starts at the origin (i.e. (0.0)) and consists of of segments of the form $(x,y) \to (x+1,y)$ or $(x,y) \to (x,y+1)$ where $x,y \in \mathbb{Z}$. In words, a monotone path is a path in the integer grid that always moves up or to the right. How many monotone paths from the origin to the point (a,b) are there?

7. A word over the alphabet $\{a,b,c,\ldots,z\}$ is called *increasing* if its letters, apart from repetitions, appear in alphabetical order. For example *aabcc* is increasing but *abzc* is not. How many increasing words are there of length n?

8. Determine the number of permutations of $\{1,2,\ldots,8\}$ in which no even integer is in its natural position.

9. Prove the inequality

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n} \geq \log(n+1).$$

From the text: 1.8.26, 1.8.31, 1.8.33, 1.8.34, 2.1.13, 2.3.1, 2.5.5, 2.5.6, 3.2.3, 3.4.2, 3.7.2, 3.8.6, 3.8.8, 3.8.10, 3.8.12, 3.8.13, 3.8.14.

Sample Questions: The test will include some questions in the following format:

For each of the following statements, say whether the statement is true or false and give a short justification for your answer.

1. $\binom{n}{10} \sim n^{10}/(10!)$.
2. If $n, k \in \mathbb{N}^+$ and $0 \le k \le n$ then
$$\binom{n}{k} \le 2^n.$$
3. More than half of the numbers in the set $\{0, 1, 2, \ldots, 10^{10} - 1\}$ contain the digit 9 (in their decimal expansion).
4. There exists $n \in \mathbb{N}^+$ such that
$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \ge 2\sqrt{n}$$
5. If $S(a, b)$ is a Stirling number of the second kind then
$$S(a, b) = \binom{a + b - 1}{b - 1}.$$
6. If $n \ge 3$ and $1 < k < n$ then $S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$.