1. Suppose \( n \geq 4 \) and \( F \) is a collection of \( n \)-element subsets of some ground set \( X \). Prove that if
\[
|F| < \frac{4^{n-1}}{3^n}
\]
then there is a coloring of \( X \) with 4 colors such that every color appears at least once in every set in \( F \).

2. Prove that for any real number \( p \) such that \( 0 \leq p \leq 1 \) and any integer \( n \) we have
\[
R(k, \ell) > n - \binom{n}{k} p^{(\ell)} - \binom{n}{\ell} (1-p)^{(\ell)}.
\]

3. Let \( A \) and \( B \) be sets such \(|A| = m\) and \(|B| = n\). Let \( \Omega \) be the set of all functions from \( A \) to \( B \). (I.e. \( \Omega = B^A \).) Let \( f \) be function drawn uniformly at random from \( \Omega \). Let the random variable \( X \) be the cardinality of set of elements of \( B \) that are not in the image of \( f \). Formally,
\[
X = |\{b \in B : \forall a \in A \ f(a) \neq b\}|.
\]
(a) Determine \( E[X] \).
(b) Prove that if \( m > n(\log n + 3) \) then \( Pr(X = 0) > 0.9 \).

A graph \( G = (V, E) \) is bipartite if there is a partition of the vertex set \( V = X \cup Y \) such that neither \( X \) nor \( Y \) contains an edge. (I.e. \( E \cap \binom{X}{2} = \emptyset \) and \( E \cap \binom{Y}{2} = \emptyset \).) In other words, every edge has one vertex in \( X \) and one vertex in \( Y \).

4. Let \( G = (V, E) \) be a graph. Show that there is a bipartite graph \( H = (V, F) \) such that \( |F| \geq |E|/2 \). In words \( G \) contains a bipartite subgraph that includes at least half of the edges of \( G \).

\textit{Hint: Consider a random bipartition.}

5. Suppose we color the edge set of \( K_n \) with 2 colors uniformly at random (so, we have \(|\Omega| = 2^{\binom{n}{2}}\) and the uniform distribution). Let \( X \) be the number of monochromatic triangles in the random coloring. Determine \( Var(X) \).

Random variables \( X \) and \( Y \) defined on the same probability space \( \Omega \) are \textit{independent} if the events \( \{X = \alpha\} \) and \( \{Y = \beta\} \) are independent for every \( \alpha, \beta \).

6. Let \( X \) and \( Y \) be independent random variables defined on the same finite probability space. Suppose \( X \) and \( Y \) take values in \( \{0, 1, 2, \ldots\} \).
(a) Prove \( E[XY] = E[X]E[Y] \).
(b) Prove \( Var(X + Y) = Var[X] + Var[Y] \).