21-228 Discrete Mathematics
Assignment # 7
Due: Friday, April 4

1. For $n, m \geq 1$ let $G_{n,m}$ be the graph with vertex set

$$\{(x, y) : 1 \leq x \leq n \text{ and } 1 \leq y \leq m\}$$

and an edge connecting vertices (x, y) and (u, v) if

$$(|x-u| = 1 \text{ and } |y-v| = 2) \qquad \text{or} \qquad (|x-u| = 2 \text{ and } |y-v| = 1)\,.$$

In words, the vertices in $G_{n,m}$ are the squares on an $n \times m$ chessboard and we connect two squares with an edge if there is a 'knight's move' that takes one to the other.

   (a) Exhibit a Hamiltonian path in $G_{3,4}$.

   (b) Show that $G_{4,4}$ does not have a Hamiltonian cycle.
   Hint: Which squares come before and after the corner squares in a Hamiltonian cycle?

2. Prove that if $G = (V, E)$ is a graph such that $|V| = n$ and

$$|E| \geq \frac{(n-1)(n-2)}{2} + 1$$

then G is connected. Give an example of a disconnected graph on n vertices with $(n-1)(n-2)/2$ edges.

3. Prove that in every tree, any two paths with maximum length have a vertex in common. Show that this is not true if we consider two maximal paths (a path is maximal if it cannot be extended to a longer path).

4. Let T be a tree with n vertices, $n \geq 2$. For each positive integer i let p_i be the number of vertices of T of degree i. Prove

$$p_1 - p_3 - 2p_4 - \cdots - (n-3)p_{n-1} = 2.$$

5. Let $V = \{1, 2, \ldots, n\}$ and let $G = (V, E)$ be a tree. Let $A_1, A_2, \ldots, A_n$ be subsets of a finite set Ω. Prove that

$$\left|\bigcup_{i=1}^{n} A_i\right| \leq \sum_{i=1}^{n} |A_i| - \sum_{\{u,v\} \in E} |A_u \cap A_v|\,.$$