21-228 Discrete Mathematics
Assignment # 7, revised
Due: Friday, April 5

1. A collection of n teams plays a round robin tournament. (I.e. a tournament in which every team plays every other team exactly once. There will be $\binom{n}{2}$ games in this tournament). We say that the tournament has property S_k if for every set S of k teams there is some team that beats all teams in the set S. For example if we have 3 teams then the tournament would have property S_1 if, for example, team 1 beats team 2, team 2 beats team 3 and team 3 beats team 1. Prove that if
$$\binom{n}{k}(1-2^{-k})^{n-k} < 1$$
then it is possible that a tournament with n teams has property S_k.

2. Which of the following statements are true for any random variable X defined on a finite probability space with the uniform distribution? Provide proofs for the true statements and counterexamples for the false statements.

   (a)
$$E[1/X] = 1/E[X].$$

   (b)
$$E[X]^2 \leq E[X^2].$$

   (c)
$$\sqrt{E[X]} \geq E[\sqrt{X}].$$

3. Let $p_2, p_2, \ldots$ be a sequence of real numbers such that $0 < p_n < 1$ for all n, and consider the following sequence of probability spaces. The set Ω_n consists of all graphs on vertex set V_n where $|V_n| = n$, and for every graph $G = (V_n, E)$ we set
$$\mathbb{P}(G) = p_n^{|E|}(1-p_n)^{\binom{n}{2}-|E|}.$$
Note that this probability function is given by using $\binom{n}{2}$ mutually independent biased coin flips to determine whether or not each element of $\binom{V_n}{2}$ appears as an edge. (This object is called the *binomial random graph* or the *Erdős-Rényi random graph.*)

   (a) Let the random variable $X_n(G)$ count the number of copies of K_3 in the graph G. Determine $E[X_n]$.
   (b) Suppose $p_n = 1/n$. Prove $\mathbb{P}(X_n = 0) > 1/2$ for all n.

4. Consider the binomial random graph (as defined in the previous problem) with $p_n = \frac{\log(n)}{n}$, and again let the random variable X_n count the number of copies of K_3. Use Chebyshev's inequality to prove
$$\lim_{n\to\infty} \mathbb{P}(X_n = 0) = 0.$$
Hint. Use indicator random variables for both the expectation and the variance.

5. Suppose the random variable X is distributed as $Bi(2m, 1/2)$ where m is a positive integer.

   (a) Use Chebyshev's inequality to bound
   $$\mathbb{P}\left(X \leq m - m^{3/4}\right).$$

   (b) Use Claim 24 from the first review sheet to find a another bound on this probability that is substantially stronger when m is large.

   Hint. Use the estimate
   $$\sum_{i=0}^{m-t} \binom{2m}{i} \leq m \binom{2m}{m-t}.$$

6. If C is a cycle and e is an edge connecting two non-consecutive vertices in C then we call e a *chord* of C.

   (a) Prove that if every vertex in a graph G has degree at least 2 then G contains a cycle.

   (b) Prove that if every vertex of a graph G has degree at least 3 then G contains a cycle with a chord.

   Hint: Consider a maximal path.