21-228 Discrete Mathematics
Assignment # 6
Due: Friday, March 29

1. Suppose $6n$ fair dice are rolled. Define a probability space that describes this experiment. Let p_n be the probability that every number (i.e. the numbers 1 through 6) appears exactly n times in one roll of the collection of dice. Give an exact expression for p_n. Find a simpler function f_n such that $p_n \sim f_n$.

2. Prove that if there is a number p such that $0 \leq p \leq 1$ and

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$

   then the Ramsey number $R(k,t)$ satisfies $R(k,t) > n$.

3. Suppose we color the edge set of K_n with 2 colors uniformly at random (so, we have $|\Omega| = 2^{\binom{n}{2}}$ and the uniform distribution).

   (a) Let X be the number of monochromatic triangles in the random coloring. Determine $E(X)$.

   (b) Conclude that there exists a coloring of the edge set of K_n that has at most $\frac{1}{4}\binom{n}{3}$ monochromatic triangles.

4. A collection of k people enter an elevator in a building that has $n+1$ floors. Each of the k people has a destination floor that is chosen uniformly at random from floors $2, \dots, n+1$. What is the expected number of stops that the elevator makes?

5. Consider the probability space on the finite set Ω with the uniform distribution. Let X be a random variable that takes values in the set $\{0, 1, 2, \dots, M\}$ such that $E[X] = M - a$. Prove that for any $1 \leq b \leq M$ we have

$$\mathbb{P}(X \geq M - b) \geq \frac{b-a}{b}.$$

6. If f is a permutation of the set $X = \{1, 2, \dots, n\}$ then a **cycle** in f is a sequence of distinct elements $a_1, a_2, \dots, a_k$ of X with the property that $f(a_i) = a_{i+1}$ for $i = 1, \dots, k-1$ and $f(a_k) = a_1$. The number of elements in the sequence is the **length** of the cycle. Note that every element of X is in a unique cycle of f and that an element $x \in X$ such that $f(x) = x$ is a cycle of length 1.

   Let Ω be the set of permutations of the set $\{1, 2, \dots, n\}$. Consider the probability space on Ω given by the uniform distribution. Determine the expected length of the cycle containing 1 in a permutation chosen at random from this probability space.