1. Suppose 6n fair dice are rolled. Define a probability space that describes this experiment. Let \( p_n \) be the probability that every number (i.e. the numbers 1 through 6) appears exactly \( n \) times in one roll of the collection of dice. Give an exact expression for \( p_n \). Find a simpler function \( f_n \) such that \( p_n \sim f_n \).

2. A woman walks randomly on the \( n \times n \) grid \( \{(x, y) : x, y \in \{1, \ldots, n\} \} \) starting at the point \((1,1)\) (i.e. the lower left corner). Each minute the woman moves either to the right or up (i.e. a move of the form \((a, b) \to (a + 1, b)\) or a move of the form \((a, b) \to (a, b + 1)\)). Her walk ends when she reaches the upper right corner, the point \((n,n)\). At each stage in which the woman has a choice of 2 moves she flips a fair coin to determine her next move. (If the woman is on the right edge (i.e. \((x,y)\) such that \(x = n\)) she automatically moves up and if she is on the top edge (i.e. \((x,y)\) such that \(y = n\)) she automatically moves right.) Define a probability space that describes this random walk. What is the probability that the woman reaches the top row of the grid before reaching \((n,n)\)? Explain your answer.

3. Let \( M \) be a \( n \times n \), 0-1 matrix chosen uniformly at random from the set of all such matrices.

   (a) Let \( k < n \) be a positive integer. What is the probability that a fixed \( k \times k \) sub-matrix of \( M \) is the all 1’s matrix?

   (b) Use the Union Bound to that if \( k \geq 4 \log_2 n \) then the probability that \( M \) has a \( k \times k \) sub-matrix of all 1’s goes to zero as \( n \) goes to infinity.

4. Consider a probability space on a finite set \( \Omega \) with the uniform distribution. Suppose \(|\Omega|\) is prime and let \( A \) and \( B \) be events such that \( 0 < P(A) < 1 \) and \( 0 < P(B) < 1 \). Prove that \( A \) and \( B \) are not independent.

5. We say that a graph \( G = (V,E) \) is connected if for every partition \( A, B \) of the vertex set into two nonempty parts there is an edge that connects \( A \) and \( B \) (i.e. there is an edge with one vertex in \( A \) and one vertex in \( B \).) For each \( n \geq 2 \) let \( V_n \) be a vertex set of cardinality \( n \) and consider a graph chosen uniformly at random from the collection of all graphs on vertex set \( V_n \). (N.b. This is equivalent to flipping an independent fair coin to determine whether or not each element of \( \binom{V_n}{2} \) appears in the graph.) Let \( C_n \) be the event that this graph is connected. Use the union bound to prove

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\lim_{n \to \infty} P(C_n) = 1.
\]