21-228 Discrete Mathematics
Assignment # 5
Due: Friday, March 2

1. Prove that any edge coloring of the edge set of $K_{17}$ with the colors Red, Blue and Green has a monochromatic triangle.

2. Let $k \geq 3$ and $n = (k - 1)^2$. Give an explicit 2-coloring of the edges of $K_n$ that does not have a monochromatic $K_k$.

3. Prove $R(3, 5) \leq 14$.

4. If $G = (V, E)$ is a graph and $v \in V$ then the degree of $v$, denoted $d(v)$, is the number of edges in $G$ that contain $v$ (e.g. the degree of every vertex in the complete graph $K_n$ is $n - 1$).

   Let $n \geq 2$ be an integer. Does there exist a graph with vertex set $V = \{v_1, \ldots, v_n\}$ such that $d(v_i) = i - 1$ for $i = 1, \ldots, n$?

5. A graph $G = (V, E)$ is bipartite if there exists a partition $V = A \cup B$ such that $E \cap \binom{A}{2} = \emptyset$ and $E \cap \binom{B}{2} = \emptyset$.

   In other words, every edge has one vertex in $A$ and one vertex in $B$. The sets $A$ and $B$ are the parts of the bipartition of $G$.

   A graph $G = (V, E)$ is $d$-regular if every vertex in $G$ has degree $d$.

   Let $G$ be a $d$-regular bipartite graph with parts $A, B$. Prove that if $d \geq 1$ then $|A| = |B|$.

6. A cycle in a graph $G = (V, E)$ is a sequence of vertices $v_0, v_1, v_2, \ldots, v_k$ such that

   - $v_0, \ldots, v_{k-1}$ are distinct,
   - $v_0 = v_k$, and
   - $\{v_i, v_{i+1}\} \in E$ for $i = 0, 1, \ldots, k - 1$.

   The cycle $v_0, v_1, v_2, \ldots, v_k$ is a Hamilton cycle if every vertex in $V$ appears exactly once in the sequence $v_0, v_1, \ldots, v_{k-1}$.

   Consider a 2-coloring of the edge set of the complete graph $K_n$ (with $n \geq 3$). Prove that there is a Hamilton cycle that is monochromatic or a Hamilton cycle with two monochromatic arcs. (I.e. there are $a < b$ such that $\{v_a, v_{a+1}\}, \{v_{a+1}, v_{a+2}\}, \ldots, \{v_{b-1}, v_b\}$ are blue while $\{v_b, v_{b+1}\}, \{v_{b+1}, v_{b+1}\}, \ldots, \{v_{k-1}, v_k\}, \{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{a-1}, v_a\}$ are green.)