21-228 Discrete Mathematics
Assignment # 5
Due: Friday, February 29

1. Let $n < m$ be positive integers. Let $\mathcal{H}$ be a collection of n-element subsets of $\{1, \dots, m\}$ (i.e. $\mathcal{H} \subseteq \binom{\{1,\dots,m\}}{n}$). Suppose the number of sets in $\mathcal{H}$ is less than $4^{n-1}/3^n$. Show that there exists a coloring of $\{1, \dots, m\}$ with the colors B, G, R, Y such that for every set $X \in \mathcal{H}$ and color $C \in \{B, G, R, Y\}$ there exists an element of X that has color C.

   Hint: choose a coloring at random and apply Boole's inequality to some set of bad events.

2. Let ω be a $n \times n$, 0-1 matrix chosen uniformly at random from the set of all such matrices.

   (a) Let $k < n$ be a positive integer. What is the probability that a *fixed* $k \times k$ sub-matrix of ω is the all 1's matrix?

   (b) Using Boole's inequality, show that if $k \geq 4 \log_2 n$ then the probability that ω has a $k \times k$ sub-matrix of all 1's goes to zero as n goes to infinity.

3. Consider a poker hand: a set of five cards drawn at random from a standard deck.

   (a) What is the probability space for this random experiment?

   Suppose now that two cards out of a five card hand have been revealed (the other three remain hidden). Determine the following conditional probabilities.

   (b) The probability of a flush if the first two cards are hearts.

   (c) The probability of a full house if the first two cards are 2's.

4. A bowl contains n cherries, exactly m of which have had their stones removed. A pig eats p cherries chosen at random without announcing how many contain stones. Subsequently, a cherry is picked at random from the bowl.

   (a) What is the probability that this cherry contains a stone?

   (b) Given that this cherry does indeed contain a stone, what is the probability that the pig consumed at least one stone?

5. Adapt the proof of Inclusion/Exclusion given in class to prove the follows version of the Principle of Inclusion/Exclusion.

   If $A_1, \dots, A_n$ are events in a probability space defined on the set Ω then

$$Pr\left(\Omega \setminus \left(\bigcup_{i=1}^{n} A_i\right)\right) = \sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|} Pr\left(A_I\right)$$

   where

$$A_I = \begin{cases} \cap_{i \in I} A_i & \text{if } I \neq \emptyset \\ \Omega & \text{if } I = \emptyset. \end{cases}$$