21-228 Discrete Mathematics
Homework 4
Due: Friday, February 23

1. Let $h_0, h_1, h_2, \ldots$ be the sequence defined by $h_n = \binom{n}{3}$. Determine the generating function for this sequence.

2. Let B_n be those strings in $\{1, 2\}^n$ which do not contain $1, 2, 2$ as a sub-string (in consecutive positions). Let $b_n = |B_n|$.

   (a) Establish a recurrence for the sequence $b_0, b_1, b_2, \ldots$

   (b) Determine the generating function for this sequence.

3. Consider a collection of n circles drawn in the plane such that:

   (i) Each pair of circles intersects in two distinct points, and

   (ii) the intersection of any three circles is empty (i.e. no point lies at the intersection of three circles).

   Let h_n be the number of regions in the plane created by the collection of intersecting circles (e.g. $h_1 = 2$ as there is a region inside the circle and a region outside the circle, $h_2 = 4$ and $h_3 = 8$).

   (a) For $n = 2, 3, \ldots$ write h_n as a function of $h_1, h_2, \ldots, h_{n-1}$ and n.

   (b) Use your answer to (a) to determine the generating function for the sequence $h_0, h_1, \ldots$.

   (c) Use your answer to (b) to give a closed form expression for h_n.

4. There are $2n$ points on a circle. We want to divide them into pairs and connect each pair with a segment (i.e. a chord) in such a way that these segments do not intersect. Show that the number of ways to do this is given by a Catalan number.

5. Suppose five points are chosen inside an equilateral triangle with side length 1. Show that there is at least one of pair of points whose distance apart is at most $1/2$.

6. Prove that every set of $n + 1$ integers chosen from $\{1, 2, \ldots, 2n\}$ contains two numbers such that one divides the other.

   Hint: Consider powers of 2.