1. Use the binomial theorem to simplify the following sum:

\[ 1 + \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} + \cdots + \frac{1}{n+1} \binom{n}{n}. \]

Note. This is NOT an approximation question.

2. How many integral solutions of the equation

\[ x_1 + x_2 + x_3 + x_4 = 18 \]

satisfy the inequalities \(0 \leq x_1 \leq 5, 0 \leq x_2 \leq 10, 0 \leq x_3 \leq 7\) and \(0 \leq x_4 \leq 13\)?

Hint: Begin with the set of integral solutions such that each \(x_i\) is non-negative.

3. How many positive integers \(n < 100\) are not divisible by a square of any integers greater than 1?

4. Prove that for any two finite sets \(I \subseteq J\) we have

\[ \sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J. \end{cases} \]

5. Let \(V = \{1, 2, \ldots, n\}\). A graph on vertex set \(V\) is an ordered pair \(G = (V, E)\) where \(E \subseteq \binom{V}{2}\). The set \(E\) is called the edge set of \(G\). The degree of a vertex \(v\) in the graph \(G\) is the number edges in \(E\) that contain \(v\).

Find an expression for the number of graphs on vertex set \(V\) that have no vertices of degree 0.

6. (a) Use the binomial theorem to prove the following identity. If \(n \in \mathbb{N}^+\) and \(n \geq 2\) then

\[ \sum_{i=1}^{n} i \binom{n}{i} (-1)^{i-1} = 0 \]

(b) Let \(\Omega\) be a finite set. Let \(A_1, \ldots, A_n\) be finite subsets of \(\Omega\). For \(S\) a nonempty subset of \(\{1, 2, \ldots, n\}\) define

\[ A_S = \bigcap_{i \in S} A_i. \]

Let \(Y\) be the set of elements of \(\Omega\) that are in exactly one of the sets \(A_1, A_2, \ldots, A_n\). (Following the notation from lecture \(Y = \{x \in \Omega : |N_x| = 1\}\).) Prove

\[ |Y| = \sum_{S \subseteq \{1, 2, \ldots, n\}, S \neq \emptyset} (-1)^{|S|-1} |S| \cdot |A_S|. \]