21-228 Discrete Mathematics Assignment # 3

Due: Friday, September 19

1. Use the binomial theorem and integration to simplify the following sum:

$$1 + \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} + \dots + \frac{1}{n+1} \binom{n}{n}.$$

Note. This is NOT an approximation question.

2. (a) How many integral solutions of the the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

satisfy the inequalities $0 \le x_1 \le 5$, $0 \le x_2 \le 5$, and $0 \le x_3 \le 5$?

(b) How many integral solutions of the the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

satisfy the inequalities $0 \le x_1 \le 5$, $0 \le x_2 \le 5$, $0 \le x_3 \le 5$ and $0 \le x_4 \le 10$? Hint: Begin with the set of integral solutions such that each x_i is non-negative.

3. Prove that for any two finite sets $I \subseteq J$ we have

$$\sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J. \end{cases}$$

- 4. Let $V = \{1, 2, ..., n\}$. A graph on vertex set V is an ordered pair G = (V, E) where $E \subseteq \binom{V}{2}$. The set E is called the edge set of G. The degree of a vertex v in the graph G is the number edges in E that contain v.
 - (a) How many graphs on vertex V are there?
 - (b) Find an expression for the number of graphs on vertex set V that have no vertices of degree 0.
- 5. Use the binomial theorem to prove the following identity. If $n \in \mathbb{N}^+$ and $n \geq 2$ then

$$\sum_{i=1}^{n} i \binom{n}{i} (-1)^{i-1} = 0$$

6. Let Ω be a finite set and let A_1, \ldots, A_n be finite subsets of Ω . For S a nonempty subset of $\{1, 2, \ldots, n\}$ define

$$A_S = \bigcap_{i \in S} A_i.$$

Let Y be the set of elements of Ω that are in exactly one of the sets A_1, A_2, \ldots, A_n . (Following the notation from lecture $Y = \{x \in \Omega : |N_x| = 1\}$.) Use the equation in the previous question to prove

$$|Y| = \sum_{S \subseteq \{1,2,\dots,n\}, S \neq \emptyset} (-1)^{|S|-1} |S| \cdot |A_S|.$$