21-228 Discrete Mathematics
Assignment # 3
Due: Friday, February 9

1. Use the binomial theorem and integration to simplify the following sum:

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n}.$$

   Note. This is NOT an approximation question.

2. Let k, n be positive integers such that $k < n$.

   (a) Use the binomial theorem to prove the following:

$$0 < x < 1 \quad \Rightarrow \quad \sum_{i=0}^{k} \binom{n}{i} \leq \frac{(1+x)^n}{x^k}.$$

   (b) Use part (a) to conclude

$$\sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{ne}{k}\right)^k.$$

3. Prove that for any two finite sets $I \subseteq J$ we have

$$\sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J. \end{cases}$$

4. Let $V = \{1, 2, \ldots, n\}$. A *graph on vertex set* V is an ordered pair $G = (V, E)$ where $E \subseteq \binom{V}{2}$. The set E is called the *edge set* of G. The *degree* of a vertex v in the graph G is the number edges in E that contain v.

   (a) How many graphs on vertex V are there?

   (b) Find an expression for the number of graphs on vertex set V that have no vertices of degree 0.

5. Let $n \in \mathbb{N}^+$ such that $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$ where $p_1, \ldots, p_4$ are primes and $a_1, \ldots, a_4 \in \mathbb{N}^+$. Let $\varphi(n)$ be the number of integers in the set $\{1, 2, \ldots, n\}$ that are coprime to n. Prove

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right)\left(1 - \frac{1}{p_4}\right).$$

6. Use the binomial theorem to prove the following identity. If $n \in \mathbb{N}^+$ and $n \geq 2$ then

$$\sum_{i=1}^{n} i\binom{n}{i}(-1)^{i-1} = 0$$