1. (a) Give a combinatorial proof of the following equation
\[ \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}. \]

If you make use of a partition, explicitly state the equivalence relation that gives the partition.
(b) Use (a) to calculate \( \sum_{i=1}^{n} i^2 \). *Hint: Consider \( k = 2 \).*
(c) Use (a) to calculate \( \sum_{i=1}^{n} i^3 \).

2. Let \( k, n \in \mathbb{N}^+ = \{x \in \mathbb{N} : x > 0\} = \{1, 2, 3, \ldots \} \). In this problem we consider labelings of \( k \) indistinguishable objects with \( n \) distinguishable colors. (I.e. we consider bags of M&M’s such that there are \( k \) pieces of candy in each bag and each M&M is one of \( n \) colors.)

(a) Assume \( n \geq k \). How many labeling are there such that each label is used at most once?
(b) Assume \( n \leq k \). How many labeling are there such that each label is used at least once?

3. Let \( n, k \) be positive integers such that \( 1 \leq k \leq n - 1 \). Prove the following inequalities:
\[ \frac{n^k}{k!} \leq \frac{n}{k} \leq \frac{n^k}{k!}. \]

4. Let \( n \in \mathbb{N}^+ \). Show that
\[ 2^{n/2} \cos \left( \frac{n\pi}{4} \right) = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots. \]
*Hint: Expand \((1 + i)^n\) using the binomial theorem (where \( i^2 = -1 \)). Use the identities \( e^{i\theta} = \cos(\theta) + i\sin(\theta) \)*

5. Prove the following bound:
\[ n! \leq e\sqrt{n} \left( \frac{n}{e} \right)^n. \]

6. (a) Prove
\[ 1 - x \leq e^{-x} \]
for all real numbers \( x \).
(b) Prove
\[ 1 - x \geq e^{-x-x^2} \]
for \( 0 \leq x \leq 0.5 \).
(c) Let $X$ and $Y$ be sets such that $|X| = 2n$ and $|Y| = n$. Let $a$ be a particular element of $X$. Show that the number of functions $f : Y \to X$ such that $a$ is not in the image of $f$ is asymptotic to $\frac{(2n)^n}{e^{1/2}}$. In other words, show

$$\lim_{n \to \infty} \frac{|\{f \in X^Y : f(y) \neq a \text{ for all } y \in Y\}|}{(2n)^n} = \frac{1}{e^{1/2}}.$$