21-228 Discrete Mathematics Assignment # 2

Due: Friday, September 12

1. Use integration to show that if $n \in \mathbb{N}$ then

$$\frac{n^4}{4} \le \sum_{i=1}^n i^3 \le \frac{(n+1)^4}{4}.$$

2. (a) Give a combinatorial proof of the following equation

$$\binom{k}{k} + \binom{k+1}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}.$$

If you make use of a partition, explicitly state the equivalence relation that gives the partition.

- (b) Use (a) to calculate $\sum_{i=1}^{n} i^2$. Hint: Consider k=2. (c) Use (a) to calculate $\sum_{i=1}^{n} i^3$.
- 3. Prove that for $n \in \mathbb{N}^+$ we have

$$2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \le 2\sqrt{n} - 1.$$

- 4. Let n, k be positive integers such that $1 \le k \le n-1$.
 - (a) Prove the following inequalities:

$$\frac{n^k}{k^k} \le \binom{n}{k} \le \frac{n^k}{k!}.$$

(b) Prove that if k is fixed and $n \to \infty$ then

$$\binom{n}{k} \sim \frac{n^k}{k!}$$
.

5. Let $n \in \mathbb{N}$. Show that

$$2^{n/2}\cos\left(\frac{n\pi}{4}\right) = \sum_{0 \le k \le n: 2|k} (-1)^{k/2} \binom{n}{k} = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots$$

Hint: Expand $(1+i)^n$ using the binomial theorem (where $i^2=-1$). Use the identities $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$.

6. (a) Prove

$$1 - x < e^{-x}$$

for all real numbers x.

(b) Prove

$$1 - x \ge e^{-x - x^2}$$

for 0 < x < 0.5.