# 21-228 Discrete Mathematics
## Assignment # 2
## Due: Friday, February 4

1. Use a combinatorial argument to show that if $n$ is a positive integer then $(n!)^3$ divides $(3n)!$.

2. (a) Give a combinatorial proof of the following equation

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

   If you make use of a partition, explicitly state the equivalence relation that gives the partition.

   (b) Use (a) to calculate $\sum_{i=1}^{n} i^2$. *Hint: Consider* $k = 2$.

   (c) Use (a) to calculate $\sum_{i=1}^{n} i^3$.

3. Let $n, k$ be positive integers such that $1 \leq k \leq n-1$.

   (a) Prove the following inequalities:

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{n^k}{k!}.$$

   (b) Prove that if $k$ is fixed and $n \to \infty$ then

$$\binom{n}{k} \sim \frac{n^k}{k!}.$$

4. Prove that for $n \in \mathbb{N}^+$ we have

$$2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.$$

5. Prove the following bound:

$$n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n.$$

6. (a) Prove

$$1 - x \leq e^{-x}$$

   for all real numbers $x$.

   (b) Prove

$$1 - x \geq e^{-x-x^2}$$

   for $0 \leq x \leq 0.5$.