21-228 Discrete Mathematics
Assignment # 1
Due: Friday, January 26

1. Find all positive integers $a > b > c$ for which

$$\binom{a}{b}\binom{b}{c} = 2\binom{a}{c}.$$

Give a combinatorial justification for your answer (i.e. appeal to the fact that $\binom{a}{b}$ gives the number of subsets of an a-element set having exactly b elements).

2. If m indistinguishable 4-sided die are rolled how many distinguishable outcomes are there? (E.g. if $m = 2$ then there are 10 possibilities.) How many outcomes are there in which each of the four numbers appear at least once?

3. Let p and q be positive integers. How many sequences of p 1's and q 0's are there with the property that there are at least two 0's between every pair of 1's?

4. A function $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is **monotone** if $i < j$ implies $f(i) \le f(j)$. How many monotone functions $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ are there?

5. Let $\mathcal{F}$ be a collection of k-element subsets of the set X, where $|X| = n$. The **shadow** of $\mathcal{F}$ is defined to be the set

$$\partial\mathcal{F} = \left\{ B \in \binom{X}{k-1} : \exists A \in \mathcal{F} \text{ such that } B \subset A \right\}.$$

Show

$$|\partial(\mathcal{F})| \ge \frac{k|\mathcal{F}|}{n-k+1}.$$

Hint. Count ordered pairs (A, B) such that $|A| = k$, $|B| = k-1$, $B \subset A$ and $A \in \mathcal{F}$.

6. Use a **combinatorial** argument to prove the following: for all positive integers m_1, m_2, n we have

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}.$$