RESEARCH STATEMENT

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1. Overview

My research is focused on Dynamical Systems and Partial Differential Equations. My recent work has spanned several areas, including the motion of dislocations in elastic materials, KAM methods for finding invariant tori in volume-preserving systems, transitions from sliding to pinned motions in epitaxial deposits on quasi-crystals, stability analysis for ODEs with stochastic parametric forcing, determining surface instability for stochastically forced viscous capillary gravity waves, and the rigorous derivation of linearized models for nematic elastomers. My earlier work involved extensions of Aubry-Mather theory to Partial Differential Equations and Pseudo-differential Equations, with applications to the equilibrium shape of crystals. I have produced these varied results by adapting my skills to the interesting problems I find and by using tools from multiple areas (e.g. Functional Analysis, Geometry, Hard Analysis, Stochastics, Numerics). My publications and preprints can be found at `http://math.cmu.edu/~tblass/research`.

In the paragraphs below, I give brief summaries of each of these projects. More detailed explanations are found in the following sections: dislocation dynamics in § 2, theory and numerical work on KAM problems in § 3, stochastic forcing in ODEs and surface waves in § 4, Γ-convergence and applications to elastomers in § 5, and Aubry-Mather for PDEs in § 6. I present my future research projects in § 7.

Dislocation Dynamics. [12]

Dislocations are imperfections in the lattice structure of crystalline materials. The modeling of dislocations is of great interest in materials science because important material properties, such as rigidity and conductivity, can be strongly affected by the presence of dislocations. For example, large collections of dislocations can result in plastic deformations in solids under applied loads.

In joint work with Irene Fonseca, Giovanni Leoni, and Marco Morandotti [12], we focus on a continuum model for screw dislocations, which are line defects in the material. These dislocations generate a strain field that in turn exerts a force on the other dislocations (both directly and through the strain-response at the boundary of the material). One of our main contributions is to give a rigorous foundation to a heuristic model introduced by Cermeli and Gurtin [16]. By introducing a variational setting, we derive a renormalized energy for a system of N dislocations, as well as a formula for the forces on the dislocations and the associated evolution law (assuming dislocations move in a way that maximally dissipates energy). The situation is complicated by crystallographic restrictions on the directions of motion (dislocation lines can only move in a discrete set of planes, called glide-planes). This leads to an ordinary differential equation with a discontinuous vector field. We prove existence of solutions, give conditions for uniqueness, and show some qualitative properties of solutions.

Volume-preserving systems. [10]

Kolmogorov-Arnold-Moser (KAM) theory is a collection of results regarding the existence of invariant sets in dynamical systems. Understanding the structure of the invariant sets is an important part of organizing the behavior of dynamical systems. The first results in KAM theory were only applicable to Hamiltonian systems, however, KAM techniques have proven fruitful in other settings. In joint work with Rafael de la Llave [10] we have developed a method for proving the existence (and local uniqueness) of codimension-one invariant tori in volume-preserving systems with one action variable and n angle variables. These invariant surfaces separate

the phase space into regions without mixing. Many physical systems are modeled by volume-preserving maps, including models for fluid flows and plasmas, where the invariant tori act as transport barriers. Thus, KAM methods for volume-preserving systems are of great interest to both pure and applied mathematicians, and have been studied theoretically and numerically (e.g. by Cheng [17], Herman [40], Meiss [29], Xia [39]).

We prove an *a posteriori* theorem stating that if there exists an approximately invariant torus with Diophantine rotation vector and if some non-degeneracy conditions are satisfied, then the system has a true invariant torus with the same rotation vector near the approximately invariant torus. This is a vast improvement on prior results, which are applicable only to perturbations of integrable systems. Our result also applies to secondary tori (i.e. tori with contractible directions) and leads to efficient numerical algorithms, which have already been implemented successfully in [23]. Our method is robust, and we also provide extensions to flows, almost-volume-preserving systems, small-twist systems, and systems with quasi-periodic forcing.

Analyticity breakdown in quasi-periodic Frenkel-Kontorova models. [9]

The classical Frenkel-Kontorova (FK) model associates a formal energy to an infinite system of particles (one-dimensional chain of atoms); the atoms interact with their nearest neighbors via a quadratic potential energy and with a substrate via a periodic potential energy. The FK model has been applied to several areas in solid-state physics, such as the modeling of crystal dislocations, domain walls in ferromagnetic materials, twinning boundaries, and epitaxial monolayers on crystal surfaces. From a mathematical point of view, the FK model is closely related to twist mappings, which arise naturally in Hamiltonian systems, and have been studied since the work of Poincaré. The set of equilibrium configurations is of particular interest and has been well-studied since the work of Aubry and Mather [6, 28].

In joint work with Rafael de la Llave [9], we present numerical findings for a generalized FK model where the periodic potential energy is replaced by a quasi-periodic potential energy. Physically, this corresponds to an epitaxial model with a quasi-crystal substrate. A theoretical framework for quasi-periodic energies was developed by Su and de la Llave, [36, 37], where equilibrium configurations are determined by "hull functions". We implement their method for finding hull functions numerically and push the limits in parameter space to find the breakdown of analyticity. This breakdown is believed to correspond to the transition from sliding to pinned modes of the material.

Stability of ODEs with colored noise forcing. [13]

Parametrically forced ordinary differential equations are used to model systems where the parameters of the system are allowed to vary. Such equations are used to model physical systems in a variety of settings, including celestial mechanics (Hill's equation), quantum mechanics (Schrödinger equation), and fluid mechanics (dispersion relations for water waves). A solution of a parametrically forced ODE is said to be stable if it is bounded for all time. These systems often have a periodic forcing term, and Floquet theory is used to determine the stability of solutions. Some physical systems call for an understanding of the stability of solutions when the forcing is stochastic. The stability of solutions has been well-understood in the periodic case for some time, and there has been some work in the stochastic setting (e.g. Adams and Bloch [1], van Kampen [38]), but not in the case of colored noise until my joint work with L. A. Romero in [13].

We considered general linear systems with parametric forcing by colored noise (filtered white noise). We use the structure of the Fokker-Planck operator associated to vector Ornstein-Uhlenbeck processes to derive a perturbation method for determining moment-stability of solutions. The perturbation expansion is done in terms of the magnitude of the forcing, which is assumed to be small. Our motivation came from a stability analysis of stochastically forced viscous capillary gravity waves. For low viscosity, the Mathieu equation is a reasonable approximation to the equation governing the height of the fluid, and in [13] we apply our results to the stochastic Mathieu equation to determine the stability of solutions.

Stability of stochastically forced viscous Faraday waves. [14]

Faraday was the first to report ripples on a fluid surface that form when a container is vibrated up and down. Such free-surface waves are now typically referred to as Faraday waves, and are an example of parametrically forced vibrations. In this case, the parameter that is varying is the gravitational acceleration. Normally, the downward acceleration due to gravity, written as g, is a constant $g = g_0$ (on the Earth's surface), but in the frame of a vertically vibrating container with forcing $\varepsilon g_s(t)$, the acceleration of gravity is $g = g_0 + \varepsilon g_s(t)$. The ripples appear when the flat steady-state solution (i.e. constant surface height) becomes unstable. To determine the stability of the surface, one can derive an equation for the surface height from the governing equations for viscous capillary gravity waves.

For fluids with low viscosity, the surface-height equation can be modeled by the Mathieu equation. In this setting, our results from [13] can be applied to determine the stability of the free surface. However, if the viscosity is not small, this approximation breaks down. In joint work with L. A. Romero and J. R. Torczynski [14], we analyze the stability of stochastically forced Faraday waves using the full linearized Navier-Stokes equations. We use the perturbation series derived in [13] for ODEs and apply it to the PDE case (where some finite sums are replaced with infinite sums). We show that the results for the full system match the results for the Mathieu approximation in the low-viscosity setting, and that for larger viscosity the results diverge from those predicted by the Mathieu equation.

Γ-limit of multiwell energies with weak coercivity. [3]

In many physical applications, it is useful to model a material body as a continuum rather than a discrete lattice (e.g. when considering macroscopic or mesoscopic properties rather than atomic-level properties). Thus, we represent the material as a bounded set $\Omega \subset \mathbb{R}^3$. When a force acts on a body, the body deforms, and this is described mathematically by a function $v(x) = x + u(x)$, where $u(x)$ is the displacement (u measures how much the body is deformed from its reference configuration). An energy, E, is associated to the deformation gradient ∇v, and is often written in terms of an integral functional of the energy density W. That is, $E(v) = \int_\Omega W(\nabla v(x))\, dx$. The deformation of the body is described by a function v that minimizes E (with appropriate boundary conditions). Existence, uniqueness, and properties of minimizers are of particular interest, and often in the "small-strain regime", which is when $v(x) = x + \varepsilon u(x)$ for a small parameter $\varepsilon \ll 1$. A simplified problem is obtained by looking at the $\varepsilon \to 0$ limit, for which the limit energy is called the "linearized energy". Linearized energies are often simpler to work with and yield useful information about the physical problem being modeled.

In joint work with Virginia Agostiniani and Konstantinos Koumatos [3], we use the framework of Γ-convergence to rigorously derive the linearized energies for families of energy functionals (where the energy densities also depend on ε, $\{W_\varepsilon\}$) and show that the the minimizers of the energy at level ε converge to a minimizer of the linearized problem as $\varepsilon \to 0$. We consider multiwell energies with weak coercivity conditions, which allows us to apply our results to a class of rubber-like materials called nematic elastomers. This extends the existing framework for deriving linearized energies, which was begun by Dal Maso, et al. [18], to a larger class of energy functionals.

Aubry-Mather Theory for PDE and ΨDE. [11, 8]

In the early 1980s, S. Aubry and J. Mather independently developed techniques for identifying invariant sets of twist mappings of the annulus, [6], [28]. Soon after, J. Moser showed how the methods and results from their work generalized to a higher dimensional setting [31]. The minimal energy trajectories and invariant sets from Aubry-Mather theory were replaced by minimal energy codimension-one surfaces and foliations of the torus in Moser's work. Specifically, one is interested in formal energies of the form $S(u) = \int_{\mathbb{R}^d} F(x, u, \nabla u)\, dx$, and in a special class of local minimizers for S called "Birkhoff minimizers." Here $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is periodic in its first $d+1$ variables, and satisfies typical regularity and growth conditions so that the standard tools from elliptic PDE theory can be applied to the associated Euler-Lagrange equation. For each Birkhoff

minimizer, u, there is a vector $\omega \in \mathbb{R}^d$ such that $u(x) - \omega \cdot x \in L^\infty(\mathbb{R}^d)$. The slope, ω, associated to u is the analogue of a rotation number associated to an invariant circle in Aubry-Mather theory. These solutions have been considered within the context of plane-like minimizers in periodic media by Caffarelli and de la Llave [15], and by many others. In [19], gradient descent methods were used to prove existence of Birkhoff minimizers for a broad class of functionals. This technique relies on the gradient semi-flow admitting a comparison principle to guarantee that the set of Birkhoff functions is preserved under the action of the semi-flow.

In joint work with Rafael de la Llave and Enrico Valdinoci, [11], we prove a comparison principle for a gradient flow in fractional Sobolev spaces. This allows us to prove existence of Birkhoff minimizers for energy functionals involving fractional powers of elliptic operators. In joint work with Rafael de la Llave, [8], we use the Sobolev-gradient method to derive an efficient numerical scheme for computing Birkhoff minimizers. We also develop a perturbation theory approach for energy densities F of the form $F(x, u, \nabla u) = \nabla u \cdot a(x) \nabla u + \varepsilon V(x, u)$ where $\varepsilon \ll 1$. Using these two methods, we compute a related function called the "minimal average energy," and investigate its differentiability properties.

2. Screw Dislocations Under Antiplane Shear: Renormalized Energy and Dynamics

A body $B \subset \mathbb{R}^3$ is undergoing *antiplane shear* if the displacement from the reference to deformed configuration only involves motion in one direction. We consider displacements in the x_3 direction, and bodies of the form $B = \Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. Therefore, we consider deformations $G : B \to B$ of the form $G(x_1, x_2, x_3) = (x_1, x_2, x_3 + u(x_1, x_2))$. On the atomic level, a dislocation is a mismatch in the lattice structure of the material. On the continuum level, this is modeled as a singularity in the strain field. When no dislocations are present, the strain field, denoted by $\mathbf{h}$, is identified with the gradient of the displacement: $\mathbf{h} = \nabla u$. However, the presence of dislocations causes the strain field to be singular at the dislocations, and $\mathbf{h}$ cannot be written as a gradient. The assumption of antiplane shear allows us to reduce the three-dimensional problem to a two-dimensional problem by considering strain fields that are defined on the cross-section Ω, taking values in $\mathbb{R}^2$.

A screw dislocation is a line singularity. In the antiplane shear setting, this line is parallel to the x_3 axis; in the cross-section Ω a screw dislocation is therefore represented as a point singularity. Thus, a screw dislocation is characterized by a position $\mathbf{z} \in \Omega$ and a vector $\mathbf{b} \in \mathbb{R}^3$, called the Burgers vector (in antiplane shear $\mathbf{b} = b\mathbf{e}_3$ is parallel to the x_3-axis, $b \in \mathbb{R}$ is called the Burgers modulus). The sign of b determines the orientation of the dislocation. A strain field associated with a system of N screw dislocations at the positions $\{\mathbf{z}_1, \dots, \mathbf{z}_N\}$ with corresponding Burgers vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_N\}$ with each $\mathbf{b}_i = b_i \mathbf{e}_3$ satisfies the relation

$$\operatorname{curl} \mathbf{h} = \sum_{i=1}^{N} b_i \delta_{\mathbf{z}_i} \quad \text{in } \Omega, \qquad b_i = \int_{\ell_i} \mathbf{h} \cdot d\mathbf{x}, \tag{2.1}$$

in the sense of distributions, where $\operatorname{curl} \mathbf{h} = \partial_1 h_2 - \partial_2 h_1$, and ℓ_i is any counterclockwise loop surrounding the dislocation point $\mathbf{z}_i$ and no other dislocation points.

We denote by $E(\mathbf{h}) = \int_\Omega W(\mathbf{h}(\mathbf{x}))\, d\mathbf{x}$ the energy associated to a strain field $\mathbf{h} : \Omega \to \mathbb{R}^2$. Working within the context of linear elasticity, we assume $W(\mathbf{h}) = \frac{1}{2}\mathbf{h} \cdot \mathbf{L}\mathbf{h}$ and $\mathbf{L}$ is a symmetric, positive-definite matrix. The physically realized strains should minimize E, and thus solve the equilibrium equation $\operatorname{div} \mathbf{L}\mathbf{h} = 0$ in Ω. It is not hard to show that the condition (2.1) is incompatible with energy minimization because any $\mathbf{h}$ solving (2.1) necessarily has $E(\mathbf{h}) = +\infty$. The problem lies in the singularities of $\mathbf{h}$, and if one computes the energy of $\mathbf{h}$ in Ω with balls of size ε removed around each $\mathbf{z}_i$, then

$$E_\varepsilon(\mathbf{h}) := \int_{\Omega \setminus \bigcup_i B(\mathbf{z}_i, \varepsilon)} W(\mathbf{h})\, d\mathbf{x} = O(|\log \varepsilon|). \tag{2.2}$$

We overcome this by introducing a variational setting where the cores $B(\mathbf{z}_i, \varepsilon)$ are removed from Ω, but the functional E_ε is minimized over a space that retains the information about the dislocations, namely the integral condition in (2.1) is enforced. We show that E_ε admits a unique minimizer $\mathbf{h}_\varepsilon$ and $\mathbf{h}_\varepsilon \to \mathbf{h}_0$, where $\mathbf{h}_0$ satisfies

(2.1) and $\operatorname{div}\mathbf{L}\mathbf{h}_0 = 0$. In fact, $\mathbf{h}_0(\mathbf{x}) = \sum_{i=1}^N \mathbf{k}_i(\mathbf{x};\mathbf{z}_i) + \nabla u_0(\mathbf{x})$ where the functions $\mathbf{k}_i$ are given explicitly, and the function u_0 solves the Neumann problem

$$\left\{ \begin{array}{ll} \operatorname{div}(\mathbf{L}\nabla u) = 0 & \text{in } \Omega \\ \mathbf{L}\left(\nabla u + \sum_{i=1}^N \mathbf{k}_i(\cdot;\mathbf{z}_i)\right)\cdot \hat{\mathbf{n}} \ = 0 & \text{on } \partial\Omega \end{array} \right\}, \quad \mathbf{k}_i(\mathbf{x};\mathbf{z}_i) = \frac{b_i}{2\pi}\nabla_{\mathbf{x}} \arctan\left(\frac{x_2 - z_{i,2}}{\lambda(x_1 - z_{i,1})}\right). \tag{2.3}$$

The number λ is a Lamé parameter related to $\mathbf{L}$. Using all of this, we show that $E_\varepsilon(\mathbf{h}_\varepsilon) = C\log\varepsilon + U(\mathbf{z}_1,\dots,\mathbf{z}_N) + O(\varepsilon)$, where the function U is called the *renormalized energy*. The renormalized energy is the physically relevant energy for describing the forces on each dislocation.

The strain field generated by the dislocations, $\mathbf{h}_0$, exerts a force on each $\mathbf{z}_\ell$, called the Peach-Köhler force, which is denoted $\mathbf{j}_\ell$. Eshelby showed that $\mathbf{j}_\ell = \lim_{\varepsilon\to 0}\int_{\partial B(\mathbf{z}_\ell,\varepsilon)}(W(\mathbf{h}_0)\mathbf{I} + \mathbf{h}_0 \otimes \mathbf{L}\mathbf{h}_0)\mathbf{n}\,ds$, (cf. [21, 25]. Using the form we derived for $\mathbf{h}_0$, we establish

Theorem 2.1. *The force on $\mathbf{z}_\ell$ is given by*

$$\mathbf{j}_\ell = -\nabla_{\mathbf{z}_\ell} U(\mathbf{z}_1,\dots,\mathbf{z}_N) = b_\ell \mathbf{R}\mathbf{L}\left(\nabla u_0(\mathbf{z}_\ell) + \sum_{i\neq\ell}\mathbf{k}_i(\mathbf{z}_\ell;\mathbf{z}_i)\right), \qquad \mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.4}$$

This gives us a practical method for computing the force on the dislocations, and thereby investigating their motions. We follow the model proposed in [16], where the dislocations move in a manner that maximally dissipates energy. This would mean that they move along the force, in accordance with (2.4). However, the lattice structure of crystalline materials results in certain "allowed directions" of motion, called *glide planes*. In the antiplane shear setting, these planes are spanned by the $\mathbf{e}_3$ vector and some other vector in the x_1x_2-plane. Thus, in the cross-section Ω, the allowed directions of motion are given by some unit vectors in $\mathbb{R}^2$. We denote the set of these *glide directions* as $\mathcal{G}$, which is assumed to be a finite collection of unit vectors with the requirement that if $\mathbf{g}\in\mathcal{G}$ then $-\mathbf{g}\in\mathcal{G}$. The maximal dissipation criterion then implies that the dislocations will move in whichever glide direction is most closely aligned with the force. We can write this as $\dot{\mathbf{z}}_\ell = (\mathbf{j}_\ell\cdot\boldsymbol{\tau}_\ell)\,\boldsymbol{\tau}_\ell$ for $\ell = 1,\dots,N$, where the $\mathbf{j}_\ell$ and $\boldsymbol{\tau}_\ell$ are functions of each $\mathbf{z}_1,\dots,\mathbf{z}_N$, and $\boldsymbol{\tau}_\ell$ takes values in $\mathcal{G}$. In particular, $\boldsymbol{\tau}_\ell(\mathbf{z}_1,\dots,\mathbf{z}_N)$ is the glide direction most closely aligned with $\mathbf{j}_\ell(\mathbf{z}_1,\dots,\mathbf{z}_N)$. This presents two difficulties in defining the dynamics for the system. First, $\boldsymbol{\tau}_\ell(\mathbf{z}_1,\dots,\mathbf{z}_N)$ is not well-defined if $\mathbf{j}_\ell$ bisects its closest neighboring glide directions. Second, the fact that $\mathcal{G}$ is discrete means that $\boldsymbol{\tau}_\ell$ changes discontinuously as the force aligns with new glide directions. Fortunately, this situation fits into the framework of discontinuous ODE as developed by Filippov [22].

We replace the differential equation with a differential inclusion, and consider each $\boldsymbol{\tau}_\ell$ as a set-valued function whose value is the set of glide directions most closely aligned with $\mathbf{j}_\ell$. Using a compact notation $\mathbf{Z} = (\mathbf{z}_1,\dots,\mathbf{z}_N)\in\mathbb{R}^{2N}$, and $\mathbf{F}(\mathbf{Z}) = ((\mathbf{j}_1(\mathbf{Z})\cdot\boldsymbol{\tau}_1(\mathbf{Z}))\boldsymbol{\tau}_1(\mathbf{Z}),\dots,(\mathbf{j}_N(\mathbf{Z})\cdot\boldsymbol{\tau}_N(\mathbf{Z}))\boldsymbol{\tau}_N(\mathbf{Z}))$, the system can be written as $\dot{\mathbf{Z}}\in\operatorname{co}\mathbf{F}(\mathbf{Z})$, where co$\mathbf{F}$ is the convex hull of $\mathbf{F}$. In regions where each $\boldsymbol{\tau}_\ell$ is single-valued, this is the original ODE system, but along the sets $\mathcal{A}_\ell = \{\mathbf{Z}\in\mathbb{R}^{2N} : \boldsymbol{\tau}_\ell(\mathbf{Z}) \text{ is double-valued}\}$, the motion can abruptly change direction or move along the sets $\mathcal{A}_\ell$. The introduction of the convex hull is what allows for motions tangent to the $\mathcal{A}_\ell$ (if $\mathcal{A}_\ell$ is smooth). These surfaces mark the discontinuities of the vector field. In our context, $\mathcal{A}_\ell$ is where the choice of glide direction for $\mathbf{z}_\ell$ is ambiguous (the direction $\boldsymbol{\tau}_\ell(\mathbf{Z})$ is double-valued), so we refer to their union $\mathcal{A} = \bigcup_\ell \mathcal{A}_\ell$ as the "ambiguity set." A motion that crosses from one side of a discontinuity surface to the other abruptly changes direction, and this motion is called *cross-slip*. A motion that hits a discontinuity surface but cannot cross it because the vector field on the other side of the surface is directed back to the surface is forced to move along the surface. Such motions are called *fine cross-slip*. Fine cross-slip can be viewed as the limit of motions that cross-slip from one side of a surface to the other and back again on finer and finer scales, the limiting motion being along the surface. We prove the following result.

Theorem 2.2. *Suppose $\partial\Omega$ is a C^2 curve. Given an initial value $\mathbf{Z}_0\in\Omega^N\subset\mathbb{R}^{2N}$ for which $\mathbf{z}_i\neq\mathbf{z}_j$ and each $\mathbf{z}_i\notin\partial\Omega$, there exists a $T>0$ and an absolutely continuous function $\mathbf{Z}:[0,T]\to\Omega^N$ such that $\mathbf{Z}(0) = \mathbf{Z}_0$*

and $\dot{\mathbf{Z}}(t) \in \text{co}\mathbf{F}(\mathbf{Z}(t))$ *for all* $t \in [0,T]$*. Moreover, if* $\mathbf{Z}_0$ *is not a source point, then* $\mathbf{Z}$ *is the unique solution to* $\dot{\mathbf{Z}} \in \text{co}\mathbf{F}(\mathbf{Z})$*, and remains unique until a source point is reached, or a point where* $\mathcal{A}$ *fails to be* C^1*.*

To quantify a source point requires some work, but heuristically, a source point is any point $\mathbf{X} \in \mathcal{A}_\ell$ where the vector field in a neighborhood of $\mathbf{X}$ points away from $\mathcal{A}_\ell$, leading to non-uniqueness. Additionally, fine cross-slip is motion tangent to $\mathcal{A}$, so if the surface does not have a well-defined tangent, then uniqueness is lost. Finite existence times are expected because oppositely oriented dislocations attract each other (i.e. if $b_i b_j < 0$ then there is an attractive force between $\mathbf{z}_i$ and $\mathbf{z}_j$). At a collision, the vector field is singular and existence is lost. Similarly, the boundary $\partial\Omega$ is naturally attractive, so even in the case where all dislocations have the same orientation (and are thus repulsive) or if there is a single dislocation, then a collision with the boundary will occur in finite time. The vector field is again singular at boundary collisions because the Neumann problem is not well-defined for $\mathbf{z}_i \in \partial\Omega$. In fact, ∇u_0 becomes unbounded as any $\mathbf{z}_i$ approaches $\partial\Omega$.

3. KAM

I have worked on two different projects involving KAM techniques. The first is a general theory for proving the existence of codimension-one invariant tori in volume-preserving systems with one action and n angle variables [10]. The second is a numerical study of the breakdown of analytic tori that correspond to energy minima in quasi-periodic Frenkel-Kontorova models [9]. Though very different in many aspects, these two projects both fall under the label "KAM theory" because they both employ Newton-like iteration schemes (on function spaces) in order to find a system's invariant tori. Some details of these are described in §3.1 and §3.2.

3.1. **KAM Theory for Volume-Preserving Maps.** [10]

KAM techniques were originally employed to prove the existence of n-dimensional (Lagrangian) tori in $2n$-dimensional Hamiltonian systems [26, 4, 30]. These methods, devised for showing that the Hamiltonian flow has invariant tori, are also applicable to maps (discrete-time dynamical systems) that preserve a geometric structure. In the case of Hamiltonian systems (and the related "twist maps") that geometric structure is a symplectic form. Not surprisingly, KAM techniques were successfully applied to volume-preserving systems, where the role of the symplectic form is replaced by a volume form [17, 39, 40]. These authors managed to prove that volume-preserving maps that are "close to integrable" have a large set of codimension-one invariant tori (these tori are graphs over the angle variables). In [10], joint work with Rafael de la Llave, we improve on those results in several ways. We discuss these improvements after the statement of Thm. 3.1, below.

Let $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$. We consider spaces $\mathcal{M} = \mathbb{T}^n \times \mathbb{R}$ consisting of n angle dimensions and one action dimension, and we write the coordinates on $\mathcal{M}$ as (φ, A), where $\varphi = (\varphi_1, \ldots, \varphi_n)$ are the angles and A is the action. We denote the volume form $\Omega = d\varphi_1 \wedge \cdots \wedge d\varphi_n \wedge dA$, and consider families of diffeomorphisms $f_\lambda : \mathcal{M} \to \mathcal{M}$ that preserve volume in the sense $f_\lambda^* \Omega = \Omega$ (f_λ^* is the pullback of f_λ).

For a fixed value of λ, the map f_λ has an invariant torus with rotation vector $\omega \in \mathbb{R}^n$ if there exists a (smooth) embedding $K : \mathbb{T}^n \to \mathcal{M}$, satisfying

$$f_\lambda(K(\theta)) = K(\theta + \omega), \quad \forall \theta \in \mathbb{T}^n. \tag{3.1}$$

The role of the parameter $\lambda \in \mathbb{R}^n$ is quite complicated, but it is important to note that an invariant torus for f_λ is a pair (K, λ) satisfying (3.1). Even for small perturbations of integrable maps, one cannot guarantee the existence of K if both λ and ω are specified. Indeed, one must be able to "adjust parameters" to accomodate a fixed frequency, and therefore we require the freedom to adjust λ (and also some non-degeneracy conditions on f_λ). It is possible to "trade parameters" by considering λ as a function of ω and solving for ω via the (finite dimensional) implicity function theorem. Adjusting parameters is also necessary to accomodate geometric constraints (e.g. zero mean flux), but I will not discuss this here.

Any embedding K satisfying (3.1) is a codimension-one surface in the $(n+1)$-dimensional space $\mathcal{M}$, and therefore it separates $\mathcal{M}$ into two connected components, acting as a transport barrier for the dynamics of f.

If, for example, f_λ were the time-one map of an incompressible fluid flow, then the presence of a codimension-one torus would imply that the fluid in one component will not mix with the fluid in the other component as time progresses.

If K satisfies (3.1) then so does $K(\cdot + \sigma)$ for any fixed $\sigma \in \mathbb{R}^n$, but K and $K(\cdot + \sigma)$ represent the same invariant torus, so solutions to (3.1) are never unique. This non-uniqueness needs to be avoided, so we fix a normalization for solutions of (3.1) in terms of a "reference torus." We assume we have an approximately invariant torus K_0 that is our initial guess for the Newton method, and use this as a reference. In particular, we require solutions to satisfy $\left\langle M_0^{-1} K \right\rangle = 0$, where M_0 is a matrix defined in terms of K_0 and $\langle \cdot \rangle$ denotes the average over $\mathbb{T}^n$.

As mentioned earlier, Newton's method is an indispensable tool in KAM theory. If we define $F[K, \lambda] := f_\lambda \circ K - K \circ T_\omega$, where T_ω is just a shift by ω (i.e. $T_\omega(\theta) = \theta + \omega$), then an invariant torus for f_λ would correspond to a pair (K, λ), such that $F[K, \lambda] = 0$. We solve this equation using a modified Newton method. As is common in KAM problems, when trying to solve the linearized equations for the Newton step, we encounter "small divisors." That is, if we represent the functions in terms of Fourier series, we encounter sums of the form $\sum_{k \in \mathbb{Z}^n \setminus \{0\}} a_k / (1 - \exp(2\pi i \omega \cdot k))$ The coefficients a_k come from an analytic function, so they decay exponentially, but the denominators become arbitrarily close to zero (or equal zero if ω has rational components) and the convergence of the series is compromised. This leads us to consider a restricted set of frequency vectors (though still of full measure in $\mathbb{R}^n$). A vector $\omega \in \mathbb{R}^n$ is *Diophantine of type* (ν, τ), written as $\omega \in \mathcal{D}(\tau, \nu)$ if

$$|\omega \cdot k - m| \geq \nu |k|^{-\tau}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}, \ \forall m \in \mathbb{Z}. \tag{3.2}$$

There is a loss of smoothness in this process because the convergence of the series with small divisors is slower than for a_k. This is unavoidable, but the quadratic convergence of the Newton method is powerful enough to converge to a solution that is still smooth.

Our main result is an *a posteriori* theorem.

Theorem 3.1. *Let $p \in \mathbb{R}$ and $\omega \in \mathcal{D}(\tau, \nu)$, and suppose λ_0 and K_0 are given. Define the initial error $E_0 := f_{\lambda_0} \circ K_0 - K_0 \circ T_\omega$. Then, if f_λ, K_0, λ_0 satisfy some non-degeneracy conditions, and if $\|E_0\|$ is small enough, then there exist λ_∞ and K_∞ satisfying $f_{\lambda_\infty} \circ K_\infty = K_\infty \circ T_\omega$, and $|\lambda_0 - \lambda_\infty|$, $\|K_0 - K_\infty\|_{\rho/2}$ are bounded in terms of $\|E_0\|_\rho$. Moreover, K_∞ is unique if we require the appropriate normalization.*

This result does not depend on the maps f_λ being close to integrable maps. It also leads to smooth dependence on extra parameters (e.g. if f_λ also depended smoothly on some parameter ε, then λ_∞ and K_∞ would as well). The method of proof leads to an efficient algorithm for numerically computing the invariant tori, which has already been implemented successfully in [23]. Solving the linearized equations is limited in speed only by switching between real space and Fourier space, so the algorithm's running time is $O(N \log N)$ if we use FFT, where N is the number of modes. The *a posteriori* format also provides a method of verification for the numerics (since the non-degeneracy conditions and smallness condition on the initial error can be computed explicitly). The result applies to contractible tori as well as non-contractible. In previous results for volume-preserving systems, only non-contractible tori were obtained. With minor adjustments to the case of maps, the method also extends to flows, to quasi-periodically forced systems, and to "small twist" systems.

3.2. **Analyticity Breakdown in Quasi-periodic Frenkel-Kontorova Models.** [9]

Frenkel-Kontorova (FK) models are commonly used in problems arising in solid-state physics. In joint work with Rafael de la Llave, [9], we focused on FK models with quasi-periodic potentials. The simplest way to motivate such models is to consider the problem of deposition of a material over a one-dimensional quasi-periodic substratum. If u_i represents the position of the i^{th} particle of the deposited material, the state of the system is given by the configuration $\{u_i\}_{i \in \mathbb{Z}}$, the sequence giving the position of all the deposited particles (note that $u_i \in \mathbb{R}$). The main example we consider is be the quasi-periodic Frenkel-Kontorova model given by

the formal energy functional

$$S(\{u_i\}_{i\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} \frac{1}{2}(u_n - u_{n+1})^2 - \hat{V}(\alpha u_n) \tag{3.3}$$

where $\hat{V} : \mathbb{T}^d \to \mathbb{R}$ and $\alpha \in \mathbb{R}^d$ is a Diophantine irrational vector (cf (3.2)). The term $\frac{1}{2}(u_n - u_{n+1})^2$ represents the interaction between nearest neighbors and the term $\hat{V}(\alpha u_n)$ represents the interaction with the substratum at the position u_n. Other physical interpretations of the model are possible, for example, the classical model introduced by Frenkel and Kontorova in [24] was used to describe the motion of planar dislocations in a crystal (they considered the case of periodic potentials with $d = 1$).

The sum in (3.3) is formal, because in general it will not be finite. However, minimizers with respect to compactly supported variations are well-defined. A sequence $\{u_i\}$ is a minimizer of S if $S(\{u_i\} + \{\varphi_i\}) - S(\{u_i\}) \geq 0$ for all sequences $\{\varphi_i\}$ that are nonzero for only finitely many values of $i \in \mathbb{Z}$ (hence the sum $S(\{u_i\} + \{\varphi_i\}) - S(\{u_i\})$ has only finitely many nonzero terms). A physically realized state is expected to be an energy minimizer, so we look at the the equilibrium equations. A configuration $\{u_i\}_{i\in\mathbb{Z}}$ is in equilibrium when $\partial_{u_i} S(\{u_i\}_{i\in\mathbb{Z}}) = 0$, and in the case (3.3), the equilibrium equations become:

$$\alpha \cdot \nabla \hat{V}(\alpha u_i) + u_{i+1} + u_{i-1} - 2u_i = 0 \tag{3.4}$$

A configuration is given by a *hull function* when we can find $h : \mathbb{T}^d \to \mathbb{R}$ and $\omega \in \mathbb{R}$ such that

$$u_n = \omega n + h(n\omega\alpha). \tag{3.5}$$

Substituting (3.5) into (3.4), we see that an equilibrium configuration is given by a hull function if and only if

$$h(n\omega\alpha + \omega\alpha) + h(n\omega\alpha - \omega\alpha) - 2h(n\omega\alpha) + (\alpha \cdot \nabla)\, \hat{V}(n\omega\alpha + \alpha h(n\omega\alpha)) = 0. \tag{3.6}$$

Our interest is in the case where equilibrium configurations can be represented by analytic hull functions. To explain this, we first note that if $\{n\omega\alpha\}_{n\in\mathbb{Z}}$ is dense on $\mathbb{T}^d$ (i.e. $k \cdot \omega\alpha \notin \mathbb{Z}$ for $k \in \mathbb{Z}^d \setminus \{0\}$), then for a continuous hull function h, (3.6) is equivalent to

$$h(\theta + \omega\alpha) + h(\theta - \omega\alpha) - 2h(\theta) + (\alpha \cdot \nabla)\, \hat{V}(\theta + \alpha h(\theta)) = 0, \quad \forall \theta \in \mathbb{T}^d. \tag{3.7}$$

The existence of a continuous solution of (3.7) implies that there is a continuous family of equilibrium configurations. For any $\sigma \in \mathbb{T}^d$, h_σ given by $h_\sigma(\theta) = h(\theta + \sigma)$ is also a solution of (3.7). The existence of a continuous family of configurations which are equilibria indicates that there is no energetic impediment to the solutions making small transitions and sliding. Hence, one expects that the solutions can be modified easily. In contrast, if the set of equilibrium configurations is discontinuous, there may be an energy barrier (the Peierls-Nabarro barrier) to perform jumps from a configuration to the nearest one and the solutions are pinned down. In fact, the smoothness of h is of particular interest, and it is expected that the transition from pinned to sliding states happens when the hull function fails to be analytic [5]. This transition, which is called the *breakdown of analyticity*, is the focus of our work.

We assume $\omega\alpha$ satisfies the Diophantine property (3.2). We consider potential functions $\hat{V}$ that depend on parameters (e.g. $V_\varepsilon(\phi_1, \phi_2) = -\frac{1}{2\pi}\varepsilon_1 \cos(2\pi\phi_1) - \frac{1}{2\pi}\varepsilon_2 \cos(2\pi\phi_2)$) and study numerically the set of parameters for which (3.7) has an analytic solution, paying special attention to the behavior of these solutions near the boundary of existence. This is done by implementing some recent numerical algorithms [36, 37] that are efficient and backed up by rigorous results so that we can compute with confidence even close to the breakdown. (In fact, these algorithms come from *a posteriori* results, similar in spirit to the one discussed in §3.1). We uncovered three phenomena that require further investigation: (I) The boundary of the domain of analyticity is a smooth curve. (II) The norm of the hull functions exhibit scaling behaviors near the boundary of analyticity (the Sobolev norms blow up in a specific way). (III) The blow up of the norms is very anisotropic.

Statement (I) refers to the region in parameter space (e.g. in $(\varepsilon_1, \varepsilon_2)$-space for the example function V_ε above) where analytic hull functions exist. This region is called the domain of analyticity for V_ε. The boundary of the domain of analyticity being smooth is analogous to what happens in phase transitions. The fact that

the transitions happen locally in a smooth manifold is one of the predictions of renormalization group theory for periodic FK models. At present, however, there is no renormalization group theory for quasi-periodic FK models.

Statement (II) refers to the fact that near the boundary of analyticity, we observe that the H^r Sobolev norms of h blow up like the distance to the boundary raised to a power. That is, $||h_\varepsilon||_r \approx C|\varepsilon - \varepsilon_c|^{-\beta r+\gamma}$ where β (and perhaps γ too) is a universal number, and ε_c lies on the boundary. Statement (III) is a refinement of (II), which says that anisotropic Sobolev norms blow up at different rates. In particular, the derivatives of h in the direction $\omega\alpha$ grow much faster than the derivatives in the direction perpendicular to $\omega\alpha$. If we define $\|h\|_{r,||} = \|(\omega\alpha\cdot\nabla)^r h\|_{L^2}$ and $\|h\|_{r,\perp} = \|(\omega\alpha^\perp\cdot\nabla)^r h\|_{L^2}$, then near the breakdown $\|h_\varepsilon\|_{r,||} \approx C|\varepsilon-\varepsilon_c|^{-\beta_{||}r+\gamma_{||}}$, and when $\varepsilon\to\varepsilon_c$, we have $\|h_\varepsilon\|_{r,\perp}/\|h_\varepsilon\|_{r,||}\to 0$.

All of these phenomena require theoretical explanations. For more details on this work and to see plots of the hull functions, domains of analyticity, and scaling behaviors for specific examples please see `http://math.cmu.edu/~tblass/research`.

4. Stochastic Forcing and Faraday Waves

4.1. **Parametrically Forced ODEs.** [13]

Faraday wave models, in the low-viscosity regime, lead to an equation for the surface height, h, of the form

$$\ddot{h} + \gamma\dot{h} + (\omega_0^2 + \varepsilon f(t))h = 0. \tag{4.1}$$

This is a parametrically forced equation because a parameter (in this case, the natural frequency ω_0) is changing with time. The amplitude ε of the forcing is assumed to be small. When f is periodic, (4.1) is known as the Mathieu equation (or Hill's equation) and the stability of solutions has been studied using Floquet theory [27], and applied to periodically forced capillary gravity waves (with no viscosity) by B. Benjamin and F. Ursell [7]. Here, a solution is stable if it is bounded for all time. Much less has been done in the case where f is colored noise. In joint work with L. A. Romero [13], we consider equations of the form

$$\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} + \varepsilon f(t)\mathbf{A}_1\mathbf{x}, \tag{4.2}$$

generalizing (4.1), where $\mathbf{A}_0$ and $\mathbf{A}_1$ are constant coefficient matrices. When f is stochastic, the stability of solutions to (4.2) needs to be understood in a statistical sense. A common approach is to consider a solution to be stable if its pth-moments are stable (and typically, $p=2$ is used). This is the approach we take, considering a solution of (4.2) to be stable if $\langle\langle \mathbf{x}^\alpha(t)\rangle\rangle$ is bounded for all time, where α is a multi-index of order p, and $\langle\langle\cdot\rangle\rangle$ denotes the sample average.

We consider processes f that have "realistic" power spectral densities. Specifically, we avoid using white noise, but instead use colored noise, which is obtained by passing white noise through a linear filter. This leads to f of the form $f(t) = \mathbf{a}\cdot\mathbf{s}(t)$, where $\mathbf{a}\in\mathbb{R}^N$ and $\dot{\mathbf{s}} = \mathbf{H}\mathbf{s} + \boldsymbol{\xi}(t)$, $\mathbf{H}$ is a constant matrix, and $\boldsymbol{\xi}$ is a weighted white noise. That is, $\left\langle\left\langle \boldsymbol{\xi}(t+\tau)\boldsymbol{\xi}^T(t)\right\rangle\right\rangle = \mathbf{B}\delta(\tau)$, where $\mathbf{B}$ is a positive, semi-definite, constant matrix (thus, $\mathbf{s}$ is a vector Ornstein-Uhlenbeck process). We use filtered white noise because we wanted to take advantage of the Fokker-Planck equation associated to $\mathbf{s}$ (the Fokker-Planck equation is a PDE that describes the evolution of the probability density function for $\mathbf{s}(t)$). However, one can only rigorously derive a Fokker-Planck equation for a stochastic differential equation only if the randomness in the equation comes from white noise. By selecting $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{a}$, we are able to build a large class of interesting processes f (there are some mild requirement on $\mathbf{H}$ and $\mathbf{B}$ in order to use our results).

We turn the question of moment-stability of solutions to (4.2) into an eigenvalue problem (i.e. determining the sign of an eigenvalue λ), and apply perturbation theory to compute the eigenvalues under the assumption that ε is small. Our main result is a method for computing the eigenvalue as a series in ε up to any order in ε, and for any choice of pth-moments.

Theorem 4.1. *For any f defined as described above, we can determine the moment-stability of solutions to (4.2) by checking if $\lambda(\varepsilon) < 0$, where λ solves an eigenvalue problem. We can compute $\lambda(\varepsilon) = \sum_{k\geq 0} \varepsilon^k \lambda_k$ to any order in ε.*

In fact, we have a simple formula up to the ε^4 term in the series for λ. It is not hard to show that the odd terms vanish, and λ_0 is just the unperturbed eigenvalue, which is given by $\lambda_0 = \sigma_q + \sigma_r$, for some q, r, where σ_j are eigenvalues of $\mathbf{A}_0$. Thus, the λ_2 term is of particular importance, and we show that

$$\lambda_2 = \frac{8}{1+\delta_{qr}} \sum_{j,k=1}^{N} C_{jkqr} C_{qrjk} G(\sigma_q + \sigma_r - \sigma_j - \sigma_k). \tag{4.3}$$

The function G is an *extended power spectral density* of f, and is given by the Laplace transform of the auto-correlation function for f. In particular, it depends only on the choices of $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{a}$. The coefficients $C_{jk\ell m}$ are determined entirely by the matrices $\mathbf{A}_0$ and $\mathbf{A}_1$.

Our analysis is based on a ladder operator approach to determining the spectrum of the Fokker-Planck operator associated to $\mathbf{s}$. The density function for $\mathbf{s}$ is $P(s_1, \ldots, s_n, t)$ and satisfies the Fokker-Planck PDE $\partial_t P = \mathcal{D}P$, and $\mathcal{D}$ is the Fokker-Planck operator, given by $\mathcal{D}P = \frac{1}{2}\mathrm{div}(\mathbf{B}\nabla P) - \mathrm{div}(\mathbf{H}\mathbf{s}P)$. The ladder operators associated to $\mathcal{D}$ are first-order differential operators satisfying the commutator relations $[\mathcal{D}, \mathcal{L}] = \mu\mathcal{L}$, for $\mu \in \mathbb{C}$. These operators provide a method of carrying out perturbation theory for the moment-stability of $\mathbf{x}(t)$. The paper is available here: `http://math.cmu.edu/~tblass/research`.

4.2. **Stochastically Induced Viscous Faraday Waves.** [14]

We analyze the stability of stochastically forced viscous Faraday waves, also called capillary gravity waves because the effects of surface tension and gravity are taken into account. These waves appear as standing waves on a surface of a vertically shaken container that holds a viscous fluid and a gas. The fluid-gas interface can become unstable, and we are interested in the onset of instability when the forcing is given by a random signal. This is an example of a parametric instability. In joint work with L. A. Romero and J. R. Torczynski [14], we apply the results of our earlier work [13] to the full linearized Navier-Stokes (LNS) equations, not making any assumptions about the damping (by viscosity) being small. This requires us to make some modifications and extensions of our previous work.

When discretizing the LNS equations, one obtains a system similar to (4.2) except the vector $\mathbf{x}$ represents coefficients in an expansion for the solution of the LNS, and thus (4.2) represents an infinite dimensional ODE. Also, the $\dot{\mathbf{x}}$ term is replaced by $\mathbf{M}\dot{\mathbf{x}}$, where $\mathbf{M}$ is a singular mass-matrix, so we have a differential-algebraic equation (DAE). However, our results can be made to accommodate DAE, and the infinite dimensional extension leads to expressions like (4.3), except with infinite sums replacing the finite sums. An important fact is that the coefficients in (4.3) can be computed explicitly for the LNS, and we have a practical method for computing the first terms in the series for $\lambda(\varepsilon)$.

For small damping (low viscosity) and a container of infinite depth, the Mathieu equation is a reasonable approximation for the behavior of the free surface. Our results apply to any depth and any viscosity, so we compare our results to the Mathieu approximation by taking the depth parameter to $-\infty$, and considering different values of the viscosity. (In the infinite-depth limit, the real eigenvalues fill an interval, i.e. become a continuous spectrum, and part of the sum in (4.3) is replaced by an integral over the continuous spectrum.) We find that our analysis for the full LNS equations matches the analysis for the Mathieu equation in the infinite-depth limit with small viscosity, but the results diverge for large viscosity, indicating the need to use the full LNS for such cases.

5. Linearized Energies via Γ-convergence

For a homogeneous and hyperelastic body, occupying in its reference configuration a bounded domain $\Omega \subset \mathbb{R}^d$, deformations of the body are described by mappings $v : \Omega \to \mathbb{R}^d$, where $v(x)$ denotes the deformed

position of the material point $x \in \Omega$. The total elastic energy corresponding to the deformation v is given by

$$\int_\Omega W(\nabla v(x))dx - \int_\Omega l(x) \cdot u(x)\, dx,$$

where $\nabla v \in \mathbb{R}^{d\times d}$ is the deformation gradient, $W : \mathbb{R}^{d\times d} \to \mathbb{R}$ is a frame-indifferent energy density associated with the material, $u(x) = v(x) - x$ is the displacement, and $l(x)$ is a load applied at $x \in \Omega$.

In joint work with Virginia Agostiniani and Konstantinos Koumatos [3], we derive linearized models from nonlinear energies with a multiwell structure, i.e. W is minimized on a set $\mathcal{U}$ of the form $SO(d)U$, U ranging in a compact subset of positive definite, symmetric matrices. Also, we weaken the standard coercivity condition to $W(F) \geq c\,\mathrm{dist}^p(F,\mathcal{U})$, $1 < p < 2$, for F far away from $\mathcal{U}$ ($p = 2$ is the standard condition); we assume the coercivity remains quadratic near $\mathcal{U}$. The linearized models are derived via Γ-convergence with respect to a small parameter ε ($v(x) = x + \varepsilon u(x)$). We prove convergence of the infima of the energy at the level ε (denoted $\mathscr{E}_\varepsilon$) to the infimum of the Γ-limit energy ($\mathscr{E}$), as well as convergence of almost-minimizers of $\mathscr{E}_\varepsilon$ to a minimizer of $\mathscr{E}$. The derivation of linearized models from nonlinear energies via Γ-convergence began with work by G. Dal Maso, M. Negri, and D. Percivale [18] on single-well energies with quadratic growth. This was extended to multiwell energies with quadratic growth by B. Schmidt [33], and to single-well energies with weak growth conditions by V. Agostiniani, G. Dal Maso and A. DeSimone [2].

Energies of the type we consider (i.e. multiwell with weak coercivity) arise naturally in a large class of compressible models for rubber-like materials, including nematic elastomers, which are materials consisting of networks of polymer chains with embedded liquid crystalline molecules. In [20], some nonlinear compressible models for nematic elastomers are considered together with their formally derived small-strain theories. These nonlinear models satisfy our assumptions and our results rigorously justify their geometrically linear counterparts.

In order to derive small-strain limiting theories, we introduce a small parameter ε and we consider a family of densities $\{W_\varepsilon\}$ with corresponding energy wells

$$\mathcal{U}_\varepsilon := \{RU_\varepsilon : R \in SO(d),\, U_\varepsilon = U_\varepsilon^T = I + \varepsilon U + o(\varepsilon) : U \in \mathcal{M}\}, \tag{5.1}$$

$\mathcal{M}$ being a compact subset of symmetric matrices. We assume that

$$W_\varepsilon \geq c\,\mathrm{dist}^2(\cdot,\mathcal{U}_\varepsilon) \ \text{ near } \mathcal{U}_\varepsilon, \qquad W_\varepsilon \geq C\,\mathrm{dist}^p(\cdot,\mathcal{U}_\varepsilon) \ \text{ away from } \mathcal{U}_\varepsilon, \tag{5.2}$$

and we investigate the limiting behavior, as $\varepsilon \to 0$, of the rescaled functionals

$$\mathscr{E}_\varepsilon(u) := \frac{1}{\varepsilon^2}\int_\Omega W_\varepsilon(I + \varepsilon\nabla u(x))\, dx - \mathscr{L}(u)$$

and their (almost) minimizers, where $\mathscr{L}(u)$ is a continuous linear functional on $W^{1,p}$ representing the applied loads. We define the rescaled densities $f_\varepsilon(F) = \frac{1}{\varepsilon^2}W(I + \varepsilon\,\mathrm{sym}\,F)$, where $\mathrm{sym}\,F = \frac{1}{2}(F + F^T)$, and assume that the f_ε have some limit f (uniformly on compact sets). Our main is result is the following theorem.

Theorem 5.1. *Let $1 < p \leq 2$, suppose that $f_\varepsilon \to f$ uniformly on compact subsets of $\mathbb{R}^{d\times d}$, and suppose that f satisfies $0 \leq f(F) \leq c(1 + |F|^2)$ for every $F \in \mathbb{R}^{d\times d}$ and some constant $c > 0$. If*

$$m_\varepsilon := \inf_{u\in W_h^{1,p}} \{\mathscr{E}_\varepsilon(u)\}, \quad m := \min_{u\in W_h^{1,2}} \left\{\int_\Omega f^{qc}(\nabla u)dx - \mathscr{L}(u)\right\}. \tag{5.3}$$

and if $\{u_\varepsilon\}$ is a sequence such that $\lim_{\varepsilon\to 0}\{\mathscr{E}_\varepsilon(u)\} = \lim_{\varepsilon\to 0} m_\varepsilon$, then, up to a subsequence, $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega,\mathbb{R}^d)$, where u attains the minimum m in (5.3). Moreover, $m_\varepsilon \to m$.

The integrand f^{qc} obtained in the limit is the *quasiconvexification* of f. The proof of Theorem 5.1 is based on two intermediate results: a compactness result following from equicoercivity, and a Γ-convergence result. In the case of nematic elastomers, the assumptions on W_ε are satisfied. In particular, the inequalities in (5.2) are satisfied with $p = 3/2$. For more details please see `http://math.cmu.edu/~tblass/research`.

6. AUBRY-MATHER THEORY FOR PDES AND ΨDES

6.1. **Comparison Principle.** In joint work with R. de la Llave and E. Valdinoci, [11], we showed how gradient descent in the fractional Sobolev spaces H^β with non-standard inner products can be used to prove existence of plane-like minimizers of variational problems of the form

$$S(u) = \frac{1}{2}\langle u, Au\rangle_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} V(x,u)\,dx = \int_{\mathbb{R}^d} \frac{1}{2}\left(a(x)\nabla u\cdot\nabla u\right) + V(x,u)\,dx. \tag{6.1}$$

A is a self-adjoint, uniformly elliptic operator with smooth coefficients, $Au = -\mathrm{div}(a(x)\nabla u)$, and $V \in C^2(\mathbb{R}^d \times \mathbb{R})$. A function $u \in H^1_{loc}(\mathbb{R}^d)$ is a plane-like minimizer if there is an $\omega \in \mathbb{R}^d$ such that $u(x) - \omega\cdot x \in L^\infty(\mathbb{R}^d)$ and u minimizes S on each compact subset of $\mathbb{R}^d$. Descent methods commonly lead to stiffness problems when implemented numerically, so we used the so-called Sobolev gradient method instead of a standard technique, [32].

The first existence result in this area appeared in [31]. When $a(x)$ and $V(x,y)$ are periodic in all variables, the process of minimizing (6.1) can be recast as a variational problem on the torus $\mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d$. In this setting, for each rational direction $\omega \in \frac{1}{N}\mathbb{Z}^d$, one seeks periodic functions v such that $v(x) = u(x) - \omega\cdot x$, where u is a minimizer. v will solve the Euler-Lagrange equation

$$-Av(x) = V_u(x, \omega\cdot x + v(x)), \quad x \in N\mathbb{T}^d,\ \omega \in \frac{1}{N}\mathbb{Z}^d. \tag{6.2}$$

Equation (6.2) is a type of Ginzberg-Landau-Allen-Cahn equation, used as a model for phase transitions. In general, this can be seen as the cell problem for a non-linear homogenization problem. The large-scale behavior of the solution is planar, determined by the frequency vector ω. The small-scale behavior is determined by v solving (6.2). For irrational frequencies, solutions are obtained by taking a sequence of rational solutions whose rational frequency vectors converge to the irrational one.

If H is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$, and $DS(u)$ is the derivative of S at u, then the gradient of S with respect to $(H, \langle\cdot,\cdot\rangle_H)$ is the unique element of H such that $DS(u)\eta = \langle \nabla S(u), \eta\rangle_H$ for all $\eta \in H$. We considered the Sobolev spaces H^β, $\beta \in (0,1)$ with inner products $\langle u, v\rangle_{H^\beta} = \langle (\gamma I + A)^\beta u, v\rangle_{L^2}$, $\gamma > 0$. These inner products depend on γ, β and A. In this setting, the descent equation is

$$\partial_t u = -(\gamma I + A)^{1-\beta}u + (\gamma I + A)^{-\beta}\left(\gamma u - V_u(x,u)\right) \tag{6.3}$$

In [11] we show that with suitable assumptions on V, for each $\beta \in (0,1)$ the flow defined by (6.3) is well-defined for initial conditions $u(0,t) = u_0(x) \in L^\infty$. The main theorem states that if $u(x,t)$ and $\tilde{u}(x,t)$ solve (6.3) for initial data u_0 and $\tilde{u}_0$, and if $u_0 \geq \tilde{u}_0$, then $u(x,t) \geq \tilde{u}(x,t)$. A corollary of this comparison principle for (6.3) is that if we take initial data $u_0(x) = \omega\cdot x$, $\omega \in \frac{1}{N}\mathbb{Z}^d$, then for each $t > 0$, $u(x,t)$ is plane-like with slope ω. The plane-like nature of the minimizers provides the *a priori* bounds that are needed in the construction of minimizers with irrational slopes as limits of the rational minimizers.

Our analysis extends to operators more general than A, and we were able to prove the same results for variational principles like (6.1) with A replaced by A^α, $\alpha \in (0,1)$. The stationary points of the descent equation for these variational principles solve

$$-A^\alpha v = V_u(x, \omega\cdot x + v), \quad x \in N\mathbb{T}^d, \omega \in \frac{1}{N}\mathbb{Z}^d. \tag{6.4}$$

Thus, using this descent method we can solve a nonlocal, nonlinear homogenization problem where the global behavior is plane-like.

Sobolev gradients often produce useful numerical techniques, and that is the case here. In the special case that $A = -\Delta$ the descent equation (6.3) lends itself to Fourier methods and the descent equation for the Fourier transform of v is

$$\partial_t \widehat{v}(t,\xi) = -(\gamma + |\xi|^2)^{1-\beta}\widehat{v}(t,\xi) + (\gamma + |\xi|^2)^{-\beta}(\gamma\widehat{v}(t,\xi) - (V_u(x,\omega\cdot x + v))\widehat{\ }(t,\xi)), \tag{6.5}$$

which is effectively an ODE in frequency space. The inherent stiffness in the numerical approximation to (6.5) (i.e. the difficulties arising from large frequencies ξ) is greatly reduced by taking values of $\beta \approx 1$. This method is straightforward to implement numerically, and converges extremely fast.

6.2. **Perturbation Methods.** In joint work with Rafael de la Llave, [8], we consider the special case of (6.1) given by

$$S_\varepsilon(u) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2 + \varepsilon V(x,u)\,dx, \tag{6.6}$$

and develop a perturbation theory for plane-like solutions to the associated Euler-Lagrange equation

$$\Delta u = \varepsilon V_u(x,u), \tag{6.7}$$

for $u(x) = \omega \cdot x + v(x)$ with $\omega \in \frac{1}{N}\mathbb{Z}^d$ and v N-periodic. Representing the solution u_ε as a Lindstedt series in the parameter ε, $u_\varepsilon = \sum_{j\geq 0} \varepsilon^j u_j$, we must solve the equations

$$\Delta u_j = [V_u(x,u_\varepsilon)]_{j-1} = [V_u(x,u^{<j})]_{j-1}, \quad x \in \mathbb{T}^d, \tag{6.8}$$

for each $j \geq 0$. Here $[\cdot]_j$ refers to the j-th coefficient of the power series in ε: $V_u(x,u_\varepsilon) = V_u(x, u_0 + \varepsilon u_1 + \ldots) = \sum_{j\geq 0}[V_u(x,u_\varepsilon)]_j \varepsilon^j$. $u^{<j}$ represents the first j terms in the Lindstedt series: $u^{<j} = u_0 + \ldots + \varepsilon^{j-1}u_{j-1}$. Under some non-degeneracy conditions on V, we prove that for each $j \geq 0$ we can solve (6.8) and that the resulting series converges. The convergence is done using a Newton method for $F_\varepsilon(u) = -\Delta u + \varepsilon V_u(x,u)$. By showing that the Newton method for F_ε converges uniformly in ε and that each iteration is analytic in ε, we show that the method converges to a solution of $F_\varepsilon(u) = 0$ and this must have the series expansion given by the Lindstedt series.

This framework also extends to the case of the fractional Laplacian for plane-like solutions:

$$-(-\Delta)^\alpha v = \varepsilon V_u(x, \omega \cdot x + v). \tag{6.9}$$

Thus we have a perturbation approach to solving the nonlinear cell problem for the Laplacian and fractional Laplacian.

There are solutions to (6.7) of the form $u(x) = \omega \cdot x + v(x)$ where v is not periodic, but only asymptotically periodic, sometimes called "heteroclinic". We show in [8] how to compute asymptotic series for these heteroclinic solutions as well, and use them together with the periodic solutions to investigate the *minimal average energy* $A_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ defined as

$$A_\varepsilon(\omega) = \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} \frac{1}{2}|\nabla u|^2 + \varepsilon V(x,u)\,dx, \tag{6.10}$$

where u is any plane-like minimizer of S_ε with slope ω. This quantity is well-defined and related to the Wulff-shape for crystals [35]. The derivative of A_ε is of particular interest, and in [34] there is a formula for computing $D_\omega A_\varepsilon(\omega)$ in terms of the periodic and heteroclinic minimizers of S_ε. We use this along with our series solutions to compute $D_\omega A_\varepsilon(\omega)$ as a function of ε, which agrees with numerical computations of $D_\omega A_\varepsilon(\omega)$ using the Sobolev gradient method described above.

7. FUTURE DIRECTIONS

7.1. **Dislocations.** We currently lack a method for following dislocation dynamics up to a collision between dislocations or a collision between a dislocation and the boundary. I would like to find a way to extend the dynamics to collisions, which would then allow a "restarting" of the system with fewer dislocations. I would also like to investigate the motion of edge dislocations using a similar approach to [12]. A natural generalization of our work that interests me is to incorporate both screw and edge dislocations, and to remove the antiplane shear assumption.

The proof of uniqueness for solutions to the dislocation dynamics is dependent on regularity assumptions on the ambiguity set $\mathcal{A}$ that are reasonable, but possibly unnecessary. I would like to remove these assumptions.

7.2. **KAM.** The algorithm for computing invariant tori in volume-preserving systems was implemented in [23] for finding non-contractible tori. The method applies to contractible tori, so it would be interesting to implement it for that case as well. I would also like to compare the of the performance of our method with that of known methods for quasi-periodically forced systems and small twist systems.

The breakdown of analyticity could be computed for more general systems than the Frenkel-Kontorova model. For instance, the algorithms in [36, 37] apply to systems with non-nearest neighbor interactions. I am also interested in finding an explanation for the three phenomena we uncovered in [9], as described in §3.2.

7.3. **Stochastically Forced Systems.** Other applications of stability for stochastically forced equations, such as to the beam equation would be interesting. This falls into a larger project of understanding how our results in [13] apply generally to PDEs. Some of our results in [14] indicate that there should be a natural extension beyond the water wave equations.

It would also be interesting to know if formulas like (4.3) would apply for a more general class of stochastic processes (not just for outputs of linear filters).

7.4. **Aubry-Mather Theory for PDEs.** Little is known about plane-like minimizers in non-periodic media (e.g. quasi-periodic, or stationary ergodic), and I would like to extend my results in that direction. It may be possible to extend the minimal average energy characterization by Senn [34] to the non-periodic case, as well. Extentions to energies involving the fractional Laplacian but in the periodic case is also of interest.

7.5. **Dynamics of Microstructure.** Though I do not have current work on the topic, I am interested in the dynamics of microstructure. For example, in the nucleation and propagation of martensitic twins during an austenite-martensite phase transition. This is a very difficult problem, and even simple visco-elastic models produce very complicated behavior. I would like to investigate (both theoretically and numerically) how different viscosity terms lead to phase motion and pattern formation in microstructure models.

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