Math 830 : Advanced Topics in PDE
Optimal Transport, Variational and PDE techniques for machine learning
Spring 2021

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Lectures MWF 4:30pm on Zoom
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Learning Objectives. The objective of the course is to cover a number of mathematical techniques relevant to foundations of machine learning. The first part of the course will deal with optimal transportation, Wasserstein gradient flows, and extensions. The second part will focus on some tools from PDE and calculus of variations. Particular attention will be given to nonlocal equation and functionals. The third part of the course will discuss discrete models on point clouds and their connection to the continuum models. In particular we will investigate models on random samples of a measure which arise in machine learning.
The objective of the course it to learn about applications of variational and PDE techniques in data science. This includes learning about tasks of data analysis such as clustering, classification, semi-supervised learning dimensional reduction. We will explore how these techniques are used in modeling, analysis and computation.

Prerequisites. knowledge of measure and integration, differential equations, advanced real analysis, some familiarity with probability, calculus of variations and differential geometry.

Evaluation. The course grade is based on exercises and the final project. Exercises will be posted continuously throughout the course. They will include both theoretical problems and implementing numerical schemes. There will be no more than 24 exercises. To get a full score on the exercise part you need to solve at least 2/3 of the problems posted. All of the exercises are due on the last day of classes. I would recommend, however, that you solve them as soon as they are posted.
The final project involves writing a document of at least 8 pages and giving a 25 minute oral presentation. There are two options for the final project:

1. *Explore a research idea and report the results.* There are truly many questions in data analysis that can be formulated in terms of variational and PDE problems that have not been explored. Your task would be to (try to) explore one of them. This can include modeling (that is giving the particular data analysis task a variational or PDE description), computational experiments (to learn how the model works), and analysis (establishing the mathematical properties of objects, for example proving that the variational problem has a unique C^1 solution).

2. *Read published research on a topic and explain its main ideas.* Typically this would constitute of reading a paper on a topic and explaining it to your peers in both written form (the report) and as an oral presentation. Reading on the subject can include getting familiar with the background material and the exploring the related publications on the topic.

Approximate Outline [1]

I. Optimal transportation and related metrics

- Optimal transportation (OT maps, OT plans, Kantorovich duality existence of transportation plans, existence of transportation potentials, Brenier's theorem, Monge-Ampere equation, geometry of optimal transportation, Benamou-Brenier theorem, entropy regularized optimal transport)
- Numerical approximation of OT, Sinkhorn algorithm
- Other transportation based distances (unbalanced transport, Hellinger–Kantorovich, Wasserstein-Fisher-Rao, TL^p, Stein divergence, MMD)
- Quantization and complexity of probability measures with respect to the above distances
- Gradient flows with respect to Wasserstein distance, Displacement convexity, stability, asymptotics. JKO scheme and particle based methods.

II. Calculus of variations and PDE background

- Brief review of the spectral theory of the Laplace operator
- Viscosity solutions for elliptic and parabolic PDE
- Total variation, BV functions
- Nonlocal PDE and variational problems
- Γ-convergence and examples

III. PDE and variational problems based on point clouds

- Graph Laplacian, spectral clustering, laplacian eigemnaps, diffusion maps
- Graph cut based clustering
- Evolutionary PDE on graphs
- Connection to continuum problems, consistency and error estimates
- Optimal transportation on graphs

[1] The topics may change depending on interest etc.