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STABILITY OF STOCHASTIC HEREDITARY SYSTEMS WITH MARKOV SWITCHING

Many processes in automatic regulation, physics, mechanics, biology, economy, ecology etc. can be modelled by functional differential equations (see, e.g. Kolmanovskii and Shaikhet (1996), Kolmanovskii and Myshkis (1992), Kolmanovskii and Nosov (1981, 1986)). One of the main problems of the theory of stochastic functional differential equations and their applications is connected with stability (see, e.g. Kolmanovskii and Myshkis (1992), Kolmanovskii and Nosov(1981, 1986). Many stability results were obtained by the construction of appropriate Lyapunov functionals. At present the method is proposed allowing, in some sense, to formalize the procedure of the corresponding Lyapunov functionals construction (Kolmanovskii (1993), Kolmanovskii and Shaikhet (1993a, 1993b, 1994, 1995), Shaikhet (1995)). In this work by virtue of proposed procedure the sufficient conditions of asymptotic mean square stability for stochastic differential equation with delay and Markov switching are obtained.

1. The statement of the problem

Consider the stochastic differential equation

$$\dot{x}(t) = a(t)Ax(t) + Bx(t-h) + \sigma(t, x_t)\dot{\xi}(t), \qquad x_0 = \varphi_0 \in H. \tag{1}$$

Let $\{\Omega, \sigma, \mathsf{P}\}$ be a probability space with a current of σ-algebras, $f_t \subset \sigma$, $t \geq 0$, $\xi(t) \in \mathbf{R}^N$ be a f_t-measurable standard Wiener process, H be a set of piecewise continuous f_0-measurable functions $\varphi(s) \in \mathbf{R}^n$, $s \leq 0$, with norm $\|\varphi\| = \sup_{s\leq 0}(\mathsf{E}\,|\varphi(s)|^2)^{1/2}$, A and B be square $n \times n$-matrices, $\sigma(t,\varphi)$ be $n \times N$-matrix functional defined by $t \geq 0$, $\varphi \in H$, $h \geq 0$, $x_t = x(t+s)$, $s \leq 0$, and a(t) be an independent of the process $\xi(t)$ f_t-measurable scalar Markov process with denumerable set of states $(a_1, a_2, \dots)$ and probabilities of transition

$$p_{ij}(t) = \mathsf{P}\{a(\tau+t) = a_j / a(\tau) = a_i\}, \qquad t, \tau \geq 0.$$

Let us assume that the limits

$$\lambda_{ij} = \lim_{t\to 0} \frac{p_{ij}(t)}{t}, \qquad i \neq j,$$

there exist and the conditions

$$\begin{aligned} &| \sigma(t,\varphi) | \leq \int_0^\infty | \varphi(-s) | \, dK(s), \qquad dK(s) \geq 0, \\ &\sigma_0 = \int_0^\infty dK(s) < \infty, \qquad \int_0^\infty s\, dK(s) < \infty \end{aligned} \tag{2}$$

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hold.

Let us represent an arbitrary functional $V(t,\varphi,a)$ defined by $t \geq 0$, $\varphi \in H$, $a \in \mathbf{R}^1$ in the form $V(t,\varphi,a) = V(t,\varphi(0),\varphi(\tau),a)$, $\tau < 0$ and let us set

$$V_\varphi(t,x,a) = V(t,\varphi,a) = V(t,x,x(t+s),a),$$
$$s < 0, \qquad \varphi = x_t, \qquad x = \varphi(0) = x(t).$$

Let D be a class of functionals $V(t,\varphi,a)$ for which the function $V_\varphi(t,x,a)$ is twice continuously differentiable with respect to x and once continuously differentiable for almost all $t \geq 0$. On the functionals from D the generator L of the equation (1) is defined by the formula

$$\begin{aligned} LV(t,x_t,a_k) = & \frac{\partial V_\varphi(t,x,a_k)}{\partial t} + (a_k Ax + Bx(t-h))' \frac{\partial V_\varphi(t,x,a_k)}{\partial x} \\ & + \frac{1}{2} Tr[\sigma'(t,x_t) \frac{\partial^2 V_\varphi(t,x,a_k)}{\partial x^2} \sigma(t,x_t)] \\ & + \sum_{i \neq k} (V(t,x_t,a_i) - V(t,x_t,a_k))\lambda_{ki}. \end{aligned} \quad (3)$$

Definition. The zero solution of the equation (1) is called mean square stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathsf{E}|x(t)|^2 < \epsilon$ if $\|\varphi_0\|^2 < \delta$. If, in addition, $\lim_{t\to\infty} \mathsf{E}|x(t)|^2 = 0$ then the zero solution of the equation (1) is called asymptotically mean square stable.

It is well known (see, e.g. Kolmanovskii and Myshkis (1992), Kolmanovskii and Nosov (1981,1986)) that asymptotic mean square stability conditions can be obtained by construction of some positive definite (or positive semidefinite for differential equations of neutral type) Lyapunov functionals $V(t,\varphi,a)$ for which the inequality

$$LV(t,\varphi,a) \leq -c \mid \varphi(0) \mid^2, \qquad c > 0, \quad (4)$$

hold. Below the sufficient asymptotic mean square stability conditions of the equation (1) the zero solution are obtained by general method of Lyapunov functional construction (Kolmanovskii (1993), Kolmanovskii and Shaikhet (1993a, 1993b, 1994, 1995), Shaikhet (1995)). Particular cases of this results were represented earlier (see Shaikhet (1994a,1994b)).

2. Asymptotic mean square stability conditions

Theorem 2.1. *Let the states of the process $a(t)$ satisfy the inequalities*

$$|a_1| \geq |a_2| \geq \cdots \geq c > 0, \quad (5)$$

and let matrices

$$R_k = a_k \mid a_k \mid (A + A') + (f_k + (\mid a_k \mid + \mid a_1 \mid) \mid B \mid + \sigma_0^2 \mid a_1 \mid)I \quad (6)$$

be negative definite uniformly with respect to $k = 1, 2, \ldots$. Here I is identity matrix,

$$f_k = \sum_{i \neq k} (\mid a_i \mid - \mid a_k \mid)\lambda_{ki}. \quad (7)$$

Then the zero solution of the equation (1) *is asymptotically mean square stable.*

Proof. In accordance with the method of Lyapunov functionals construction (Kolmanovskii (1993), Kolmanovskii and Shaikhet (1993a, 1993b, 1994, 1995), Shaikhet (1995)) we will construct the functional V in the form $V = V_1 + V_2$, where the functional V_1 must be a Lyapunov function for auxiliary system without delay $\dot{y}(t) = a(t)Ay(t)$. Let be

$$V_1(t, x_t, a) = |a| |x(t)|^2 . \tag{8}$$

Using (3), (8), (1), (2), (7), we obtain

$$\begin{aligned} LV_1(t, x_t, a_k) &= a_k|a_k|x'(t)(A + A')x(t) + f_k|x(t)|^2 + 2|a_k|x'(t)Bx(t-h) \\ &+ |a_k||\sigma(t, x_t)|^2 \leq a_k \mid a_k \mid x'(t)(A + A')x(t) + (f_k + |a_k||B|)|x(t)|^2 \\ &+ |a_k|(|B||x(t-h)|^2 + \sigma_0 \int_0^\infty |x(t-s)|^2 \, dK(s)). \end{aligned} \tag{9}$$

Let us choose the functional V_2 in the form

$$V_2(t, x_t, a_k) = |a_1| \left(|B| \int_{t-h}^t |x(s)|^2 \, ds + \sigma_0 \int_0^\infty dK(s) \int_{t-s}^t |x(\tau)|^2 \, d\tau \right).$$

Then

$$\begin{aligned} &LV_2(t, x_t, a_k) \\ &= |a_1|(|B|(|x(t)|^2 - |x(t-h)|^2) + \sigma_0 \int_0^\infty dK(s)(|x(t)|^2 - |x(t-s)|^2)) \\ &= |a_1|(|B| + \sigma_0^2)|x(t)|^2 - |a_1|(|B||x(t-h)|^2 + \sigma_0 \int_0^\infty |x(t-s)|^2 \, dK(s)). \end{aligned} \tag{10}$$

From (9), (10), (5) (6) for $V = V_1 + V_2$ it follows $LV(t, x_t, a_k) \leq x'(t)R_k x(t) \leq -c|x(t)|^2$.

It means (Kolmanovskii and Nosov (1981,1986)) that the zero solution of the equation (1) is asymptotically mean square stable. Theorem is proved. □

Theorem 2.2. *Let the inequalities* (5) *and* $|B|h < 1$ *hold and let matrices*

$$Q_k = |a_k| (a_k(A + A') + B + B') + (f_k + (g_k + \rho)h + \sigma_0^2 |a_1|)I \tag{11}$$

are negative definite uniformly with respect to $k = 1, 2, \ldots$. *Here* I *is identity matrix,* f_k *is defined by* (7),

$$g_k = ||a_k|(a_k A + B)'B + f_k B|, \qquad \rho = \sup_k (g_k + |f_k||B'B|h). \tag{12}$$

Then the zero solution of the equation (1) *is asymptotically mean square stable.*

Proof. Reduce the equation (1) to the form of a stochastic differential neutral type equation

$$\frac{d}{dt}(x(t) + \int_{t-h}^t Bx(s) \, ds) = (a(t)A + B)x(t) + \sigma(t, x_t)\dot{\xi}(t). \tag{13}$$

We will construct the Lyapunov functional V in the form $V = V_1 + V_2$ again. But in this case the functional V_1 must be a Lyapunov function for another auxiliary system without delay

$$\dot{y}(t) = (a(t)A + B)y(t), \qquad y(t) = x(t) + \int_{t-h}^t Bx(s) \, ds.$$

Therefore let be

$$V_1(t, x_t, a) = |a| \left| x(t) + \int_{t-h}^t Bx(s) \, ds \right|^2 . \tag{14}$$

Using (3), (14) (13), (2), (7), (12), we obtain

$$
\begin{aligned}
LV_1(t,x_t,a_k) &= 2|a_k|x'(t)(a_kA+B)'\left(x(t)+\int_{t-h}^{t} Bx(s)\,ds\right)\\
&\quad + |a_k|\left|\sigma(t,x_t)|^2 + f_k|x(t)+\int_{t-h}^{t} Bx(s)\,ds\right|^2\\
&\le |a_k|x'(t)(a_k(A+A')+B+B')x(t)+f_k|x(t)|^2\\
&\quad + 2x'(t)(|a_k|(a_kA+B)'B+f_kB)\int_{t-h}^{t} x(s)\,ds\\
&\quad + |f_k||B'B|h\int_{t-h}^{t}|x(s)|^2\,ds+\sigma_0|a_k|\int_0^{\infty}|x(t-s)|^2\,dK(s)\\
&\le |a_k|x'(t)(a_k(A+A')+B+B')x(t)+(f_k+g_kh)|x(t)|^2\\
&\quad + \rho\int_{t-h}^{t} \mid x(s)\mid^2\,ds+\sigma_0\mid a_k\mid\int_0^{\infty}\mid x(t-s)\mid^2\,dK(s).
\end{aligned}
$$

Let us choose the functional V_2 in the form

$$
V_2(t,x_t,a_k)=\rho\int_{t-h}^{t}(s-t+h)\mid x(s)\mid^2\,ds+\sigma_0\mid a_1\mid\int_0^{\infty}dK(s)\int_{t-s}^{t}\mid x(\tau)\mid^2\,d\tau.
$$

Then using (11) for $V=V_1+V_2$ we get $LV(t,x_t,a_k)\le x'(t)Q_kx(t)\le -c\mid x(t)\mid^2$.

Remark that the necessity of the condition $\mid B\mid h<1$ follows from the theorem about asymptotic mean square stability for stochastic differential equations of neutral type (Kolmanovskii and Nosov (1981)). Theorem is proved. □

3. Example

Consider the scalar stochastic differential equation

$$
\dot{x}(t)=a(t)x(t)+bx(t-h)+\sigma x(t-\tau)\dot{\xi}(t). \tag{15}
$$

Let us assume the Markov process a(t) has two states (a_1,a_2) such that $a_2<0$, $a_1>|a_2|>b>-a_1$.

From Theorem 2.1 it follows that the asymptotic mean square stability conditions for the zero solution of the equation (15) have a form $\frac{a_1^2+a_1|b|+a_1\sigma^2/2}{\lambda_{12}}<\frac{a_1+a_2}{2}<\frac{a_2^2-(a_1-a_2)|b|/2-a_1\sigma^2/2}{\lambda_{21}}$.

From Theorem 2.2 we have other asymptotic mean square stability conditions: $|b|h<1$,

$$
\frac{a_1^2+a_1b+a_1\sigma^2/2+(d_1+\rho_0)\mid b\mid h/2}{\lambda_{12}}<\frac{a_1+a_2}{2}
<\frac{a_2^2+a_2b-a_1\sigma^2/2-(d_2+\rho_0)\mid b\mid h/2}{\lambda_{21}}.
$$

Here $d_k=||a_k|(a_k+b)+f_k|$, $\rho_0=\sup_k(d_k+|bf_k|h)$.

Consider in particular the scalar differential equation $\dot{x}(t)=a(t)x(t)$. Let us assume the Markov process a(t) has two states (a_1,a_2) such that $a_2<0$, $a_1>|a_2|$. If λ_{12} is so large and λ_{21} is so small that the inequalities $\frac{a_1^2}{\lambda_{12}}<\frac{a_1+a_2}{2}<\frac{a_2^2}{\lambda_{21}}$ hold then the zero solution of the above equation is asymptotically mean square stable.

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