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Method of Lyapunov functionals construction in stability of delay evolution equations

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Abstract

The investigation of stability for hereditary systems is often related to the construction of Lyapunov functionals. The general method of Lyapunov functionals construction which was proposed by V. Kolmanovskii and L. Shaikhet and successfully used already for functional differential equations, for difference equations with discrete time, for difference equations with continuous time, is used here to investigate the stability of delay evolution equations, in particular, partial differential equations.

Keywords: Method of Lyapunov functionals construction; Evolution equations; Stability; Partial differential equations; 2D Navier–Stokes model with delays

1. Introduction

The study of functional differential equations is motivated by the fact that when one wants to model some evolution phenomena arising in Physics, Biology, Engineering, etc., some hereditary characteristics such as aftereffect, time lag and time delay can appear in the variables.

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Typical examples arise from the researches of materials with thermal memory, biochemical reactions, population models, etc. (see, for instance, [1–5,9,11–15,18,23–26,32–37] and references therein). On the other hand, one important and interesting problem in the analysis of functional differential equations is the stability, the theory of which has been greatly developed over the last years. There exist many works dealing with the construction of Lyapunov functionals for a wide range of equations containing some kind of hereditary properties.

As it is well known, in the case without any hereditary features, Lyapunov's technique is available to obtain sufficient conditions for the stability of solutions of (partial) differential equations. However, in the case of differential equations with hereditary properties, for instance, even in the case of constant time delays, Lyapunov's method becomes difficult to apply effectively as N.N. Krasovskii [21] pointed out. The main reason is that it is much more difficult (or even impossible in some cases) to construct proper Lyapunov functions (or functionals) for functional differential equations than for those without any hereditary characteristics.

Our interest in this paper is to investigate the stability of dynamical systems modelled by delay evolution equations, in particular, by partial differential equations with delays, using the general method of Lyapunov functionals construction that was proposed by V. Kolmanovskii and L. Shaikhet and successfully used already for functional differential equations, for difference equations with discrete time, and for difference equations with continuous time [16,17,19,20, 27–31].

Taking into account that many interesting problems from applications have main operators which satisfy some kind of coercivity assumption, we will exploit this idea here and will be interested in this class of operators.

1.1. Notations and definitions

Let U and H be two real separable Hilbert spaces such that $U \subset H \equiv H^* \subset U^*$, where the injections are continuous and dense. Let $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ be the norms in U, H and U^* respectively, $((\cdot,\cdot))$ and $(\cdot,\cdot)$ be the scalar products in U and H respectively, and $\langle\cdot,\cdot\rangle$ the duality product between U and U^*. We assume that

$$|u| \leqslant \beta\|u\|, \quad u \in U. \tag{1.1}$$

Let $C(-h,0,H)$ be the Banach space of all continuous functions from $[-h,0]$ to H, $x_t \in C(-h,0,H)$ for each $t \in [0,\infty)$, be the function defined by $x_t(s) = x(t+s)$ for all $s \in [-h,0]$. The space $C(-h,0,U)$ is similarly defined.

Let $A(t,\cdot): U \to U^*$, $f_1(t,\cdot): C(-h,0,H) \to U^*$ and $f_2(t,\cdot): C(-h,0,U) \to U^*$ be three families of nonlinear operators defined for $t > 0$, $A(t,0) = 0$, $f_1(t,0) = 0$, $f_2(t,0) = 0$.

Consider the equation

$$\begin{aligned} &\frac{du(t)}{dt} = A\big(t,u(t)\big) + f_1(t,u_t) + f_2(t,u_t), \quad t > 0, \\ &u(s) = \psi(s), \quad s \in [-h,0]. \end{aligned} \tag{1.2}$$

Let us denote by $u(\cdot;\psi)$ the solution of Eq. (1.2) corresponding to the initial condition ψ.

Definition 1.1. The trivial solution of Eq. (1.2) is said to be stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $|u(t;\psi)| < \epsilon$ for all $t \geqslant 0$, if $|\psi|_{C_H} = \sup_{s\in[-h,0]} |\psi(s)| < \delta$.

Definition 1.2. The trivial solution of Eq. (1.2) is said to be exponentially stable if it is stable and there exists a positive constant λ such that for any $\psi \in C(-h, 0, U)$ there exists C (which may depend on ψ) such that $|u(t; \psi)| \leqslant Ce^{-\lambda t}$ for $t > 0$.

1.2. Lyapunov type stability theorem

Let us now prove a theorem which will be crucial in our stability investigation.

Theorem 1.1. *Assume that there exists a functional $V(t, u_t)$ such that the following conditions hold for some positive numbers c_1, c_2 and λ:*

$$V(t, u_t) \geqslant c_1 e^{\lambda t} \left|u(t)\right|^2, \quad t \geqslant 0, \tag{1.3}$$

$$V(0, u_0) \leqslant c_2 |\psi|_{C_H}^2, \tag{1.4}$$

$$\frac{d}{dt} V(t, u_t) \leqslant 0, \quad t \geqslant 0. \tag{1.5}$$

Then the trivial solution of Eq. (1.2) *is exponentially stable.*

Proof. Integrating (1.5) we obtain $V(t, u_t) \leqslant V(0, u_0)$. From here and (1.3), (1.4) it follows

$$c_1 \left|u(t)\right|^2 \leqslant e^{-\lambda t} V(0, u_0) \leqslant c_2 |\psi|_{C_H}^2.$$

The inequality $c_1 |u(t)|^2 \leqslant c_2 |\psi|_{C_H}^2$ means that the trivial solution of Eq. (1.2) is stable. Besides, from the inequality $c_1 |u(t)|^2 \leqslant e^{-\lambda t} V(0, u_0)$, it follows that the trivial solution of Eq. (1.2) is exponentially stable. □

Note that Theorem 1.1 implies that the stability investigation of Eq. (1.2) can be reduced to the construction of appropriate Lyapunov functionals. A formal procedure to construct Lyapunov functionals is described below.

1.3. Procedure of Lyapunov functionals construction

The procedure consists of four steps.

Step 1. To transform Eq. (1.2) into the form

$$\frac{dz(t, u_t)}{dt} = A_1\big(t, u(t)\big) + A_2(t, u_t) \tag{1.6}$$

where $z(t, \cdot)$ and $A_2(t, \cdot)$ are families of nonlinear operators, $z(t, 0) = 0$, $A_2(t, 0) = 0$, operator $A_1(t, \cdot)$ only depends on t and $u(t)$, but does not depend on the previous values $u(t + s)$, $s < 0$.

Step 2. Assume that the trivial solution of the auxiliary equation without memory

$$\frac{dy(t)}{dt} = A_1\big(t, y(t)\big)\, dt \tag{1.7}$$

is exponentially stable and therefore there exists a Lyapunov function $v(t, y(t))$, which satisfies the conditions of Theorem 1.1.

Step 3. A Lyapunov functional $V(t, u_t)$ for Eq. (1.6) is constructed in the form $V = V_1 + V_2$, where $V_1(t, u_t) = v(t, z(t, u_t))$. Here the argument y of the function $v(t, y)$ is replaced on the functional $z(t, x_t)$ from the left-hand part of Eq. (1.6).

Step 4. Usually, the functional $V_1(t, u_t)$ almost satisfies the conditions of Theorem 1.1. In order to fully satisfy these conditions, it is necessary to calculate $\frac{d}{dt} V_1(t, u_t)$ and estimate it. Then, the additional functional $V_2(t, u_t)$ can be chosen in a standard way.

Note that the representation (1.6) is not unique. This fact allows, using different representations type of (1.6) or different ways of estimating $\frac{d}{dt} V_1(t, u_t)$, to construct different Lyapunov functionals and, as a result, to get different sufficient conditions of exponential stability.

2. Construction of Lyapunov functionals for equations with time-varying delay

Consider the following evolution equation

$$\frac{du(t)}{dt} = A\big(t, u(t)\big) + F\big(u\big(t - h(t)\big)\big), \tag{2.1}$$

where $A(t, \cdot), F : U \to U^*$ are appropriate partial differential operators (see conditions below), which is a particular case of Eq. (1.2).

We will apply the method described above to construct Lyapunov functionals for Eq. (2.1), and, as a consequence, to obtain sufficient conditions ensuring the stability of the trivial solution.

We will use two different constructions which will provide different stability regions for the parameters involved in the problem.

2.1. *The first way of Lyapunov functionals construction*

First we consider a quite general situation for the operators involved in Eq. (2.1).

Theorem 2.1. *Assume that operators in Eq.* (2.1) *satisfy the conditions*:

$$\begin{aligned} &\big\langle A(t, u), u\big\rangle \leqslant -\gamma \|u\|^2, \quad \gamma > 0, \\ &F\colon U \to U^*, \qquad \big\|F(u)\big\|_* \leqslant \alpha \|u\|, \quad u \in U, \end{aligned} \tag{2.2}$$

and

$$h(t) \in [0, h_0], \quad \dot{h}(t) \leqslant h_1 < 1. \tag{2.3}$$

If

$$\gamma > \frac{\alpha}{\sqrt{1 - h_1}}, \tag{2.4}$$

then the trivial solution of Eq. (2.1) *is exponentially stable.*

Proof. Owing to the procedure of Lyapunov functionals construction, let us consider the auxiliary equation without memory

$$\frac{d}{dt} y(t) = A\big(t, y(t)\big). \tag{2.5}$$

The function $v(t,y)=e^{\lambda t}|y|^2$, $\lambda>0$, is a Lyapunov function for Eq. (2.5), i.e. it satisfies the conditions of Theorem 1.1. Actually, it is easy to see that conditions (1.3), (1.4) hold for the function $v(t,y(t))$. Besides, since $\gamma>0$, there exists $\lambda>0$ such that $2\gamma>\lambda\beta^2$. Using (2.5), (1.1) and (2.2), we obtain

$$\frac{d}{dt}v\big(t,y(t)\big)=\lambda e^{\lambda t}\big|y(t)\big|^2+2e^{\lambda t}\big\langle A\big(t,y(t)\big),y(t)\big\rangle\leqslant -e^{\lambda t}\big(2\gamma-\lambda\beta^2\big)\big\|y(t)\big\|^2\leqslant 0.$$

According to the procedure, we now construct a Lyapunov functional V for Eq. (2.1) in the form $V=V_1+V_2$, where $V_1(t,u_t)=e^{\lambda t}|u(t)|^2$. For Eq. (2.1) we obtain

$$\begin{aligned}\frac{d}{dt}V_1(t,u_t)&=\lambda V_1(t,u_t)+2e^{\lambda t}\big\langle A\big(t,u(t)\big)+F\big(u\big(t-h(t)\big)\big),u(t)\big\rangle\\&\leqslant e^{\lambda t}\big[\lambda\big|u(t)\big|^2+2\big(-\gamma\big\|u(t)\big\|^2+\alpha\big\|u\big(t-h(t)\big)\big\|\big\|u(t)\big\|\big)\big]\\&\leqslant e^{\lambda t}\left[\lambda\beta^2\big\|u(t)\big\|^2-2\gamma\big\|u(t)\big\|^2+\alpha\left(\epsilon\big\|u\big(t-h(t)\big)\big\|^2+\frac{1}{\epsilon}\big\|u(t)\big\|^2\right)\right]\\&=e^{\lambda t}\left[\left(\lambda\beta^2-2\gamma+\frac{\alpha}{\epsilon}\right)\big\|u(t)\big\|^2+\epsilon\alpha\big\|u\big(t-h(t)\big)\big\|^2\right].\end{aligned}$$

Set now

$$V_2(t,u_t)=\frac{\epsilon\alpha}{1-h_1}\int_{t-h(t)}^{t}e^{\lambda(s+h_0)}\big\|u(s)\big\|^2\,ds.$$

Then

$$\begin{aligned}\frac{d}{dt}V_2(t,u_t)&=\frac{\epsilon\alpha}{1-h_1}\big(e^{\lambda(t+h_0)}\big\|u(t)\big\|^2-\big(1-\dot h(t)\big)e^{\lambda(t-h(t)+h_0)}\big\|u\big(t-h(t)\big)\big\|^2\big)\\&\leqslant\frac{\epsilon\alpha e^{\lambda t}}{1-h_1}\big(e^{\lambda h_0}\big\|u(t)\big\|^2-(1-h_1)e^{\lambda(h_0-h(t))}\big\|u\big(t-h(t)\big)\big\|^2\big)\\&\leqslant\epsilon\alpha e^{\lambda t}\left(\frac{e^{\lambda h_0}}{1-h_1}\big\|u(t)\big\|^2-\big\|u\big(t-h(t)\big)\big\|^2\right).\end{aligned}$$

Thus, for $V=V_1+V_2$ we have

$$\frac{d}{dt}V(t,u_t)\leqslant\left[\lambda\beta^2-2\gamma+\alpha\left(\frac{1}{\epsilon}+\frac{\epsilon e^{\lambda h_0}}{1-h_1}\right)\right]e^{\lambda t}\big\|u(t)\big\|^2.$$

Rewrite the expression in square brackets as

$$-2\gamma+\alpha\left(\frac{1}{\epsilon}+\frac{\epsilon}{1-h_1}\right)+\lambda\beta^2+\epsilon\alpha\frac{e^{\lambda h_0}-1}{1-h_1}.$$

To minimize this expression in the brackets, choose $\epsilon=\sqrt{1-h_1}$. As a consequence we obtain

$$\frac{d}{dt}V(t,u_t)\leqslant-\left[2\left(\gamma-\frac{\alpha}{\sqrt{1-h_1}}\right)-\rho(\lambda)\right]e^{\lambda t}\big\|u(t)\big\|^2\tag{2.6}$$

with

$$\rho(\lambda)=\lambda\beta^2+\alpha\frac{e^{\lambda h_0}-1}{\sqrt{1-h_1}}.$$

Since $\rho(0) = 0$, then by condition (2.4) there exists $\lambda > 0$ small enough such that

$$2\left(\gamma - \frac{\alpha}{\sqrt{1-h_1}}\right) \geqslant \rho(\lambda).$$

From here and (2.6) it follows that $\frac{d}{dt}V(t,u_t) \leqslant 0$. So, the functional $V(t,u_t)$ constructed above satisfies the conditions in Theorem 1.1. This means that the trivial solution of Eq. (2.1) is exponentially stable. □

Note, in particular, if $h(t) \equiv h_0$ then $h_1 = 0$ and condition (2.4) takes the form $\gamma > \alpha$.

2.2. The second way of Lyapunov functionals construction

We now establish a second result which implies that the operator F must be less general than in Theorem 2.1. However, as we will show later in the applications section, the stability regions provided by this theorem will be better than the ones given by Theorem 2.1.

Theorem 2.2. *Suppose that operators in Eq.* (2.1) *satisfy the following conditions*:

$$\begin{aligned}
&\langle A(t,u) + F(u), u\rangle \leqslant -\gamma\|u\|^2, \quad \gamma > 0,\\
&\|A(t,u) + F(u)\|_* \leqslant \alpha_1\|u\|,\\
&F : U \to U, \qquad \|F(u)\| \leqslant \alpha_2\|u\|, \quad u \in U,
\end{aligned} \tag{2.7}$$

and

$$h(t) \in [0, h_0], \quad \dot{h}(t) \leqslant h_1 < 1, \quad |\dot{h}(t)| \leqslant h_2. \tag{2.8}$$

If

$$\gamma > \alpha_1\alpha_2 h_0 + (1 + \alpha_2 h_0)\frac{\alpha_2 h_2}{\sqrt{1-h_1}}, \tag{2.9}$$

then the trivial solution of Eq. (2.1) *is exponentially stable.*

Proof. To use the procedure of Lyapunov functionals construction, let us first transform Eq. (2.1) as

$$\frac{d}{dt}z(t,u_t) = A\big(t,u(t)\big) + F\big(u(t)\big) + \dot{h}(t)F\big(u\big(t-h(t)\big)\big), \tag{2.10}$$

where

$$z(t,u_t) = u(t) + \int_{t-h(t)}^{t} F\big(u(s)\big)\,ds. \tag{2.11}$$

Consider the auxiliary equation without memory in the form

$$\frac{d}{dt}y(t) = A\big(t,y(t)\big) + F\big(y(t)\big). \tag{2.12}$$

The function $v(t,y) = e^{\lambda t}|y|^2$ is a Lyapunov function for Eq. (2.12). Actually, since $\gamma > 0$ then there exists $\lambda > 0$ such that $2\gamma > \lambda\beta^2$. Using (2.12), (1.1), (2.7), we obtain

$$\begin{aligned}\frac{d}{dt}v\big(t,y(t)\big)&=\lambda e^{\lambda t}\big|y(t)\big|^2+2e^{\lambda t}\big\langle A\big(t,y(t)\big)+F\big(y(t)\big),y(t)\big\rangle\\&\leqslant -e^{\lambda t}\big(2\gamma-\lambda\beta^2\big)\big\|y(t)\big\|^2.\end{aligned}$$

Next, we construct a Lyapunov functional V for Eqs. (2.10), (2.11) in the form $V=V_1+V_2$, where

$$V_1(t,u_t)=e^{\lambda t}\big|z(t,u_t)\big|^2, \tag{2.13}$$

and $z(t,u_t)$ is defined by (2.11). Using (2.7) for Eqs. (2.10) and (2.11) we have

$$\begin{aligned}\frac{d}{dt}V_1(t,u_t)&=\lambda V_1(t,u_t)+2e^{\lambda t}\big\langle A\big(t,u(t)\big)+F\big(u(t)\big)+\dot h(t)F\big(u\big(t-h(t)\big)\big),z(t,u_t)\big\rangle\\
&=\lambda V_1(t,u_t)+2e^{\lambda t}\Big\langle A\big(t,u(t)\big)+F\big(u(t)\big)+\dot h(t)F\big(u\big(t-h(t)\big)\big),\\
&\quad u(t)+\int_{t-h(t)}^{t}F\big(u(s)\big)\,ds\Big\rangle\\
&=\lambda V_1(t,u_t)+2e^{\lambda t}\Big\langle A\big(t,u(t)\big)+F\big(u(t)\big),u(t)+\int_{t-h(t)}^{t}F\big(u(s)\big)\,ds\Big\rangle\\
&\quad+2e^{\lambda t}\dot h(t)\Big(F\big(u\big(t-h(t)\big)\big),u(t)+\int_{t-h(t)}^{t}F\big(u(s)\big)\,ds\Big)\\
&\leqslant \lambda V_1(t,u_t)+2e^{\lambda t}\Big[-\gamma\big\|u(t)\big\|^2+\alpha_1\alpha_2\int_{t-h(t)}^{t}\big\|u(t)\big\|\big\|u(s)\big\|\,ds\Big]\\
&\quad+2e^{\lambda t}\big|\dot h(t)\big|\Big(\alpha_2\big\|u\big(t-h(t)\big)\big\|\big\|u(t)\big\|+\alpha_2^2\int_{t-h(t)}^{t}\big\|u\big(t-h(t)\big)\big\|\big\|u(s)\big\|\,ds\Big)\\
&\leqslant \lambda V_1(t,u_t)+e^{\lambda t}\Big[-2\gamma\big\|u(t)\big\|^2+\alpha_1\alpha_2\int_{t-h(t)}^{t}\Big(\frac{1}{\epsilon_1}\big\|u(t)\big\|^2+\epsilon_1\big\|u(s)\big\|^2\Big)\,ds\Big]\\
&\quad+e^{\lambda t}\big|\dot h(t)\big|\Big[\alpha_2\Big(\epsilon_2\big\|u\big(t-h(t)\big)\big\|^2+\frac{1}{\epsilon_2}\big\|u(t)\big\|^2\Big)\\
&\quad+\alpha_2^2\int_{t-h(t)}^{t}\Big(\epsilon_3\big\|u\big(t-h(t)\big)\big\|^2+\frac{1}{\epsilon_3}\big\|u(s)\big\|^2\Big)\,ds\Big]\\
&=\lambda V_1(t,u_t)+e^{\lambda t}\Big[\Big(-2\gamma+\frac{1}{\epsilon_1}\alpha_1\alpha_2h(t)+\frac{1}{\epsilon_2}\alpha_2\big|\dot h(t)\big|\Big)\big\|u(t)\big\|^2\\
&\quad+\alpha_2\big(\epsilon_2+\epsilon_3\alpha_2h(t)\big)\big|\dot h(t)\big|\big\|u\big(t-h(t)\big)\big\|^2\end{aligned}$$

$$+\alpha_2\left(\epsilon_1\alpha_1+\frac{1}{\epsilon_3}\alpha_2\left|\dot{h}(t)\right|\right)\int_{t-h(t)}^{t}\left\|u(s)\right\|^2ds\Bigg].$$

From (2.13) and (2.10) it follows

$$\begin{aligned}
e^{-\lambda t}V_1(t,u_t)&=\left|u(t)\right|^2+2\int_{t-h(t)}^{t}\big(u(t),F\big(u(s)\big)\big)\,ds+\left|\int_{t-h(t)}^{t}F\big(u(s)\big)\,ds\right|^2\\
&\leqslant\left|u(t)\right|^2+2\int_{t-h(t)}^{t}\left|u(t)\right|\left|F\big(u(s)\big)\right|ds+h(t)\int_{t-h(t)}^{t}\left|F\big(u(s)\big)\right|^2ds\\
&\leqslant\left|u(t)\right|^2+\alpha_2\beta^2\int_{t-h(t)}^{t}\left(\epsilon_4\left\|u(t)\right\|^2+\frac{1}{\epsilon_4}\left\|u(s)\right\|^2\right)ds\\
&\quad+\alpha_2^2h(t)\beta^2\int_{t-h(t)}^{t}\left\|u(s)\right\|^2ds\\
&\leqslant\big(1+\epsilon_4\alpha_2h(t)\big)\beta^2\left\|u(t)\right\|^2+\alpha_2\beta^2\left(\frac{1}{\epsilon_4}+\alpha_2h(t)\right)\int_{t-h(t)}^{t}\left\|u(s)\right\|^2ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{dt}V_1(t,u_t)&\leqslant e^{\lambda t}\left[\lambda\beta^2\big(1+\epsilon_4\alpha_2h(t)\big)-2\gamma+\frac{1}{\epsilon_1}\alpha_1\alpha_2h(t)+\frac{1}{\epsilon_2}\alpha_2\left|\dot{h}(t)\right|\right]\left\|u(t)\right\|^2\\
&\quad+e^{\lambda t}\alpha_2\big(\epsilon_2+\epsilon_3\alpha_2h(t)\big)\left|\dot{h}(t)\right|\left\|u\big(t-h(t)\big)\right\|^2\\
&\quad+e^{\lambda t}\alpha_2\left[\epsilon_1\alpha_1+\frac{\alpha_2}{\epsilon_3}\left|\dot{h}(t)\right|+\lambda\beta^2\left(\frac{1}{\epsilon_4}+\alpha_2h(t)\right)\right]\int_{t-h(t)}^{t}\left\|u(s)\right\|^2ds\\
&\leqslant e^{\lambda t}\left[\lambda\beta^2(1+\epsilon_4\alpha_2h_0)-2\gamma+\frac{1}{\epsilon_1}\alpha_1\alpha_2h_0+\frac{1}{\epsilon_2}\alpha_2h_2\right]\left\|u(t)\right\|^2\\
&\quad+e^{\lambda t}(\epsilon_2+\epsilon_3\alpha_2h_0)\alpha_2h_2\left\|u\big(t-h(t)\big)\right\|^2\\
&\quad+e^{\lambda t}\alpha_2\left[\epsilon_1\alpha_1+\frac{\alpha_2}{\epsilon_3}h_2+\lambda\beta^2\left(\frac{1}{\epsilon_4}+\alpha_2h_0\right)\right]\int_{t-h_0}^{t}\left\|u(s)\right\|^2ds.
\end{aligned}$$

Put now

$$\begin{aligned}
V_2(t,u_t)&=\frac{(\epsilon_2+\epsilon_3\alpha_2h_0)\alpha_2h_2}{1-h_1}\int_{t-h(t)}^{t}e^{\lambda(s+h_0)}\left\|u(s)\right\|^2ds\\
&\quad+\alpha_2\left[\epsilon_1\alpha_1+\frac{\alpha_2}{\epsilon_3}h_2+\lambda\beta^2\left(\frac{1}{\epsilon_4}+\alpha_2h_0\right)\right]\int_{t-h_0}^{t}e^{\lambda(s+h_0)}(s-t+h_0)\left\|u(s)\right\|^2ds.
\end{aligned}$$

Then

$$\begin{aligned}\frac{d}{dt}V_2(t,u_t) &= (\epsilon_2+\epsilon_3\alpha_2h_0)\alpha_2h_2\left[\frac{e^{\lambda(t+h_0)}}{1-h_1}\|u(t)\|^2 - e^{\lambda(t-h(t)+h_0)}\|u\big(t-h(t)\big)\|^2\right]\\ &\quad+\alpha_2\left[\epsilon_1\alpha_1+\frac{\alpha_2}{\epsilon_3}h_2+\lambda\beta^2\left(\frac{1}{\epsilon_4}+\alpha_2h_0\right)\right]\\ &\quad\times\left[e^{\lambda(t+h_0)}h_0\|u(t)\|^2-\int_{t-h_0}^{t}e^{\lambda(s+h_0)}\|u(s)\|^2\,ds\right].\end{aligned}$$

Since $e^{\lambda t}\leqslant e^{\lambda(s+h_0)}$ for $s\geqslant t-h_0$, then for $V=V_1+V_2$, we obtain

$$\begin{aligned}\frac{d}{dt}V(t,u_t) &\leqslant e^{\lambda t}\left[\lambda\beta^2(1+\epsilon_4\alpha_2h_0)-2\gamma+\frac{1}{\epsilon_1}\alpha_1\alpha_2h_0\right.\\ &\quad\left.+\frac{1}{\epsilon_2}\alpha_2h_2+\alpha_2h_2(\epsilon_2+\epsilon_3\alpha_2h_0)\frac{e^{\lambda h_0}}{1-h_1}\right]\|u(t)\|^2\\ &\quad+e^{\lambda(t+h_0)}\alpha_2h_0\left[\epsilon_1\alpha_1+\frac{1}{\epsilon_3}\alpha_2h_2+\lambda\beta^2\left(\frac{1}{\epsilon_4}+\alpha_2h_0\right)\right]\|u(t)\|^2\\ &= e^{\lambda t}\left[\lambda\beta^2(1+\epsilon_4\alpha_2h_0)-2\gamma+\frac{1}{\epsilon_1}\alpha_1\alpha_2h_0+\frac{1}{\epsilon_2}\alpha_2h_2\right.\\ &\quad+\alpha_2h_2(\epsilon_2+\epsilon_3\alpha_2h_0)\frac{e^{\lambda h_0}}{1-h_1}\\ &\quad\left.+e^{\lambda h_0}\alpha_2h_0\left[\epsilon_1\alpha_1+\frac{1}{\epsilon_3}\alpha_2h_2+\lambda\beta^2\left(\frac{1}{\epsilon_4}+\alpha_2h_0\right)\right]\right]\|u(t)\|^2\\ &= \left[-2\gamma+\alpha_1\alpha_2h_0\left(\frac{1}{\epsilon_1}+\epsilon_1\right)+\alpha_2h_2\left(\frac{1}{\epsilon_2}+\frac{\epsilon_2}{1-h_1}\right)\right.\\ &\quad\left.+\alpha_2^2h_0h_2\left(\frac{1}{\epsilon_3}+\frac{\epsilon_3}{1-h_1}\right)+\rho(\lambda)\right]e^{\lambda t}\|u(t)\|^2,\end{aligned}\tag{2.14}$$

where

$$\begin{aligned}\rho(\lambda) &= \lambda\left[\beta^2(1+\epsilon_4\alpha_2h_0)+e^{\lambda h_0}\alpha_2h_0\beta^2\left(\frac{1}{\epsilon_4}+\alpha_2h_0\right)\right]\\ &\quad+\left(e^{\lambda h_0}-1\right)\left[\alpha_2h_0\left(\epsilon_1\alpha_1+\frac{\alpha_2h_2}{\epsilon_3}\right)+\frac{(\epsilon_2+\epsilon_3\alpha_2h_0)\alpha_2h_2}{1-h_1}\right].\end{aligned}\tag{2.15}$$

To minimize the right-hand side of inequality (2.14) we choose $\epsilon_1=1$, $\epsilon_2=\epsilon_3=\sqrt{1-h_1}$. Then, inequality (2.14) takes the form

$$\frac{d}{dt}V(t,u_t)\leqslant-\left[2\left(\gamma-\alpha_1\alpha_2h_0-(1+\alpha_2h_0)\frac{\alpha_2h_2}{\sqrt{1-h_1}}\right)-\rho(\lambda)\right]e^{\lambda t}\|u(t)\|^2.\tag{2.16}$$

From (2.15) it follows that $\rho(0)=0$. Thus, there exists $\lambda>0$ small enough such that from condition (2.7) we deduce that

$$2\left(\gamma-\alpha_1\alpha_2h_0-(1+\alpha_2h_0)\frac{\alpha_2h_2}{\sqrt{1-h_1}}\right)\geqslant\rho(\lambda).\tag{2.17}$$

This and (2.16) imply that $\frac{d}{dt}V(t,u_t) \leqslant 0$ and, as a consequence, the functional $V(t,u_t)$ constructed above satisfies conditions (1.4), (1.5). However, we cannot ensure that Theorem 1.1 holds true since the functional $V(t,u_t)$ does not satisfy the condition (1.3). Then, we will proceed in a different way.

From (2.16), (2.17) it follows that there exists $c > 0$ such that

$$V(t,u_t) - V(0,u_0) \leqslant -c\int_0^t e^{\lambda s}\left\|u(s)\right\|^2 ds.$$

Therefore,

$$\int_0^\infty e^{\lambda s}\left\|u(s)\right\|^2 ds \leqslant \frac{V(0,u_0)}{c}, \qquad V(t,u_t) \leqslant V(0,u_0). \tag{2.18}$$

Note also that

$$\begin{aligned}
\left|z(t,u_t)\right|^2 &= \left|u(t) + \int_{t-h(t)}^t F\big(u(s)\big)\,ds\right|^2 \\
&\geqslant \left|u(t)\right|^2 - 2\int_{t-h(t)}^t \left|u(t)\right|\left|F\big(u(s)\big)\right| ds \\
&\geqslant \left|u(t)\right|^2 - 2\alpha_2\beta\int_{t-h(t)}^t \left|u(t)\right|\left\|u(s)\right\| ds \\
&\geqslant \left|u(t)\right|^2 - \alpha_2\left(\left|u(t)\right|^2 h(t) + \beta^2\int_{t-h(t)}^t \left\|u(s)\right\|^2 ds\right) \\
&\geqslant (1-\alpha_2 h_0)\left|u(t)\right|^2 - \alpha_2\beta^2\int_{t-h_0}^t \left\|u(s)\right\|^2 ds.
\end{aligned} \tag{2.19}$$

From (2.7) it follows that

$$\gamma\|u\|^2 \leqslant -\big\langle A(t,u) + F(u), u\big\rangle \leqslant \left\|A(t,u) + F(u)\right\|_* \|u\| \leqslant \alpha_1\|u\|^2,$$

i.e. $\gamma \leqslant \alpha_1$. Using (2.9) we have $\alpha_2 h_0 < \gamma\alpha_1^{-1} \leqslant 1$. So, from (2.19) we obtain

$$\left|u(t)\right|^2 \leqslant \frac{|u(t) + \int_{t-h(t)}^t F(u(s))\,ds|^2 + \alpha_2\beta^2\int_{t-h_0}^t \|u(s)\|^2 ds}{1-\alpha_2 h_0}. \tag{2.20}$$

Since

$$V(0,u_0) \geqslant V(t,u_t) \geqslant V_1(t,u_t) = e^{\lambda t}\left|u(t) + \int_{t-h(t)}^t F\big(u(s)\big)\,ds\right|^2,$$

then

$$\left|u(t)+\int_{t-h(t)}^{t} F\big(u(s)\big)\,ds\right|^2 \leqslant e^{-\lambda t} V(0,u_0). \tag{2.21}$$

It is easy to see that there exists $C>0$ such that $V(0,u_0)\leqslant C\|u_0\|^2$. Now, from (2.19)–(2.21) it follows that

$$\big|u(t)\big|^2 \leqslant K\|u_0\|^2, \quad K=\frac{C+\alpha_2(h_0+\frac{C}{c})}{1-\alpha_2 h_0}.$$

Therefore, the trivial solution of Eq. (2.1) is stable.

Thanks to (2.18) we have that there exists $C_1>0$ such that

$$e^{\lambda(t-h_0)}\int_{t-h_0}^{t}\big\|u(s)\big\|^2\,ds \leqslant \int_{t-h_0}^{t} e^{\lambda s}\big\|u(s)\big\|^2\,ds \leqslant \int_{-h_0}^{\infty} e^{\lambda s}\big\|u(s)\big\|^2\,ds \leqslant C_1.$$

Hence,

$$\int_{t-h_0}^{t}\big\|u(s)\big\|^2\,ds \leqslant C_1 e^{\lambda h_0} e^{-\lambda t} \tag{2.22}$$

and from (2.20)–(2.22) it follows that, by conditions (2.7)–(2.9), the trivial solution of Eq. (2.1) is exponentially stable. □

Note that if, in particular, $h(t)=h_0$, then $h_1=h_2=0$ and condition (2.9) takes the form $\gamma>\alpha_1\alpha_2 h_0$.

3. Some applications

In this section we will show some interesting applications to illustrate how our results work.

3.1. Application to a 2D Navier–Stokes model

We first consider a 2D Navier–Stokes model with delays. Although this model has already been analysed in details in [6–8], there are some situations which still have not been considered in those works. We aim to provide some additional results on this model as well as to improve some sufficient conditions established in [7] by applying Theorem 2.1.

Let $\Omega\subset\mathbf{R}^2$ be an open and bounded set with regular boundary Γ, $T>0$ given, and consider the following functional Navier–Stokes problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t}-\nu\Delta u+\sum_{i=1}^{2}u_i\frac{\partial u}{\partial x_i}=-\nabla p+g(t,u_t) \quad \text{in } (0,T)\times\Omega,\\
&\operatorname{div} u=0 \quad \text{in } (0,T)\times\Omega,\\
&u=0 \quad \text{on } (0,T)\times\Gamma,\\
&u(0,x)=u_0(x), \quad x\in\Omega,\\
&u(t,x)=\psi(t,x), \quad t\in(-h,0),\ x\in\Omega,
\end{aligned} \tag{3.1}$$

where we assume that $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, u_0 the initial velocity field, g is an external force containing some hereditary characteristic and ψ the initial datum in the interval of time $(-h, 0)$, where h is a positive fixed number.

To begin with we consider the following usual abstract spaces (see [10,22] for more details):

$$\mathcal{U} = \{u \in (C_0^\infty(\Omega))^2\colon \operatorname{div} u = 0\},$$

$H =$ the closure of $\mathcal{U}$ in $(L^2(\Omega))^2$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$, where for $u, v \in (L^2(\Omega))^2$,

$$(u, v) = \sum_{j=1}^{2} \int_\Omega u_j(x) v_j(x)\, dx,$$

$U =$ the closure of $\mathcal{U}$ in $(H_0^1(\Omega))^2$ with the norm $\|\cdot\|$, and associated scalar product $((\cdot,\cdot))$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}\, dx.$$

It follows that $U \subset H \equiv H^* \subset U^*$, where the injections are dense and compact. Now we denote $a(u, v) = ((u, v))$, and define the trilinear form b on $U \times U \times U$ by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j\, dx, \quad \forall u, v, w \in U.$$

Assume that the delay term is given by

$$g(t, u_t) = Gu(t - h(t)),$$

where $G \in \mathcal{L}(U, U^*)$ is a self-adjoint linear operator, and the delay function $h(t)$ satisfies the assumptions in Theorem 2.1. Then problem (3.1) can be set in the abstract formulation (see [6–8,10,26] for a detailed description)

$$\begin{aligned}
&\text{To find} \quad u \in L^2(-h, T; U) \cap L^\infty(0, T; H) \quad \text{such that for all } v \in U,\\
&\frac{d}{dt}(u(t), v) + \nu a(u(t), v) + b(u(t), u(t), v) = (Gu(t - h(t)), v),\\
&u(0) = u_0, \qquad u(t) = \psi(t), \quad t \in (-h, 0),
\end{aligned} \tag{3.2}$$

where the equation in (3.2) must be understood in the sense of $\mathcal{D}'(0, T)$.

Observe that Eq. (3.2) can be rewritten as Eq. (2.1) by denoting $A(t,\cdot), F : U \to U^*$ the operators defined as

$$A(t, u) = -\nu a(u, \cdot) - b(u, u, \cdot), \quad F(u) = Gu, \quad u \in U.$$

By arguing as in case (3) from [6, p. 2448], it is not difficult to check that conditions in Theorem 2.1 hold provided $\nu > \|G\|_{\mathcal{L}(U,U^*)}$, and we can therefore ensure that there exists a unique solution to this problem (3.2) which, in addition, satisfies $u \in C^0(0, T; H)$ for any $T > 0$. As G is linear, then we have that 0 is a stationary solution to our model and we can analyse its stability. When G maps U or H into H (in other words, G is a first or zero-order linear partial differential operator), Theorems 3.3 and 3.5 in [7] guarantee the exponential stability of the trivial solution provided the viscosity parameter ν is large enough. For instance, in the case that G maps H into H, the null solution of Eq. (3.2) is exponentially stable if

$$2\nu\lambda_1 > \frac{(2-h_1)\|G\|_{\mathcal{L}(H,H)}}{1-h_1}, \tag{3.3}$$

where λ_1 is the first eigenvalue of the Stokes operator (see also Corollary 3.7 in [7] for another sufficient condition when G maps U into H). However, the results obtained in [7] do not cover the more general situation in which G may contain second-order partial derivatives. This is why we consider this situation.

Thus, in the present situation, i.e. for the operator $G \in \mathcal{L}(U, U^*)$ and the function $g(t, u_t) = Gu(t-h(t))$ defined above, we have that $\gamma = \nu$, $\alpha = \|G\|_{\mathcal{L}(U,U^*)}$, $\beta = \lambda_1^{-1/2}$ and assumptions in Theorem 2.1 hold assuming that

$$\nu > \frac{\|G\|_{\mathcal{L}(U,U^*)}}{\sqrt{1-h_1}}.$$

Remark 3.1. Observe that if $G \in \mathcal{L}(H, H)$, then $G \in \mathcal{L}(U, U^*)$ and, in addition, we have that

$$\|G\|_{\mathcal{L}(U,U^*)} \leqslant \lambda_1^{-1}\|G\|_{\mathcal{L}(H,H)},$$

so, if we assume that

$$\nu\lambda_1 > \frac{\|G\|_{\mathcal{L}(H,H)}}{\sqrt{1-h_1}} \tag{3.4}$$

it also follows that

$$\nu > \frac{\|G\|_{\mathcal{L}(U,U^*)}}{\sqrt{1-h_1}}$$

and, consequently, we have the exponential stability of the trivial solution. It is worth pointing out that condition (3.4) improves the condition established in [7], which is (3.3).

3.2. Application to some reaction–diffusion equations

In this subsection we will consider three different reaction–diffusion equations to show how we can obtain different stability regions for the parameters involved in the equation.

Let us then consider the following three problems:

$$\frac{\partial u(t,x)}{\partial t} = \nu\frac{\partial^2 u(t,x)}{\partial x^2} + \mu\frac{\partial^2 u(t-h(t),x)}{\partial x^2}, \tag{3.5}$$

$$\frac{\partial u(t,x)}{\partial t} = \nu\frac{\partial^2 u(t,x)}{\partial x^2} + \mu\frac{\partial u(t-h(t),x)}{\partial x}, \tag{3.6}$$

$$\frac{\partial u(t,x)}{\partial t} = \nu\frac{\partial^2 u(t,x)}{\partial x^2} + \mu u\big(t-h(t),x\big) \tag{3.7}$$

with conditions

$$t \geqslant 0, \quad x \in [a,b], \quad u(t,a) = u(t,b) = 0,$$
$$h(t) \in [0,h_0], \quad \dot{h}(t) \leqslant h_1 < 1, \quad \big|\dot{h}(t)\big| \leqslant h_2,$$

where $\nu > 0$ and μ is an arbitrary constant. Note that in all of these situations we can consider $U = H_0^1([a,b])$ and $H = L^2([a,b])$. The constant β for the injection $U \subset H$ equals $\beta = \lambda_1^{-1/2}$, where $\lambda_1 = \pi(b-a)^{-1}$ is the first eigenvalue of the operator $-\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions. We can therefore apply Theorem 2.1 to all these examples yielding the following sufficient stability conditions.

For Eq. (3.5)

$$\nu > \frac{|\mu|}{\sqrt{1-h_1}},$$

for Eq. (3.6)

$$\nu > \frac{|\mu|}{\sqrt{\lambda_1(1-h_1)}},$$

for Eq. (3.7)

$$\nu > \frac{|\mu|}{\lambda_1\sqrt{1-h_1}}. \tag{3.8}$$

Note that in the particular case $[a,b] = [0,\pi]$ it holds $\lambda_1 = 1$ and these three conditions given by Theorem 2.1 are the same.

Observe that Theorem 2.2 can be applied only to Eq. (3.7). For this equation the parameters of Theorem 2.2 are $\gamma = \alpha_1 = \nu - \mu\lambda_1^{-1}$, $\alpha_2 = |\mu|\lambda_1^{-1/2}$. It gives the following sufficient stability condition:

$$\nu > \frac{\mu}{\lambda_1} + \frac{|\mu|h_2}{\sqrt{\lambda_1(1-h_1)}}\left(\frac{\sqrt{\lambda_1}+|\mu|h_0}{\sqrt{\lambda_1}-|\mu|h_0}\right), \quad |\mu| < \frac{\sqrt{\lambda_1}}{h_0}. \tag{3.9}$$

Figure 1 shows the stability regions for Eq. (3.7) given by conditions (3.8) (the bound (1)) and (3.9) (the bound (2)) for the following values of the parameters $h_0 = 1$, $h_1 = h_2 = 0.1$,

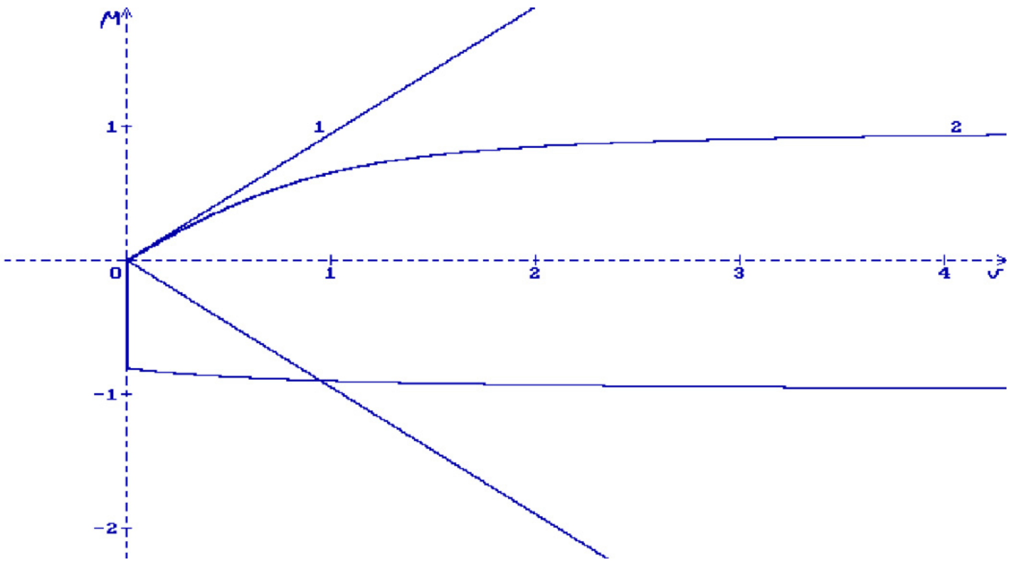

Fig. 1.

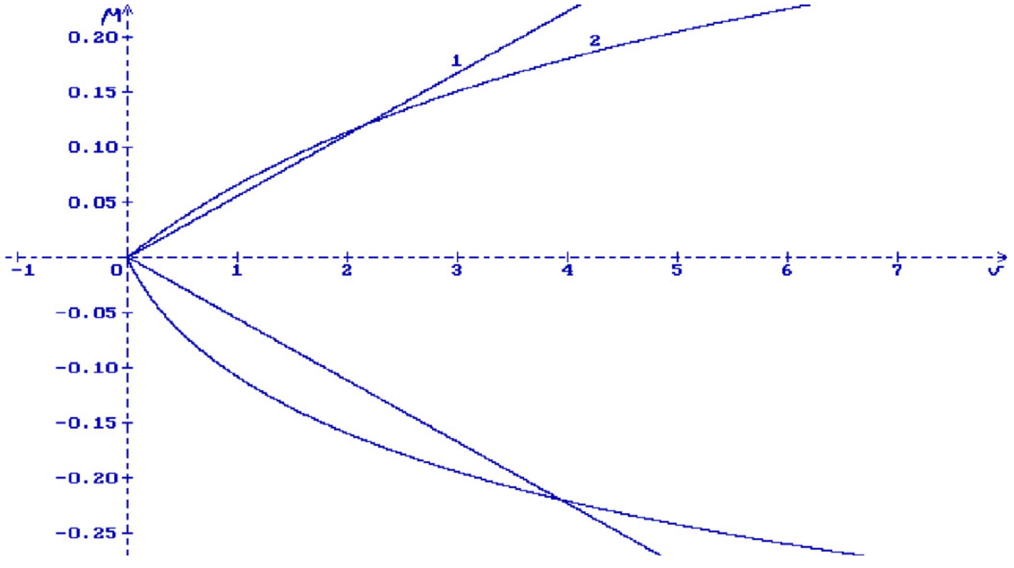

Fig. 2.

$\lambda_1 = 1$. One can see that for some negative μ condition (3.9) gives an additional part of stability region, i.e. for some negative μ condition (3.9) is better than (3.8). It is easy to show also that if $\sqrt{1-h_1} + h_2\sqrt{\lambda_1} < 1$ then condition (3.9) is better than (3.8) and for some positive μ. For example, Fig. 2 shows the stability regions for $h_0 = 1$, $h_1 = h_2 = 0.95$, $\lambda_1 = 0.25$. If $\mu = 0.06$ then the right-hand part of inequality (3.8) equals 1.073 but the right-hand part of inequality (3.9) equals 0.889, i.e. is less than 1.073.

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