



# Optimal Control of Volterra Type Stochastic Difference Equations

N. KUCHKINA AND L. SHAIKHET

Department of Mathematics, Informatics and Computing

Donetsk State Academy of Management

Chelyuskintsev, 163-a, Donetsk 340015, Ukraine

**Abstract**—Many processes in automatic regulation, physics, etc. can be modelled by stochastic difference equations. One of the main problems of the theory of difference equations and their applications is connected with stability and optimal control [1]. In this paper we discuss the optimal control of second-kind Volterra type stochastic difference equations. In [2–9] for Volterra type stochastic integral equations, analogous results were obtained. ©1998 Published by Elsevier Science Ltd. All rights reserved.

## 1. PROBLEM STATEMENT

Let  $\{\Omega, \sigma, P\}$  be a probability space,  $i \in Z = \{0, 1, \dots, N\}$  means discrete time,  $f_i \subset \sigma$ ,  $i \in Z$ , is a family of  $\sigma$ -algebras,  $M$  denotes the mathematical expectation,  $H$  is the space of  $f_i$ -adapted functions  $x(i) \in \mathbb{R}^n$ ,  $i \in Z$ , such that

$$\|x\|^2 = \max_{i \in Z} M|x(i)|^2 < \infty.$$

Consider the problem of the optimal control  $u(j)$  of the trajectory  $x(i)$  of the motion, given by the equation

$$x(i+1) = \eta(i+1) + \sum_{j=0}^i a(i,j)x(j) + \sum_{j=0}^i b(i,j)u(j), \quad (1.1)$$

and cost functional  $J(u)$  which must be minimized

$$J(u) = M \left[ x'(N)Fx(N) + \sum_{j=0}^{N-1} u'(j)G(j)u(j) \right]. \quad (1.2)$$

Here  $\eta(i)$  is an  $f_i$ -adapted function from  $H$ ,  $x(i) \in \mathbb{R}^n$ ,  $u(i) \in \mathbb{R}^m$ ,  $a(i,j)$  and  $b(i,j)$  are deterministic  $n \times n$  and  $n \times m$  matrices,  $F$  is a nonnegative definite matrix and  $G(j)$ ,  $j \in Z$ , are positive definite matrices.

Arbitrary  $f_i$ -adapted function  $u(i) \in \mathbb{R}^m$  with finite norm,  $\|u\| < \infty$ , is called admissible control,  $U$  is the set of the admissible controls.

The control  $u_0$ , which has minimal cost functional, i.e.,  $J(u_0) = \inf_{u \in U} J(u)$ , is called optimal control.

Let  $u_\varepsilon$ ,  $\varepsilon > 0$ , be an admissible control, for which there exists the limit

$$J_0(u_0) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [J(u_\varepsilon) - J(u_0)]. \quad (1.3)$$

Obviously, the inequality  $J_0(u_0) \geq 0$  is a necessary condition for the optimality of control  $u_0$ . The existence of the limit (1.3) and the necessary optimality condition depend on the form of the control  $u_\varepsilon$ . We will consider controls  $u_\varepsilon$  in the following form:

$$u_\varepsilon(i) = u_0(i) + \varepsilon v(i), \quad u_0, v \in U. \quad (1.4)$$

For controls of the form (1.4), the necessary optimality condition to be optimal using the limit relation (1.3) will be solved for the case (1.1),(1.2) and will be given in Section 2.

A matrix  $b(i, j)$  is called resolvent of the kernel  $a(i, j)$  if the solution of the equation

$$x(i+1) = \eta(i+1) + \sum_{j=0}^i a(i, j)x(j) \quad (1.5)$$

can be represented in the form

$$x(i+1) = \eta(i+1) + \sum_{j=0}^i b(i, j)\eta(j). \quad (1.6)$$

The kernel and the resolvent are connected by the relations [9]:

$$b(i, j) = a(i, j) + \sum_{k=j+1}^i b(i, k)a(k-1, j) = a(i, j) + \sum_{k=j+1}^i a(i, k)b(k-1, j).$$

## 2. MAIN RESULT

**THEOREM 2.1.** *For the optimal control problem given by (1.1),(1.2), the limit (1.3) exists assuming the form (1.4) and we have*

$$J_0(u_0) = 2M \left[ x'_0(N)Fq_0(N) + \sum_{j=0}^{N-1} u'_0(j)G(j)v(j) \right], \quad (2.1)$$

where  $x_0$  is the solution of equation (1.1) with the control  $u_0$ , and  $q_0(i)$  is the solution of the equation

$$q_0(i+1) = \sum_{j=0}^i a(i, j)q_0(j) + \sum_{j=0}^i b(i, j)v(j), \quad q_0(0) = 0. \quad (2.2)$$

**PROOF.** Let  $x_\varepsilon(i)$  be a solution of equation (1.1) with control  $u_\varepsilon$  and let

$$q_\varepsilon(i) = \frac{1}{\varepsilon} [x_\varepsilon(i) - x_0(i)].$$

From (1.2),(1.4), it follows that

$$\begin{aligned} \frac{1}{\varepsilon} [J(u_\varepsilon) - J(u_0)] &= \frac{1}{\varepsilon} M \left\{ x'_\varepsilon(N)Fx_\varepsilon(N) - x'_0(N)Fx_0(N) \right. \\ &\quad \left. + \sum_{j=0}^{N-1} [(u_0(j) + \varepsilon v(j))'G(j)(u_0(j) + \varepsilon v(j)) - u'_0(j)G(j)u_0(j)] \right\} \\ &= M \left\{ q'_\varepsilon(N)Fx_\varepsilon(N) + x'_0(N)Fq_\varepsilon(N) \right. \\ &\quad \left. + \sum_{j=0}^{N-1} [2v'(j)G(j)u_0(j) + \varepsilon v'(j)G(j)v(j)] \right\}. \end{aligned} \quad (2.3)$$

Further, from (1.1) it follows that  $q_\varepsilon(i)$  is the solution of the equation

$$q_\varepsilon(i+1) = \sum_{j=0}^i a(i,j)q_\varepsilon(j) + \sum_{j=0}^i b(i,j)v(j), \quad q_\varepsilon(0) = 0. \quad (2.4)$$

From (2.2),(2.4), it is easy to see that  $q_\varepsilon(i) = q_0(i)$  and  $\lim_{\varepsilon \rightarrow 0} x_\varepsilon(i) = x_0(i)$ . Therefore, from (2.3), (2.1) follows. The theorem is proved. ■

LEMMA 2.1. *The necessary condition for control  $u_0$  to be optimal,  $J_0(u_0) \geq 0$ , in the problem (1.1),(1.2) ensures the unique solution*

$$u_0(j) = -G^{-1}(j)\psi'(N-1, j, b(\cdot, j))FM_jx_0(N). \quad (2.5)$$

Here

$$\psi(i, j, f(\cdot)) = f(i) + \sum_{k=j+1}^i R(i, k)f(k-1),$$

where  $R(i, k)$  is the resolvent of the kernel  $a(i, k)$ ,  $f(k)$  is an arbitrary function on  $Z$ ,  $M_j = M\{\cdot/f_j\}$  denotes the conditional expectation.

PROOF. Using (1.5),(1.6), the solution of equation (2.2) can be represented in the form

$$\begin{aligned} q_0(i+1) &= \sum_{j=0}^i b(i,j)v(j) + \sum_{j=0}^i R(i,j) \sum_{k=0}^{j-1} b(j-1,k)v(k) \\ &= \sum_{j=0}^i \left[ b(i,j) + \sum_{k=j+1}^i R(i,k)b(k-1,j) \right] v(j) \\ &= \sum_{j=0}^i \psi(i, j, b(\cdot, j))v(j). \end{aligned} \quad (2.6)$$

Substituting (2.6) into (2.1) we obtain

$$J_0(u_0) = 2M \sum_{j=0}^{N-1} [M_jx'_0(N)F\psi(N-1, j, b(\cdot, j)) + u'_0(j)G(j)]v(j). \quad (2.7)$$

Expression (2.7) is nonnegative for all  $v \in U$  if and only if the expression in square brackets equals zero. Thus, the optimal control has the form (2.5). The lemma is proved. ■

THEOREM 2.2. *The optimal solution of the control problem (1.1),(1.2) can be represented in the form*

$$\begin{aligned} u_0(j+1) &= \alpha(j+1) + p(j+1)\psi(N-1, j, I)x_0(j+1) \\ &\quad + \sum_{k=0}^j \gamma(j, k)x_0(k), \quad j = 0, 1, \dots, N-2, \\ u_0(0) &= \alpha(0) + p(0) \left( 1 + \sum_{k=0}^{N-1} R(N-1, k) \right) x_0(0). \end{aligned} \quad (2.8)$$

Here  $I$  is the identity matrix, and

$$\begin{aligned} p(j) &= -G^{-1}(j)\psi'(N-1, j, b(\cdot, j))F \\ &\quad \times \left[ I + \sum_{k=j}^{N-1} \psi(N-1, k, b(\cdot, k))G^{-1}(k)\psi'(N-1, k, b(\cdot, k))F \right]^{-1}, \end{aligned} \quad (2.9)$$

$$\begin{aligned} \alpha(j+1) &= p(j+1)\psi(N-1, j\beta(\cdot, j+1)) \\ &+ \sum_{k=0}^j Q(j, k)p(k)\psi(N-1, k-1, \beta(\cdot, k)), \end{aligned} \quad (2.10)$$

$$\begin{aligned} \gamma(j, k) &= p(j+1)\psi(N-1, j, a_j(\cdot, k)) + Q(j, k)p(k)\psi(N-1, k-1, I) \\ &+ \sum_{l=k+1}^j Q(j, l)p(l)\psi(N-1, l-1, a_{l-1}(\cdot, k)), \end{aligned} \quad (2.11)$$

$Q(j, k)$  is the resolvent of the kernel  $p(j+1)\psi(N-1, j, b_j(\cdot, k))$ ,

$$\begin{aligned} a_j(i, k) &= a(i, k) - a(j, k), & b_j(i, k) &= b(i, k) - b(j, k), \\ \beta(i, j) &= M_j [\eta(i+1) - \eta(j)]. \end{aligned}$$

PROOF. Let us calculate the conditional expectation  $M_j x_0(N)$ . From (1.1) for  $i > j$  we get

$$\begin{aligned} x_0(i+1) - x_0(j+1) &= \eta(i+1) - \eta(j+1) + \sum_{k=0}^j a_j(i, k)x_0(k) \\ &+ \sum_{k=j+1}^i a(i, k)x_0(k) + \sum_{k=0}^j b_j(i, k)u_0(k) + \sum_{k=j+1}^i b(i, k)u_0(k). \end{aligned}$$

Putting

$$\begin{aligned} \xi(i, j+1) &= x_0(j+1) + \beta(i, j+1) + \sum_{k=0}^j a_j(i, k)x_0(k) \\ &+ \sum_{k=0}^j b_j(i, k)u_0(k) + \sum_{k=j+1}^i b(i, k)M_{j+1}u_0(k), \end{aligned} \quad (2.12)$$

from here we obtain

$$M_{j+1}x_0(i+1) = \xi(i, j+1) + \sum_{k=j+1}^i a(i, k)M_{j+1}x_0(k).$$

From (1.5),(1.6), it follows that  $M_{j+1}x_0(i+1)$  has the representation

$$M_{j+1}x_0(i+1) = \xi(i, j+1) + \sum_{k=j+1}^i R(i, k)\xi(k-1, j+1). \quad (2.13)$$

Substituting (2.12) into (2.13), we obtain

$$\begin{aligned} M_{j+1}x_0(i+1) &= x_0(j+1) + \beta(i, j+1) + \sum_{k=0}^j a_j(i, k)x_0(k) \\ &+ \sum_{k=0}^j b_j(i, k)u_0(k) + \sum_{k=j+1}^i b(i, k)M_{j+1}u_0(k) \\ &+ \sum_{m=j+1}^i R(i, m) \left[ x_0(j+1) + \beta(m-1, j+1) + \sum_{k=0}^j a_j(m-1, k)x_0(k) \right. \\ &\left. + \sum_{k=0}^j b_j(m-1, k)u_0(k) + \sum_{k=j+1}^{m-1} b(m-1, k)M_{j+1}u_0(k) \right] \end{aligned}$$

$$\begin{aligned}
 &= \left[ I + \sum_{m=j+1}^i R(i, m) \right] x_0(j+1) + \beta(i, j+1) + \sum_{m=j+1}^i R(i, m)\beta(m-1, j+1) \\
 &+ \sum_{k=0}^j \left[ a_j(i, k) + \sum_{m=j+1}^j R(i, m)a_j(m-1, k) \right] x_0(k) \\
 &+ \sum_{k=0}^j \left[ b_j(i, k) + \sum_{m=j+1}^i R(i, m)b_j(m-1, k) \right] u_0(k) \\
 &+ \sum_{k=j+1}^i \left[ b(i, k) + \sum_{m=k+1}^i R(i, m)b(m-1, k) \right] M_{j+1}u_0(k) \\
 &= \psi(i, j, I)x_0(j+1) + \psi(i, j, \beta(\cdot, j+1)) \\
 &+ \sum_{k=0}^j \psi(i, j, a_j(\cdot, k))x_0(k) + \sum_{k=0}^j \psi(i, j, b_j(\cdot, k))u_0(k) \\
 &+ \sum_{k=j+1}^i \psi(i, k, b(\cdot, k))M_{j+1}u_0(k).
 \end{aligned}$$

Let  $i = N - 1$  and

$$\begin{aligned}
 \xi_0(j+1) &= \psi(N-1, j, I)x_0(j+1) + \psi(N-1, j, \beta(\cdot, j+1)) \\
 &+ \sum_{k=0}^j \psi(N-1, j, a_j(\cdot, k))x_0(k).
 \end{aligned}$$

Then,

$$\begin{aligned}
 M_{j+1}x_0(N) &= \xi_0(j+1) + \sum_{k=0}^j \psi(N-1, j, b_j(\cdot, k))u_0(k) \\
 &+ \sum_{k=j+1}^{N-1} \psi(N-1, k, b(\cdot, k))M_{j+1}u_0(k).
 \end{aligned} \tag{2.14}$$

Note that from (2.5) for  $j < k$  we have

$$M_{j+1}u_0(k) = -G^{-1}(k)\psi'(N-1, k, b(\cdot, k))FM_{j+1}x_0(N). \tag{2.15}$$

Therefore, substituting (2.15) into (2.14) we obtain the following expression for  $M_{j+1}x_0(N)$ :

$$\begin{aligned}
 M_{j+1}x_0(N) &= \left[ I + \sum_{k=j+1}^{N-1} \psi(N-1, k, b(\cdot, k))G^{-1}(k)\psi'(N-1, k, b(\cdot, k))F \right]^{-1} \\
 &\times \left[ \xi_0(j+1) + \sum_{k=0}^j \psi(N-1, j, b_j(\cdot, k))u_0(k) \right].
 \end{aligned} \tag{2.16}$$

Substituting (2.16) into (2.5) and using (2.9), we obtain

$$u_0(j+1) = p(j+1) \left[ \xi_0(j+1) + \sum_{k=0}^j \psi(N-1, j, b_j(\cdot, k))u_0(k) \right]. \tag{2.17}$$

Thus, the optimal control  $u_0(j)$  satisfies equation (2.17). Hence, it admits the representation

$$u_0(j+1) = p(j+1)\xi_0(j+1) + \sum_{k=0}^j Q(j, k)p(k)\xi_0(k). \tag{2.18}$$

Substituting  $\xi_0(j)$  into (2.18), we obtain

$$\begin{aligned} u_0(j+1) &= p(j+1) \left[ \psi(N-1, j, I)x_0(j+1) + \psi(N-1, j, \beta(\cdot, j+1)) \right. \\ &\quad \left. + \sum_{k=0}^j \psi(N-1, j, a_j(\cdot, k))x_0(k) \right] + \sum_{k=0}^j Q(j, k)p(k) \left[ \psi(N-1, k-1, I)x_0(k) \right. \\ &\quad \left. + \psi(N-1, k-1, \beta(\cdot, k)) + \sum_{l=0}^{k-1} \psi(N-1, k-1, a_{k-1}(\cdot, l))x_0(l) \right] \\ &= p(j+1)\psi(N-1, j, \beta(\cdot, j+1)) + \sum_{k=0}^j Q(j, k)p(k)\psi(N-1, k-1, \beta(\cdot, k)) \\ &\quad + p(j+1)\psi(N-1, j, I)x_0(j+1) + p(j+1) \sum_{k=0}^j \psi(N-1, j, a_j(\cdot, k))x_0(k) \\ &\quad + \sum_{k=0}^j Q(j, k)p(k)\psi(N-1, k-1, I)x_0(k) \\ &\quad + \sum_{k=0}^j Q(j, k)p(k) \sum_{l=0}^{k-1} \psi(N-1, k-1, a_{k-1}(\cdot, l))x_0(l). \end{aligned}$$

Changing the order of the summation in the last expression,

$$\begin{aligned} &\sum_{k=0}^j Q(j, k)p(k) \sum_{l=0}^{k-1} \psi(N-1, k-1, a_{k-1}(\cdot, l))x_0(l) \\ &= \sum_{l=0}^{j-1} \left[ \sum_{k=l+1}^j Q(j, k)p(k)\psi(N-1, k-1, a_{k-1}(\cdot, l)) \right] x_0(l) \\ &= \sum_{k=0}^j \left[ \sum_{l=k+1}^j Q(j, l)p(l)\psi(N-1, l-1, a_{l-1}(\cdot, k)) \right] x_0(k), \end{aligned}$$

and using (2.10),(2.11), we obtain (2.8). The theorem is proved. ■

REMARK 2.1. If the process  $\eta(i)$  is a martingale [10], then  $\beta(i, j) = 0, i \geq j$  and therefore  $\alpha(i) = 0, i \in Z$ .

REMARK 2.2. Analogously we can obtain the optimal control for a system with noise at the control. For example, consider the scalar equation

$$x(i+1) = \eta(i+1) + \sum_{j=0}^i [\beta(i, j) + \gamma(i, j)u(j)] \xi(j),$$

with the cost functional

$$J(u) = M \left[ x^2(N) + \lambda \sum_{i=0}^{N-1} u^2(i) \right].$$

Here  $\lambda > 0, \xi(j)$  are  $f_{j+1}$ -adapted random variables,  $j \in Z$ , which are independent from each other and from  $\eta(i), M\xi(j) = 0, M\xi^2(j) = 1$ .

In this case,

$$\begin{aligned} J_0(u_0) &= 2M \left[ x_0(N)q_0(N) + \lambda \sum_{i=0}^{N-1} u_0(i)v(i) \right], \\ q_0(i+1) &= \sum_{j=0}^i \gamma(i, j)v(j)\xi(j), \quad q_0(0) = 0. \end{aligned}$$

Since

$$\begin{aligned}
 Mx_0(N)q_0(N) &= M \left\{ \left[ \eta(N) + \sum_{j=0}^{N-1} [\beta(N-1, j) + \gamma(N-1, j)u_0(j)] \xi(j) \right] \sum_{j=0}^{N-1} \gamma(N-1, j)\xi(j)v(j) \right\} \\
 &= \sum_{j=0}^{N-1} [\beta(N-1, j) + \gamma(N-1, j)u_0(j)] \gamma(N-1, j)v(j),
 \end{aligned}$$

then,

$$J_0(u_0) = 2 \sum_{j=0}^{N-1} [(\beta(N-1, j) + \gamma(N-1, j)u_0(j)) \gamma(N-1, j) + \lambda u_0(j)] v(j).$$

It follows from the necessary condition of the optimality that the optimal control  $u_0$  has the form

$$[\beta(N-1, j) + \gamma(N-1, j)u_0(j)] \gamma(N-1, j) + \lambda u_0(j) = 0.$$

From here

$$u_0(j) = -\beta(N-1, j)\gamma(N-1, j) [\lambda + \gamma^2(N-1, j)]^{-1}.$$

### 3. EXAMPLES

EXAMPLE 3.1. Let us consider the scalar equation

$$x(i+1) = \eta(i+1) + \sum_{j=0}^i a(i, j)x(j) + \sum_{j=0}^i b(i, j)u(j) \tag{3.1}$$

and cost functional

$$J(u) = M \left[ Fx^2(N) + \sum_{j=0}^{N-1} G(j)u^2(j) \right]. \tag{3.2}$$

This problem is a particular case of the original problem (1.1),(1.2). The optimal control of it is represented in the form (2.8)–(2.11). In this case, the function  $\psi(N-1, j, b(\cdot, j))$  in (2.9) can be represented in another form. For this, let us rewrite equation (2.2) for  $q_0(i)$  in matrix form

$$Q_0 = AQ_0 + BV. \tag{3.3}$$

Here  $A$  and  $B$  are  $(N+1) \times (N+1)$ -dimensional matrices and have the form

$$\begin{aligned}
 A &= \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ a(0,0) & 0 & 0 & \dots & 0 & 0 \\ a(1,0) & a(1,1) & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a(N-1,0) & a(N-1,1) & a(N-1,2) & \dots & a(N-1, N-1) & 0 \end{pmatrix}, \\
 B &= \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ b(0,0) & 0 & 0 & \dots & 0 & 0 \\ b(1,0) & b(1,1) & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b(N-1,0) & b(N-1,1) & b(N-1,2) & \dots & b(N-1, N-1) & 0 \end{pmatrix},
 \end{aligned}$$

the vectors are  $Q'_0 = (q_0(0), \dots, q_0(N))$ ,  $V' = (v(0), \dots, v(N))$ . So, as  $\det(I - A) = 1$ , the inverse matrix  $D = (I - A)^{-1}$  exists.

The solution of equation (3.3) is

$$Q_0 = DBV. \quad (3.4)$$

Comparing (2.6) and (3.4), we obtain

$$\psi(N-1, j, b(\cdot, j)) = r(N, j),$$

where  $r(N, j)$ ,  $j = 0, 1, \dots, N$ , are the elements of the last row of the matrix  $DB$ . Then, expression (2.9) can be written in the form

$$p(j) = -G^{-1}(j)r(N, j)F \left[ 1 + \sum_{k=j}^{N-1} r^2(N, k)G^{-1}(k)F \right]^{-1}. \quad (3.5)$$

REMARK 3.1. For calculation  $r(N, j)$ , we need only the last row of the matrix  $D$ .

EXAMPLE 3.2. Let us consider the scalar equation

$$x(i+1) = \eta(i+1) + \sum_{j=0}^i ax(j) + bu(i), \quad \eta(0) = x(0), \quad (3.6)$$

with the cost functional

$$J(u) = M \left[ x^2(N) + \lambda \sum_{i=0}^{N-1} u^2(i) \right].$$

Here  $\lambda > 0$ ,  $\eta(i+1) = x(0) + \sum_{j=0}^i \sigma(j)\xi(j)$ ,  $\sigma(j)$  are arbitrary constants,  $\xi(j)$  are  $f_{j+1}$ -adapted independent random variables,  $j \in Z$ ,  $M\xi(j) = 0$ ,  $M\xi^2(j) = 1$ .

In this case,  $\eta(i)$ ,  $i \in Z$ , is a martingale, therefore  $\alpha(i) = 0$ ,  $i = 0, \dots, N-1$ . Moreover, the last row of matrix  $DB$  is

$$(ab(a+1)^{N-2}, ab(a+1)^{N-3}, \dots, ab, b, 0).$$

Since  $a_j(i, k) = 0$  for  $k \leq j < i$ , and

$$b_j(i, k) = \begin{cases} 0, & k \neq j \text{ or } k = j = N-1, \\ -b, & k = j \neq N-1, \end{cases}$$

one can get  $\psi(N-1, j, a_j(\cdot, k)) = 0$  for  $j \geq k$  and

$$Q(j, k) = \begin{cases} 0, & k \neq j \text{ or } k = j = N-1, \\ -b(a+1)^{N-2-j}p(j+1), & k = j \neq N-1. \end{cases}$$

Then,

$$\gamma(j, k) = \begin{cases} 0, & k \neq j \text{ or } k = j = N-1, \\ Q(j, j)p(j)(a+1)^{N-j}, & k = j \neq N-1. \end{cases}$$

As a result, we have

$$u_0(0) = p(0)(a+1)^N x_0(0),$$

$$u_0(j+1) = p(j+1)(a+1)^{N-1-j} [x_0(j+1) - bp(j)(a+1)^{N-1-j} x_0(j)], \quad j = 0, 1, \dots, N-2.$$

Here

$$p(j) = \frac{ab(a+2)(a+1)^{N-2-j}}{\lambda(a+2) + b^2 [a(a+1)^{2(N-1-j)} + 2]}, \quad j = 0, 1, \dots, N-2,$$

$$p(N-1) = -\frac{b}{(\lambda + b^2)}, \quad p(N) = 0.$$



## REFERENCES

1. V.B. Kolmanovskii and L.E. Shaikhet, General method of Lyapunov functionals construction for stability investigations of stochastic difference equations, *Dynamic Systems and Applications*, WSSIAA 4, 397–439 (1995).
2. V.B. Kolmanovskii and L.E. Shaikhet, *Control of Systems with Aftereffect* (in Russian), Translations of Mathematical Monographs, Volume 157, American Mathematical Society, Providence, RI, (1996).
3. L.E. Shaikhet, On optimal control of Volterra's equations, *Problems of Control and Information Theory* 13 (3), 141–152 (1984).
4. L.E. Shaikhet, On optimal control of integral-functional equations, (in Russian), *Prikladnaya Matematika i Mekhanika* 49 (6), 923–934 (1985).
5. L.E. Shaikhet, On a necessary condition of optimality for stochastic systems with noise by control, *Problems of Control and Information Theory* 15 (1), 35–46 (1986).
6. L.E. Shaikhet, Optimal control of stochastic integral equations, *Lecture Notes in Control and Information Science*, IASAS, USA 81, 178–187 (1986).
7. L.E. Shaikhet, On an optimal control problem of partly observable stochastic Volterra process, *Problems of Control and Information Theory* 16 (6), 439–448 (1987).
8. V.B. Kolmanovskii and L.E. Shaikhet, Some problems of optimal control and estimation for stochastic hereditary systems, *Differential Equations and Control Theory*, 129–141 (1995).
9. N.V. Kuchkina and L.E. Shaikhet, Optimal control problem for nonlinear stochastic difference second kind Volterra equations, *Computers Math. Applic.* 34 (9), 65–73 (1997).
10. R.S. Liptser and R.N. Shiriyayev, *Statistics of Stochastic Processes*, (in Russian), Nauka, Moscow, (1972).