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Editors

Vladimir Korolyuk
Yurii Lin'kov
Nikolai Portenko

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LEONID E. SHAIKHET

STABILITY OF SYSTEMS OF STOCHASTIC LINEAR DIFFERENCE EQUATIONS WITH VARYING DELAYS

One method of Lyapunov functionals construction is used here for constructing of asymptotic mean square stability conditions for systems of stochastic linear difference equations with varying delays. Stability conditions are formulated in terms of existence of positive definite solutions of some matrix equations.

1. Problem statement

Lyapunov functionals are used for investigation of hereditary systems in problems of stability and optimal control (see e.g. Andreyeva, Kolmanovskii and Shaikhet (1992), Kolmanovskii and Myshkis (1992), Kolmanovskii and Nosov (1986), Kolmanovskii and Shaikhet (1996a), Sverdan and Tsarkov (1994)). One method of Lyapunov functionals construction has been proposed and developed for differential and difference equations (Beretta, Kolmanovskii and Shaikhet (1998), Ford, Edwards, Roberts and Shaikhet (1997), (Kolmanovskii and Rodionov (1995), Kolmanovskii and Shaikhet (1993a, 1993b, 1994, 1995a, 1995b, 1996b, 1997a, 1997b, 1998a, 1998b), Shaikhet and Shaikhet (1998), Shaikhet (1995a, 1995b, 1996a, 1996b, 1997a, 1997b, 1997c, 1998)). We use it here for constructing of asymptotic mean square stability conditions for systems of stochastic linear difference equations with varying delays. Stability conditions are formulated in terms of existence of positive definite solutions of some matrix equations.

Let $\{\Omega, \sigma, \mathsf{P}\}$ be a probability space, i be a discrete time, $i \in Z_0 \cup Z$, $Z = \{0, 1, ...\}$, $Z_0 = \{-h, ..., 0\}$, $h \geq 0$, $\{f_i \in \sigma\}$ be a sequence of σ-algebras, $\xi_0, \xi_1, ...$ be a sequence of mutually independent scalar random variables, ξ_i be f_{i+1}-adapted and independent on f_i, E be a mathematical expectation, $\mathsf{E}\xi_i = 0$, $\mathsf{E}\xi_i^2 = 1$.

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Consider the stochastic difference equation

$$x_{i+1} = F(i, x_{-h}, ..., x_i) + G(i, x_{-h}, ..., x_i)\xi_i, \qquad i \in Z, \tag{1}$$

$$x_i = \varphi_i, \qquad i \in Z_0.$$

Here $x_i \in \mathbf{R}^n$, the functions F and G are defined on $Z * S$, where S is a space of sequences with elements from $\mathbf{R}^n$. It is assumed that $F(i, ...)$ and $G(i, ...)$ does not depend on x_j for $j > i$, $F(i, 0, ..., 0) = 0, G(i, 0, ..., 0) = 0$.

Definition. The zero solution of the equation (1) is called mean square stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathsf{E}\,|x_i|^2 < \epsilon$, $i \in Z$, if $\|\varphi\|^2 = \sup_{i\in Z_0} \mathsf{E}\,|\varphi_i|^2 < \delta$. If, besides, $\lim_{i\to\infty} \mathsf{E}\,|x_i|^2 = 0$ for every initial function φ, then the equation (1) zero solution is called asymptotically mean square stable.

Theorem 1.1. *(Kolmanovskii and Shaikhet (1995a)) Let there exists the non-negative functional $V_i = V(i, x_{-h}, ..., x_i)$, $i \in Z$, which satisfies the conditions*

$$\mathsf{E}\,V(0, x_{-h}, ..., x_0) \le c_1\|\varphi\|^2,$$

$$\mathsf{E}\,\Delta V_i \le -c_2\,\mathsf{E}\,|x_i|^2, \qquad i \in Z,$$

where $\Delta V_i = V_{i+1} - V_i$, $c_1 > 0$, $c_2 > 0$. Then the equation (1) zero solution is asymptotically mean square stable.

From Theorem 1.1 it is follows that the construction of stability conditions of the equation (1) zero solution can be reduced to the construction of appropriate Lyapunov functionals. Following the method of Lyapunov functionals construction (Kolmanovskii and Shaikhet (1995a)), we will obtain the stability conditions for linear systems of stochastic difference equations with varying delays.

2. Formal procedure of Lyapunov functionals construction

The proposed procedure of Lyapunov functionals construction consists of four steps.

Step 1. Represent the functions F and G at the right-hand side of the equation (1) in the form

$$F(i, x_{-h}, ..., x_i) = F_{1i} + F_{2i} + \Delta F_{3i}, \qquad G(i, x_{-h}, ..., x_i) = G_{1i} + G_{2i},$$

$$F_{1i} = F_1(i, x_{i-\tau}, ..., x_i), \qquad F_{2i} = F_2(i, x_{-h}, ..., x_i),$$

$$F_{3i} = F_3(i, x_{-h}, ..., x_i), \qquad \Delta F_{3i} = F_{3,i+1} - F3i,$$

$$G_{1i} = G_1(i, x_{i-\tau}, ..., x_i), \qquad G_{2i} = G_2(i, x_{-h}, ..., x_i),$$

$$F_1(i, 0, ..., 0) = F_2(i, 0, ..., 0) = F_3(i, 0, ..., 0) =$$

$$= G_1(i, 0, ..., 0) = G_2(i, 0, ..., 0) = 0,$$

$\tau \ge 0$ is a given integer. This way the equation (1) take the form

$$x_{i+1} = F_{1i} + F_{2i} + \Delta F_{3i} + (G_{1i} + G_{2i})\xi_i. \tag{2}$$

Step 2. Let the function $v(i, y_{i-\tau}, ..., y_i)$ be a Lyapunov function for the auxiliary difference equation

$$y_{i+1} = F_1(i, y_{i-\tau}, ..., y_i) + G_1(i, y_{i-\tau}, ..., y_i)\xi_i, \quad i \in Z,$$

and satisfies the conditions of Theorem 1.1.

Step 3. Let us construct the Lyapunov functional V_i in the form $V_i = V_{1i} + V_{2i}$, where the main component V_{1i} has the form

$$V_{1i} = v(i, x_{i-\tau}, ..., x_i - F_3(i, x_{-h}, ..., x_i)).$$

Step 4. In order to satisfy the conditions of Theorem 1.1 the additional component V_{2i} must be chosen usually by a standard way.

3. System with nonincreasing delays

Using the procedure described above we will construct asymptotic mean square stability conditions for the stochastic linear difference equation

$$x_{i+1} = Ax_i + Bx_{i-k(i)} + Cx_{i-m(i)}\xi_i. \tag{3}$$

It is supposed that $h = \max(k(0), m(0))$ and delays $k(i)$ and $m(i)$ satisfy the inequalities

$$k(i) \geq k(i+1), \qquad m(i) \geq m(i+1), \qquad i \in Z. \tag{4}$$

Below two ways of Lyapunov functionals construction for the equation (3) are considered. The stability conditions are formulated in the form of matrix Riccati equations.

3.1. Using the representation (2) let us put (Step 1) $\tau = 0$,

$$F_{1i} = Ax_i, \quad F_{2i} = Bx_{i-k(i)}, \quad F_{3i} = G_{1i} = 0, \quad G_{2i} = Cx_{i-m(i)}.$$

In this case the auxiliary equation (Step 2) has the form

$$x_{i+1} = Ax_i. \tag{5}$$

Let for some positive definite matrix Q the matrix equation

$$A'DA - D = -Q \tag{6}$$

has a positive definite solution D. Then the function $v_i = x_i'Dx_i$ is a Lyapunov function for the equation (5). Really, calculating Δv_i and using (6) we get

$$\Delta v_i = x_{i+1}'Dx_{i+1} - x_i'Dx_i = x_i'A'DAx_i - x_i'Dx_i = -x_i'Qx_i.$$

We will construct (Step 3) the Lyapunov functional V_i for the equation (3) in the form $V_i = V_{1i} + V_{2i}$, where $V_{1i} = v_i = x_i'Dx_i$. Calculating $\mathsf{E}\Delta V_{1i}$ for the equation (3) we get

$$\mathsf{E}\,\Delta V_{1i} = \mathsf{E}[x_{i+1}'Dx_{i+1} - x_i'Dx_i] =$$

$$= \mathsf{E}[(Ax_i + Bx_{i-k(i)} + Cx_{i-m(i)}\xi_i)'D(Ax_i + Bx_{i-k(i)} + Cx_{i-m(i)}\xi_i) - x_i'Dx_i] =$$

$$= \mathsf{E}[x_i'(A'DA - D)x_i + x_i'A'DBx_{i-k(i)} + x_{i-k(i)}'B'DAx_i +$$

$$+ x_{i-k(i)}'B'DBx_{i-k(i)} + x_{i-m(i)}'C'DCx_{i-m(i)}].$$

Note that the inequality

$$a'b + b'a \leq a'Ra + b'R^{-1}b \tag{7}$$

hold for every positive definite matrix R.

Using (7) for $a = Ax_i$, $b = DBx_{i-k(i)}$, we get

(8) $x_i'A'DBx_{i-k(i)} + x_{i-k(i)}'B'DAx_i \le x_i'A'RAx_i + x_{i-k(i)}'B'DR^{-1}DBx_{i-k(i)}$.

Therefore

$$\mathsf{E}\,\Delta V_{1i} \le \mathsf{E}[x_i'(A'DA - D + A'RA)x_i + y_{i-k(i)} + z_{i-m(i)}],$$

where

$$y_j = x_j'B'(DR^{-1}D + D)Bx_j, \quad z_j = x_j'C'DCx_j.$$

Choose the functional V_{2i} (Step 4) in the form

$$V_{2i} = \sum_{j=i-k(i)}^{i-1} y_j + \sum_{j=i-m(i)}^{i-1} z_j.$$

Then using (4) we obtain

$$\Delta V_{2i} = \sum_{j=i+1-k(i+1)}^{i} y_j + \sum_{j=i+1-m(i+1)}^{i} z_j - V_{2i} =$$

$$= y_i + z_i + \sum_{j=i+1-k(i+1)}^{i-1} y_j + \sum_{j=i+1-m(i+1)}^{i-1} z_j -$$

$$- \sum_{j=i+1-k(i)}^{i-1} y_j - \sum_{j=i+1-m(i)}^{i-1} z_j - y_{i-k(i)} - z_{i-m(i)} \le$$

$$\le y_i + z_i - y_{i-k(i)} - z_{i-m(i)}.$$

As a result for the functional $V_i = V_{1i} + V_{2i}$ we have

$$\mathsf{E}\,\Delta V_i \le -\,\mathsf{E}\,x_i'Qx_i, \tag{9}$$

where

$$-Q = A'DA - D + A'RA + B'DR^{-1}DB + B'DB + C'DC. \tag{10}$$

Remark 3.1. Note that using (7) for $a = DAx_i$, $b = Bx_{i-k(i)}$, instead (8) we obtain the inequality

$$x_i'A'DBx_{i-k(i)} + x_{i-k(i)}'B'DAx_i \le x_i'A'DRDAx_i + x_{i-k(i)}'B'R^{-1}Bx_{i-k(i)}.$$

In this case we obtain (9) where

$$-Q = A'DA - D + A'DRDA + B'R^{-1}B + B'DB + C'DC. \tag{11}$$

Using (7) for other representations of a and b we can obtain other representations of Q.

This way we get

Theorem 3.1. *Let for some positive definite matrices Q and R there exists the positive definite solution of the matrix Riccati equation (10) (or (11)). Then the equation (3) zero solution is asymptotically mean square stable.*

Remark 3.2. Note that in scalar case the positive definite solution of the equation (10) (or (11)) there exists if and only if

(12) $$(|A| + |B|)^2 + C^2 < 1.$$

3.2. Consider other way of Lyapunov functional construction. Let us use the representation (2) (Step 1) by $\tau = 0$,

$$F_{1i} = (A + B)x_i, \quad G_{1i} = 0, \quad G_{2i} = Cx_{i-m(i)},$$

$$F_{2i} = -\sum_{j=i+1-k(i)}^{i-k(i+1)} Bx_j, \quad F_{3i} = -\sum_{j=i-k(i)}^{i-1} Bx_j.$$

In this case the auxiliary equation (Step 2) has the form

(13) $$x_{i+1} = (A + B)x_i.$$

Let for some positive definite matrix Q the matrix equation

$$(A + B)'D(A + B) - D = -Q$$

has a positive definite solution D. Then the function $v_i = x_i'Dx_i$ is a Lyapunov function for the equation (13).

We will construct Lyapunov functional V_i for the equation (3) in the form $V_i = V_{1i} + V_{2i}$, where (Step 3) $V_{1i} = (x_i - F_{3i})'D(x_i - F_{3i})$. Calculating $\mathsf{E}\,\Delta V_{1i}$ we get

$$\begin{aligned}\mathsf{E}\,\Delta V_{1i} &= \mathsf{E}[(x_{i+1} - F_{3,i+1})'D(x_{i+1} - F_{3,i+1}) - V_{1i} = \\ &= \mathsf{E}[(F_{1i} - x_i + F_{2i} + G_{2i}\xi_i)'D(F_{1i} + x_i + F_{2i} - 2F_{3i} + G_{2i}\xi_i) = \\ &= \mathsf{E}[x_i'((A + B)'D(A + B) - D)x_i + \sum_{j=1}^{5} I_j],\end{aligned}$$

where

$$I_1 = 2F_{1i}'DF_{2i}, \quad I_2 = F_{2i}'DF_{2i}, \quad I_3 = -2(F_{1i} - x_i)'DF_{3i},$$
$$I_4 = 2F_{2i}'DF_{3i}, \quad I_5 = G_{2i}'DG_{2i}.$$

Put

$$k_0 = \sup_{i \in Z}(k(i) - k(i + 1)).$$

Let R be positive definite matrix. Using (7) for $a = DBx_j$, $b = (A + B)x_i$, we get

$$\begin{aligned}I_1 &= -\sum_{j=i+1-k(i)}^{i-k(i+1)} (x_j'B'D(A + B)x_i + x_i'(A + B)'DBx_j) \leq \\ &\leq \sum_{j=i+1-k(i)}^{i-k(i+1)} (x_j'B'DRDBx_j + x_i'(A + B)'R^{-1}(A + B)x_i) \leq\end{aligned}$$

$$\leq k_0 x_i'(A+B)'R^{-1}(A+B)x_i + \sum_{j=i+1-k(i)}^{i-k(i+1)} x_j'B'DRDBx_j.$$

It is easy to see that

$$I_2 = \sum_{l,j=i+1-k(i)}^{i-k(i+1)} x_j'B'DBx_l = \Big|\sum_{j=i+1-k(i)}^{i-k(i+1)} D^{1/2}Bx_j\Big|^2 \leq$$

$$\leq k_0 \sum_{j=i+1-k(i)}^{i-k(i+1)} |D^{1/2}Bx_j|^2 = k_0 \sum_{j=i+1-k(i)}^{i-k(i+1)} x_j'B'DBx_j.$$

Let P be positive definite matrix. Using (7) for $a = DBx_j$, $b = (A+B-I)x_i$, we get

$$I_3 = -\sum_{j=i-k(i)}^{i-1} (x_j'B'D(A+B-I)x_i + x_i'(A+B-I)'DBx_j) \leq$$

$$\leq \sum_{j=i-k(i)}^{i-1} (x_j'B'DPDBx_j + x_i'(A+B-I)'P^{-1}(A+B-I)x_i) \leq$$

$$\leq k(0)x_i'(A+B-I)'P^{-1}(A+B-I)x_i + \sum_{j=i-k(i)}^{i-1} x_j'B'DPDBx_j.$$

Let S be positive definite matrix. Using (7) for $a = Bx_j$, $b = DBx_l$, we get

$$I_4 = -\sum_{l=i-k(i)}^{i-1} \sum_{j=i+1-k(i)}^{i-k(i+1)} (x_j'B'DBx_l + x_l'B'DBx_j) \leq$$

$$\leq \sum_{l=i-k(i)}^{i-1} \sum_{j=i+1-k(i)}^{i-k(i+1)} (x_j'B'SBx_j + x_l'B'DS^{-1}DBx_l) \leq$$

$$\leq k(0) \sum_{j=i+1-k(i)}^{i-k(i+1)} x_j'B'SBx_j + k_0 \sum_{l=i-k(i)}^{i-1} x_l'B'DS^{-1}DBx_l.$$

Let $u_i = x_i'C'DCx_i$. Then $I_5 = u_{i-m(i)}$. As a result we have

$$\mathsf{E}\Delta V_{1i} \leq \mathsf{E}\,[x_i'((A+B)'D(A+B) - D +$$

$$+ k_0(A+B)'R^{-1}(A+B) + k(0)(A+B-I)'P^{-1}(A+B-I))x_i +$$

$$+ \sum_{j=i+1-k(i)}^{i-k(i+1)} y_j + \sum_{j=i-k(i)}^{i-1} z_j + u_{i-m(i)}\Big],$$

where

$$y_j = x_j'(B'DRDB + k_0B'DB + k(0)B'SB)x_j,$$

$$z_j = x_j'(B'DPDB + k_0B'DS^{-1}DB)x_j.$$

Denote $k_m = \inf_{i\in Z} k(i)$ and choose the functional V_{2i} (Step 4) in the form

$$V_{2i} = \sum_{l=i}^{i+k_m-2} \sum_{j=l+1-k(0)}^{l-k_m} y_j + \sum_{j=i+k_m-k(0)}^{i-1} (j-i-k_m+k(0)+1)y_j +$$
$$+ \sum_{j=i-k(0)}^{i-1} (j-i+k(0)+1)z_j + \sum_{j=i-m(i)}^{i-1} u_j.$$

Calculating ΔV_{2i}, we have

$$\Delta V_{2i} = \sum_{l=i+1}^{i+k_m-1} \sum_{j=l+1-k(0)}^{l-k_m} y_j + \sum_{j=i+1+k_m-k(0)}^{i} (j-i-k_m+k(0))y_j +$$
$$+ \sum_{j=i+1-k(0)}^{i} (j-i+k(0))z_j + \sum_{j=i+1-m(i+1)}^{i} u_j - V_{2i} =$$
$$= (k(0)-k_m)y_i + k(0)z_i + v_i - v_{i-m(i)} +$$
$$- \sum_{j=i+1-k(0)}^{i-k_m} y_j - \sum_{j=i-k(0)}^{i-1} z_j + \sum_{j=i+1-m(i+1)}^{i-1} u_j - \sum_{j=i+1-m(i)}^{i-1} u_j.$$

Using that $k(0) \geq k(i) \geq k_m$ and $m(i) \geq m(i+1)$ for functional $V_i = V_{1i} + V_{2i}$ we have the inequality (9), where

$$-Q = (A+B)'D(A+B) - D + k_0(A+B)'R^{-1}(A+B)+$$
$$+k(0)(A+B-I)'P^{-1}(A+B-I) + C'DC+ \tag{14}$$
$$+(k(0)-k_m)B'(DRD + k_0D + k(0)S)B + k(0)B'D(P + k_0S^{-1})DB.$$

Theorem 3.2. *Let for some positive definite matrices Q, P, R and S there exists the positive definite solution of the matrix Riccati equation (14). Then the equation (3) zero solution is asymptotically mean square stable.*

Remark 3.3. Analogously with Remark 3.1 we can show that instead of the equation (14) can be used other matrix Riccati equations, for example

$$-Q = (A+B)'D(A+B) - D + k_0(A+B)'DR^{-1}D(A+B)+$$
$$+k(0)(A+B-I)'DP^{-1}D(A+B-I) + C'DC+ \tag{15}$$
$$+(k(0)-k_m)B'(R + k_0D + k(0)S)B + k(0)B'(P + k_0DS^{-1}D)B.$$

Remark 3.4. Let $k(i) = k = const.$ In this case the equation (14) has the form

$$-Q = (A+B)'D(A+B) - D + C'DC+$$
$$+k(A+B-I)'P^{-1}(A+B-I) + kB'DPDB,$$

the equation (15) has the form

$$-Q = (A+B)'D(A+B) - D + C'DC+$$

$$+k(A+B-I)'DP^{-1}D(A+B-I)+kB'PB.$$

Remark 3.5. It is easy to get that in scalar case a positive definite solution of the equation (14) (or (15)) there exists if and only if

$$(|A+B|+\gamma|B|)^2+2k(0)|B|(1-A-B+\gamma|B|)+C^2<1,$$
$$|A+B|<1, \quad \gamma=\sqrt{k_0(k(0)-k_m)}.$$

If $k(i)=k=const$ then $\gamma=0$ and this condition can be rewrite in the form

$$C^2<(1-A-B)(1+A+B-2k|B|), \quad |A+B|<1. \tag{16}$$

4. System with unbounded delays

Construct now asymptotic mean square stability conditions for the stochastic linear difference equation

$$x_{i+1}=\sum_{j=0}^{k(i)}\alpha_j A_j x_{i-j}+\sum_{j=0}^{m(i)}\beta_j B_j x_{i-j}\xi_i. \tag{17}$$

Here A_j and B_j are $n*n$-matrices, α_j and β_j are scalars. It is supposed that $h=max(k(0),m(0))$ and delays $k(i)$ and $m(i)$ satisfy the inequalities

$$k(i+1)-k(i)\leq 1, \quad m(i+1)-m(i)\leq 1 \tag{18}$$

and

$$\hat{k}=\sup_{i\in Z}k(i)\leq\infty, \quad \hat{m}=\sup_{i\in Z}m(i)\leq\infty. \tag{19}$$

4.1. Let us represent (Step 1) the equation (17) in the form (2) by $\tau=0$,

$$F_{1i}=\alpha_0A_0x_i, \qquad F_{2i}=\sum_{j=1}^{k(i)}\alpha_jA_jx_{i-j}, \qquad F_{3i}=0,$$

$$G_{1i}=\beta_0B_0x_i, \qquad G_{2i}=\sum_{j=1}^{m(i)}\beta_jB_jx_{i-j}.$$

In this case the auxiliary equation (Step 2) has the form

$$x_{i+1}=\alpha_0A_0x_i+\beta_0B_0x_i\xi_i.$$

Let for some positive definite matrix Q the equation

$$\alpha_0^2A_0'DA_0+\beta_0^2B_0'DB_0-D=-Q$$

has a positive solution D. Then the function $v_i=x_i'Dx_i$ is a Lyapunov function for the auxiliary equation. Really, calculating $\mathsf{E}\,\Delta v_i$ we get

$$\mathsf{E}\,\Delta v_i=\mathsf{E}(x_{i+1}'Dx_{i+1}-x_i'Dx_i)=$$
$$=\mathsf{E}[(\alpha_0A_0x_i+\beta_0B_0x_i\xi_i)'D(\alpha_0A_0x_i+\beta_0B_0x_i\xi_i)-x_i'Dx_i]=$$
$$=\mathsf{E}[\alpha_0^2x_i'A_0'DA_0x_i+\beta_0^2x_i'B_0'DB_0x_i-x_i'Dx_i]=-\mathsf{E}\,x_i'Qx_i.$$

We will construct a Lyapunov functional for the equation (17) in the form $V_i = V_{1i} + V_{2i}$, where (Step 3) $V_i = x_i' D x_i$. Let

$$\alpha = \sum_{l=0}^{\hat{k}} |\alpha_l|, \quad \beta = \sum_{l=0}^{\hat{m}} |\beta_l|.$$

Calculating $\mathsf{E}\,\Delta V_{1i}$ for the equation (17) we get

$$\mathsf{E}\,\Delta V_{1i} = \mathsf{E}[(\sum_{j=0}^{k(i)} \alpha_j A_j x_{i-j} + \sum_{j=0}^{m(i)} \beta_j B_j x_{i-j}\xi_i)' D (\sum_{j=0}^{k(i)} \alpha_j A_j x_{i-j} +$$

$$+ \sum_{j=0}^{m(i)} \beta_j B_j x_{i-j}\xi_i) - x_i' D x_i] =$$

$$= \mathsf{E}\left[\Big| \sum_{j=0}^{k(i)} \alpha_j D^{\frac{1}{2}} A_j x_{i-j} \Big|^2 + \Big| \sum_{j=0}^{m(i)} \beta_j D^{\frac{1}{2}} B_j x_{i-j} \Big|^2 - x_i' D x_i \right] \le$$

$$\le \mathsf{E}\left[\sum_{l=0}^{k(i)} |\alpha_l| \sum_{j=0}^{k(i)} |\alpha_j| |D^{\frac{1}{2}} A_j x_{i-j}|^2 + \sum_{l=0}^{m(i)} |\beta_l| \sum_{j=0}^{m(i)} |\beta_j| |D^{\frac{1}{2}} B_j x_{i-j}|^2 - x_i' D x_i \right] \le$$

$$\le \mathsf{E}[x_i'(K_0 + L_0 - D) x_i + J_{1i} + J_{2i}],$$

where

$$J_{1i} = \sum_{j=1}^{k(i)} x_{i-j}' K_j x_{i-j}, \qquad J_{2i} = \sum_{j=1}^{m(i)} x_{i-j}' L_j x_{i-j}, \tag{20}$$

$$K_j = \alpha |\alpha_j| A_j' D A_j, \quad L_j = \beta |\beta_j| B_j' D B_j.$$

Let us choose the functional V_{2i} (Step 4) in the form $V_{2i} = V_{2i}^{(1)} + V_{2i}^{(2)}$, where

$$V_{2i}^{(1)} = \sum_{j=1}^{k(i)} x_{i-j}' Z_j^{(1)} x_{i-j}, \qquad Z_j^{(1)} = \sum_{l=j}^{\hat{k}} K_l, \tag{21}$$

$$V_{2i}^{(2)} = \sum_{j=1}^{m(i)} x_{i-j}' Z_j^{(2)} x_{i-j}, \qquad Z_j^{(2)} = \sum_{l=j}^{\hat{m}} L_l. \tag{22}$$

Then using (18) we get

$$\Delta V_{2i}^{(1)} = \sum_{j=1}^{k(i+1)} x_{i+1-j}' Z_j^{(1)} x_{i+1-j} - \sum_{j=1}^{k(i)} x_{i-j}' Z_j^{(1)} x_{i-j} =$$

$$= \sum_{j=0}^{k(i+1)-1} x_{i-j}' Z_{j+1}^{(1)} x_{i-j} - \sum_{j=1}^{k(i)} x_{i-j}' Z_{j+1}^{(1)} x_{i-j} - J_{1i} =$$

$$= x_i' Z_1^{(1)} x_i - J_{1i} + \sum_{j=1}^{k(i+1)-1} x_{i-j}' Z_{j+1}^{(1)} x_{i-j} - \sum_{j=1}^{k(i)} x_{i-j}' Z_{j+1}^{(1)} x_{i-j} \le$$

$$\le x_i' Z_1^{(1)} x_i - J_{1i}. \tag{23}$$

Analogously

$$\Delta V_{2i}^{(2)} \le x_i' Z_1^{(2)} x_i - J_{2i}. \tag{24}$$

As a result for the functional V_i we have the inequality (9), where

$$-Q = \alpha \sum_{j=1}^{\hat{k}} |\alpha_j| A_j' D A_j + \beta \sum_{j=1}^{\hat{m}} |\beta_j| B_j' D B_j - D. \tag{25}$$

Theorem 4.1. *Let for some positive definite matrix Q there exists the positive definite solution of the matrix equation (25). Then the equation (17) zero solution is asymptotically mean square stable.*

Remark 4.1. Let $A_j = A = const$, $B_j = B = const$. Then the equation (25) take the form

$$-Q = \alpha^2 A' D A + \beta^2 B' D B - D.$$

If $A = B = I$ then the asymptotic mean square stability condition has the form

$$\alpha^2 + \beta^2 < 1. \tag{26}$$

It is easy to see that the condition (12) is the partial case of the condition (26).

Remark 4.2. From the inequalities (18) it follows that

$$k(i) \le k(0) + i, \quad m(i) \le m(0) + i.$$

4.2. Consider now another way of Lyapunov functionals construction. It is supposed that the stochastic difference equation has the form

$$x_{i+1} = \sum_{j=0}^{k(i)} A_j x_{i-j} + \sum_{j=0}^{m(i)} B_j x_{i-j} \xi_i, \tag{27}$$

and delays satisfy the conditions (18), (19).

Let us represent the equation (27) (Step 1) in the form (2), where $\tau = 0$,

$$F_{1i} = W_0 x_i, \quad F_{2i} = -\sum_{j=k(i+1)}^{k(i)} W_{j+1} x_{i-j}, \quad F_{3i} = -\sum_{j=1}^{k(i)} W_j x_{i-j},$$

$$W_l = \sum_{j=l}^{\hat{k}} A_j, \quad G_{1i} = 0, G_{2i} = \sum_{j=0}^{m(i)} B_j x_{i-j}.$$

Really, it is easy to see that

$$W_0x_i - \sum_{j=k(i+1)}^{k(i)} W_{j+1}x_{i-j} - \sum_{j=1}^{k(i+1)} W_jx_{i+1-j} + \sum_{j=1}^{k(i)} W_jx_{i-j} =$$

$$= \sum_{j=0}^{k(i)} W_jx_{i-j} - \sum_{j=k(i+1)}^{k(i)} W_{j+1}x_{i-j} - \sum_{j=0}^{k(i+1)-1} W_{j+1}x_{i-j} =$$

$$\sum_{j=0}^{k(i)}(W_j - W_{j+1})x_{i-j} = \sum_{j=0}^{k(i)} A_jx_{i-j}.$$

In this case the auxiliary equation (Step 2) has the form $x_{i+1} = W_0x_i$. Let for some positive definite matrix Q the equation $W_0'DW_0 - D = -Q$ has a positive definite solution D. Then the function $v_i = x_i'Dx_i$ is a Lyapunov function for the auxiliary equation.

We will construct the Lyapunov functional V_i in the form $V_i = V_{1i} + V_{2i}$, where (Step 3) $V_{1i} = (x_i - F_{3i})'D(x_i - F_{3i})$. Calculating $\mathsf{E}\,\Delta V_{1i}$ and using the representations for x_{i+1} and F_{1i} we get

$$\mathsf{E}\,\Delta V_{1i} = \mathsf{E}[(x_{i+1} - F_{3,i+1})'D(x_{i+1} - F_{3,i+1}) - V_{1i}] =$$

$$= \mathsf{E}(x_{i+1} - F_{3,i+1} - x_i + F_{3i})'D(x_{i+1} - F_{3,i+1} + x_i - F_{3i}) =$$

$$= \mathsf{E}((W_0 - I)x_i + F_{2i} + G_{2i}\xi_i)'D((W_0 + I)x_i + F_{2i} - 2F_{3i} + G_{2i}\xi_i) =$$

$$= \mathsf{E}[x_i'(W_0 - I)'D(W_0 + I)x_i + \sum_{j=1}^{4} I_j],$$

where

$$I_1 = 2F_{2i}'D(W_0x_i - F_{3i}), \quad I_2 = F_{2i}'DF_{2i},$$
$$I_3 = 2x_i'(I - W_0)'DF_{3i}, \quad I_4 = G_{2i}'DG_{2i}.$$

Using (7) for some positive definite matrices P_{jl} we get

$$I_1 = \sum_{j=k(i+1)}^{k(i)} \sum_{l=0}^{k(i)} (x_{i-j}'W_{j+1}'DW_lx_{i-l} + x_{i-l}'W_l'DW_{j+1}x_{i-j}) \leq$$

$$\leq \sum_{j=k(i+1)}^{k(i)} \sum_{l=0}^{k(i)} (x_{i-j}'W_{j+1}'P_{jl}W_{j+1}x_{i-j} + x_{i-l}'W_l'DP_{jl}^{-1}DW_lx_{i-l}) =$$

$$= \sum_{j=k(i+1)}^{k(i)} x_{i-j}'W_{j+1}' \sum_{l=0}^{k(i)} P_{jl}W_{j+1}x_{i-j} + \sum_{j=0}^{k(i)} x_{i-j}'W_j'D \sum_{l=k(i+1)}^{k(i)} P_{lj}^{-1}DW_jx_{i-j}.$$

Using (7) for some positive definite matrices R_{jl} we get

$$I_2 = \frac{1}{2} \sum_{j=k(i+1)}^{k(i)} \sum_{l=k(i+1)}^{k(i)} (x_{i-j}'W_{j+1}'DW_{l+1}x_{i-l} + x_{i-l}'W_{l+1}'DW_{j+1}x_{i-j}) \leq$$

$$\leq \frac{1}{2} \sum_{j=k(i+1)}^{k(i)} \sum_{l=k(i+1)}^{k(i)} (x'_{i-j} W'_{j+1} R_{jl} W_{j+1} x_{i-j} + x'_{i-l} W'_{l+1} D R_{jl}^{-1} D W_{l+1} x_{i-l}) =$$

$$= \frac{1}{2} \sum_{j=k(i+1)}^{k(i)} \sum_{l=k(i+1)}^{k(i)} (x'_{i-j} W'_{j+1} R_{jl} W_{j+1} x_{i-j} + x'_{i-j} W'_{j+1} D R_{lj}^{-1} D W_{j+1} x_{i-j}) =$$

$$= \frac{1}{2} \sum_{j=k(i+1)}^{k(i)} x'_{i-j} W'_{j+1} \sum_{l=k(i+1)}^{k(i)} (R_{jl} + D R_{lj}^{-1} D) W_{j+1} x_{i-j}.$$

Using (7) for some positive definite matrices S_j we get

$$I_3 = \sum_{j=1}^{k(i)} (x'_i (I - W_0)' D W_j x_{i-j} + x'_{i-j} W'_j D (I - W_0) x_i) \leq$$

$$\leq \sum_{j=1}^{k(i)} (x'_{i-j} W'_j S_j W_j x_{i-j} + x'_i (I - W_0)' D S_j^{-1} D (I - W_0) x_i).$$

Using (7) for some positive definite matrices T_{jl} we get

$$I_4 = \frac{1}{2} \sum_{j=0}^{m(i)} \sum_{l=0}^{m(i)} (x'_{i-j} B'_j D B_l x_{i-l} + x'_{i-l} B'_l D B_j x_{i-j}) \leq$$

$$\leq \frac{1}{2} \sum_{j=0}^{m(i)} \sum_{l=0}^{m(i)} (x'_{i-l} B'_l T_{jl} B_l x_{i-l} + x'_{i-j} B'_j D T_{jl}^{-1} D B_j x_{i-j}) =$$

$$= \frac{1}{2} \sum_{j=0}^{m(i)} x'_{i-j} B'_j \sum_{l=0}^{m(i)} (T_{jl} + D T_{lj}^{-1} D) B_j x_{i-j}.$$

As a result we have

$$\mathsf{E}\, \Delta V_{1i} \leq \mathsf{E}[x'_i ((W_0 - I)' D (W_0 + I) + W'_0 D \sum_{l=k_m}^{\hat{k}} P_{l0}^{-1} D W_0 +$$

$$(28) \qquad + (I - W_0)' D \sum_{l=1}^{\hat{k}} S_l^{-1} D (I - W_0) + \frac{1}{2} \sum_{l=0}^{\hat{m}} B'_0 (T_{0l} + D T_{l0}^{-1} D) B_0) x_i +$$

$$+ J_{1i} + J_{2i} + J_{3i}],$$

where J_{1i}, J_{2i} are defined by (20) with

$$(29) \qquad K_j = W'_j (S_j + D \sum_{l=k_m}^{\hat{k}} P_{lj}^{-1} D) W_j, \qquad L_j = \frac{1}{2} \sum_{l=0}^{\hat{m}} B'_j (T_{jl} + D T_{lj}^{-1} D) B_j,$$

and

$$J_{3i} = \sum_{j=k_m}^{k(i)} x'_{i-j} M_j x_{i-j},$$

where

$$M_j = W'_{j+1}\Big(\sum_{l=1}^{\hat{k}} P_{jl} + \frac{1}{2}\sum_{l=k_m}^{\hat{k}} (R_{jl} + DR_{lj}^{-1}D)\Big)W_{j+1}.$$

Choose the functional V_{2i} (Step 4) in the form $V_{2i} = V_{2i}^{(1)} + V_{2i}^{(2)} + V_{2i}^{(3)}$. Here the functionals $V_{2i}^{(1)}$, $V_{2i}^{(2)}$ are defined by (21), (22), (29). The functional $V_{2i}^{(3)}$ has the form

$$V_{2i}^{(3)} = \sum_{j=k_m}^{k(i)} x'_{i-j} Z_j^{(3)} x_{i-j} + \sum_{j=1}^{k_m-1} x'_{i-j} Z_{k_m}^{(3)} x_{i-j}, \qquad Z_j^{(3)} = \sum_{l=j}^{\hat{k}} M_l$$

if $k_m > 1$ and

$$V_{2i}^{(3)} = \sum_{j=1}^{k(i)} x'_{i-j} Z_j^{(3)} x_{i-j} \qquad Z_j^{(3)} = \sum_{l=j}^{\hat{k}} M_l$$

if $0 \le k_m \le 1$.

The functionals $V_{2i}^{(1)}$, $V_{2i}^{(2)}$ are satisfying the inequalities (23), (24). Let us obtain a similar condition for $V_{2i}^{(3)}$.

Firstly suppose that $k_m > 1$. Then using (18) we get

$$\Delta V_{2i}^{(3)} = \sum_{j=k_m}^{k(i+1)} x'_{i+1-j} Z_j^{(3)} x_{i+1-j} + \sum_{j=1}^{k_m-1} x'_{i+1-j} Z_{k_m}^{(3)} x_{i+1-j} -$$

$$- \sum_{j=k_m}^{k(i)} x'_{i-j} Z_j^{(3)} x_{i-j} - \sum_{j=1}^{k_m-1} x'_{i-j} Z_{k_m}^{(3)} x_{i-j} =$$

$$= x'_i Z_{k_m}^{(3)} x_i + \sum_{j=k_m}^{k(i+1)-1} x'_{i-j} Z_{j+1}^{(3)} x_{i-j} - \sum_{j=k_m}^{k(i)} x'_{i-j} Z_j^{(3)} x_{i-j} +$$

$$+ x'_{i+1-k_m} Z_{k_m}^{(3)} x_{i+1-k_m} + \sum_{j=1}^{k_m-2} x'_{i-j} Z_{k_m}^{(3)} x_{i-j} - \sum_{j=1}^{k_m-1} x'_{i-j} Z_{k_m}^{(3)} x_{i-j} =$$

$$= x'_i Z_{k_m}^{(3)} x_i + \sum_{j=k_m}^{k(i+1)-1} x'_{i-j} Z_{j+1}^{(3)} x_{i-j} - \sum_{j=k_m}^{k(i)} x'_{i-j} Z_{j+1}^{(3)} x_{i-j} - J_{3i} \le$$

$$\le x'_i Z_{k_m}^{(3)} x_i - J_{3i}.$$

Let now $0 \le k_m \le 1$. Then analogously to (23) we have

$$\Delta V_{2i}^{(3)} \le x'_i Z_1^{(3)} x_i - \sum_{j=1}^{k(i)} x'_{i-j} M_j x_{i-j} = x'_i Z_{k_m}^{(3)} x_i - J_{3i}.$$

This way for $k_m \geq 0$ we have

$$\Delta V_{2i}^{(3)} \leq x_i' Z_{k_m}^{(3)} x_i - J_{3i}. \tag{30}$$

As a result from (28), (23), (24), (30) for the functional $V_i = V_{1i} + V_{2i}$ we obtain

$$\mathsf{E}\,\Delta V_{1i} \leq \mathsf{E}\, x_i'((W_0 - I)'D(W_0 + I) + W_0'D \sum_{l=k_m}^{\hat{k}} P_{l0}^{-1} DW_0 +$$

$$+(I - W_0)'D \sum_{j=1}^{\hat{k}} S_j^{-1} D(I - W_0) + \frac{1}{2} \sum_{l=0}^{\hat{m}} B_0'(T_{0l} + DT_{l0}^{-1}D)B_0 + Z_1^{(1)} + Z_1^{(2)} + Z_{k_m}^{(3)})x_i$$

or (9) with

$$-Q = (W_0 - I)'D(W_0 + I) + (I - W_0)'D \sum_{j=1}^{\hat{k}} S_j^{-1} D(I - W_0) +$$

$$+ \sum_{j=1}^{k} W_j' S_j W_j + \sum_{j=0}^{\hat{k}} W_j' D \sum_{l=k_m}^{\hat{k}} P_{lj}^{-1} DW_j + \frac{1}{2} \sum_{j=0}^{\hat{m}} \sum_{l=0}^{\hat{m}} B_j'(T_{jl} + DT_{lj}^{-1} D)B_j +$$

$$+ \sum_{j=k_m}^{\hat{k}} W_{j+1}' (\sum_{l=0}^{\hat{k}} P_{jl} + \frac{1}{2} \sum_{l=k_m}^{\hat{k}} (R_{jl} + DR_{lj}^{-1} D))W_{j+1}. \tag{31}$$

From here and Theorem 1.1 it follows

Theorem 4.2. *Let for some positive definite matrices Q, P_{jl}, R_{jl}, S_j and T_{jl} there exists the positive definite solution of the matrix Riccati equation (31). Then the equation (27) zero solution is asymptotically mean square stable.*

Remark 4.3. Analogously with Remarks 3.1 and 3.3 we can show that instead of the equation (31) can be used other matrix Riccati equations.

Remark 4.4. Let $k(i) = k = const$. In this case the equation (31) has the form

$$-Q = (W_0 - I)'D(W_0 + I) + (I - W_0)'D \sum_{j=1}^{k} S_j^{-1} D(I - W_0) +$$

$$+ \sum_{j=1}^{k} W_j' S_j W_j + \frac{1}{2} \sum_{j=0}^{\hat{m}} \sum_{l=0}^{\hat{m}} B_j'(T_{jl} + DT_{lj}^{-1} D)B_j.$$

Remark 4.5. It is easy to get that in scalar case a positive definite solution of the equation (31) there exists if and only if

$$2\alpha_0\alpha_{k_m+1} + \alpha_{k_m+1}^2 + \sigma^2 < (1 - W_0)(1 + W_0 - 2\alpha_1), \qquad |W_0| < 1, \tag{32}$$

where

$$\alpha_l = \sum_{j=l}^{k} |W_j|, \qquad \sigma = \sum_{j=0}^{\hat{m}} |B_j|.$$

Note that if $k(i) = k = const$ then $\alpha_{k_m+1} = 0$. For the equation (3) in this case we have $W_0 = A + B$, $\alpha_1 = k|B|$, $\sigma = |C|$ and the condition (32) coincide with (16).

Example 4.1. Let in the equation (27) $k(i) = [qi]$, where $i \in Z$, $0 \le q \le 1$, $[x]$ is the integral part of a number x. It is easy to see that the function $k(i)$ satisfies the condition (18), $k_m = 0$, $\hat{k} = \infty$.

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Department of Mathematics, Informatics and Computing, Donetsk State Academy of Management, Chelyuskintsev 163-a, 340015 Donetsk, Ukraine
E-mail: leonid@dsam.donetsk.ua, leonid.shaikhet@usa.net