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Stability of equilibrium states of a nonlinear delay differential equation with stochastic perturbations

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SUMMARY

The nonlinear delay differential equation with exponential and quadratic nonlinearities is considered. It is assumed that the equation is exposed to stochastic perturbations of the white noise type, which are directly proportional to the deviation of the system state from the equilibrium point. Sufficient conditions for stability in probability of the zero and positive equilibriums of the considered system under stochastic perturbations are obtained. The research results are illustrated by numerical simulations. The proposed investigation procedure can be applied for arbitrary nonlinear stochastic delay differential equations with an order of nonlinearity higher than one. Copyright © 2016 John Wiley & Sons, Ltd.

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KEY WORDS: stability; nonlinear; delay; stochastic

1. INTRODUCTION

A lot of mathematical models in different applications are described via nonlinear difference or delay differential equations [1–4]. Models of such type are enough popular among researchers, for example, growth models in economics [5–12] , different types of epidemic models [13–18], the known Mackey–Glass model [19], models of population dynamics, in particular, predator–prey models [20–23], well-known Nicholsons blowflies equation [24–30], and many other [31–33]. It is known that nonlinear equations can have several equilibria, so, one of the main directions in investigation of the nonlinear models is stability analysis of these equilibria, in particular, stability positive or zero equilibria under stochastic perturbations [3, 4, 10–13, 15–18, 23, 28, 30].

In the succeeding sections, it is shown that investigation of stability in probability of equilibria of nonlinear stochastic differential equation can be reduce to investigation of asymptotic mean square stability of some linear stochastic differential equation. The idea of the proposed method of stability investigation is similar to the stability method of the first approximation, namely, the nonlinear equation is centralized and linearized around an equilibrium point, and the zero solution of the obtained linear equation is investigated on asymptotic mean square stability. Obtained asymptotic mean square stability conditions of the zero solution of the linear equation at the same time are conditions for stability in probability of corresponding equilibrium of the initial nonlinear equation. This method can be applied for arbitrary nonlinear differential equation with the order of nonlinearity higher than one.

1.1. Description of the considered model

The nonlinear delay differential equation

$$\dot{M}(t) = re^{-\alpha M(t-\tau)}e^{-\mu\tau}M(t-\tau) - pM(t) - kM^2(t), \tag{1}$$

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with a positive initial function

$$M(s) = \phi(s), \quad s \in [-\tau, 0], \tag{2}$$

describes [20] a population growth for adult female mosquitoes. Here, $M(t)$ is the population size of female mosquitoes at time t; r, d, and k are per capita birth rate, per capita death rate, and intraspecific competition rate, respectively; μ is the natural death rate of mosquitoes during the preadult stages; τ is the maturation time; and $\frac{1}{\alpha}$ is used to measure the maximum blood meal resource that the maximum eggs reproductivity can be reached. It is supposed that the parameters satisfy the conditions

$$r > p > 0, \quad k \geqslant 0, \quad \mu > 0, \quad \tau \geqslant 0, \quad \alpha \geqslant 0, \quad \alpha^2 + k^2 \neq 0. \tag{3}$$

In the case $\mu = k = 0$, the equation (1) describes the population dynamics of well-known Nicholsons blowflies, which is one of the most important mathematical models in ecology [24–30].

In [20], it is shown that for a strictly positive initial function (2), the solution $M(t)$ of the equation (1) remains positive and bounded. In this paper, sufficient conditions for stability in probability of the zero and positive equilibriums of the equation (1) under stochastic perturbations are obtained.

1.2. Equilibrium points

It is easy to see that for arbitrary values of the parameters (3), the equation (1) has the zero equilibrium. Putting in the equation (1) $M(t) = M^*$ and supposing that $M^* \neq 0$, we obtain that the positive equilibrium point of this equation is defined by the algebraic equation

$$re^{-\alpha M^*} e^{-\mu\tau} = p + kM^*. \tag{4}$$

Via (4) for $M^* > 0$, we obtain

$$e^{-\alpha M^*} = \frac{p + kM^*}{r} e^{\mu\tau} \leqslant 1. \tag{5}$$

From this, it follows that if

$$\tau \leqslant \tau_m = \frac{1}{\mu} \ln \frac{r}{p}, \tag{6}$$

then the positive equilibrium M^* exists and via (5) satisfies the condition

$$M^* \leqslant \frac{re^{-\mu\tau} - p}{k} \leqslant \frac{r - p}{k}. \tag{7}$$

Note that in the case $\alpha = 0$ from (5), we have

$$M^* = \frac{re^{-\mu\tau} - p}{k}. \tag{8}$$

Consider the positive equilibrium M^* as a function of the delay τ, that is, $M^* = M^*(\tau)$. Representing (4) in the form

$$re^{-\mu\tau} = (p + kM^*(\tau))e^{\alpha M^*(\tau)}, \tag{9}$$

one can see that $M^*(\tau)$ is a strictly decreasing function. Substituting τ_m from (6) into (4), we obtain

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$$p \geqslant pe^{-\alpha M^*(\tau_m)} = p + kM^*(\tau_m) \geqslant p.$$

From this (and also from (8) in the case $\alpha = 0$), it follows that $M^*(\tau_m) = 0$.

2. STOCHASTIC PERTUBATIONS AND LINEARIZATION

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a basic probability space, $\{\mathfrak{F}_t, t \geqslant 0\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, that is, $\mathfrak{F}_{t_1} \subset \mathfrak{F}_{t_2}$ for $t_1 < t_2$, $\mathbf{P}\{\cdot\}$ be the probability of the event enclosed in the braces, and $\mathbf{E}$ be the mathematical expectation.

Let us suppose that the equation (1) is influenced by stochastic perturbations of the white noise type that are directly proportional to the deviation of the solution $M(t)$ from the equilibrium M^*. Then (1) takes the form of the Ito stochastic differential equation [34]

$$\begin{aligned} dM(t) = \Big[& re^{-\alpha M(t-\tau)} e^{-\mu\tau} M(t-\tau) \\ & -pM(t) - kM^2(t)\Big] dt + \sigma(M(t) - M^*) dw(t), \end{aligned} \tag{10}$$

where $w(t)$ is the standard $\mathfrak{F}_t$-adapted Wiener process [34, 36].

Note that the proposed type of stochastic perturbations was first considered in [13] and used later in a lot of different mathematical models (see [35, 36] and references therein). In that case, the equilibrium M^* of the deterministic differential equation (1) is also the solution of the stochastic differential equation (10).

Putting in (10), $M(t) = x(t) + M^*$, that is

$$\begin{aligned} dx(t) = \Big[& re^{-\alpha(x(t-\tau)+M^*)} e^{-\mu\tau} \left(x(t-\tau) + M^*\right) \\ & -p\left(x(t) + M^*\right) - k\left(x(t) + M^*\right)^2\Big] dt + \sigma x(t) dw(t), \end{aligned} \tag{11}$$

and using (4), we obtain for $x(t)$ the nonlinear equation

$$\begin{aligned} dx(t) = \Big[& \left(p + kM^*\right) e^{-\alpha x(t-\tau)} x(t-\tau) + M^*\left(p + kM^*\right)(e^{-\alpha x(t-\tau)} - 1) \\ & - \left(p + 2kM^*\right) x(t) - kx^2(t)\Big] dt + \sigma x(t) dw(t). \end{aligned} \tag{12}$$

It is easy to see that stability of the equilibrium M^* of the equation (10) is equivalent to stability of the zero solution of the equation (12).

Together with the equation (12), we will consider the linearization of this equation around the zero solution

$$dy(t) = \left[\left(p + kM^*\right)\left(1 - \alpha M^*\right) y(t-\tau) - \left(p + 2kM^*\right) y(t)\right] dt + \sigma y(t) dw(t). \tag{13}$$

From (11), it follows that for the zero equilibrium of the equation (1), the equations similar to (12) and (13) are respectively

$$dx(t) = \left[re^{-\alpha x(t-\tau)} e^{-\mu\tau} x(t-\tau) - px(t) - kx^2(t)\right] dt + \sigma x(t) dw(t). \tag{14}$$

and

$$dy(t) = \left[re^{-\mu\tau} y(t-\tau) - py(t)\right] dt + \sigma y(t) dw(t). \tag{15}$$

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3. SOME DEFINITIONS AND AUXILIARY STATEMENTS

Definition 3.1
The zero solution of the equation (12) (or (14)) is called stable in probability if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that the solution $x(t,\phi)$ of (12) (or (14)) satisfies the condition $\mathbf{P}\{\sup_{t \geqslant 0} |x(t,\phi)| > \varepsilon_1 | \mathfrak{F}_0\} < \varepsilon_2$ for any initial function ϕ such that $\mathbf{P}\{\sup_{s\in[-\tau,0]} |\phi(s)| < \delta\} = 1$.

Definition 3.2
The zero solution of the equation (13) (or (15)) is called mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|y(t,\phi)|^2 < \varepsilon$, $t \geqslant 0$, provided that $\sup_{s\in[-\tau,0]} \mathbf{E}|\phi(s)|^2 < \delta$; asymptotically mean square stable if it is mean square stable and for each initial function ϕ the solution $y(t)$ of (13) (or (15)) satisfies the condition $\lim_{t\to\infty} \mathbf{E}|y(t)|^2 = 0$.

Remark 3.1
As it follows from [36], in order to obtain sufficient conditions for stability in probability of the zero solution of the nonlinear differential equation (12) (or (14)) with the order of nonlinearity higher than one (Appendix), it is enough to obtain sufficient conditions for asymptotic mean square stability of the zero solution of the linear equation (13) (or (15)).

Consider now the linear Ito stochastic differential equation

$$dx(t) = [Ax(t) + Bx(t-\tau)]\,dt + \sigma x(t-h)dw(t), \tag{16}$$

where A, B, σ, $\tau \geqslant 0$, $h \geqslant 0$ are known constants. It is easy to see that the equations (13), (15) are particular cases of the equation (16).

Lemma 3.1
[36, p. 44] A necessary and sufficient condition for asymptotic mean square stability of the zero solution of the equation (16) is

$$A + B < 0, \qquad G^{-1} > \frac{1}{2}\sigma^2, \tag{17}$$

where

$$G = \begin{cases} \frac{Bq^{-1}\sin(q\tau)-1}{A+B\cos(q\tau),} & B + |A| < 0,\ q = \sqrt{B^2 - A^2}, \\ \frac{1}{2}\left(\tau + \frac{1}{|A|}\right), & B = A < 0, \\ \frac{Bq^{-1}\sinh(q\tau)-1}{A+B\cosh(q\tau)}, & A + |B| < 0,\ q = \sqrt{A^2 - B^2}, \end{cases} \tag{18}$$

$\sinh(q\tau)$ and $\cosh(q\tau)$ in (18) are the hyperbolic sine and hyperbolic cosine, respectively.

Via Remark 3.1, to obtain sufficient condition for stability in probability of the zero solution of the nonlinear equation (12) (or (14)), it is enough to apply the conditions (17), (18) for asymptotic mean square stability of the zero solution of the linear equation (13) (or (15)), respectively.

4. MAIN RESULT. STABILITY OF THE EQUILIBRIUMS

4.1. Zero equilibrium.

Theorem 4.1
Let be

$$\tau > \tau_m = \frac{1}{\mu}\ln\frac{r}{p}. \tag{19}$$

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The zero solution of the stochastic differential equation (15) is asymptotically mean square stable if and only if

$$G^{-1} > \frac{1}{2}\sigma^2, \tag{20}$$

where

$$G = \frac{1 - re^{-\mu\tau}q^{-1}\sinh(q\tau)}{p - re^{-\mu\tau}\cosh(q\tau)}, \quad q = \sqrt{p^2 - r^2e^{-2\mu\tau}}. \tag{21}$$

Proof

For the equation (15), we have $A = -p < 0$, $B = re^{-\mu\tau} > 0$. From (19), we obtain $p > re^{-\mu\tau}$, that is, $A + |B| < 0$. So, the representation (21) for G follows from the last line of (18). The proof is completed. □

4.2. *Positive equilibrium*

Let us suppose that the condition (6) holds and therefore the positive equilibrium M^* of the equation (1) there exists and satisfies the condition (7). Denote

$$\hat{M} = \begin{cases} \frac{2}{\alpha} & \text{if } k = 0, \\ \frac{1}{2}\left[\frac{3}{\alpha} - \frac{p}{k} + \sqrt{\left(\frac{3}{\alpha} - \frac{p}{k}\right)^2 + \frac{8p}{\alpha k}}\right] & \text{if } k > 0. \end{cases} \tag{22}$$

Theorem 4.2

Let M^* be the positive root of the equation (4). The zero solution of the stochastic differential equation (13) is asymptotically mean square stable if and only if the condition (13) holds, where G is defined as follows: for $\alpha > 0$ if $M^* > \hat{M}$, then

$$\begin{aligned} G &= \frac{1 + (p + kM^*)(\alpha M^* - 1)q^{-1}\sin(q\tau)}{p + 2kM^* + (p + kM^*)(\alpha M^* - 1)\cos(q\tau)}, \\ q &= \sqrt{(p + kM^*)^2(\alpha M^* - 1)^2 - (p + 2kM^*)^2}, \end{aligned} \tag{23}$$

if $M^* = \hat{M}$, *then*

$$G = \frac{1}{2}\left(\tau + \frac{1}{p + 2kM^*}\right), \tag{24}$$

if $M^* < \hat{M}$, *then*

$$\begin{aligned} G &= \frac{1 - (p + kM^*)|1 - \alpha M^*|q^{-1}\sinh(q\tau)}{p + 2kM^* - (p + kM^*)|1 - \alpha M^*|\cosh(q\tau)}, \\ q &= \sqrt{(p + 2kM^*)^2 - (p + kM^*)^2(1 - \alpha M^*)^2}; \end{aligned} \tag{25}$$

for $\alpha = 0$ G *is defined by (25) with* $\alpha = 0$.

Proof

Via Lemma 3.1 for the equation (13), we have

$$A = -(p + 2kM^*) < 0, \qquad B = (p + kM^*)(1 - \alpha M^*). \tag{26}$$

So, the first condition (17) for the linear equation (13) has the form

$$(p + kM^*)(1 - \alpha M^*) < p + 2kM^*, \tag{27}$$

and evidently holds for arbitrary values of the parameters given in (3).

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To define G for the condition (20), let us consider first the case $\alpha M^* > 1$, that is, $B < 0$. In this case, via (18), we obtain that G is defined by (23) if $|B| > |A|$ and by (24) if $|B| = |A|$. From this and (26), it follows that G is defined by (23) or (24) if the condition $(p + kM^*)(\alpha M^* - 1) \geqslant p + 2kM^*$ or that is the same

$$\alpha k(M^*)^2 - (3k - \alpha p)M^* - 2p \geqslant 0, \tag{28}$$

holds. Using (22), we obtain the solution of the inequality (28) in the form $M^* \geqslant \hat{M}$. If $|A| > |B|$, then via (18) and the inequality contrary to (28), we obtain that G is defined by (25) if

$$\hat{M} > M^* > \alpha^{-1}. \tag{29}$$

Let now $0 < \alpha M^* \leqslant 1$, that is, $B \geqslant 0$. Via (18), in this case, G is defined by (25) if $|A| > B$ that coincides with (27) and holds for all $M^* \leqslant \alpha^{-1}$. Uniting this inequality with (29), we obtain that G is defined by (25) if $M^* < \hat{M}$. If $\alpha = 0$, then via (3), $k > 0$, and via (26), $|A| > B > 0$. So, G is defined by (25) with $\alpha = 0$ too. The proof is completed. □

5. NUMERICAL ANALYSIS

Consider a dependence of the positive equilibrium $M^*(\tau)$ on the delay $\tau \in [0, \tau_m]$, using (6) and solving the equation (9). In Figure 1, the graph of the function $M^*(\tau)$ (in thousands of pieces) is shown on the interval $[0, \tau_m]$ for the following values of the parameters [20]:

$$r = 42, \quad p = 1/21, \quad k = 0.0001, \quad \mu = 0.28, \quad \alpha = 0.0001. \tag{30}$$

Calculations show that $M^*(0) = 27198$, $\tau_m = 24.222$, $M^*(\tau_m) = 0$.

Putting in (9), $M^* = \alpha^{-1}$, we obtain the value τ_1 of the delay τ for which the condition $\alpha M^*(\tau_1) = 1$ holds:

$$\tau_1 = \frac{1}{\mu}\left(\ln\frac{r}{p + \alpha^{-1}k} - 1\right).$$

From this for the values of the parameters given in (30), we obtain $\tau_1 = 9.61$, $M^*(\tau_1) = 10000$. The points

$$\begin{aligned} A(0, M^*(0)) &= A(0, 27198), \\ S(\tau_1, M^*(\tau_1)) &= S(9.61, 10000), \\ B(\tau_m, M^*(\tau_m)) &= B(24.222, 0) \end{aligned}$$

are shown in Figure 1.

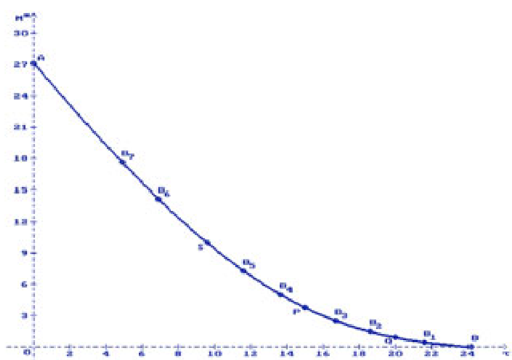

Figure 1. Graph of the function $M^* = M^*(\tau)$ and stability regions.

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Substituting the values of the parameters given in (30) into (22), we obtain $\hat{M} = 29527$. Using that $M^*(\tau)$ is a decreasing function, we have $M^*(\tau) \leqslant M^*(0) = 27198 < 29527 = \hat{M}$, $\tau \in [0, \tau_m]$. From this, it follows that for the parameters values (30), the condition $M^* \geqslant \hat{M}$ is impossible, and therefore, via Theorem 4.2, G cannot be defined by (23) or (24).

Thus, if the condition (20) with G defined by (25) holds, then the zero solution of the equation (13) with the values of the parameters (30) is asymptotically mean square stable, and therefore the positive equilibrium M^* of the equation (10) is stable in probability.

Besides, of the points A, S, B, in Figure 1, the part $[A, B_i]$ of the function $M^*(\tau)$ graph with $A(0, M^*(0))$ and B_i, $i = 1, ..., 7$, obtained by virtue of the conditions (20), (25) for different values of $\varepsilon = \frac{1}{2}\sigma^2$:

$$
\begin{aligned}
&1)\ \varepsilon = 0.1, \quad B_1(21.6, 470), \quad 2)\ \varepsilon = 0.3, \quad B_2(18.6, 1502),\\
&3)\ \varepsilon = 0.5, \quad B_3(16.7, 2555), \quad 4)\ \varepsilon = 1.0, \quad B_4(13.7, 5014),\\
&5)\ \varepsilon = 1.5, \quad B_5(11.6, 7349), \quad 6)\ \varepsilon = 2.8, \quad B_6(6.9, 14212),\\
&7)\ \varepsilon = 3.3, \quad B_7(4.9, 17688),
\end{aligned}
$$

shows the points $(\tau, M^*(\tau))$, for which the zero solution of the equation (13) is asymptotically mean square stable, and therefore, the positive equilibrium of the equation (10) is stable in probability.

The following results of numerical simulation of solutions of the equations (10) and (14) are represented. Trajectories of the Wiener process are simulated via the algorithm described in [36, p. 30].

In Figure 2, 100 trajectories of the equation (10) solution are shown in the point $P(15, 3822)$ with the initial function $M(s) = 4000(1 + 0.7\cos(s))$, $s \in [-15.0]$, the delay $\tau = 15$, and $\sigma = 1$. In this case, $\varepsilon = 0.5$ and $P(15, 3822) \in [A, B_3]$, that is, the point $P(15, 3822)$ is a point of stability in

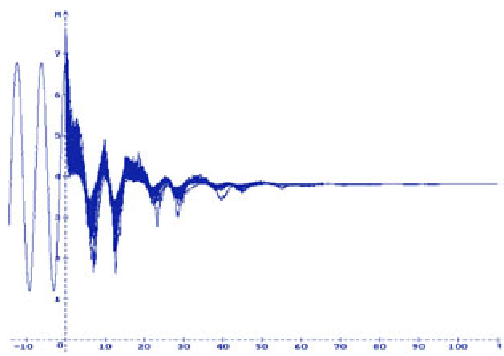

Figure 2. One hundred trajectories of the equation (10) solution in the point $P(15, 3822)$.

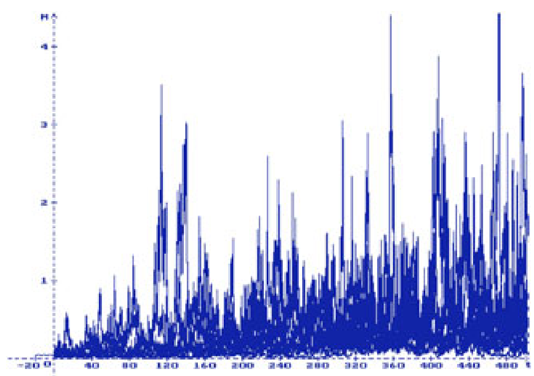

Figure 3. Ten trajectories of the equation (14) solution in the point $Q(20, 938)$.

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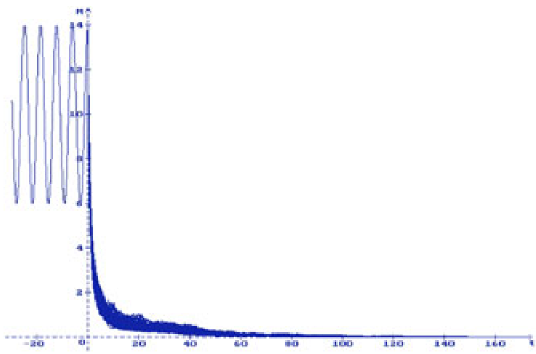

Figure 4. One hundred trajectories of the equation (14) solution, $\tau = 30$.

probability for the equilibrium $M^*(15) = 3822$ of the equation (10), and all trajectories converge to this equilibrium.

In Figure 3, 10 trajectories of the equation (14) solution are shown in the point $Q(20, 938)$ (Figure 2) with the initial function $M(s) = 10(5 + \sin(s))$, $s \in [-20.0]$, the delay $\tau = 20$, and $\sigma = 1$. From (19), via $\tau = 20 < \tau_m = 24.222$, it follows that the point $Q(20, 938)$ is not a point of stability in probability for the zero solution of the equation (14). So, the trajectories with a small enough initial function do not converge to zero and fill with itself whole space.

In Figure 4, 100 trajectories of the equation (14) solution are shown with the initial function $M(s) = 10000(1 + 0.4\sin(s))$, $s \in [-30.0]$, the delay $\tau = 30$, and $\sigma = 0.25$. Calculations show that $\tau = 30 > \tau_m = 24.222$, $G^{-1} = 0.04049 > \frac{1}{2}\sigma^2 = 0.03125$. So, the conditions (19)–(18) hold, the zero solution of the equation (14) stable in probability, and all trajectories converge to zero.

6. CONCLUSIONS

In this paper, a nonlinear differential equation with exponential and quadratic nonlinearities is considered. It is supposed that the equation is exposed to stochastic perturbations that are directly proportional to the deviation of the solution from one of two (the zero or positive) equilibrium points. The special procedure is proposed, which allows to obtain sufficient conditions for stability in probability of the equation equilibriums. The obtained results are illustrated by numerical simulations of solutions of the considered equation. The proposed investigation procedure can be applied for arbitrary nonlinear differential equation with an order of nonlinearity higher than one.

APPENDIX

Consider the nonlinear Ito stochastic differential equation

$$dx(t) = \left(\int_0^\infty dK(s)x(t-s) + g(t, x_t) \right) dt + \sigma x(t) dw(t), \tag{A1}$$

where the functional $g(t, \varphi)$ satisfies the condition

$$|g(t,\varphi)| \leqslant \int_0^\infty |\varphi(-s)|^\nu dr(s), \qquad \sup_{s \leqslant 0} |\varphi(s)| \leqslant \delta, \qquad \nu > 1, \tag{A2}$$

δ is small enough, $r(s)$ is nondecreasing integrable function, and the integrals are understood as Stieltjes integrals.

Understanding that in the neighborhood of zero a nonlinear function with the order of nonlinearity higher that one (the condition (A2)) tends to zero sooner than a linear function explains why asymptotic mean square stability of the zero solution of the equation (A1) without $g(t, x_t)$ guarantees stability in probability of the zero solution of the nonlinear equation (A1). In details and in more general case, it is shown in [36].

REFERENCES

1. Gyori I, Trofimchuk S. Global attractivity in $\dot{x}(t) = -\delta x(t) + pf(x(t-\tau))$. *Dynamic Systems and Applications* 1999; **8**:197–210.
2. El-Metwally E, Grove EA, Ladas G, Levins R, Radin M. On the difference equation $x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$. *Nonlinear Analysis* 2001; **47**:4623–4634.
3. Papaschinopoulos G, Ellina G, Papadopoulos KB. Asymptotic behavior of the positive solutions of an exponential type system of difference equations. *Applied Mathematics and Computation* 2014; **245**:181–190. (Available from: http://dx.doi.org/10.1016/j.amc.2014.07.074).
4. Shaikhet L. Stability of equilibrium states for a stochastically perturbed exponential type system of difference equations. *Journal of Computational and Applied Mathematics* 2015; **290**:92–103. (Available from: http://dx.doi.org/10.1016/j.cam.2015.05.002).
5. Solow R. A contribution to the theory of economic growth. *Quarterly Journal of Economics* 1956; **70**:65–94.
6. Swan T. Economic growth and capital accumulation. *Economic Record* (1956); **32**:334–361.
7. Day R. The emergence of chaos from classical economic growth. *Quarterly Journal of Economics* 1983; **98**:203–213.
8. Day R. *Complex Economic Dynamics: An Introduction to Dynamical Systems and Mar- ket Mechanism*. MIT Press: Cambridge, 1994.
9. Puu T. *Bifurcations and Chaos: Nonlinear Phenomena in Economics* (2nd edn). Springer: Berlin, 2003.
10. Bischi GI, Chiarella C, Kopel M, Szidarovszky F. *Nonlinear Oligopolies: Stability and Bifurcation*. Springer: Berlin, (2010).
11. Matsumoto A, Szidarovszky F. Asymptotic behavior of a delay differential neoclassical growth model. *Sustainability* 2013; **5**:440–455.
12. Chen W, Wang W. Global exponential stability for a delay differential neoclassical growth model. *Advances in Difference Equations* 2014; **2014**:325.
13. Beretta E, Kolmanovskii V, Shaikhet L. Stability of epidemic model with time delays influenced by stochastic perturbations. *Mathematics and Computers in Simulation* 1998; **5**(34):269–277. (Special Issue Delay Systems). 45 (34) (1998) 269–277. (Special Issue Delay Systems).
14. Lahrouz A, Omari L, Kiouach D. Global analysis of a deterministic and stochastic nonlinear SIRS epidemic model. *Nonlinear Analysis: Modelling and Control* 2011; **16**:59–76.
15. Jumpen W, Orankitjaroen S, Boonkrong P, Wiwatanapataphee B. SEIQR-SIS epidemic network model and its stability. *International Journal of Mathematics and Computers in Simulation* 2011; **5**:326–333.
16. Fan X, Wang Z, Xu X. Global stabilityof two-group epidemic models with distributed delays and random perturbation. *Abstract and Applied Analysis* 2012; **2012**:132095. 12 pages. http://dx.doi.org/10.1155/2012/132095.
17. Beretta E, Hara T, Ma W, Takeuchi Y. Global asymptotic stability of an SIR epidemic model with distributed time delay. *Nonlinear Analysis* 2001; **47**:4107–4115.
18. Buonomo B, Rionero S. On the Lyapunov stability for SIRS epidemic models with general nonlinear incidence rate. *Applied Mathematics and Computation* 2010; **217**(8):4010–4016.
19. Mackey MC, Glass L. Oscillation and chaos in physiological control system. *Science* 1997; **197**:287–289.
20. Wan H, Zhu H. A new model with delay for mosquito population dynamics. *Mathematical Biosciences and Engineering* 2014; **11**(6):1395–1410.
21. Chunyan J, Jiang D, Xiaoyue L. Qualitative analysis of a stochastic ratio-dependent predator–prey system. *Journal of Computational and Applied Mathematics* 2011; **235**:1326–1341.
22. Arino O, Elabdllaoui A, Mikaram J, Chattopadhyay J. Infection on prey population may act as a biological control in a ratio-dependent predator–prey model. *Nonlinearity* 2004; **17**:1101–1116.
23. Bandyopadhyay M, Chattopadhyay J. Ratio dependent predator–prey model: effect of environmental fluctuation and stability. *Nonlinearity* 2005; **18**:913–936.
24. Nicholson AJ. An outline of the dynamics of animal populations. *Australian Journal of Zoology* 1954; **2**:9–65.
25. Yi T, Zou X. Global attractivity of the diffusive Nicholson blowflies equation with Neumann boundary condition: a non-monotone case. *Journal of Differential Equations* 2008; **245**:3376–3388.
26. Berezansky L, Braverman E, Idels L. Nicholsons blowflies differential equations revis-ited: main results and open problems. *Applied Mathematical Modelling* 2010; **34**:1405–1417.
27. Chen W, Liu B. Positive almost periodic solution for a class of Nicholsons blowflies model with multiple time-varying delays. *Journal of Computational and Applied Mathematics* 2011; **235**:2090–2097.
28. Wang W, Wang L, Chen W. Existence and exponential stability of positive almost periodic solution for Nicholson-type delay systems. *Nonlinear Analysis: Real World Applications* 2011; **12**:1938–1949.
29. Shu H, Wang L, Wu J. Global dynamics of Nicholsons blowflies equation revisited: onset and termination of nonlinear oscillations. *Journal of Differential Equations* 2013; **255**:2565–2586.

DOI: 10.1002/rnc

30. Liu B. Global exponential stability of positive periodic solutions for a delayed Nicholsons blowflies model. *Journal of Mathematical Analysis and Applications* 2014; **412**:212–221.
31. Carletti M. On the stability properties of a stochastic model for phage-bacteria interaction in open marine environment. *Mathematical Biosciences* 2002; **175**:117–131.
32. Bandyopadhyay M, Saha T, Pal R. Deterministic and stochastic analysis of a delayed allelopathic phytoplankton model within fluctuating environment. *Nonlinear Analysis Hybrid Systems* 2008; **2**:958–970.
33. Burkeand MA, Heiland F. Social dynamics of obesity. *Economic Inquiry* 2007; **45**:571–591.
34. Gikhman II, Skorokhod AV. *Stochastic Differential Equations*. Springer: Berlin, 1972.
35. Shaikhet L. *Lyapunov Functionals and Stability of Stochastic Difference Equations*. Springer: London, Dordrecht, Heidelberg, New York, 2011.
36. Shaikhet L. *Lyapunov Functionals and Stability of Stochastic Functional Differential Equations*. Springer: Dordrecht, Heidelberg, New York, London, 2013.

DOI: 10.1002/rnc