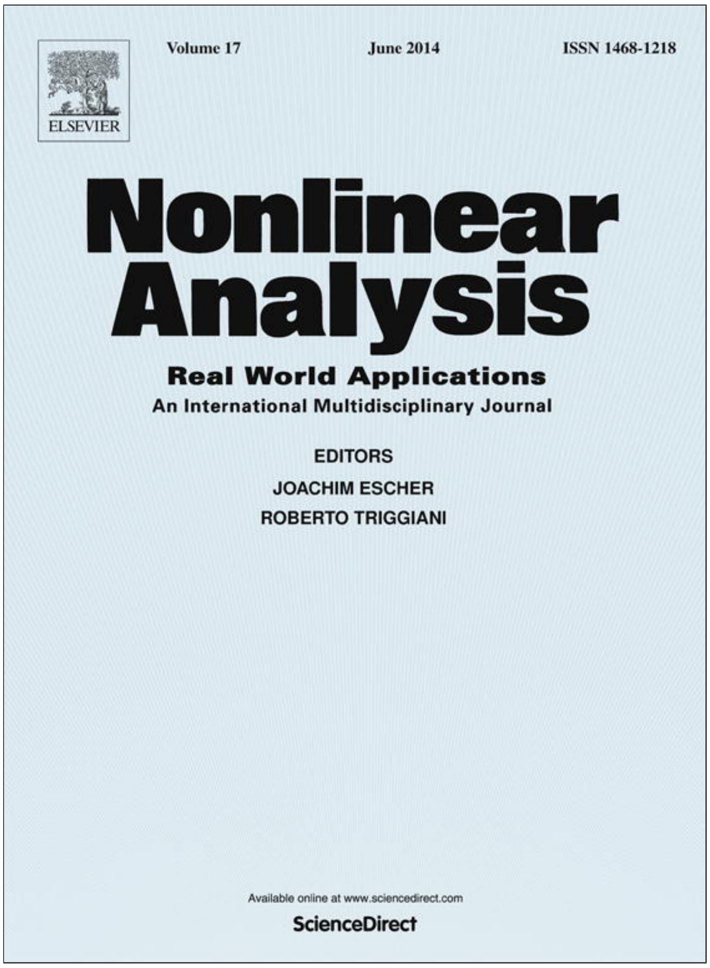
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Probabilistic stability analysis of social obesity epidemic by a delayed stochastic model

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ABSTRACT

Sufficient conditions for stability in probability of the equilibrium point of a social obesity epidemic model with distributed delay and stochastic perturbations are obtained. The obesity epidemic model is demonstrated on the example of the Region of Valencia, Spain. The considered nonlinear system is linearized in the neighborhood of the positive point of equilibrium and a sufficient condition for asymptotic mean square stability of the zero solution of the constructed linear system is obtained.

1. Introduction

Social obesity epidemic models are popular with researchers (see, for instance, [1–8]). In this paper the known nonlinear social obesity epidemic model [8] is generalized on the system with distributed delay. It is supposed also that this nonlinear system is exposed to additive stochastic perturbations of the type of the white noise that are directly proportional to the deviation of the system state from the equilibrium point. Such type of stochastic perturbations was first proposed in [9,10] and successfully used later in a many other works (see, for instance, [11–17]). The considered nonlinear system is linearized in the neighborhood of the positive point of equilibrium and a sufficient condition for asymptotic mean square stability of the zero solution of the constructed linear system is obtained. Since the order of nonlinearity is higher than 1 this condition is also a sufficient one [15,16] for stability in probability of the initial nonlinear system by stochastic perturbations.

This type of stability investigation was successfully used for investigation of different nonlinear mathematical models (SIR epidemic and some other medical models [9,13,14,18], predator–prey model [10,15–17], Nicholson blowfly model [15,16], inverted pendulum [15,16]) and can be an interesting tool, in particular, to policy makers.

1.1. Description of the considered model

To build the mathematical obesity model [8] the 24- to 65-year-old population is divided into three subpopulations based on their body mass index (BMI $=$ Weight/Height^2). The classes or subpopulations are: individuals at a normal weight (BMI $<$ 25 kg/m^2) $N(t)$, people who are overweight (25 kg/m^2 $\leq$ BMI $<$ 30 kg/m^2) $S(t)$ and obese individuals (BMI $\geq$ 30 kg/m^2) $O(t)$.

The transitions between the different subpopulations are determined as follows: once an adult starts an unhealthy lifestyle he/she becomes addicted to the unhealthy lifestyle and starts a progression to being overweight $S(t)$ because of this lifestyle. If this adult continues with his/her unhealthy lifestyle he/she can become an obese individual $O(t)$. In both these classes individuals can stop his/her unhealthy lifestyle and then move to classes $N(t)$ and $S(t)$, respectively.

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The transitions between the subpopulations $N(t)$, $S(t)$ and $O(t)$ are governed by terms proportional to the sizes of these subpopulations. Conversely, the transitions from normal to overweight occurs through the transmission of an unhealthy lifestyle from the overweight and obese subpopulations to the normal-weight subpopulation, depending on the meetings among them. It is assumed that when an individual at a normal weight is *infected* by the transmission of an unhealthy lifestyle from the overweight and obese subpopulations, there is a time s during which the infection develops and it is only after that time that the infected individual (an individual at a normal weight with an unhealthy lifestyle) becomes an infectious individual (an overweight individual). This transit is modeled using the term

$$\beta N(t)\int_0^\infty (S(t-s)+O(t-s))dK(s),$$

where $K(s)$ is a nondecreasing function such that

$$\int_0^\infty dK(s)=1, \tag{1.1}$$

the integral being understood in the Stieltjes sense. The subpopulations' sizes and their behaviors with time determine the dynamic evolution of adulthood excess weight.

Thus, under the above assumptions, the following non-linear system of integro-differential equations is obtained:

$$\begin{aligned}
\dot N(t) &= \mu N_0-\mu N(t)-\beta N(t)\int_0^\infty (S(t-s)+O(t-s))dK(s)+\rho S(t),\\
\dot S(t) &= \mu S_0+\beta N(t)\int_0^\infty (S(t-s)+O(t-s))dK(s)-(\mu+\gamma+\rho)S(t)+\varepsilon O(t),\\
\dot O(t) &= \mu O_0+\gamma S(t)-(\mu+\varepsilon)O(t).
\end{aligned} \tag{1.2}$$

Remark 1.1. Note that the solution of the differential equation $\dot x(t)=a(t)-b(t)x(t)$, with $x(0)\ge 0$, $a(t)\ge 0$ and $b(t)\ge 0$, has the representation

$$x(t)=\left(x(0)+\int_0^t a(\tau)e^{\int_0^\tau b(s)ds}d\tau\right)e^{-\int_0^t b(s)ds}$$

and therefore $x(t)\ge 0$. Since each equation of (1.2) has the form of this equation then the system has non-negative solution.

The time-invariant parameters of this system of equations are:

- ε, the rate at which an obese adult with a healthy lifestyle becomes an overweight individual;
- μ, average stay time in the system of 24- to 65-year-old adults. Note that this parameter is not a birth rate and/or a death rate. In this case it is assumed as a recruitment and exit rate and its value is the same for entering and leaving the system and inversely proportional to the mean time spent by an adult in the system;
- ρ, the rate at which an overweight individual moves to the normal-weight subpopulation;
- β, transmission rate because of social pressure to adopt an unhealthy lifestyle (TV, friends, family, job and so on).
- γ, the rate at which an overweight 24- to 65-year-old adult becomes an obese individual because of unhealthy lifestyle;
- N_0, proportion of normal weight coming from the 23-year-old age group;
- S_0, proportion of overweight coming from the 23-year-old age group;
- O_0, proportion of obese coming from the 23-year-old age group.

Here the parameters ε, μ, ρ, β, γ are nonnegative numbers and N_0, S_0 O_0 are nonnegative numbers that satisfy the condition

$$N_0+S_0+O_0=1. \tag{1.3}$$

Taking into account these conditions and summing the equations of the system (1.2) and putting $Q(t)=N(t)+S(t)+O(t)$, $Q_0=N_0+S_0+O_0$, we obtain the equation $\dot Q(t)=-\mu(Q(t)-Q_0)$ with the initial condition $Q(0)=Q_0$. It is easy to see that this equation has the unique solution $Q(t)=Q_0$. So, from the assumption (1.3), i.e., $Q_0=1$, it follows that $Q(t)=1$ and the system (1.2) can be simplified to the system of two equations:

$$\begin{aligned}
\dot N(t) &= \mu N_0-\mu N(t)-\beta N(t)\int_0^\infty (1-N(t-s))dK(s)+\rho S(t),\\
\dot S(t) &= \mu S_0+\beta N(t)\int_0^\infty (1-N(t-s))dK(s)-(\mu+\gamma+\rho)S(t)+\varepsilon(1-N(t)-S(t)).
\end{aligned} \tag{1.4}$$

1.2. Existence of the equilibrium point

The equilibrium point (N^*, S^*) of the system (1.4) is defined by the conditions $\dot{N}(t) = 0, \dot{S}(t) = 0, N(t) = N^*, S(t) = S^*$ and via (1.4), (1.1) is a solution of the system of the algebraic equations:

$$\begin{aligned}&\mu N_0 - \mu N^* - \beta N^*(1 - N^*) + \rho S^* = 0,\\ &\mu S_0 + \beta N^*(1 - N^*) - (\mu + \gamma + \rho)S^* + \varepsilon(1 - S^* - N^*) = 0.\end{aligned} \tag{1.5}$$

From Eqs. (1.5) it follows that

$$\begin{aligned}&S^* = \rho^{-1}[\mu(N^* - N_0) + \beta N^*(1 - N^*)],\\ &S^* = k\rho^{-1}[\mu S_0 + \beta N^*(1 - N^*) + \varepsilon(1 - N^*)],\end{aligned} \tag{1.6}$$

where

$$k = \rho(\mu + \gamma + \rho + \varepsilon)^{-1} < 1. \tag{1.7}$$

Via (1.6), (1.7) we obtain that N^* is a root of the quadratic equation

$$\beta(1 - k)(N^*)^2 - (\mu + k\varepsilon + \beta(1 - k))N^* + \mu(N_0 + kS_0) + k\varepsilon = 0. \tag{1.8}$$

Lemma 1.1. *Assume that $N_0 + kS_0 < 1$. If $\beta > 0$ then Eq. (1.8) has two real roots: $N_1^* \in (0, 1)$ and $N_2^* > 1$. If $\beta = 0$ and $\mu k\varepsilon > 0$ then Eq. (1.8) has one root, $N^* \in (N_0 + kS_0, 1)$.*

Proof. Via $N_0 + kS_0 < 1$ and $\beta > 0$ we have

$$\begin{aligned}D &= \sqrt{(\mu + k\varepsilon + \beta(1 - k))^2 - 4\beta(1 - k)(\mu(N_0 + kS_0) + k\varepsilon)}\\ &> \sqrt{(\mu + k\varepsilon + \beta(1 - k))^2 - 4\beta(1 - k)(\mu + k\varepsilon)} = |\mu + k\varepsilon - \beta(1 - k)|,\end{aligned} \tag{1.9}$$

i.e., $D > |\mu + k\varepsilon - \beta(1 - k)| \geq 0$ and therefore the quadratic equation (1.8) has two real roots:

$$N_1^* = \frac{\mu + k\varepsilon + \beta(1 - k) - D}{2\beta(1 - k)}, \qquad N_2^* = \frac{\mu + k\varepsilon + \beta(1 - k) + D}{2\beta(1 - k)}. \tag{1.10}$$

If $\mu + k\varepsilon < \beta(1 - k)$ then

$$N_1^* < \frac{\mu + k\varepsilon}{\beta(1 - k)} < 1, \qquad N_2^* > 1.$$

If $\mu + k\varepsilon \geq \beta(1 - k)$ then

$$N_1^* < 1, \qquad N_2^* > \frac{\mu + k\varepsilon}{\beta(1 - k)} \geq 1.$$

If $\beta = 0$ then from (1.8) it follows that

$$1 > N^* = \frac{\mu(N_0 + kS_0) + k\varepsilon}{\mu + k\varepsilon} > N_0 + kS_0.$$

The proof is completed.

Lemma 1.2. *Assume that $N_0 = 1$. If $\mu + k\varepsilon < \beta(1 - k)$ then Eq. (1.8) has two roots on the interval $(0, 1]$: $N_1^* \in (0, 1)$ and $N_2^* = 1$. If $\mu + k\varepsilon \geq \beta(1 - k)$ then Eq. (1.8) has one root only on the interval $(0, 1]$: $N_1^* = 1$.*

Proof. Via $N_0 = 1$ and (1.3) we have $S_0 = 0$. Then similar to (1.9) $D = |\mu + k\varepsilon - \beta(1 - k)|$. If $\mu + k\varepsilon < \beta(1 - k)$ then $D = \beta(1 - k) - (\mu + k\varepsilon)$ and via (1.10) we obtain

$$N_1^* = \frac{\mu + k\varepsilon}{\beta(1 - k)} < 1, \qquad N_2^* = 1.$$

If $\mu + k\varepsilon > \beta(1 - k)$ then $D = \mu + k\varepsilon - \beta(1 - k)$ and via (1.10) we have

$$N_1^* = 1, \qquad N_2^* = \frac{\mu + k\varepsilon}{\beta(1 - k)} > 1.$$

If $\mu + k\varepsilon = \beta(1 - k)$ then $D = 0$ and $N_1^* = N_2^* = 1$.
The proof is completed.

Example 1.1. Following [8] put

$$\mu = 0.000469, \quad \gamma = 0.0003, \quad \varepsilon = 0.000004, \quad \rho = 0.000035,$$
$$\beta = 0.00085, \quad N_0 = 0.704, \quad S_0 = 0.25, \quad O_0 = 0.046.$$

Then via (1.10), (1.6), (1.3) we obtain

$$N^* = 0.3311, \quad S^* = 0.3814, \quad O^* = 0.2875.$$

Putting $\beta = 0$ with the same values of the other parameters, in accordance with Lemma 1.2 we obtain

$$N^* = 0.7149 > N_0 + kS_0 = 0.7148, \quad S^* = 0.1465, \quad O^* = 0.1386.$$

Put now $N_0 = 1$, $S_0 = O_0 = 0$. In accordance with Lemma 1.2 if $\beta = 0.00085$, i.e., if $\beta > 0$, then $\beta > (\mu + k\varepsilon)(1-k)^{-1} = 0.00049$ and $N^* = 0.5770$, $S^* = 0.2588$, $O^* = 0.1642$. If $\beta = 0$ then $N^* = 1$, $S^* = O^* = 0$.

Note that here and below all information used to define the parameters of the model is based on a sample of the Valencian population.

2. Stochastic perturbations, centralization and linearization

Let us suppose that the system (1.4) is exposed to stochastic perturbations which can be considered as metering errors. We will suppose that these stochastic perturbations are of the type of white noise $(\dot{w}_1(t), \dot{w}_2(t))$ and are directly proportional to the deviation of the system (1.4) state $(N(t), S(t))$ from the equilibrium point (N^*, S^*), i.e.,

$$\dot{N}(t) = \mu N_0 - \mu N(t) - \beta N(t) \int_0^\infty (1 - N(t-s))dK(s) + \rho S(t) + \sigma_1 (N(t) - N^*)\dot{w}_1(t) \tag{2.1}$$
$$\dot{S}(t) = \mu S_0 + \beta N(t) \int_0^\infty (1 - N(t-s))dK(s) - (\mu + \gamma + \rho)S(t) + \varepsilon(1 - N(t) - S(t)) + \sigma_2 (S(t) - S^*)\dot{w}_2(t).$$

Here $w_1(t), w_2(t)$ are the mutually independent standard Wiener processes, and the stochastic differential equations of the system (2.1) are understood in the Ito sense [16]. Note that the equilibrium point (N^*, S^*) of the system (1.4) is the solution of the system (2.1) too.

To centralize the system (2.1) in the equilibrium point put now $x_1 = N - N^*$, $x_2 = S - S^*$. Then via (2.1), (1.7) we obtain

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \beta N^* I(x_{1t}) + \beta x_1(t) I(x_{1t}) + \sigma_1 x_1(t)\dot{w}_1(t),$$
$$\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) - \beta N^* I(x_{1t}) - \beta x_1(t) I(x_{1t}) + \sigma_2 x_2(t)\dot{w}_2(t), \tag{2.2}$$

where

$$a_{11} = -\mu - \beta(1 - N^*), \quad a_{12} = \rho, \tag{2.3}$$
$$a_{21} = -\varepsilon + \beta(1 - N^*), \quad a_{22} = -k^{-1}\rho,$$
$$I(x_{1t}) = \int_0^\infty x_1(t-s)dK(s).$$

Example 2.1. Using the values of the parameters from Example 1.1 we obtain

$$a_{11} = -0.0010376, \quad a_{12} = 0.000035,$$
$$a_{21} = 0.0005646, \quad a_{22} = -0.000808.$$

It is clear that stability of the equilibrium point of the system (2.1) is equivalent to stability of the zero solution of the system (2.2). Neglecting the nonlinear terms in (2.2) we obtain the linear part of the system (2.2):

$$\dot{y}_1(t) = a_{11}y_1(t) + a_{12}y_2(t) + \beta N^* I(y_{1t}) + \sigma_1 y_1(t)\dot{w}_1(t), \tag{2.4}$$
$$\dot{y}_2(t) = a_{21}y_1(t) + a_{22}y_2(t) - \beta N^* I(y_{1t}) + \sigma_2 y_2(t)\dot{w}_2(t).$$

3. Stability of the equilibrium point

Note that the nonlinear system (2.2) has the order of nonlinearity higher than 1. Thus, as follows from [15,16], sufficient conditions for asymptotic mean square stability of the zero solution of the linear part (2.4) at the same time are sufficient conditions for stability in probability of the zero solution of the nonlinear system (2.2) and therefore are sufficient conditions for stability in probability of the solution (N^*, S^*) of the system (2.1).

To get sufficient conditions for asymptotic mean square stability of the zero solution of the system (2.4) rewrite this system in the form

$$\dot{y}(t) = Ay(t) + B(y_t) + \sigma(y(t))\dot{w}(t), \tag{3.1}$$

where

$$\begin{aligned} &y(t) = (y_1(t), y_2(t))', \\ &w(t) = (w_1(t), w_2(t))', \\ &B(y_t) = (\beta N^* I(y_{1t}) - \beta N^* I(y_{1t}))', \\ &A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \sigma(y(t)) = \begin{pmatrix} \sigma_1 y_1(t) & 0 \\ 0 & \sigma_2 y_2(t) \end{pmatrix}, \end{aligned} \tag{3.2}$$

and $a_{ij},\ i, j = 1, 2$, are defined by (2.3).

Definition 3.1. The zero solution of Eq. (3.1) with the initial condition defined by $y(s) = \phi(s)$, $s \le 0$, is called:

- mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|y(t, \phi)|^2 < \varepsilon, t \ge 0$, provided that $\sup_{s\le 0} \mathbf{E}|\phi(s)|^2 < \delta$;
- asymptotically mean square stable if it is mean square stable and for each initial function ϕ the solution $y(t)$ of Eq. (3.1) satisfies the condition $\lim_{t\to\infty} \mathbf{E}|y(t)|^2 = 0$;
- stable in probability if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that the solution $y(t, \phi)$ of (3.1) satisfies the condition $\mathbf{P}\{\sup_{t\ge 0} |y(t, \phi)| > \varepsilon_1/\mathfrak{F}_0\} < \varepsilon_2$ for any initial function ϕ such that $\mathbf{P}\{\sup_{s\le 0} |\phi(s)| < \delta\} = 1$.

Following the procedure of Lyapunov functional construction [15,16] for stability investigation of Eq. (3.1) consider the auxiliary equation without memory

$$\dot z(t) = Az(t) + \sigma(z(t))\dot w(t). \tag{3.3}$$

Remark 3.1. Via the Routh–Hurwitz criterion [19] the zero solution of the differential equation $\dot z(t) = Az(t)$, where A is a 2×2-matrix, is asymptotically stable if and only if

$$\mathrm{Tr}(A) = a_{11} + a_{22} < 0, \qquad \det(A) = a_{11}a_{22} - a_{12}a_{21} > 0. \tag{3.4}$$

For the matrix A with the elements (2.3) these conditions hold:

$$\begin{aligned} &\mathrm{Tr}(A) = -[\mu + \rho k^{-1} + \beta(1 - N^*)] < 0, \\ &\det(A) = \rho k^{-1}[\mu + k\varepsilon + \beta(1 - k)(1 - N^*)] > 0. \end{aligned}$$

Example 3.1. Using the values of the parameters from Example 1.1 we have

$$\mathrm{Tr}(A) = -0.0018456, \qquad \det(A) = 0.0000008.$$

Put

$$\delta_i = \frac{1}{2}\sigma_i^2, \qquad \hat a_{ii} = a_{ii} + \delta_i, \quad i = 1, 2. \tag{3.5}$$

Lemma 3.1. *If*

$$a_{21} \le 0 \tag{3.6}$$

and

$$\hat a_{11} < 0, \qquad \hat a_{22} < 0, \tag{3.7}$$

then the zero solution of Eq. (3.3) *is asymptotically mean square stable.*

Proof. Let the matrix P be a positive definite solution of the matrix equation

$$A'P + PA + P_\sigma = -C, \tag{3.8}$$

where

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \qquad P_\sigma = \begin{pmatrix} p_{11}\sigma_1^2 & 0 \\ 0 & p_{22}\sigma_2^2 \end{pmatrix}, \qquad C = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.9}$$

$c > 0$, the matrix A is defined in (3.2), (2.3).

Let L be the generator [16] of Eq. (3.3) and $v(z) = z'Pz$. Then via (3.8)

$$Lv = z'(A'P + PA + P_\sigma)z = -z'Cz.$$

This means [16] that if the matrix P is a positive definite matrix then the zero solution of Eq. (3.3) is asymptotically mean square stable. So, it is enough to show that the solution P of the matrix equation (3.8), (3.9) really is a positive definite one.

Note that the matrix equation (3.8), (3.9) can be represented as the system of the equations

$$\begin{aligned} &2(p_{11}a_{11}+p_{12}a_{21}+p_{11}\delta_1)=-c,\\ &2(p_{12}a_{12}+p_{22}a_{22}+p_{22}\delta_2)=-1,\\ &p_{11}a_{12}+p_{12}\mathrm{Tr}(A)+p_{22}a_{21}=0, \end{aligned} \tag{3.10}$$

with the solution

$$\begin{aligned} p_{11}&=-\frac{c+2a_{21}p_{12}}{2\hat{a}_{11}},\\ p_{22}&=-\frac{1+2a_{12}p_{12}}{2\hat{a}_{22}},\\ p_{12}&=\frac{a_{21}\hat{a}_{11}+ca_{12}\hat{a}_{22}}{2Z},\\ Z&=\mathrm{Tr}(A)\hat{a}_{11}\hat{a}_{22}-a_{12}a_{21}(\hat{a}_{11}+\hat{a}_{22}). \end{aligned} \tag{3.11}$$

Put now

$$A_i=\det(A)+a_{ii}^2,\quad i=1,2. \tag{3.12}$$

Via (3.4), (3.5), (3.11), (3.12)

$$\begin{aligned} Z+a_{12}a_{21}\hat{a}_{22}&=\mathrm{Tr}(A)\hat{a}_{11}\hat{a}_{22}-a_{12}a_{21}(\hat{a}_{11}+\hat{a}_{22})+a_{12}a_{21}\hat{a}_{22}\\ &=(\mathrm{Tr}(A)\hat{a}_{22}-a_{12}a_{21})\hat{a}_{11}=(A_2-|\mathrm{Tr}(A)|\delta_2)\hat{a}_{11} \end{aligned} \tag{3.13}$$

and similarly

$$Z+a_{12}a_{21}\hat{a}_{11}=(A_1-|\mathrm{Tr}(A)|\delta_1)\hat{a}_{22}. \tag{3.14}$$

From (3.11), (3.13) we obtain

$$\begin{aligned} p_{11}&=-\frac{cZ+a_{21}(ca_{12}\hat{a}_{22}+a_{21}\hat{a}_{11})}{2Z\hat{a}_{11}}\\ &=-\frac{c(Z+a_{12}a_{21}\hat{a}_{22})+a_{21}^2\hat{a}_{11}}{2Z\hat{a}_{11}}\\ &=-\frac{c(A_2-|\mathrm{Tr}(A)|\delta_2)+a_{21}^2}{2Z}. \end{aligned} \tag{3.15}$$

Similarly from (3.11), (3.14) it follows that

$$p_{22}=-\frac{A_1-|\mathrm{Tr}(A)|\delta_1+ca_{12}^2}{2Z}. \tag{3.16}$$

Let us show that $p_{11}>0$, $p_{22}>0$ for arbitrary $c>0$. In fact, note that via (3.12), (3.4),

$$\begin{aligned} A_1A_2&=(\det(A)+a_{11}^2)(\det(A)+a_{22}^2)\\ &=(\det(A)+a_{11}^2+a_{22}^2)\det(A)+a_{11}^2a_{22}^2\\ &=(|\mathrm{Tr}(A)|^2-(a_{11}a_{22}+a_{12}a_{21}))\det(A)+a_{11}^2a_{22}^2\\ &=|\mathrm{Tr}(A)|^2\det(A)+a_{12}^2a_{21}^2\ge|\mathrm{Tr}(A)|^2\det(A) \end{aligned}$$

or

$$\frac{|\mathrm{Tr}(A)|\det(A)}{A_1}\le\frac{A_2}{|\mathrm{Tr}(A)|},\qquad \frac{|\mathrm{Tr}(A)|\det(A)}{A_2}\le\frac{A_1}{|\mathrm{Tr}(A)|}. \tag{3.17}$$

Besides via (3.4), (3.6), (3.7), (3.12) we have

$$\begin{aligned} &|\mathrm{Tr}(A)|\det(A)-|a_{11}|A_2=-a_{12}a_{21}|a_{22}|\ge 0,\\ &|\mathrm{Tr}(A)|\det(A)-|a_{22}|A_1=-a_{12}a_{21}|a_{11}|\ge 0. \end{aligned}$$

So,

$$|a_{11}|\le\frac{|\mathrm{Tr}(A)|\det(A)}{A_2},\qquad |a_{22}|\le\frac{|\mathrm{Tr}(A)|\det(A)}{A_1}. \tag{3.18}$$

As a result, from (3.7), (3.17), (3.18) we obtain

$$\delta_1 < \frac{A_1}{|\mathrm{Tr}(A)|}, \qquad \delta_2 < \frac{A_2}{|\mathrm{Tr}(A)|}. \tag{3.19}$$

Note also that via (3.11), (3.4), (3.6), (3.7) we have $Z < 0$. From this and (3.15), (3.16), (3.19) it follows that $p_{11} > 0$, $p_{22} > 0$ for an arbitrary $c > 0$.

Let us show that $p_{11}p_{22} > p_{12}^2$. In fact, from

$$\frac{(c + 2a_{21}p_{12})(1 + 2a_{12}p_{12})}{4\hat{a}_{11}\hat{a}_{22}} > p_{12}^2$$

it follows that $4Bp_{12}^2 - 2(a_{21} + ca_{12})p_{12} < c$ by $B = \hat{a}_{11}\hat{a}_{22} - a_{12}a_{21} > 0$. Substituting into this inequality p_{12} from (3.11) we have

$$B(a_{21}\hat{a}_{11} + ca_{12}\hat{a}_{22})^2 - (a_{21} + ca_{12})(a_{21}\hat{a}_{11} + ca_{12}\hat{a}_{22})Z < cZ^2$$

or

$$c^2a_{12}^2\hat{a}_{22}(Z - B\hat{a}_{22}) + c\hat{a}_{11}\hat{a}_{22}(Z\mathrm{Tr}(A) - 2a_{12}a_{21}B) + a_{21}^2\hat{a}_{11}(Z - B\hat{a}_{11}) > 0.$$

From (3.11), (3.6), (3.7) it follows that

$$\begin{aligned}
\hat{a}_{11}(Z - B\hat{a}_{11}) &= \hat{a}_{11}(\mathrm{Tr}(A)\hat{a}_{11}\hat{a}_{22} - a_{12}a_{21}(\hat{a}_{11} + \hat{a}_{22}) - (\hat{a}_{11}\hat{a}_{22} - a_{12}a_{21})\hat{a}_{11}) \\
&= \hat{a}_{11}(\mathrm{Tr}(A)\hat{a}_{11}\hat{a}_{22} - a_{12}a_{21}\hat{a}_{22} - \hat{a}_{11}^2\hat{a}_{22}) = \hat{a}_{11}\hat{a}_{22}(\mathrm{Tr}(A)\hat{a}_{11} - a_{12}a_{21} - \hat{a}_{11}^2) \\
&= \hat{a}_{11}\hat{a}_{22}((\mathrm{Tr}(A) - \hat{a}_{11})\hat{a}_{11} - a_{12}a_{21}) = \hat{a}_{11}\hat{a}_{22}((a_{22} - \delta_1)\hat{a}_{11} - a_{12}a_{21}) > 0.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\hat{a}_{22}(Z - B\hat{a}_{22}) = \hat{a}_{11}\hat{a}_{22}((a_{11} - \delta_2)\hat{a}_{22} - a_{12}a_{21}) > 0, \\
&Z\mathrm{Tr}(A) - 2a_{12}a_{21}B > 0.
\end{aligned}$$

So, for an arbitrary $c > 0$ the matrix P with the entries (3.11) is a positive definite one. The proof is completed.

Lemma 3.2. *If*

$$a_{21} > 0 \tag{3.20}$$

and

$$\max(\delta_1, \delta_2) < \frac{\det(A)}{|\mathrm{Tr}(A)|} \tag{3.21}$$

then the zero solution of Eq. (3.3) *is asymptotically mean square stable.*

Proof. Similarly to Lemma 3.1 it is enough to show that the solution P of the matrix equation (3.8), (3.9) is a positive definite matrix.

Note that via the condition (3.20) we have

$$A_i = a_{11}a_{22} - a_{12}a_{21} + a_{ii}^2 \le a_{ii}\mathrm{Tr}(A), \quad i = 1, 2.$$

From this and (3.4), (3.12), (3.21) it follows that

$$\delta_i < \frac{\det(A)}{|\mathrm{Tr}(A)|} \le \frac{A_i}{|\mathrm{Tr}(A)|} \le |a_{ii}|, \quad i = 1, 2. \tag{3.22}$$

Via (3.11), (3.5), (3.4), (3.12), (3.21) we have

$$\begin{aligned}
Z &= \mathrm{Tr}(A)(a_{11} + \delta_1)(a_{22} + \delta_2) - a_{12}a_{21}(\mathrm{Tr}(A) + \delta_1 + \delta_2) \\
&= \mathrm{Tr}(A)\det(A) + \mathrm{Tr}(A)\delta_1 a_{22} + \mathrm{Tr}(A)\delta_2 a_{11} + \mathrm{Tr}(A)\delta_1\delta_2 - a_{12}a_{21}(\delta_1 + \delta_2) \\
&= -|\mathrm{Tr}(A)|\det(A) + A_2\delta_1 + A_1\delta_2 - |\mathrm{Tr}(A)|\delta_1\delta_2 \\
&< -|\mathrm{Tr}(A)|\det(A) + (A_1 + A_2)\frac{\det(A)}{|\mathrm{Tr}(A)|} \\
&= -(|\mathrm{Tr}(A)|^2 - A_1 - A_2)\frac{\det(A)}{|\mathrm{Tr}(A)|} = -\frac{2a_{12}a_{21}\det(A)}{|\mathrm{Tr}(A)|} < 0.
\end{aligned} \tag{3.23}$$

From (3.15), (3.16), (3.22), (3.23) we obtain that $p_{11} > 0$, $p_{22} > 0$ for an arbitrary $c > 0$.

Let us show that $p_{11}p_{22} > p_{12}^2$. In fact, via (3.11), (3.15), (3.16) this inequality takes the form

$$(c(A_2 - |\mathrm{Tr}(A)|\delta_2) + a_{21}^2)(A_1 - |\mathrm{Tr}(A)|\delta_1 + ca_{12}^2) > (ca_{12}\hat{a}_{22} + a_{21}\hat{a}_{11})^2$$

which is equivalent to the condition

$$\begin{aligned} &c^2a_{12}^2(\det(A) - |\mathrm{Tr}(A)|\delta_2 + a_{22}^2 - \hat{a}_{22}^2) + a_{21}^2(\det(A) - |\mathrm{Tr}(A)|\delta_1 + a_{11}^2 - \hat{a}_{11}^2) \\ &\quad + c[(A_1 - |\mathrm{Tr}(A)|\delta_1)(\det(A) - |\mathrm{Tr}(A)|\delta_2) + a_{22}^2(\det(A) \\ &\quad - |\mathrm{Tr}(A)|\delta_1) + (\det(A))^2 + 2a_{12}a_{21}(a_{11}a_{22} - \hat{a}_{11}\hat{a}_{22})] > 0. \end{aligned} \tag{3.24}$$

Via (3.20)–(3.22) and $|a_{ii}| \geq |\hat{a}_{ii}|$, $i = 1, 2$, the condition (3.24) holds for an arbitrary $c > 0$. So, for an arbitrary $c > 0$ the solution P of the matrix equation (3.8), (3.9) is a positive definite matrix. The proof is completed.

Remark 3.2. If the condition (3.6) holds, i.e. $a_{21} \leq 0$, then $\beta \in \left[0, (\mu+\varepsilon)(1-k)^{-1}\right]$. If $\beta > (\mu+\varepsilon)(1-k)^{-1}$ then the condition (3.20) holds, i.e. $a_{21} > 0$. For example, by the values of the parameters from Example 1.1 we have $\beta = 0.00085 > (\mu+\varepsilon)(1-k)^{-1} = 0.0004945$ and $a_{21} = 0.0005646 > 0$.

Theorem 3.1. *If the conditions (3.6), (3.7) or (3.20), (3.21) hold and for some $c > 0$ the entries (3.11) of the matrix P satisfy the condition*

$$(\beta N^*|p_{12} - p_{22}|)^2 + 2\beta N^*|p_{11} - p_{12}| < c \tag{3.25}$$

then the solution (N^*, S^*) of the system (2.1) is stable in probability.

Proof. Note that the order of nonlinearity of the system (2.1) is higher than one. Therefore [15,16] to get for this system conditions of stability in probability it is enough to get conditions for asymptotic mean square stability of the zero solution of the linear part (2.4) of this system. Following the procedure of Lyapunov functional construction [15,16] we will construct a Lyapunov functional for the system (2.4) in the form $V = V_1 + V_2$, where $V_1 = y'Py$, $y = (y_1, y_2)'$, P is the positive definite solution of the system (3.10) with the entries (3.11) and V_2 will be chosen below.

Let L be the generator [8] of the system (2.4). Then via (2.4), (3.10),

$$\begin{aligned} LV_1 &= 2(p_{11}y_1(t) + p_{12}y_2(t))(a_{11}y_1(t) + a_{12}y_2(t) + \beta N^*I(y_{1t})) + p_{11}\sigma_1^2y_1^2(t) \\ &\quad + 2(p_{12}y_1(t) + p_{22}y_2(t))(a_{21}y_1(t) + a_{22}y_2(t) - \beta N^*I(y_{1t})) + p_{22}\sigma_2^2y_2^2(t) \\ &= -cy_1^2(t) - y_2^2(t) + 2\beta N^*[(p_{11} - p_{12})y_1(t) + (p_{12} - p_{22})y_2(t)]I(y_{1t}). \end{aligned}$$

Via (1.1), (2.3) we have

$$\begin{aligned} &2y_1(t)I(y_{1t}) \leq y_1^2(t) + I(y_{1t}^2), \\ &2y_2(t)I(y_{1t}) \leq \nu y_2^2(t) + \nu^{-1}I(y_{1t}^2) \end{aligned}$$

for some $\nu > 0$ and using these inequalities, we obtain

$$\begin{aligned} LV_1 &\leq -cy_1^2(t) - y_2^2(t) + \beta N^*|p_{11} - p_{12}|(y_1^2(t) + I(y_{1t}^2)) + \beta N^*|p_{12} - p_{22}|(\nu y_2^2(t) + \nu^{-1}I(y_{1t}^2)) \\ &= (\beta N^*|p_{11} - p_{12}| - c)y_1^2(t) + (\beta N^*|p_{12} - p_{22}|\nu - 1)y_2^2(t) + qI(y_{1t}^2), \end{aligned} \tag{3.26}$$

where

$$q = \beta N^*(|p_{11} - p_{12}| + |p_{12} - p_{22}|\nu^{-1}). \tag{3.27}$$

Putting

$$V_2 = q\int_0^\infty \int_{t-s}^t y_1^2(\theta)d\theta dK(s),$$

via (1.1), (2.3) we get $LV_2 = q(y_1^2(t) - I(y_{1t}^2))$. Therefore, via (3.26), (3.27) for the functional $V = V_1 + V_2$ we have

$$LV \leq (2\beta N^*|p_{11} - p_{12}| + \beta N^*|p_{12} - p_{22}|\nu^{-1} - c)y_1^2(t) + (\beta N^*|p_{12} - p_{22}|\nu - 1)y_2^2(t).$$

Thus, if

$$2\beta N^*|p_{11} - p_{12}| + \beta N^*|p_{12} - p_{22}|\nu^{-1} < c, \qquad \beta N^*|p_{12} - p_{22}|\nu < 1, \tag{3.28}$$

then via [16] the zero solution of the system (2.4) is asymptotically mean square stable.

From (3.28) it follows that

$$\frac{\beta N^*|p_{12} - p_{22}|}{c - 2\beta N^*|p_{11} - p_{12}|} < \nu < \frac{1}{\beta N^*|p_{12} - p_{22}|}. \tag{3.29}$$

Thus, if for some $c > 0$ the condition (3.25) holds then there exists $\nu > 0$ such that the conditions (3.29) (or (3.28)) hold too and therefore the zero solution of the system (2.4) is asymptotically mean square stable. From this it follows that the zero solution of the system (2.2) and therefore the equilibrium point (N^*, S^*) of the system (2.1) is stable in probability. The proof is completed.

Example 3.2. Consider the system (1.4) with the values of the parameters $\varepsilon, \mu, \rho, \beta, \gamma$ and the equilibrium point (N^*, S^*) given in Example 1.1. We consider the levels of noises $\sigma_1 = 0.028256, \sigma_2 = 0.029031$. From (3.5) it follows that $\delta_1 = 0.0003992$, $\delta_2 = 0.0004214$ and the condition (3.20) holds: $\max(\delta_1, \delta_2) < \det(A)|\mathrm{Tr}(A)|^{-1} = 0.0004436$.

Put $c = 10$. Then via (3.11) $p_{11} = 8335.7$, $p_{12} = 569.4$, $p_{22} = 1344.7$ and the condition (3.25) holds: $(\beta N^*|p_{12}-p_{22}|)^2 + 2\beta N^*|p_{11} - p_{12}| = 4.419 < 10$. Thus, the solution of the system (1.4) is stable in probability.

Let us get now two corollaries from Theorem 3.1 which simplify a verification of the stability condition (3.25). Via (3.11) we have

$$\begin{aligned} p_{12} - p_{11} &= p_{12} + \frac{c + 2a_{21}p_{12}}{2\hat{a}_{11}} = \left(1 + \frac{a_{21}}{\hat{a}_{11}}\right) \frac{a_{21}\hat{a}_{11} + ca_{12}\hat{a}_{22}}{2Z} + \frac{c}{2\hat{a}_{11}} \\ &= \frac{(a_{21} + \hat{a}_{11})a_{12}\hat{a}_{22} + Z}{2Z\hat{a}_{11}} c + \frac{(a_{21} + \hat{a}_{11})a_{21}}{2Z} = B_0 c + B_1, \end{aligned} \tag{3.30}$$

where

$$B_0 = \frac{(\mathrm{Tr}(A) + a_{12})\hat{a}_{22} - a_{12}a_{21}}{2Z}, \qquad B_1 = \frac{(a_{21} + \hat{a}_{11})a_{21}}{2Z}. \tag{3.31}$$

And similarly

$$\begin{aligned} p_{12} - p_{22} &= p_{12} + \frac{1 + 2a_{12}p_{12}}{2\hat{a}_{22}} = \left(1 + \frac{a_{12}}{\hat{a}_{22}}\right) \frac{a_{21}\hat{a}_{11} + ca_{12}\hat{a}_{22}}{2Z} + \frac{1}{2\hat{a}_{22}} \\ &= \frac{(a_{12} + \hat{a}_{22})a_{12}c}{2Z} + \frac{(a_{12} + \hat{a}_{22})a_{21}\hat{a}_{11} + Z}{2Z\hat{a}_{22}} = D_0 c + D_1, \end{aligned} \tag{3.32}$$

where

$$D_0 = \frac{(a_{12} + \hat{a}_{22})a_{12}}{2Z}, \qquad D_1 = \frac{(\mathrm{Tr}(A) + a_{21})\hat{a}_{11} - a_{12}a_{21}}{2Z}. \tag{3.33}$$

Remark 3.3. Put

$$f(c) = (\beta N^* D_0)^2 \left(c + \frac{D_1}{D_0}\right)^2 + 2\beta N^* |B_0| \left|c + \frac{B_1}{B_0}\right| - c. \tag{3.34}$$

Taking into account (3.30)–(3.34) the condition (3.25) is equivalent to the condition $f(c) < 0$.

Put now

$$\begin{aligned} S &= (\beta N^* D_0)^2 \left(\frac{D_1}{D_0} - \frac{B_1}{B_0}\right)^2 + \frac{B_1}{B_0}, \\ R_+ &= 2\beta N^* |B_0| \left(\frac{1 - 2\beta N^*|B_0|}{2(\beta N^* D_0)^2} - \frac{D_1}{D_0} + \frac{B_1}{B_0}\right), \\ R_- &= -2\beta N^* |B_0| \left(\frac{1 + 2\beta N^*|B_0|}{2(\beta N^* D_0)^2} - \frac{D_1}{D_0} + \frac{B_1}{B_0}\right), \\ Q &= \frac{1}{4(\beta N^* D_0)^2} - \frac{D_1}{D_0} - \frac{B_0^2}{D_0^2}. \end{aligned} \tag{3.35}$$

Corollary 3.1. *If the conditions (3.6), (3.7) or (3.20), (3.21) hold and $S < 0$ then the solution (N^*, S^*) of the system (2.1) is stable in probability.*

Proof. Via (3.35) from $S < 0$ it follows that $B_1 B_0^{-1} < 0$. Putting $c_0 = -B_1 B_0^{-1} > 0$ we obtain $f(c_0) = S < 0$, i.e. the condition (3.25) holds. The proof is completed.

Corollary 3.2. *If the conditions (3.6), (3.7) or (3.20), (3.21) hold and $0 \le R_+ < Q$ then the solution (N^*, S^*) of the system (2.1) is stable in probability.*

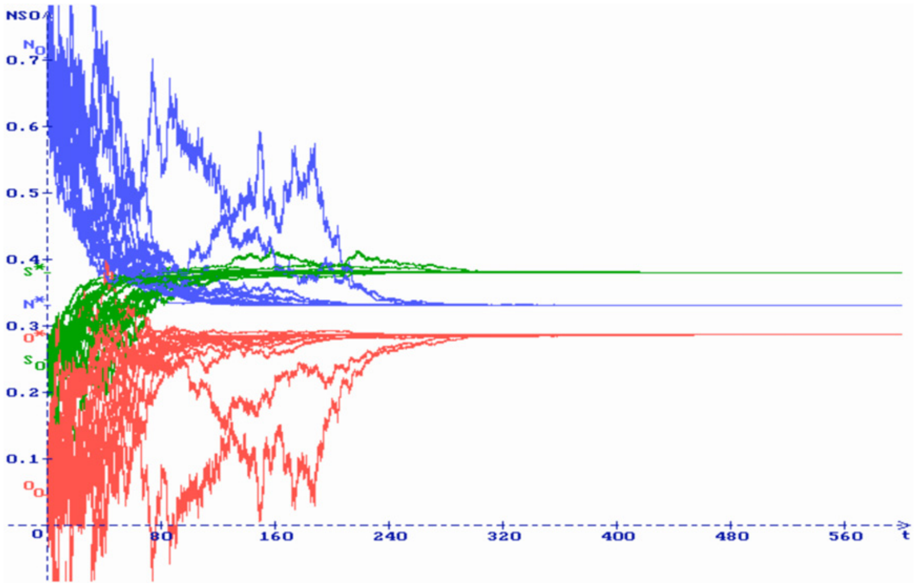

Fig. 1. 25 trajectories of the processes $N(t)$ (blue), $S(t)$ (green), $O(t)$ (red) with the values of the parameters $\mu = 0.000469$, $\gamma = 0.0003$, $\varepsilon = 0.000004$, $\rho = 0.000035$, $\beta = 0.00085$, $h = 0.1$, $\delta_1 = 0.0003992$, $\delta_2 = 0.0004214$, $N_0 = 0.704$, $S_0 = 0.25$, $O_0 = 0.046$ and the equilibrium point $N^* = 0.3311$, $S^* = 0.3814$, $O^* = 0.2875$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Proof. Let us suppose that $c + B_1 B_0^{-1} \geq 0$. Then the minimum of the function $f(c)$ is reached by

$$c_0 = \frac{1 - 2\beta N^* |B_0|}{2(\beta N^* D_0)^2} - \frac{D_1}{D_0} \geq -\frac{B_1}{B_0}.$$

Substituting c_0 into the function $f(c)$ we obtain that the condition $f(c_0) < 0$ is equivalent to the condition $0 \leq R_+ < Q$. The proof is completed.

Corollary 3.3. *If the conditions (3.6), (3.7) or (3.20), (3.21) hold and $0 < R_- < Q$ then the solution (N^*, S^*) of the system (2.1) is stable in probability.*

Proof. Let us suppose that $c + B_1 B_0^{-1} < 0$. Then the minimum of the function $f(c)$ is reached by

$$c_0 = \frac{1 + 2\beta N^* |B_0|}{2(\beta N^* D_0)^2} - \frac{D_1}{D_0} < -\frac{B_1}{B_0}.$$

Substituting c_0 into the function $f(c)$ we obtain that the condition $f(c_0) < 0$ is equivalent to the condition $0 < R_- < Q$. The proof is completed.

Example 3.3. Consider the system (2.1) with the values of the parameters from Example 1.1 and $\delta_1 = 0.0003992$, $\delta_2 = 0.0002661$. Calculating S, R_+, Q, we obtain: $S = -0.0100916 < 0$, $R_+ = 7499 < Q = 18161$. Via both Corollaries 3.1 and 3.2 the solution (N^*, S^*) of the system (2.1) is stable in probability.

Example 3.4. Consider the system (2.1) with the values of the parameters from Example 1.1 and $\delta_1 = 0.0003992$, $\delta_2 = 0.0004214$. Calculating S, R_+, Q, we obtain: $S = 0.0051611 > 0$, $R_+ = 7811 < Q = 18914$. The condition of Corollary 3.1 does not hold but from Corollary 3.2 it follows that the solution (N^*, S^*) of the system (2.1) is stable in probability.

4. Numerical simulation

Let us suppose that in (1.3) $dK(s) = \delta(s - h)ds$, where $\delta(s)$ is the Dirac delta-function, $h \geq 0$ is a delay.

In Fig. 1 trajectories of the solution of (2.1), (1.2) are shown for the values of the parameters from Examples 1.1 and 3.1: $\mu = 0.000469$, $\gamma = 0.0003$, $\varepsilon = 0.000004$, $\rho = 0.000035$, $\beta = 0.00085$, the initial values $N_0 = 0.704$, $S_0 = 0.25$, $O_0 = 0.046$, the levels of noises $\sigma_1 = 0.028256$, $\sigma_2 = 0.029031$ and the delay $h = 0.1$. One can see that all trajectories go to the equilibrium point $N^* = 0.3311$, $S^* = 0.3814$, $O^* = 0.2875$.

Putting $\beta = 0$ with the same values of the other parameters, in accordance with Example 1.1 one can see that all trajectories go to another equilibrium point $N^* = 0.7149$, $S^* = 0.1465$, $O^* = 0.1386$ (Fig. 2).

Change now the initial values on $N_0 = 1$, $S_0 = O_0 = 0$, and put again $\beta = 0.00085$. In accordance with Example 1.1 corresponding trajectories of the solution go to the equilibrium point $N^* = 0.5770$, $S^* = 0.2588$, $O^* = 0.1642$ (Fig. 3).

Numerical simulations of the processes $N(t)$, $S(t)$ and $O(t)$ were obtained with the step of discretization $\Delta = 0.01$ via the standard Euler–Maruyama scheme for stochastic differential equations.

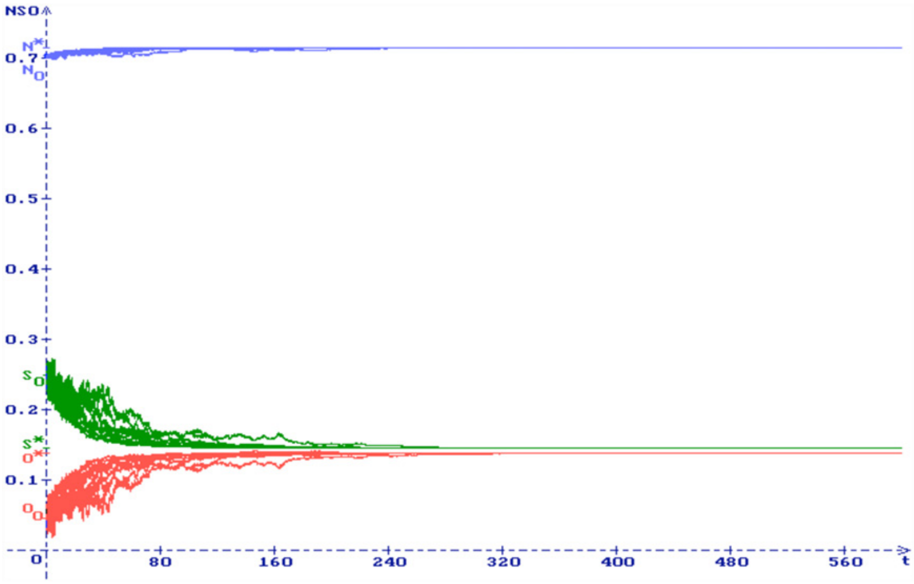

Fig. 2. 25 trajectories of the processes $N(t)$ (blue), $S(t)$ (green), $O(t)$ (red) with the values of the parameters $\mu = 0.000469$, $\gamma = 0.0003$, $\varepsilon = 0.000004$, $\rho = 0.000035$, $\beta = 0$, $h = 0.1$, $\delta_1 = 0.0003992$, $\delta_2 = 0.0004214$, $N_0 = 0.704$, $S_0 = 0.25$, $O_0 = 0.046$ and the equilibrium point $N^* = 0.7149$, $S^* = 0.1465$, $O^* = 0.1386$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

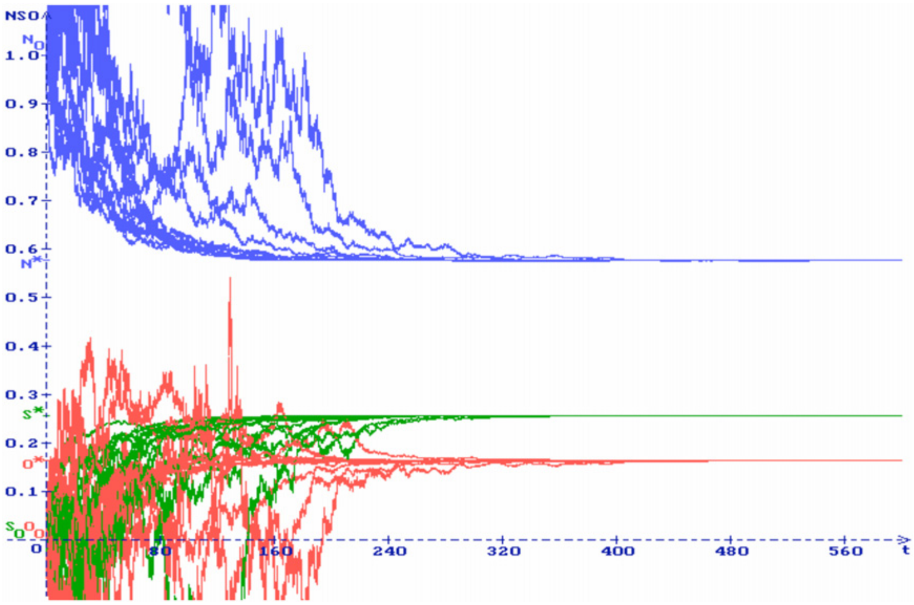

Fig. 3. 25 trajectories of the processes $N(t)$ (blue), $S(t)$ (green), $O(t)$ (red) with the values of the parameters $\mu = 0.000469$, $\gamma = 0.0003$, $\varepsilon = 0.000004$, $\rho = 0.000035$, $\beta = 0.00085$, $h = 0.1$, $\delta_1 = 0.0003992$, $\delta_2 = 0.0004214$, $N_0 = 1$, $S_0 = 0$, $O_0 = 0$ and the equilibrium point $N^* = 0.5770$, $S^* = 0.2588$, $O^* = 0.1642$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

5. Conclusions

In this work, a multidisciplinary approach based on delayed and stochastic differential equations is proposed to understand the evolution of the obesity epidemic in the Region of Valencia, Spain. Taking into account the study proposed, we note that around 70% of the Valencian population will be obese or overweight in the next years. The existence of a equilibrium point stable in probability in $N^* = 0.3311045$, $S^* = 0.3814023$, $O^* = 0.2874932$, allows us to confirm it. Note that N, S and O correspond to normal weight, overweight and obese population, respectively.

This work is an example of how delayed and stochastic models can be useful tools to model human behavior. In this case, we have worked on a specific region in Spain but this approach can be applied to any region or country. The main result of the paper is the identification of the conditions for stability in probability of the equilibrium point by stochastic perturbations. We consider that this approach can be an interesting framework for public health authorities and policy makers.

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