FUNCTIONAL DIFFERENTIAL EQUATIONS

VOLUME 24
2017, NO 1–2
PP. 57–67

ABOUT SOME ASYMPTOTIC PROPERTIES OF SOLUTION OF STOCHASTIC DELAY DIFFERENTIAL EQUATION WITH A LOGARITHMIC NONLINEARITY *

L. SHAIKHET †

Abstract. A nonlinear stochastic differential equation with delay and logarithmic nonlinearity is considered. Some properties of asymptotic behavior of the solution of this equation are discussed. In particular, the asymptotic behavior of the solution in the neighborhood of the zero and positive equilibria is described.

Key Words. Logarithmic nonlinearity, stochastic delay differential equation, asymptotic mean square stability, stability in probability.

AMS(MOS) subject classification. 34F05, 34G20, 34K20. 60G52, 60H10

Introduction. Logarithmic nonlinearity is used in a lot of applied mathematical models. For example, the dynamics of the interaction between the immune system (eector T cells) and chronic myelogenous leukaemia cancer cells in the body can be described by the following system of two ordinary nonlinear delay dierential equations [1]

$$\begin{aligned} &\dot{x}(t) = \beta_1 x(t) \ln \tfrac{K}{x(t)} - \gamma_1 x(t)y(t) - \omega\gamma_3 x(t-h(t)), \\ &\dot{y}(t) = \beta_2 \tfrac{x(t)y(t)}{\eta_1 + x(t)} - \gamma_2 x(t)y(t) + in_\alpha \gamma_4 \tfrac{y(t)y(t-\tau(t))}{\eta_2 + y(t)} - \mu_y y(t), \end{aligned} \tag{1}$$

where all parameters are positive numbers, delays $h(t)$ and $\tau(t)$ are non-negative non-increasing functions. The logarithmic term of this system has a particular interest since it vanishes both for $x(t) = 0$ and for $x(t) = K$.

A similar situation occurs also in medical models of angiogenesis [2, 3], in physics, in particular, in quantum theory [4, 5, 6] and other. But any

* Supported by the Israel Science Foundation (grant no. 1128/14) and Israeli Ministry of Absorption

† School of Electrical Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel

results about asymptotic behavior of solution of stochastic delay differential equation with a such type of nonlinearity are not found.

To show a feature of asymptotic behavior of the solution of an equation with a logarithmic nonlinearity consider for the beginning the deterministic differential equation without delay

$$\dot{x}(t) + ax(t) + \gamma x(t)\ln(x(t)) = 0, \quad x(0) > 0.$$

Using the solution of this equation

$$x(t) = \exp\left(\left(\ln x(0) + \frac{a}{\gamma}\right)e^{-\gamma t} - \frac{a}{\gamma}\right),$$

we can conclude that via $\lim\limits_{x\to+0} x\ln x = 0$ it has two equilibria: the zero and the positive $\exp\left(-\frac{a}{\gamma}\right)$, and the following statements holds.

Proposition 1. *If $\gamma > 0$ then $\lim\limits_{t\to\infty} x(t) = \exp\left(-\frac{a}{\gamma}\right)$ for arbitrary a and $x(0) > 0$. If $\gamma < 0$ then for arbitrary a: $\lim\limits_{t\to\infty} x(t) = 0$ if $x(0) \in \left(0, \exp\left(-\frac{a}{\gamma}\right)\right)$ and $\lim\limits_{t\to\infty} x(t) = \infty$ if $x(0) > \exp\left(-\frac{a}{\gamma}\right)$.*

Consider now the following delay differential equation with a logarithmic nonlinearity

$$\begin{aligned} &\dot{x}(t) + ax(t) + bx(t-h) + \gamma x(t)\ln(x(t)) = 0, \\ &t \geq 0, \quad x(s) = \phi(s) > 0, \quad s \in [-h, 0]. \end{aligned} \tag{2}$$

The equation (1.1) has both the positive and the zero equilibria

$$x^* = \exp\left(-\frac{a+b}{\gamma}\right), \quad x_0^* = 0. \tag{3}$$

Below behavior of the positive solution of the equation (2) is investigated under stochastic perturbations of the type of white noise, that are directly proportional to the deviation of the solution $x(t)$ from the positive equilibrium x^* or from the zero equilibrium, i.e., of the Ito stochastic differential equations [7]

$$\dot{x}(t) + ax(t) + bx(t-h) + \gamma x(t)\ln(x(t)) + \sigma(x(t) - x^*)\dot{w}(t) = 0 \tag{4}$$

and

$$\dot{x}(t) + ax(t) + bx(t-h) + \gamma x(t)\ln(x(t)) + \sigma x(t)\dot{w}(t) = 0, \tag{5}$$

where $x(s) = \phi(s) > 0$, $s \in [-h, 0]$, σ is a constant and $w(t)$ is the standard Wiener process [7, 8].

Note that the positive equilibrium x^* is the solution of the equation (4) and the zero equilibrium is the solution of the equation (5).

Below some generalisations of Proposition 1 for the equations (4) and (5) are obtained.

REMARK 1. *In the case* $\gamma = 0$ *the necessary and sufficient condition for asymptotic mean square stability of the zero solution of the equation* (5) *is known and has the form [8]*

$$
\begin{gathered}
a + b > 0, \qquad G^{-1} > \tfrac{1}{2}\sigma^2, \\
G = \begin{cases} \frac{1 + bq^{-1}\sin(qh)}{a + b\cos(qh)}, & b > |a|, \quad q = \sqrt{b^2 - a^2}, \\ \frac{1+ah}{2a}, & b = a > 0, \\ \frac{1 + bq^{-1}\sinh(qh)}{a + b\cosh(qh)}, & a > |b|, \quad q = \sqrt{a^2 - b^2}. \end{cases}
\end{gathered} \tag{6}
$$

REMARK 2. *In the case* $\gamma = 0$ *besides* (6) *the equation* (5) *has also a simple delay-independent sufficient stability condition: if*

$$
a > |b| + \frac{1}{2}\sigma^2 \tag{7}
$$

then the zero solution of the equation (5) *is asymptotically mean square stable [8].*

1. Stability analysis.

1.1. The equation (4).

THEOREM 1. *If*

$$
\gamma > \begin{cases} \frac{1}{2}\sigma^2 & \text{for} \quad b \le 0, \\ 2b + \frac{1}{2}\sigma^2 & \text{for} \quad b > 0, \end{cases} \tag{8}
$$

then the positive equilibrium x^* *of the equation* (4) *is stable in probability.*

Proof. Substituting $x(t) = y(t) + x^*$ into the equation (4) and using via (3) the representation

$$
\ln(y(t) + x^*) = \ln(x^*) + \ln(1 + y(t)/x^*) = -(a + b)/\gamma + \ln(1 + y(t)/x^*),
$$

we obtain

$$
\begin{aligned}
&\dot{y}(t) = -a(y(t) + x^*) - b(y(t-h) + x^*) \\
&-(y(t) + x^*)(-(a+b) + \gamma \ln(1 + y(t)/x^*)) - \sigma y(t)\dot{w}(t) \\
&= \; by(t) - by(t-h) - \gamma x^*(1 + y(t)/x^*)\ln(1 + y(t)/x^*) - \sigma y(t)\dot{w}(t)
\end{aligned}
$$

or

$$\dot{y}(t)-by(t)+by(t-h)+\gamma x^*(1+y(t)/x^*)\ln(1+y(t)/x^*)+\sigma y(t)\dot{w}(t)=0. \tag{9}$$

So, stability of the solution x^* of the equation (4) is equivalent to stability of the zero solution of the equation (9).

Using the representation $\ln(1+x)=x+o(x)$, where $\lim_{x\to 0}\frac{o(x)}{x}=0$, we obtain that the linear part of the equation (9) has the form

$$\dot{z}(t)+(\gamma-b)z(t)+bz(t-h)+\sigma z(t)\dot{w}(t)=0 \tag{10}$$

and is a particular case of the equation (5) with $\gamma=0$.

Via Remark 2 if the condition (8) holds then the zero solution of the equation (10) is asymptotically mean square stable. Taking into account that the order of a nonlinearity of the equation (9) is higher than one we can conclude that the condition (8) at the same time is a sufficient condition for stability in probability of the zero solution of the equation (9) and therefore of the solution x^* of the equation (4) [8]. The proof is completed. ∎

Remark 3. *Via Remark 1 we have also the delay-dependent condition for asymptotic mean square stability of the zero solution of the equation* (10) *and therefore the condition for stability in probability of the solution x^* of the equation* (4) *in the form*

$$\begin{gathered} G^{-1}>p=\tfrac{1}{2}\sigma^2, \\ G=\begin{cases} \frac{1+bq^{-1}\sin(qh)}{\gamma+b(\cos(qh)-1)}, & 2b>\gamma>0, \quad q=\sqrt{\gamma(2b-\gamma)}, \\ \frac{1+bh}{2b}, & \gamma=2b>0, \\ \frac{1+bq^{-1}\sinh(qh)}{\gamma+b(\cosh(qh)-1)}, & \gamma>b+|b|, \quad q=\sqrt{\gamma(\gamma-2b)}. \end{cases} \end{gathered} \tag{11}$$

Example 1. *Let be $a=-0.5$, $b=0.2$, $\gamma=0.45$, $h=1$, $\sigma=0.25$. By these values of the parameters we have $p=0.03125$, $x^*=1.9477$. So, the condition* (8) *holds ($\gamma>2b+p=0.43125$) and therefore the equilibrium x^* stable in probability. Note that the condition* (11) *holds too: $G^{-1}=0.3766>p$. In Fig.1 300 trajectories (green) of the equation* (4) *solution with the initial function $\phi(s)=x_0cos(s)$, $s\in[-h,0]$, for different values of x_0 converge to the equilibrium x^*. But if the initial function is placed far enough from the equilibrium x^* ($x_0=0.5$) then the solution does not converge to the equilibrium that is quite normal for stability in probability and some trajectories (red) can go to zero.*

1.2. The equation (5)**.** Below $\mathbf{E}$, $\mathbf{P}$ and L denote respectively the expectation, a probability and the generator of the equation (5), $p=\frac{1}{2}\sigma^2$.

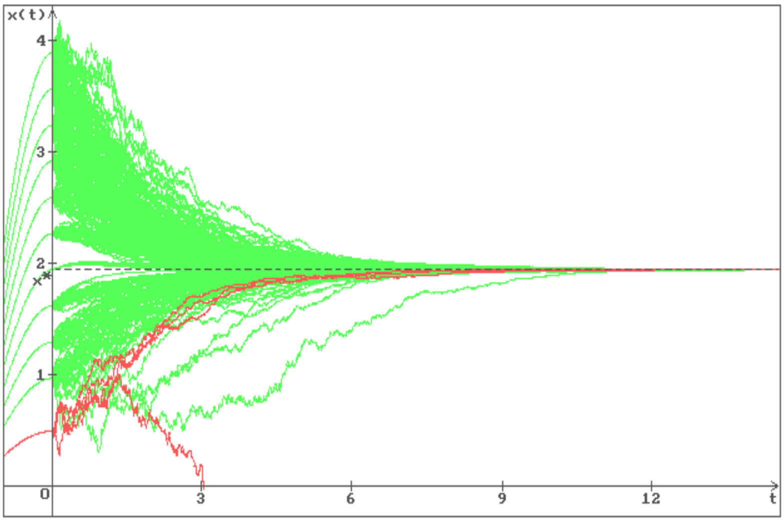

FIG. 1. *304 trajectories of the equation* (4)*:* $a = -0.5$, $b = 0.2$, $\gamma = 0.45$, $h = 1$, $\sigma = 0.25$, $x^* = 1.9477$

LEMMA 1. *If* $x(s) > 0$ *for* $s \leq 0$ *then* $x(t) > 0$ *for* $t \geq 0$.

Proof. Via (5) and Ito's formula we have

$$d\ln(x(t)) = -\left(a + b\frac{x(t-h)}{x(t)} + \gamma\ln(x(t)) + p\right)dt - \sigma dw(t)$$

and after integration [7]

$$\begin{aligned} x(t) = \exp\Big\{ &\left(\ln x(0) + \tfrac{a+p}{\gamma}\right)e^{-\gamma t} - \tfrac{a+p}{\gamma} \\ &-b\int_0^t \tfrac{x(s-h)}{x(s)}e^{-\gamma(t-s)}ds - \sigma\int_0^t e^{-\gamma(t-s)}dw(s)\Big\} > 0. \end{aligned}$$

The proof is completed. ∎

THEOREM 2. *For arbitrary* a *and* $x(s) > 0$, $s \leq 0$, *the solution of the equation* (5) *satisfies the conditions:*

(i) If $\gamma > 0$ *and* $b \geq 0$ *then* $\limsup_{t\to\infty} \mathbf{E}\ln(x(t)) \leq -\frac{a+p}{\gamma}$.

(ii) If $\gamma < 0$, $b \geq 0$ *and* $x(0) < \exp\left(-\frac{a+p}{\gamma}\right)$ *then* $\lim_{t\to\infty} \mathbf{E}\ln(x(t)) = -\infty$.

(iii) If $\gamma < 0$, $b \leq 0$ *and* $x(0) > \exp\left(-\frac{a+p}{\gamma}\right)$ *then* $\lim_{t\to\infty} \mathbf{E}\ln(x(t)) = \infty$.

Proof. For the function $\ln(x(t))$ via (5), $b \geq 0$, $x(t) > 0$ we have

$$L\ln(x(t)) = -a - b\frac{x(t-h)}{x(t)} - \gamma\ln(x(t)) - p \leq -a - p - \gamma\ln(x(t)).$$

From this via Dynkin's formula [7]

$$\frac{d}{dt}\mathbf{E}\ln(x(t)) = \mathbf{E}L\ln(x(t)) \le -a - p - \gamma\mathbf{E}\ln(x(t)).$$

Thus, $\mathbf{E}\ln(x(t)) \le y(t)$, where $y(t)$ is defined by the differential equation $\dot{y}(t) = -a - p - \gamma y(t)$ with the initial value $y(0) = \ln(x(0))$ and the solution

$$y(t) = \left(\ln(x(0)) + \frac{a+p}{\gamma}\right)e^{-\gamma t} - \frac{a+p}{\gamma}. \tag{12}$$

Via $\gamma > 0$ from (12) follows (i). If $\gamma < 0$, $b \ge 0$ and $x(0) < \exp\left(-\frac{a+p}{\gamma}\right)$ then (12) implies (ii). In the case $b \le 0$ we obtain $\mathbf{E}\ln(x(t)) \ge y(t)$. So, if $\gamma < 0$ and $x(0) > \exp\left(-\frac{a+p}{\gamma}\right)$ then (12) implies (iii). The proof is completed. ∎

REMARK 4. *In the case $b = p = 0$ Theorem 4 coincides with Proposition 1.1.*

REMARK 5. *Via Jensen's inequality $\mathbf{E}\ln(x(t)) \le \ln(\mathbf{E}x(t))$. So, from (i) generally speaking does not follow that $\limsup_{t\to\infty}\mathbf{E}x(t) \le \exp(-\frac{a+p}{\gamma})$.*

THEOREM 3. *Let be $\gamma < 0$, $b \ge 0$, $a > p$. If for some $t_0 \ge 0$ the solution $x(t)$ of the equation (5) satisfies the condition $\mathbf{P}\{x(t) \le 1\} = 1$, $t \ge t_0$, then $\lim_{t\to\infty}\mathbf{E}x^2(t) = 0$.*

Proof. Via (5) for $V(x) = x^2$ we have

$$LV(x(t)) = 2(-a + |\gamma|\ln(x(t)) + p)x^2(t) - 2bx(t)x(t-h). \tag{13}$$

From (13) and the lemma conditions it follows that $LV(x(t)) \le -2(a - p)x^2(t)$. So, via Dynkin's formula $\frac{d}{dt}\mathbf{E}x^2(t) \le -2(a-p)\mathbf{E}x^2(t)$, $t \ge t_0$, and $\mathbf{E}x^2(t) \le \mathbf{E}x^2(t_0)e^{-2(a-p)(t-t_0)} \to 0$ if $t \to \infty$. The proof is completed. ∎

THEOREM 4. *Let be $\gamma < 0$, $t_0 \ge 0$, $x_{mb} = \exp((a+b-p)/|\gamma|)$, $x_\varepsilon = \exp(\varepsilon/|\gamma|)$, ε a small enough positive number. Then the solution of the equation (5) satisfies the following conditions:*

(i) If $b \ge 0$ and $\mathbf{P}\{x(t) < x_{m0}x_\varepsilon^{-1}\} = 1$, $t \ge t_0$, then $\lim_{t\to\infty}\mathbf{E}x^2(t) = 0$.
(ii) If $b \le 0$ and $\mathbf{P}\{x(t) < x_{mb}x_\varepsilon^{-1}\} = 1$, $t \ge t_0$, then $\lim_{t\to\infty}\mathbf{E}x^2(t) = 0$.
(iii) If $b \le 0$ and $\mathbf{P}\{x(t) > x_{m0}x_\varepsilon\} = 1$, $t \ge t_0$, then $\lim_{t\to\infty}\mathbf{E}x^2(t) = \infty$.

Proof. (i) For the function $V(x) = x^2$ via (5), (13) we have

$$LV(x(t)) \le 2(-a + |\gamma|(a - p - \varepsilon)/|\gamma| + p)x^2(t) = -2\varepsilon x^2(t).$$

Via Dynkin's formula [7] we obtain $\frac{d}{dt}\mathbf{E}x^2(t) \le -2\varepsilon\mathbf{E}x^2(t)$, $t \ge t_0$, and therefore $\mathbf{E}x^2(t) \le \mathbf{E}x^2(t_0)e^{-2\varepsilon(t-t_0)} \to 0$ if $t \to \infty$.

(ii) For the function $V_1(x) = x^2$ via (5), (13) we obtain

$$LV_1(x(t)) \leq 2(-a + |\gamma|(a + b - p - \varepsilon)/|\gamma| + p)x^2(t) + |b|(x^2(t) + x^2(t-h))$$
$$= (b - 2\varepsilon)x^2(t) + |b|x^2(t-h).$$

Using the additional functional $V_2(x_t) = |b| \int_{t-h}^{t} x^2(s)ds$ with $LV_2(x_t) = |b|(x^2(t) - x^2(t-h))$, for the functional $V = V_1 + V_2$ we obtain $LV(x_t) \leq -2\varepsilon x^2(t)$ and therefore [8] $\mathbf{E}x^2(t) \to 0$ if $t \to \infty$.

(iii) For the function $V(x) = x^2$ via (5), (13) we have

$$LV(x(t)) \geq 2(-a + |\gamma|(a - p + \varepsilon)/|\gamma| + p)x^2(t) = 2\varepsilon x^2(t).$$

Therefore, $\mathbf{E}x^2(t) \geq \mathbf{E}x^2(t_0)e^{2\varepsilon(t-t_0)} \to \infty$ if $t \to \infty$. The proof is completed. $\square$

Example 2. *Let be* $a = -0.95$, $b = 0.1$, $\gamma = 0.95$, $h = 2$, $\sigma = 0.5$. *By these values of the parameters we have* $p = 0.125$, $-(a + p)/\gamma = 0.8684$. *In Fig.2 1000 trajectories (green)* $\ln(x(t))$ *of the equation* (5) *solution are shown with the initial function* $\phi(s) = 2.2$, $s \in [-h, 0]$, $\mathbf{E}\ln(x(t))$ *(red),* $\ln(\mathbf{E}x(t))$ *(yellow) and* $y(t)$ *(blue) given in* (12). *The conditions of Theorem 4 (i) hold, so,* $\mathbf{E}\ln(x(t)) \leq y(t)$. *In accordance with Remark 5 one can see that* $\mathbf{E}\ln(x(t)) \leq \ln(\mathbf{E}x(t))$. *Besides in Fig.2 one can see that* $\ln(\mathbf{E}x(t)) \leq y(t)$ *too. It was not proved here but maybe it possible to be proved too.*

Example 3. *Let be* $a = 0.05$, $b = 0.02$, $\gamma = -0.7$, $h = 0.4$, $\sigma = 0.25$. *By these values of the parameters we have* $p = 0.03125$, $x^* = 1.105$, $\exp(-(a+p)/\gamma) = 1.123$, $x_{m0} = 1.027$. *In Fig.3 1000 trajectories (green) of the equation* (5) *solution are shown with the initial function* $\phi(s) = 0.619\cos(s)$, $s \in [-h, 0]$. *The conditions of Theorem 2 (ii), Theorem 3 and Theorem 4 (i) hold, all trajectories converge to zero and* $\sqrt{\mathbf{E}x^2(t)}$ *(red) converges to zero too.*

Example 4. *Let be* $a = 1.5$, $b = 0.2$, $\gamma = -1$, $h = 0.3$, $\sigma = 0.25$. *By these values of the parameters we have* $p = 0.03125$, $x^* = 5.4739$, $\exp(-(a+p)/\gamma) = 4.6240$, $x_{m0} = 4.3438$. *In Fig.4 500 trajectories of the equation* (5) *solution are shown with* $x(s) = x_0 cos(s)$, $s \in [-h, 0]$, $x_0 = 0.75x_{m0} = 3.2579$. *The conditions of Theorem 2 (ii), Theorem 3 and Theorem 4 (i) hold and therefore* $\sqrt{\mathbf{E}x^2(t)}$ *(red) converges to zero.*

Example 5. *Putting in Example 4* $b = -0.2$ *with the same values of all other parameters we obtain* $x^* = 3.6693$, $x_{mb} = 3.5564$. *In Fig.5 500 trajectories of the equation* (5) *solution are shown with* $x(s) = x_0 cos(s)$, $s \in [-h, 0]$, $x_0 = 0.65x_m = 2.3117$. *The conditions of Theorem 3 and Theorem 4 (ii) hold and therefore* $\sqrt{\mathbf{E}x^2(t)}$ *(red) converges to zero.*

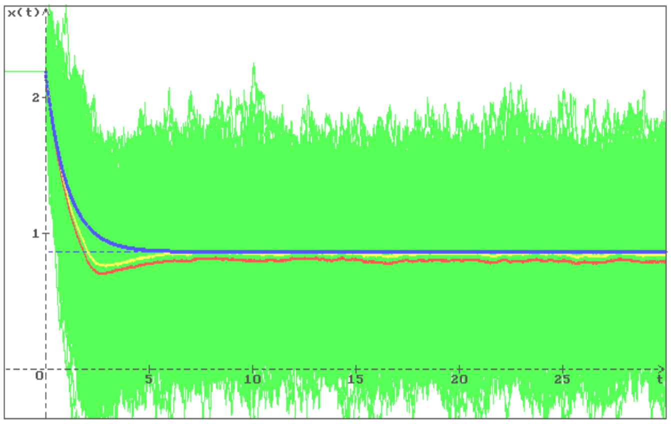

FIG. 2. *1000 trajectories (green) of* $\ln(x(t))$ *of the equation* (5), $\mathbf{E}\ln(x(t))$ *(red),* $\ln(\mathbf{E}x(t))$ *(yellow),* $y(t)$ *(blue):* $a = -0.95$, $b = 0.1$, $\gamma = 0.95$, $h = 2$, $\sigma = 0.5$, $-(a+p)/\gamma = 0.8684$

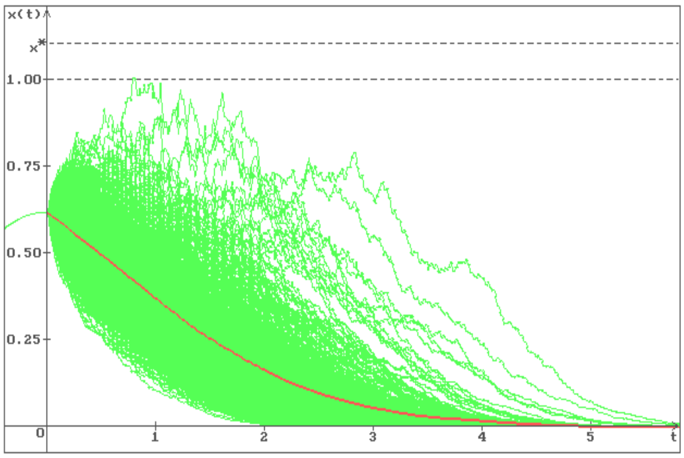

FIG. 3. *1000 trajectories (green) of the equation* (5) *solution and* $\sqrt{\mathbf{E}x^2(t)}$ *(red):* $a = 0.05$, $b = 0.02$, $\gamma = -0.7$, $h = 0.4$, $\sigma = 0.25$, $x^* = 1.105$

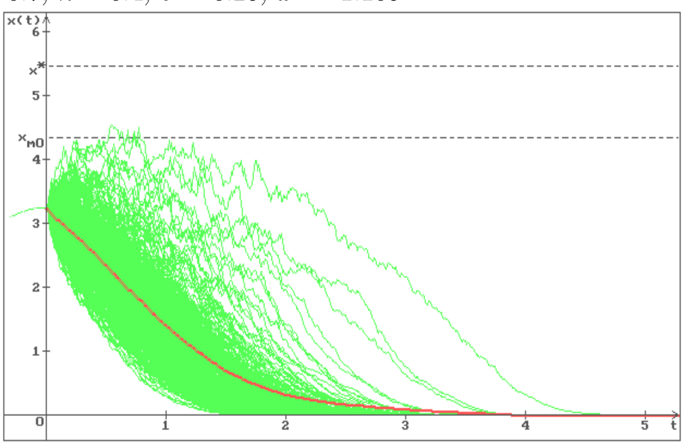

FIG. 4. *500 trajectories (green) of the equation* (5) *solution and* $\sqrt{\mathbf{E}x^2(t)}$ *(red):* $a = 1.5$, $b = 0.2$, $\gamma = -1$, $h = 0.3$, $\sigma = 0.25$, $x^* = 5.4739$, $x_{m0} = 4.3438$

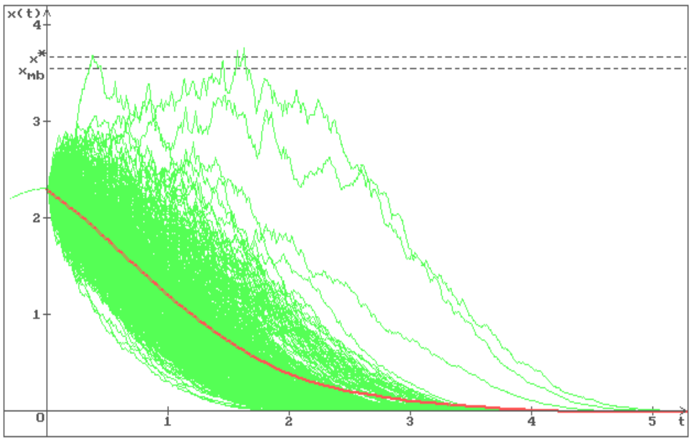

FIG. 5. *500 trajectories (green) of the equation* (5) *solution and* $\sqrt{\mathbf{E}x^2(t)}$ *(red):* $a = 1.5$, $b = -0.2$, $\gamma = -1$, $h = 0.3$, $\sigma = 0.25$, $x^* = 3.6693$, $x_{mb} = 3.5564$

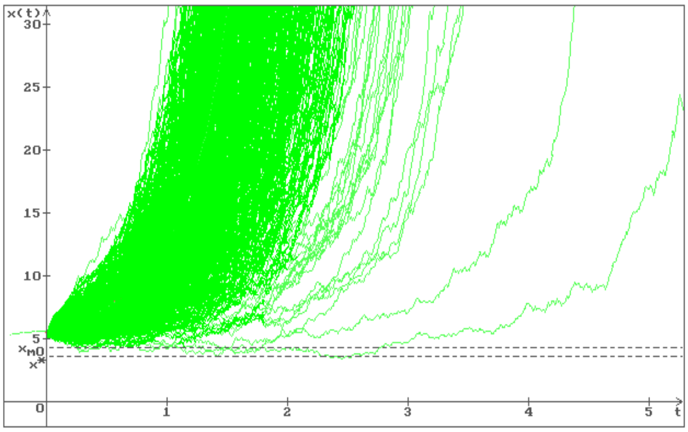

FIG. 6. *500 trajectories (green) of the equation* (5) *solution and* $\sqrt{\mathbf{E}x^2(t)}$ *(red):* $a = 1.5$, $b = -0.2$, $\gamma = -1$, $h = 0.3$, $\sigma = 0.25$, $x^* = 3.6693$, $x_{m0} = 4.3438$, $x(0) = 5.6469$

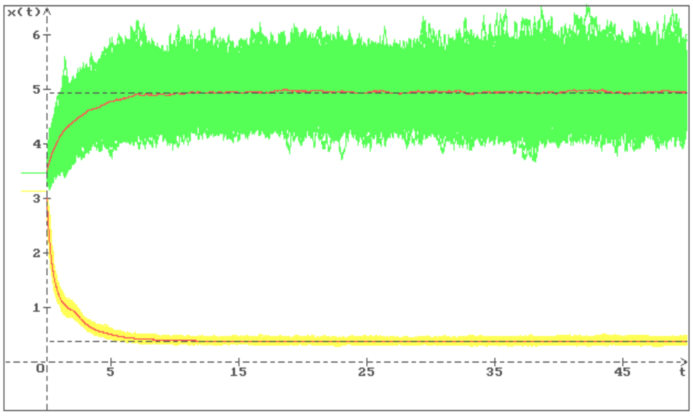

FIG. 7. *300 trajectories (green) of the equation* (5) *solution and* $\mathbf{E}x(t)$ *(red):* $b = -0.25$, $\gamma = 0.75$, $h = 2$, $\sigma = 0.1$*;* $a = 0.95$, $x^* = 0.3932$, $x(0) = 3.1459$ *(yellow);* $a = -0.95$, $x^* = 4.9530$, $x(0) = 3.4671$ *(green)*

Example 6. *For the values of the parameters from Example 5 we have* $x^* = 3.6693$, $\exp(-(a+p)/\gamma) = 4.6240$, $x_{m0} = 4.3438$. *In Fig.6 500 trajectories of the equation (1.4) solution are shown with* $x(s) = x_0 cos(s)$, $s \in [-h, 0]$, $x_0 = 1.3x_{m0} = 5.6469$. *The conditions (iii) of Theorems 2 and 4 hold and therefore* $\lim_{t\to\infty} \mathbf{E}x^2(t) = \infty$. *One can see that all 500 trajectories of the equation* (5) *solution go to the infinity.*

Remark 6. *Note that the equilibrium* x^* *is not a solution of the equation* (5). *So, the solution* $x(t)$ *of the equation* (5) *cannot converge to* x^*. *Substituting* $x(t) = z(t)+x^*$ *into the equation* (5) *and rejecting nonlinear terms, similarly to* (10) *we obtain* $\dot{z}(t) + (\gamma - b)z(t) + bz(t-h) + \sigma(z(t) + x^*)\dot{w}(t) = 0$, *and therefore the expectation* $e(t) = \mathbf{E}z(t)$ *satisfies the equation* $\dot{e}(t) + (\gamma - b)e(t) + be(t-h) = 0$. *Via (2.1) we obtain the following statement: if* $\gamma > b + |b|$ *then* $\lim_{t\to\infty} e(t) = 0$, *i.e., the solution of the equation* (5) *in the first approximation satisfies the condition* $\lim_{t\to\infty} \mathbf{E}x(t) = x^*$ *(asymptotic convergence of the mean). Via*

$$|\mathbf{E}x(t) - x^*| = |\mathbf{E}(x(t) - x^*)| \le \mathbf{E}|x(t) - x^*| \le \sqrt{\mathbf{E}|x(t) - x^*|^2} \to 0$$

asymptotic convergence of the mean follows from asymptotic convergence in the mean and asymptotic convergence in the mean square but not vice verse.

Example 7. *Let be* $a = 0.95$, $b = -0.25$, $\gamma = 0.75$, $\sigma = 0.1$, $h = 2$. *In Fig.7 300 trajectories (yellow) with* $x(0) = 3.1459$ *place around the equilibrium* $x^* = 0.3932$ *and* $\lim_{t\to\infty} \mathbf{E}x(t) = x^*$ *(red). For* $a = -0.95$ *we obtain that 300 trajectories (green) with* $x(0) = 3.4671$ *place around the equilibrium* $x^* = 4.9530$ *and* $\lim_{t\to\infty} \mathbf{E}x(t) = x^*$ *(red). So, in the both cases we obtain asymptotic convergence of the mean.*

Note that the expectations in examples above are calculated as arithmetical mean of the trajectories, i.e., $\mathbf{E}f(x(t)) = \frac{1}{n}\sum_{i=1}^{n} f(x_i(t))$, where $x_i(t)$ are trajectories of the equation (5) solution and n is the number of the considered trajectories.

2. Conclusions. In the paper differential equation with delay and logarithmic nonlinearity is considered. Asymptotic behavior of the solution of this equation is investigated in the neighborhood of the positive and zero equilibria under stochastic perturbations that are directly proportional to the deviation of the system state from the equilibrium. The conditions are obtained by which the solution converges to one of the equilibria or to the infinity. All obtained properties are illustrated by examples with numerical simulation of the solution of the considered equation. It is expected that the obtained first results in this direction can be applied to investigation of more complicated systems with a logarithmic nonlinearity.

REFERENCES

[1] L. Berezansky, S. Bunimovich-Mendrazitsky, A. Domoshnitsky, *A mathematical model with time-varying delays in the combined treatment of chronic myeloid leukemia,* Advances in Difference Equations **2012:217** (2012), 13 p.

[2] P. Amster, L. Berezansky, and L. Idels, *Stability of Hahnfeldt angiogenesis models with time lags,* Mathematical and Computer Modelling, **55**(2012), 2052–2060.

[3] L. Berezansky, E. Braverman, and L. Idels, *Effect of treatment on the global dynamics of delayed pathological angiogenesis models,* Journal of Theoretical Biology, **363**(2014), 13–21.

[4] K. Zloshchastiev, *Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences,* Gravitation and Cosmology, **16**(2010), 288–297.

[5] A. Avdeenkov and K. Zloshchastiev, *Quantum Bose liquids with logarithmic nonlinearity: self-sustainability and emergence of spatial extent,* Journal of Physics B: Atomic, Molecular and Optical Physics, **44**(2011), Number 19, 195–303.

[6] T. Yamano, *Localized traveling wave solution for a logarithmic nonlinear Schrdinger equation,* Wave Motion, **67**(2016), 116–120.

[7] I. Gikhman and A. Skorokhod, *Stochastic differential equations*, Springer, 1972.

[8] L. Shaikhet, *Lyapunov functionals and stability of stochastic functional differential equations,* Springer Science & Business Media, 2013.