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STABILIZATION OF INVERTED PENDULUM BY CONTROL WITH DELAY

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ABSTRACT: The problem of stabilization of the inverted pendulum is considered. Unlike of the classical way of stabilization in which the stabilized control is a linear combination of the states and velocities of the pendulum here another way of stabilization is proposed. It is supposed that only the trajectory of the pendulun can be observed and stabilized control depends on whole trajectory of the pendulum.

AMS (MOS) Subject Classification. 50A10, 50B20

1. STATEMENT OF THE PROBLEM

The problem of stabilization of the inverted pendulum is very popular among the researches (see, for instance, Kapitza, 1965; Levi, 1988; Blackburn et al., 1992; Acheson, 1993; Acheson & Mullin, 1993; Levi & Weckesser, 1995). The linearized mathematical model of the controlled inverted pendulum can be described by linear differential equation of second order

$$\ddot{x}(t) - ax(t) = u(t), \qquad a > 0, \qquad t \geq 0. \tag{1.1}$$

The classical way of stabilization of the system (1.1) uses the control $u(t)$ in the form $u(t) = -b_1 x(t) - b_2 \dot{x}(t)$, $b_1 > a$, $b_2 > 0$.

But this type of control which represents an instanteneous feedback is quite difficult to realize because usually we need some finite time to make measurements of the coordinates and velocoties, to treat the results of the measurements and to implement them in the control action.

Here another way of stabilization is proposed. It is supposed that only the trajectory of the pendulum is observed, control $u(t)$ depends on the previous values of the trajectory $x(s)$, $s \leq t$, and has the form

$$u(t) = \int_0^\infty dK(\tau)x(t-\tau). \tag{1.2}$$

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where kernel $K(\tau)$ is a function of bounded variation on $[0,\infty]$ and the integral is understood in the Stiltjes sense.

It means in particular that both distributed and discrete delays can be used depending on the concrete choise of the kernel $K(\tau)$.

The initial condition for the system (1.1), (1.2) has the form

$$x(s) = \varphi(s), \qquad \dot{x}(s) = \dot{\varphi}(s), \qquad s \leq 0, \tag{1.3}$$

where $\varphi(s)$ is a given continuously differentiable function.

Definition 1.1. *The zero solution of the system (1.1)-(1.3) is called stable if for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $\max\{|x(t)|, |\dot{x}(t)|\} < \epsilon$ *for all* $t \geq 0$ *if* $\|\varphi\| = \sup_{s\leq 0}(|\varphi(s)|+|\dot{\varphi}(s)|) < \delta$. *If, besides,* $\lim_{t\to\infty} x(t) = 0$ *and* $\lim_{t\to\infty} \dot{x}(t) = 0$ *for every initial function* φ, *then the zero solution of the system (1.1)-(1.3) is called asymptotically stable in the whole.*

2. STABILIZATION BY CONTROL DEPENDING ON TRAJECTORY

Let us show that the inverted pendulum (1.1) can be stabilized by the control (1.2). Substituting (1.2) into (1.1) and putting $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$ we obtain the system of differential equations

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = ax_1(t) + \int_0^\infty dK(\tau)x_1(t-\tau). \tag{2.1}$$

To prove the asymptotic stability of the system (2.1) we will use the method of Lyapunov functionals construction, which was proposed in (Kolmanovskii & Shaikhet, 1994). This method consists of four steps. Corresponding to the first step of the method transform the system (2.1) in the following way. Put

$$k_i = \int_0^\infty \tau^i dK(\tau), \quad i = 0,1, \qquad k_j = \int_0^\infty \tau^j |dK(\tau)|, \quad j = 2,3. \tag{2.2}$$

Since

$$\int_{t-\tau}^t x_2(s)ds = \int_{t-\tau}^t \dot{x}_1(s)ds = x_1(t) - x_1(t-\tau)$$

then using (2.2) we have

$$\int_0^\infty dK(\tau)x_1(t-\tau) = k_0x_1(t) - \int_0^\infty dK(\tau)\int_{t-\tau}^t x_2(s)ds.$$

Therefore from the second equation of the system (2.1) it follows

$$\dot{x}_2(t) = (a+k_0)x_1(t) - \int_0^\infty dK(\tau)\int_{t-\tau}^t x_2(s)ds. \tag{2.3}$$

Put

$$a_1 = -(a+k_0), \qquad z(t) = x_2(t) - \int_0^\infty dK(\tau)\int_{t-\tau}^t (s-t+\tau)x_2(s)ds. \tag{2.4}$$

Stabilization of Inverted Pendulum

Using (2.2) we have

$$\frac{d}{dt}\int_0^\infty dK(\tau)\int_{t-\tau}^t (s-t+\tau)x_2(s)ds = k_1x_2(t) - \int_0^\infty dK(\tau)\int_{t-\tau}^t x_2(s)ds. \quad (2.5)$$

Subtracting (2.5) from (2.3) and using (2.4) we reduce the system (2.1) to the equations

$$\dot{x}_1(t) = x_2(t), \qquad \dot{z}(t) = -a_1x_1(t) - k_1x_2(t). \quad (2.6)$$

Following the second step of the method of Lyapunov functionals construction we consider the auxiliary system of ordinary differential equations

$$\dot{y}_1(t) = y_2(t), \qquad \dot{y}_2(t) = -a_1y_1(t) - k_1y_2(t). \quad (2.7)$$

Let us assume that $a_1 > 0$, $k_1 > 0$. Remark that these inequalities represent necessary and sufficient conditions for asymptotic stability of system (2.7). Therefore there exists Lyapunov function $v = v(y_1, y_2)$, for which

$$\dot{v} = -y_1^2 - y_2^2. \quad (2.8)$$

Choosing the function v in the form $v = p_{11}y_1^2 + 2p_{12}y_1y_2 + p_{22}y_2^2$, and using equations (2.7), (2.8) we get the system of equations for p_{11}, p_{12}, p_{22}:

$$p_{11} - k_1p_{12} - a_1p_{22} = 0, \qquad 2a_1p_{12} = 1, \qquad 2(k_1p_{22} - p_{12}) = 1. \quad (2.9)$$

The solution of the system (2.9) has the form

$$p_{11} = \frac{a_1p + k_1}{2a_1}, \qquad p_{12} = \frac{1}{2a_1}, \qquad p_{22} = \frac{p}{2a_1}, \qquad p = \frac{a_1 + 1}{k_1}. \quad (2.10)$$

Following the third step of the method of Lyapunov functionals construction we will construct Lyapunov functional for the system (2.6) in the form $V = V_1 + V_2$, where

$$V_1 = p_{11}x_1^2(t) + 2p_{12}x_1(t)z(t) + p_{22}z^2(t), \quad (2.11)$$

p_{11}, p_{12}, p_{22} and $z(t)$ are described by (2.10) and (2.4).

Calculating $\dot{V}_1$ by virtue of (2.6), (2.4) we have

$$\begin{aligned}\dot{V}_1 = &- 2p_{12}a_1x_1^2(t) - 2(k_1p_{22} - p_{12})x_2^2(t) + 2(p_{11} - k_1p_{12} - a_1p_{22})x_1(t)x_2(t) + \\ &+ 2p_{22}a_1\int_0^\infty dK(\tau)\int_{t-\tau}^t (s-t+\tau)x_1(t)x_2(s)ds + \\ &+ 2(k_1p_{22} - p_{12})\int_0^\infty dK(\tau)\int_{t-\tau}^t (s-t+\tau)x_2(t)x_2(s)ds.\end{aligned}$$

By virtue of (2.9), (2.10) it is easy to get that for arbitrary $\gamma > 0$ we have

$$\dot V_1 = -x_1^2(t) - x_2^2(t) + p\int_0^\infty dK(\tau)\int_{t-\tau}^t (s-t+\tau)x_1(t)x_2(s)ds + \\ + \int_0^\infty dK(\tau)\int_{t-\tau}^t (s-t+\tau)x_2(t)x_2(s)ds \le \\ \le -x_1^2(t) - x_2^2(t) + \frac{p}{2}\int_0^\infty |dK(\tau)|\int_{t-\tau}^t (s-t+\tau)\Big(\gamma x_1^2(t) + \frac{1}{\gamma}x_2^2(s)\Big)ds + \\ + \frac{1}{2}\int_0^\infty |dK(\tau)|\int_{t-\tau}^t (s-t+\tau)(x_2^2(t) + x_2^2(s))ds = \\ = -\Big(1 - \frac{\gamma p k_2}{4}\Big)x_1^2(t) - \Big(1 - \frac{k_2}{4}\Big)x_2^2(t) + \alpha\int_0^\infty |dK(\tau)|\int_{t-\tau}^t (s-t+\tau)x_2^2(s)ds, \tag{2.12}$$

where

$$\alpha = \frac{1}{2}\Big(1 + \frac{p}{\gamma}\Big). \tag{2.13}$$

Following the fourth step of the method of Lyapunov functionals construction we choose the functional V_2 in the form

$$V_2 = \frac{\alpha}{2}\int_0^\infty |dK(\tau)|\int_{t-\tau}^t (s-t+\tau)^2 x_2^2(s)ds. \tag{2.14}$$

Then

$$\dot V_2 = \frac{\alpha k_2}{2}x_2^2(t) - \alpha\int_0^\infty |dK(\tau)|\int_{t-\tau}^t (s-t+\tau)x_2^2(s)ds. \tag{2.15}$$

Therefore for the functional $V = V_1 + V_2$ we have

$$\dot V \le -\Big(1 - \frac{\gamma p k_2}{4}\Big)x_1^2(t) - \Big(1 - \frac{k_2}{4} - \frac{\alpha k_2}{2}\Big)x_2^2(t). \tag{2.16}$$

Using representation (2.13) for α and equating the coefficients before $x_1^2(t)$ and $x_2^2(t)$ in (2.16) we get the equation $\frac{\gamma p}{2} = 1 + \frac{p}{2\gamma}$ for γ with positive root

$$\gamma = \frac{1 + \sqrt{1+p^2}}{p}. \tag{2.17}$$

Using (2.17) we have

$$\dot V \le -(1 - \frac{k_2}{4}(1 + \sqrt{1+p^2}))(x_1^2(t) + x_2^2(t)).$$

From here and (Kolmanovskii & Nosov, 1986) we get the following
Theorem 2.1. *Let*

$$k_0 < -a, \qquad k_1 > 0, \qquad k_2 < \frac{4}{1 + \sqrt{1 + (\frac{1-a-k_0}{k_1})^2}}. \tag{2.18}$$

Then the system (2.1) is asymptotically stable in the whole.

Remark 2.1. *Since the functional V is not positive definite and*

$$\left| \int_0^\infty dK(\tau) \int_{t-\tau}^t (s-t+\tau)x_2(s)ds \right| \le \frac{k_2}{2} \sup_{s\le t} |x_2(s)|$$

the condition $k_2 < 2$ must be fulfilled [8]. It is so since it follows from (2.18).

Example 2.1. Let $dK(\tau) = b\delta(\tau - h)d\tau$, $h \ge 0$, and $\delta(\tau)$ is a delta-function. In this case $k_0 = b$, $k_1 = bh$, $k_2 = |b|h^2$. The first and second conditions (2.18) give us inequalities $b < -a < 0$, $bh > 0$. Thus, we have a contradiction. It means that the conditions (2.18) are not valid.

Let us show that the system

$$\ddot{x}(t) - ax(t) = bx(t-h) \tag{2.19}$$

cannot be asymptotically stable not for any b and h. It means that the inverted pendulum cannot be stabilizated by the control of the form $u(t) = bx(t-h)$.

The corresponding to the system (2.19) characteristic equiation has the form

$$z^2 - a - be^{-hz} = 0. \tag{2.20}$$

Substituting $z = \alpha + i\beta$, where $i = \sqrt{-1}$, α and β are real numbers, into (2.20) we get $\alpha^2 + 2\alpha\beta i - \beta^2 - a - be^{-h\alpha}(\cos(h\beta) - i\sin(h\beta)) = 0$. Thus for α and β we have the system of the equations

$$\alpha^2 - \beta^2 - a - be^{-h\alpha}\cos(h\beta) = 0, \qquad 2\alpha\beta + be^{-h\alpha}\sin(h\beta) = 0. \tag{2.21}$$

Let us show that system (2.21) for every b and h has at least one solution with $\alpha \ge 0$.

Let $a + b > 0$. In this case for $\beta = 0$ we have $\alpha^2 - a = be^{-h\alpha}$. Consider $f(\alpha) = \alpha^2 - a - be^{-h\alpha}$. Since $f(0) = -(a+b) < 0$ and $\lim_{\alpha\to\infty} f(\alpha) = +\infty$ then there exists $\alpha > 0$ such that $f(\alpha) = 0$.

Let $a + b = 0$. In this case $\alpha = \beta = 0$ is the solution of the system (2.21).

Let $a + b < 0$. If $\beta = 0$ and $\alpha \le 0$ we have a contradiction. Really $0 \le \alpha^2 = a + be^{-h\alpha} \le a + b < 0$. It means that by $\beta = 0$ all roots of the equation $\alpha^2 = a + be^{-h\alpha}$ must be positive. If $\beta \ne 0$ then from the second equation of the system (2.21) we have

$$\alpha = -\frac{1}{2}hbe^{-h\alpha}\frac{\sin(h\beta)}{h\beta} \ge 0.$$

Thus, for every b and h the system (2.21) has at least one solution with $\alpha \ge 0$, i.e. equation (2.19) cannot be asymptotically stable.

Example 2.2. Let $dK(\tau) = (b_1\delta(\tau - h_1) + b_2\delta(\tau - h_2))d\tau$, $h_1, h_2 \ge 0$. In this case the conditions (2.18) have the form

$$k_0 = b_1 + b_2 < -a, \qquad k_1 = b_1h_1 + b_2h_2 > 0,$$
$$k_2 = |b_1|h_1^2 + |b_2|h_2^2 < \frac{4}{1 + \sqrt{1 + \left(\frac{1-a-b_1-b_2}{b_1h_1+b_2h_2}\right)^2}} = k_m. \tag{2.22}$$

Let us show that for any $a > 0$ there exist b_1, b_2, h_1, h_2, such that the conditions (2.22) hold and therefore the system

$$\ddot{x}(t) - ax(t) = b_1 x(t - h_1) + b_2 x(t - h_2) \tag{2.23}$$

is asymptotically stable.

Put $b_1 = b$, $b_2 = -\alpha b$, $h_1 = h$, $h_2 = \beta h$. Here b is arbitrary positive number, positive numbers α, β and h must be chosen.

Using the first condition (2.22) we have $k_0 = b - \alpha b = -(\alpha - 1)b < -a$. It holds if $\alpha > 1 + ab^{-1}$. The second condition $k_1 = bh - \alpha\beta bh = bh(1 - \alpha\beta) > 0$ holds if $\beta < \alpha^{-1}$.

Let us show that the third condition, which in this case has the form

$$bh^2(1 + \alpha\beta^2) < \frac{4}{1 + \sqrt{1 + \left(\frac{1-a+(\alpha-1)b}{bh(1-\alpha\beta)}\right)^2}}, \tag{2.24}$$

hold for all sufficiently small h. Really, let us transform (2.24) in the following way

$$bh^2(1 + \alpha\beta^2) < \frac{4bh(1 - \alpha\beta)}{bh(1 - \alpha\beta) + \sqrt{b^2h^2(1 - \alpha\beta)^2 + (1 + (\alpha - 1)b - a)^2}},$$

$$b^2h^2(1 - \alpha\beta)^2 + (1 + (\alpha - 1)b - a)^2 < \left(\frac{4(1 - \alpha\beta)}{h(1 + \alpha\beta^2)} - bh(1 - \alpha\beta)\right)^2,$$

$$(1 + (\alpha - 1)b - a)^2 + \frac{8b(1 - \alpha\beta)^2}{1 + \alpha\beta^2} < \frac{16(1 - \alpha\beta)^2}{h^2(1 + \alpha\beta^2)^2}.$$

From here it follows

$$h < \frac{4(1 - \alpha\beta)}{\sqrt{(1 + \alpha\beta^2)[(1 + \alpha\beta^2)(1 + (\alpha - 1)b - a)^2 + 8b(1 - \alpha\beta)^2]}}.$$

Thus, we have shown that parameters b_1, b_2, h_1 and h_2 can be choose such that the system (2.23) would become asymptotically stable.

3. NONLINEAR CASE

Consider the problem of stabilization of the nonlinear model of the inverted pendulum

$$\ddot{x}(t) - a \sin x(t) = u(t), \qquad a > 0, \tag{3.1}$$

using control (1.2). In Theorem 2.1 it is proved that by conditions (2.18) on the kernel $K(\tau)$ in (1.2) corresponding linearized system (2.1) is asymptotically stable in the whole. From here it follows (Kolmanovskii & Nosov, 1986) that by conditions (2.18) the zero solution of nonlinear system (3.1), (1.2) is asymptotically stable too if an initial condition (1.3) belongs to some neighborhood of origin, which is called a region of attraction. Let us construct some estimate of the region of attraction for the zero solution of the system (3.1), (1.2).

Stabilization of Inverted Pendulum

Similarly (2.6) let us represent (3.1), (1.2) in the form

$$\dot{x}_1(t) = x_2(t), \qquad \dot{z}(t) = -a_1 x_1(t) - k_1 x_2(t) + af(x_1(t)), \tag{3.2}$$

where $f(x) = \sin x - x$, k_1, a_1 and $z(t)$ are defined by (2.2), (2.4).

Following the method of Lyapunov functionals construction note that the auxiliary ordinary differential equations for the system (3.2) must be chosen in the form (2.7). Therefore Lyapunov functional for the system (3.2) we will construct again in the form $V = V_1 + V_2$, where the functional V_1 is defined by (2.11), (2.10), (2.4). Calculating $\dot{V}_1$ for the system (3.2) similarly to (2.12) we have

$$\begin{aligned}\dot{V}_1 \leq &-\Big(1 - \frac{\gamma p k_2}{4}\Big)x_1^2(t) - \Big(1 - \frac{k_2}{4}\Big)x_2^2(t) + \\ &+ \alpha \int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s - t + \tau) x_2^2(s) ds + \\ &+ 2ap_{12} x_1(t) f(x_1(t)) + 2ap_{22} x_2(t) f(x_1(t)) - \\ &- 2ap_{22} \int_0^\infty dK(\tau) \int_{t-\tau}^t (s - t + \tau) x_2(s) f(x_1(t)) ds,\end{aligned}$$

where γ, p and α are defined by (2.17), (2.10) and (2.13).

Suppose that for all $t \geq 0$ and some positive ρ

$$|x_1(t)| \leq \rho. \tag{3.3}$$

Using the inequality $|f(x)| = |\sin x - x| \leq \frac{|x|^3}{6}$, we get

$$|x_1(t) f(x_1(t))| \leq \frac{x_1^4(t)}{6} \leq \frac{\rho^2}{6} x_1^2(t),$$

$$2|x_2(t) f(x_1(t))| \leq \frac{x_1^2(t)}{6}(x_1^2(t) + x_2^2(t)) \leq \frac{\rho^2}{6}(x_1^2(t) + x_2^2(t)).$$

For arbitrary $\gamma_1 > 0$ we have

$$\begin{aligned}&2\int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s - t + \tau) |x_2(s) f(x_1(t))| ds \leq \\ &\leq \frac{\rho^2}{6} \int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s - t + \tau)\Big(\gamma_1 x_1^2(t) + \frac{1}{\gamma_1} x_2^2(s)\Big) ds \leq \\ &\leq \frac{\rho^2}{6}\Big[\frac{\gamma_1 k_2}{2} x_1^2(t) + \frac{1}{\gamma_1} \int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s - t + \tau) x_2^2(s) ds\Big].\end{aligned}$$

As a result for $\dot{V}_1$ we have

$$\begin{aligned}\dot{V}_1 \leq &-\Big[1 - \frac{\gamma p k_2}{4} - \frac{a\rho^2}{6}\Big(2p_{12} + p_{22} + \frac{\gamma_1 k_2 p_{22}}{2}\Big)\Big] x_1^2(t) - \\ &- \Big(1 - \frac{k_2}{4} - \frac{a\rho^2 p_{22}}{6}\Big) x_2^2(t) + \alpha_1 \int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s - t + \tau) x_2^2(s) ds,\end{aligned}$$

where $\alpha_1 = \alpha + \frac{a\rho^2 p_{22}}{6\gamma_1}$. Choosing V_2 in the form

$$V_2 = \frac{\alpha_1}{2}\int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s-t+\tau)^2 x_2^2(s)ds, \tag{3.4}$$

and using representations (2.13), (2.17) for α and γ we have

$$\dot V \le -\left[1 - \frac{k_2}{4}(1+\sqrt{1+p^2}) - \frac{a\rho^2}{6}\left(2p_{12} + p_{22} + \frac{\gamma_1 k_2 p_{22}}{2}\right)\right]x_1^2(t) - \\ - \left[1 - \frac{k_2}{4}(1+\sqrt{1+p^2}) - \frac{a\rho^2 p_{22}}{6}\left(1 + \frac{k_2}{2\gamma_1}\right)\right]x_2^2(t).$$

Equating the coefficients before $x_1^2(t)$, $x_2^2(t)$ we get the equation for γ_1. Using a positive root of this equation we have

$$\dot V \le -\left[1 - \frac{k_2}{4}(1+\sqrt{1+p^2}) - \frac{a\rho^2}{6}\left(p_{12} + p_{22} + \sqrt{p_{12}^2 + \frac{k_2^2 p_{22}^2}{4}}\right)\right](x_1^2(t) + x_2^2(t)).$$

Using representation (2.10) for p we get that if

$$\rho^2 < \frac{6\left[1 - \frac{k_2}{4}\left(1+\sqrt{1+4a_1^2 p_{22}^2}\right)\right]}{a\left(p_{12} + p_{22} + \sqrt{p_{12}^2 + \frac{k_2^2 p_{22}^2}{4}}\right)} \tag{3.5}$$

then $\dot V$ be negative definite, i.e. the functional $V(t)$ as a function of t is decreasing.

Let us take and fix some positive ρ satisfying inequality (3.5). Denote by $\lambda_0 > 0$ and $\lambda_1 > 0$ minimal and maximal eigen values of the positive definite matrix P with entries p_{ij} given by (2.10). Then

$$\lambda_0 x_1^2(t) \le V(t) \le V(0) \le \lambda_1(x_1^2(0) + x_2^2(0)) + \frac{\alpha_1 k_3}{6}\sup_{s\le 0} x_2^2(s).$$

Hence the domain of attraction of the zero solution of system (3.2), (1.2) contains the set of initial functions satisfying inequality

$$\frac{\lambda_1}{\lambda_0}(\varphi^2(0) + \dot\varphi^2(0)) + \frac{\alpha_1 k_3}{6\lambda_0}\sup_{s\le 0}\dot\varphi_2^2(s) \le \rho^2.$$

4. PENDULUM WITH STOCHASTIC PERTURBATIONS

Let us suppose that the parameter a in equation (1.1) is influenced by perturbations of a white noise type (Kolmanovskii & Shaikhet, 1994; Kolmanovskii & Nosov, 1986; Shaikhet, 1995)

$$\ddot x(t) - (a + \sigma\dot\xi(t))x(t) = u(t), \qquad a > 0, \qquad t \ge 0. \tag{4.1}$$

Here $\xi(t)$ is a standard Wiener process, σ is a constant.

In this case similarly to (2.6) and (3.2) the system (4.1) with control (1.2) and initial condition (1.3) can be rewritten in the form

$$\dot{x}_1(t) = x_2(t), \qquad \dot{z}(t) = -a_1 x_1(t) - k_1 x_2(t) + \sigma x_1(t)\dot{\xi}(t), \tag{4.2}$$

where k_1, a_1 and $z(t)$ are defined by (2.2), (2.4).

The system (4.2), (2.4) with initial condition

$$x_1(s) = \varphi(s), \quad x_2(s) = \dot{\varphi}(s), \quad s \le 0, \tag{4.3}$$

is a system of stochastic differential equations of neutral type. Stability theory of systems of such type was considered in (Kolmanovskii & Nosov, 1986).

Definition 4.1. *The zero solution of equation (4.2), (2.4) is called mean square stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x_1(t)|^2 + \mathbf{E}|x_2(t)|^2 < \epsilon$ for any $t \ge 0$ provided that $\|\varphi\|_1^2 = \sup_{s\le 0}\{\mathbf{E}|\varphi(s)|^2 + \mathbf{E}|\dot{\varphi}(s)|^2\} < \delta$. If, besides, $\lim_{t\to\infty}\{\mathbf{E}|x_1(t)|^2 + \mathbf{E}|x_2(t)|^2\} = 0$ for every initial function φ, then the zero solution of equation (4.2), (2.4) is called asymptotically mean square stable in the whole.*

From (Kolmanovskii & Nosov, 1986) it follows that in order to obtain conditions of asymptotic mean square stability of the zero solution of system (4.2), (2.4) it is enough to construct Lyapunov functional V, satisfying the condition

$$LV \le -c(|x_1(t)|^2 + |x_2(t)|^2), \tag{4.4}$$

where L is a generating operator of system (4.2), (2.4).

We will construct the Lyapunov functional for system (4.2), (2.4) in the form $V = V_1 + V_2$, where V_1 is defined by (2.11), (2.10). Calculating LV_1 and using (2.9) analogously to (2.12) we obtain

$$LV_1 \le -\Big(1 - \frac{\gamma p k_2}{4} - \sigma^2 p_{22}\Big)x_1^2(t) - \Big(1 - \frac{k_2}{4}\Big)x_2^2(t) + \\ + \alpha\int_0^\infty |dK(\tau)| \int_{t-\tau}^t (s - t + \tau)x_2^2(s)ds,$$

where α, p and γ are defined by (2.13), (2.10) and (2.17).

Choosing the functional V_2 in the form (2.14) for the functional $V = V_1 + V_2$ similarly to (2.16) we have

$$LV \le -\Big(1 - \frac{\gamma p k_2}{4} - \sigma^2 p_{22}\Big)x_1^2(t) - \Big(1 - \frac{k_2}{4} - \frac{\alpha k_2}{2}\Big)x_2^2(t).$$

Using (2.17), (2.10) we obtain that if the inequality

$$\sigma^2 < \frac{2a_1}{p}\Big(1 - \frac{k_2}{4}\Big(1 + \sqrt{1 + \Big(\frac{a_1 + 1}{k_1}\Big)^2}\Big)\Big) \tag{4.5}$$

holds then the functional V satisfies condition (4.4).

Thus, if the conditions (2.18) and (4.5) hold then the zero solution of system (4.2), (2.4) is asymptotically mean square stable.

Consider the nonlinear pendulum under stochastic perturbations

$$\ddot{x}(t) - (a + \sigma\dot{\xi}(t))\sin x(t) = u(t), \qquad a > 0, \qquad t \geq 0, \tag{4.6}$$

with control (1.2). Similarly to (3.2) and (4.2) rewrite the system (4.6), (1.2) in the form of the system of nonlinear stochastic differential equations of neutral type

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{z}(t) &= -a_1x_1(t) - k_1x_2(t) + af(x_1(t)) + (\sigma x_1(t) + f(x_1(t)))\dot{\xi}(t), \end{aligned} \tag{4.7}$$

with initial condition (4.3), where $f(x) = \sin x - x$, k_1, a_1 and $z(t)$ are defined by (2.2), (2.4).

Definition 4.2. *The zero solution of equation (4.7), (2.4) is called stable in probability if for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there exists $\delta > 0$ such that the solution $(x_1(t), x_2(t)) = (x_1(t,\varphi), x_2(t,\varphi))$ of equation (4.7), (2.4) satisfies the condition $\mathbf{P}\{|x_1(t,\varphi)| + |x_2(t,\varphi)| > \epsilon_1\} < \epsilon_2$ for any initial function $\varphi(s)$ such that $\mathbf{P}\{\sup_{s\leq 0}(|\varphi(s)| + |\dot{\varphi}(s)|) \leq \delta\} = 1$. Here $\mathbf{P}\{\cdot\}$ is the probability of the event enclosed in braces.*

Note that system (4.2) is the linear part of the system (4.7) and since $|f(x)| \leq \frac{x^3}{6}$ then the order of nonlinearity of the system (4.7) is 3. From (Shaikhet, 1995) it follows that if the order of nonlinearity of the system under consideration is more than one then the condition which is sufficient for asymptotic mean square stability of the linear part of this system is sufficient for stability in probability of the whole system. Thus if the conditions (2.18) and (4.5) hold then the zero solution of system (4.7), (2.4) is stable in probability. It means that the inverted pendulum under stochastic perturbations can be stabilized by control depending only on the trajectories.

5. ABOUT STABILIZATION BY CONTROL DEPENDING ON VELOCITY

Consider equation (1.1) with control

$$u(t) = \int_0^\infty dK(\tau)\dot{x}(t-\tau). \tag{5.1}$$

The characteristic equation of system (1.1), (5.1) has the form

$$z^2 - a - z\int_0^\infty dK(\tau)e^{-z\tau} = 0, \qquad z = \alpha + \beta i. \tag{5.2}$$

Let us show that this equation has at least one root z with positive real part α. Really, let $\beta = 0$, $f(\alpha) = \alpha^2 - a - \alpha\int_0^\infty dK(\tau)e^{-\alpha\tau}$. The equation (5.2) takes the form $f(\alpha) = 0$. Note that $f(0) = -a < 0$ and

$$\begin{aligned} \lim_{\alpha\to\infty} f(\alpha) &= \lim_{\alpha\to\infty}\left(\alpha^2 - a - \alpha\int_{+0}^\infty dK(\tau)e^{-\alpha\tau} - \alpha dK(0)\right) = \\ &= \lim_{\alpha\to\infty}(\alpha^2 - a - \alpha dK(0)) = \infty. \end{aligned}$$

Therefore there exists at least one $\alpha = \alpha_0 > 0$ which is a root of the equation $f(\alpha) = 0$. Thus, the inverted pendulum cannot be stabilized by control type of (5.1) depending only on velocity.

6. NUMERICAL ANALYSIS OF THE INVERTED PENDULUM

Consider the linearized equatiòn of the inverted pendulum in the form (2.23). Sufficient conditions of asymptotic stability of system (2.23) are (2.22). Remark that two first inequalities of conditions (2.22) are also necessary conditions for asymptotic stability but third inequality is only sufficient condition.

1) Let $x(0) = 6$, $\dot{x}(0) = 0$, $a = 1$, $b_1 = 1$, $b_2 = -4$, $h_1 = 0.25$, $h_2 = 0.04$. In this case $k_0 = -3$, $k_1 = 0.09$, $k_2 = 0.07$, $k_m = 0.12$. Hence the conditions (2.22) are valid and the system (2.23) is asymptotically stable. The trajectories of the system (2.23) in the spaces (x_1, x_2) and $(t, x(t))$ are shown on Fig.6.1.

2) Let $x(0) = 8$, $\dot{x}(0) = 0$, $a = 3$, $b_1 = 1$, $b_2 = -4.5$, $h_1 = 0.6$, $h_2 = 0.1$. Then $k_0 = -3.5$, $k_1 = 0.15$, $k_2 = 0.41$, $k_m = 0.36$. Hence third inequality of condition (2.22) is not valid. But the linear system (2.23) is stable (see Fig.6.2) and corresponding nonlinear system

$$\ddot{x}(t) - a \sin x(t) = b_1 x(t - h_1) + b_2 x(t - h_2) \tag{6.1}$$

is stable also (see Fig.6.3).

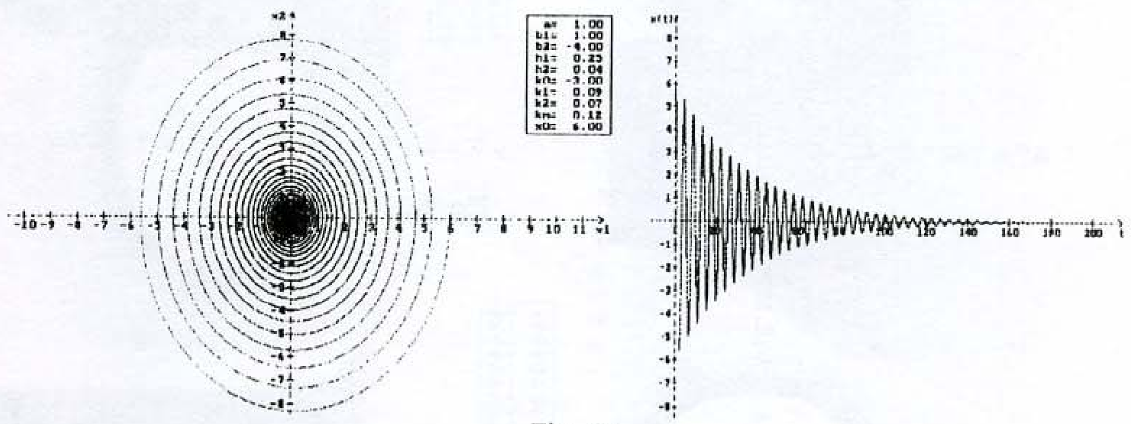

Fig. 6.1

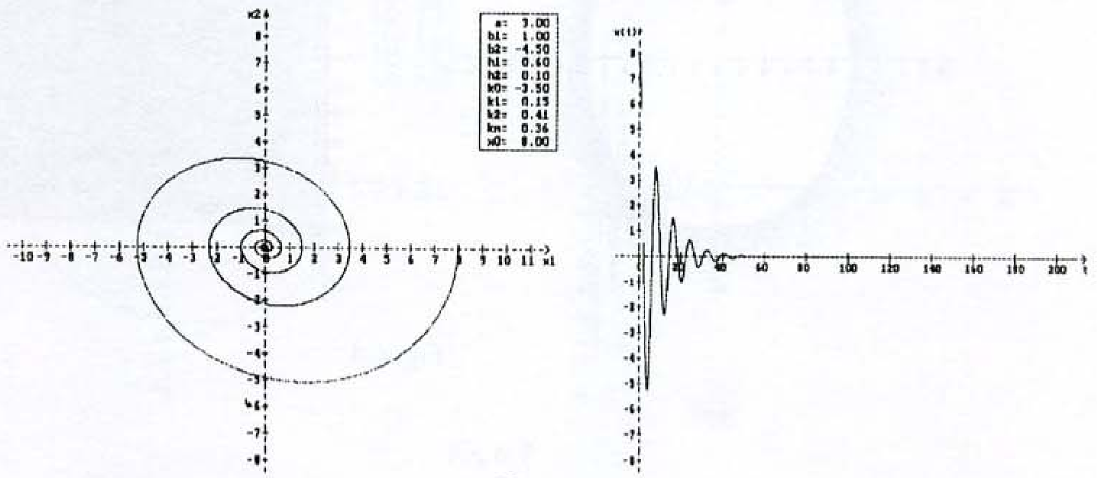

Fig.6.2

Borne, Kolmanovskii and Shaikhet

3) Let us investigate the influence of the delay h_1 on the behaviour of the pendulum. All other parameters have the same values as in the case 2). If $h_1 = 0.59$ then the linear system (2.23) is stable, but nonlinear system (6.1) has an attractor (Fig.6.4). The same phenomena take place for all values of delay h_1 from $h_1 = 0.59$ to $h_1 = 0.54$ (see Fig.6.5 illustrating nonlinear system (6.1)). If $h_1 = 0.53$ then the nonlinear system (1.6) is unstable (Fig.6.6) but the linear system (2.23) is stable (Fig.6.7).

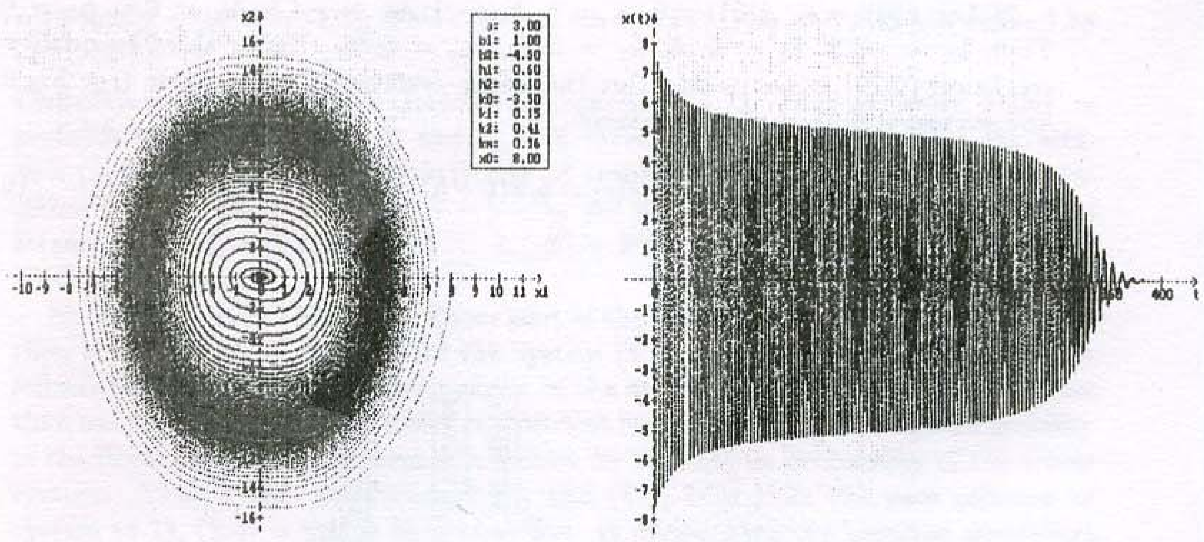

Fig. 6.3

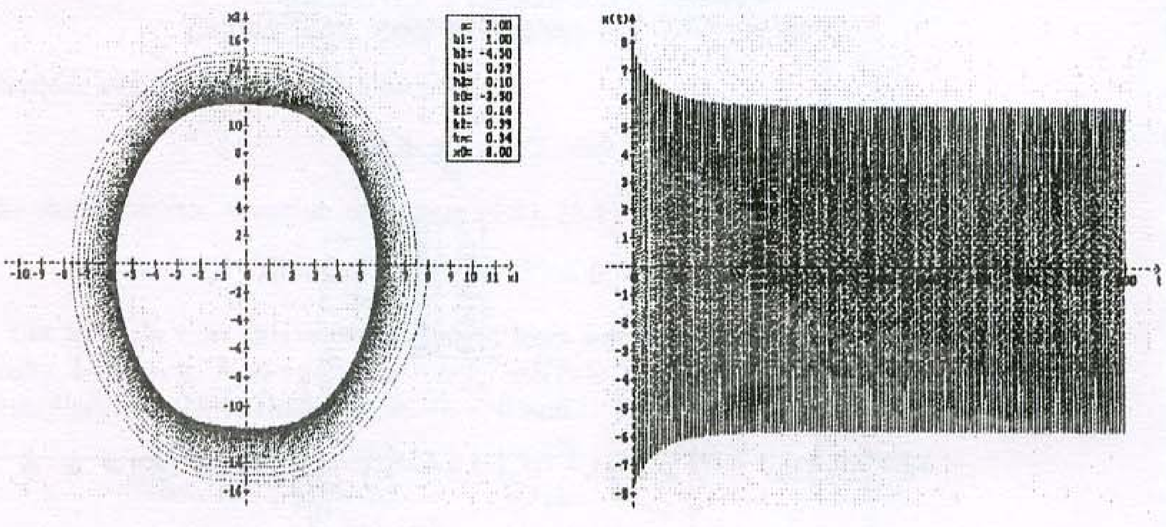

Fig. 6.4

Stabilization of Inverted Pendulum

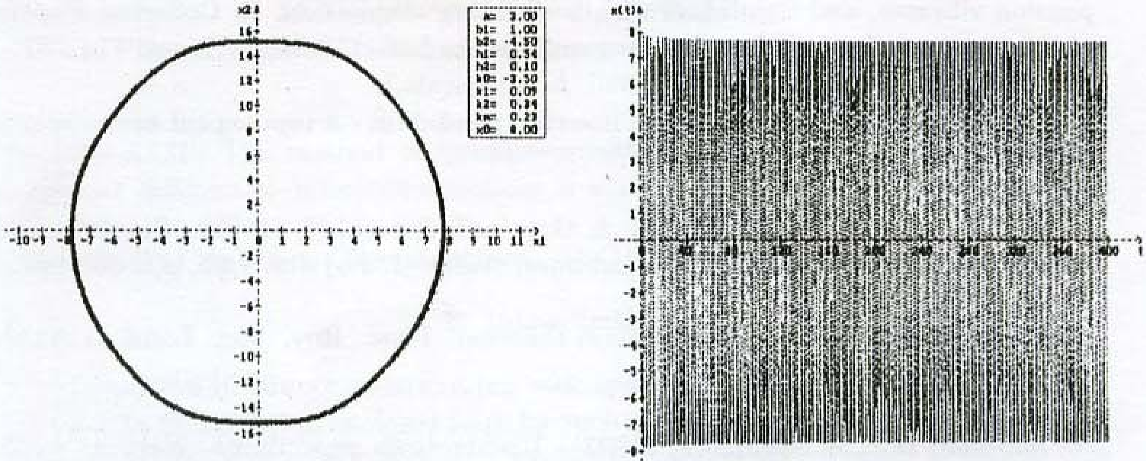

Fig.6.5

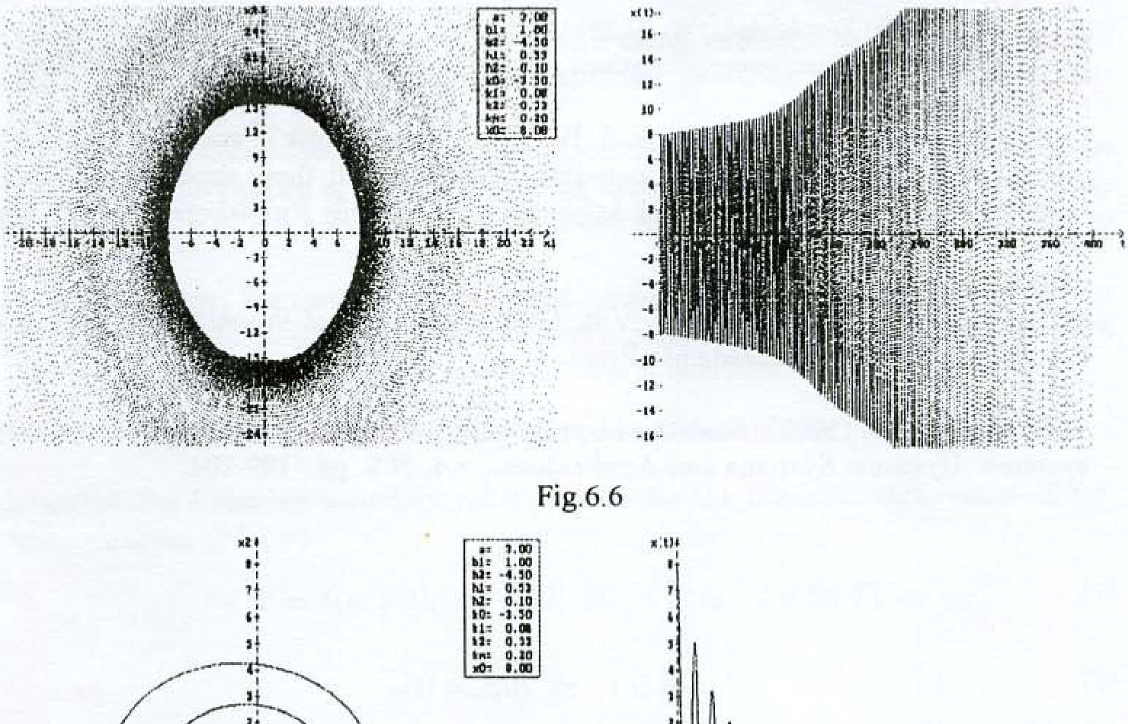

Fig.6.6

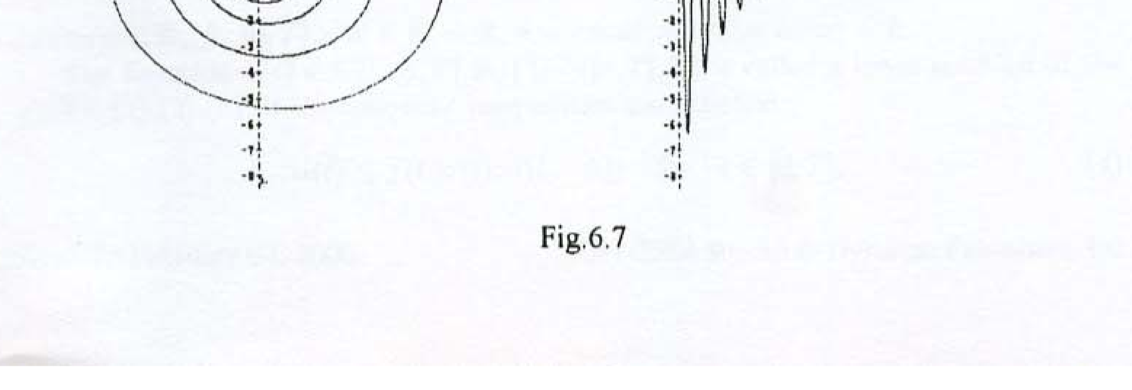

Fig.6.7

Borne, Kolmanovskii and Shaikhet

REFERENCES

1. Kapitza, P.L. (1965). Dynamical stability of a pendulum when its point of suspension vibrates, and Pendulum with a vibrating suspension. In Collected Papers of P.L.Kapitza. D.ter Haar. ed., Pergamon Press Ltd. London, v.2, pp. 714-737.

2. Levi, M. (1988). Stability of the inverted pendulum - a topological explanation. SIAM Review, v.30, pp. 639-644.

3. Blackburn, J.A., Smith, H.J.T., & Gronbech-Jensen, N. (1992). Stability and Hopf bifurcations in an inverted pendulum. Amer. J. Physics, v.60, pp. 903-908.

4. Acheson, D.J. (1993). A pendulum theorem. Proc. Roy. Soc. Lond., v.A443, pp. 239-245.

5. Acheson, D.J., & Mullin, T. (1993). Upside-down pendulums. Nature, v.366, pp. 215-216.

6. Levi, M., & Weckesser, W. (1995). Stabilization of the inverted linearized pendulum by high frequency vibrations. SIAM Review, v.37, No.2, pp. 219-223.

7. Kolmanovskii, V.B., & Shaikhet, L.E. (1994). New results in stability theory for stochastic functional differential equations (SFDEs) and their applications. Proceedings of Dynamic Systems and Applications. Dynamic Publishers Inc., v.1, pp. 167-171.

8. Kolmanovskii, V.B., & Nosov, V.R. (1986). Stability of Functional Differential Equations. New York: Academic Press.

9. Shaikhet, L.E. (1995). Stability in probability of nonlinear stochastic hereditary systems. Dynamic Systems and Applications, v.4, N.2, pp. 199-204.