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STABILITY OF EQUILIBRIUMS OF STOCHASTICALLY PERTURBED DELAY DIFFERENTIAL NEOCLASSICAL GROWTH MODEL

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ABSTRACT. The known nonlinear delay differential neoclassical growth model is considered. It is assumed that this model is influenced by stochastic perturbations of the white noise type and these perturbations are directly proportional to the deviation of the system state from the zero or a positive equilibrium. Sufficient conditions for stability in probability of the positive equilibrium and for exponential mean square stability of the zero equilibrium are obtained. Numerical calculations and figures illustrate the obtained stability regions and behavior of stable and unstable solutions of the considered model. The proposed investigation procedure can be applied for arbitrary nonlinear stochastic delay differential equations with the order of nonlinearity higher than one.

1. Introduction.

The well known delay differential neoclassical growth model is often enough discussed in mathematical economics (see, for instance, [2, 3, 4, 6, 7]). Here it is supposed that this model is influenced by stochastic perturbations of the white noise type that are directly proportional to the deviation of the system state from the one of system equilibrium. It is shown how the stability theory for stochastic delay differential equations [10] can be applied to stability investigation of equilibriums of nonlinear models under stochastic perturbations.

1.1. Statement of the problem.

Consider the delay differential neoclassical growth model

$$\begin{aligned} \dot{x}(t) &= ax^{\gamma}(t-h)e^{-bx(t-h)} - cx(t), \\ x(s) &= \phi(s), \quad s \in [-h, 0], \end{aligned} \tag{1.1}$$

where a, b, c, γ are positive parameters and $h \geq 0$.

It is easy to see that for arbitrary values of the parameters the equation (1.1) has the zero equilibrium. Besides the positive equilibrium x^* (that can be not unique) is defined by the algebraic equation

$$a(x^*)^{\gamma-1}e^{-bx^*} = c. \tag{1.2}$$

In particular, if $\gamma = 1$ then the equation (1.1) is well known Nicholson's blowflies equation [8] and (1.2) by the condition $a > c$ has the explicit positive solution $x^* = \frac{1}{b}\ln\frac{a}{c}$.

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Below we will investigate stability of both equilibriums of the equation (1.1) under stochastic perturbations of the white noise type that are directly proportional to the deviation of the solution $x(t)$ from the equilibrium point x^*. By that the equation (1.1) takes the form of stochastic differential equation [5, 10]

$$\dot{x}(t) = ax^{\gamma}(t-h)e^{-bx(t-h)} - cx(t) + \sigma(x(t) - x^*)\dot{w}(t), \tag{1.3}$$

where σ is a constant and $w(t)$ is the standard Wiener process.

In the case $\gamma = 1$ this problem was in detail investigated in [1, 9, 10], where, in particular, sufficient conditions for stability in probability both for the zero and the positive equilibriums were obtained. Below the case $\gamma \geq 1$ is considered, sufficient conditions for stability in probability of the positive equilibrium and sufficient conditions for exponential mean square stability of the zero equilibrium are obtained.

1.2. Centering and linearization. Let us center the equation (1.3) at the point of equilibrium x^* using the new variable $y(t) = x(t) - x^*$. By this from (1.3), (1.2) we obtain

$$\begin{aligned}\dot{y}(t) &= a(y(t-h) + x^*)^{\gamma}e^{-b(y(t-h)+x^*)} - c(y(t) + x^*) + \sigma y(t)\dot{w}(t) \\ &= -\, cy(t) + a(y(t-h) + x^*)^{\gamma}e^{-b(y(t-h)+x^*)} - a(x^*)^{\gamma}e^{-bx^*} + \sigma y(t)\dot{w}(t)\end{aligned}$$

or

$$\dot{y}(t) = -cy(t) + ae^{-bx^*}[(y(t-h) + x^*)^{\gamma}e^{-by(t-h)} - (x^*)^{\gamma}] + \sigma y(t)\dot{w}(t). \tag{1.4}$$

It is easy to see that stability of the zero solution of the equation (1.4) is equivalent to stability of the positive equilibrium x^* of the equation (1.3).

Along with the equation (1.4) we will consider the linear part of this equation. Using for $\gamma \geq 1$ the representations $(y + x^*)^{\gamma} = (x^*)^{\gamma} + \gamma(x^*)^{\gamma-1}y + o(y)$ and $e^{-by} = 1 - by + o(y)$, where $o(y)$ means that $\lim_{y\to 0}\frac{o(y)}{y} = 0$, we obtain

$$\begin{aligned}(y + x^*)^{\gamma}e^{-by} - (x^*)^{\gamma} &= [(x^*)^{\gamma} + \gamma(x^*)^{\gamma-1}y + o(y)][1 - by + o(y)] - (x^*)^{\gamma} \\ &= \gamma(x^*)^{\gamma-1}y - (x^*)^{\gamma}by + o(y) \\ &= -\,(x^*)^{\gamma-1}\mu y + o(y),\end{aligned}$$

where

$$\mu = bx^* - \gamma. \tag{1.5}$$

From this via (1.2) we have

$$\begin{aligned}ae^{-bx^*}[(y(t-h) + x^*)^{\gamma}e^{-by(t-h)} - (x^*)^{\gamma}] &= -\, ae^{-bx^*}(x^*)^{\gamma-1}\mu y(t-h) + o(y) \\ &= -\, c\mu y(t-h) + o(y).\end{aligned}$$

Neglecting $o(y)$, as a result we obtain the linear part of the equation (1.4) in the form

$$\dot{z}(t) = -cz(t) - c\mu z(t-h) + \sigma z(t)\dot{w}(t). \tag{1.6}$$

2. Auxiliary definitions and statements.

Lemma 2.1. *The function* $f(x) = x^{\gamma-1}e^{-bx}$, $x \geq 0$, $\gamma \geq 1$, $b > 0$, *satisfies the condition* $f(x) \leq K_0$, *where*

$$K_0 = \left(\frac{\gamma-1}{eb}\right)^{\gamma-1} \quad \text{if} \quad \gamma > 1 \quad \text{and} \quad K_0 = 1 \quad \text{if} \quad \gamma = 1. \tag{2.1}$$

For the proof it is enough to note that $\sup_{x\geq 0} f(x) = f\left(\frac{\gamma-1}{b}\right)$.

Lemma 2.2. *If*

$$c > K_0 a, \tag{2.2}$$

where K_0 *is defined in (2.1), then the equation (1.2) does not have positive solutions.*

For the proof it is enough to note that for a positive solution x^* of the equation (1.2) via the conditions (2.2), (1.2) and Lemma 2.1 we obtain the contradiction

$$K_0 < \frac{c}{a} = (x^*)^{\gamma-1}e^{-bx^*} \leq K_0.$$

Definition 2.1. The zero solution of the equation (1.4) with the initial condition defined by $y(s) = \phi(s)$, $s \in [-h, 0]$, is called:

- stable in probability if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that the solution $y(t, \phi)$ of the equation (1.4) satisfies the condition $\mathbf{P}\{\sup_{t\geq 0}|y(t,\phi)| > \varepsilon_1\} < \varepsilon_2$ for any initial function ϕ such that $\mathbf{P}\{\sup_{s\leq 0}|\phi(s)| < \delta\} = 1$;

- mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|y(t,\phi)|^2 < \varepsilon$, $t \geq 0$, provided that $\|\phi\|^2 = \sup_{s\leq 0}\mathbf{E}|\phi(s)|^2 < \delta$;

- asymptotically mean square stable if it is mean square stable and for each initial function ϕ the solution $y(t)$ of the equation (1.4) satisfies the condition $\lim_{t\to\infty}\mathbf{E}|y(t)|^2 = 0$;

- exponentially mean square stable if it is mean square stable and there exists $\lambda > 0$ such that for each initial function ϕ there exists $C > 0$ (which may depend on ϕ) such that $\mathbf{E}|y(t,\phi)|^2 \leq Ce^{-\lambda t}$ for $t > 0$.

Lemma 2.3. [9, 10] *A necessary and sufficient condition for asymptotic mean square stability of the zero solution of the linear Ito stochastic differential equation*

$$\dot{x}(t) = Ax(t) + Bx(t-h) + \sigma x(t-\tau)\dot{w}(t), \tag{2.3}$$

is

$$A + B < 0, \qquad G^{-1} > \frac{1}{2}\sigma^2, \tag{2.4}$$

where

$$G = \begin{cases} \frac{Bq^{-1}\sin(qh)-1}{A+B\cos(qh)}, & B + |A| < 0, \quad q = \sqrt{B^2 - A^2}, \\ \frac{1+|A|\tau}{2|A|}, & B = A < 0, \\ \frac{Bq^{-1}\sinh(qh)-1}{A+B\cosh(qh)}, & A + |B| < 0, \quad q = \sqrt{A^2 - B^2}. \end{cases} \tag{2.5}$$

Let L be the generator [5, 10] of the stochastic differential equation (1.4), $y(t)$ be the solution of (1.4) in the time moment t, y_t be the trajectory of the solution of (1.4) until the time moment t, $\mathbf{E}$ be the expectation.

Theorem 2.1. [10] *Let there exist a functional $V(t,\phi)$ and $\lambda > 0$ such that for a solution $y(t)$ of the equation (1.4) the following inequalities hold:*

$$\mathbf{E}V(t,y_t) \geq c_1 e^{\lambda t}\mathbf{E}|y(t)|^2, \qquad \mathbf{E}V(0,\phi) \leq c_2\|\phi\|^2, \qquad \mathbf{E}LV(t,y_t) \leq 0,$$

$t \geq 0$, $c_i > 0$, $i = 1,2$. Then the zero solution of the equation (1.4) is exponentially mean square stable.

3. Stability conditions.

3.1. Positive equilibrium. Note that the order of nonlinearity of the equation (1.3) is higher than one. In this case [10] sufficient conditions for asymptotic mean square stability of the zero solution of the linear equation (1.6) (as the linear part of the nonlinear equation (1.4)) at the same time are sufficient conditions for stability in probability of the zero solution of the equation (1.4).

Via Lemma 2.3 we obtain the following statement.

Lemma 3.1. *The zero solution of the equation (1.6), (1.5) with $\gamma \geq 1$ is asymptotically mean square stable if and only if*

$$x^* > \frac{\gamma - 1}{b} \quad \text{and} \quad G^{-1} > p = \frac{1}{2}\sigma^2, \tag{3.1}$$

where

$$G = \begin{cases} \frac{c^{-1}+\mu q^{-1}\sinh(qh)}{1+\mu\cosh(qh)}, & x^* < \frac{\gamma+1}{b}, \quad q = c\sqrt{1-\mu^2}, \\ \frac{c^{-1}+h}{2}, & x^* = \frac{\gamma+1}{b}, \\ \frac{c^{-1}+\mu q^{-1}\sin(qh)}{1+\mu\cos(qh)}, & x^* > \frac{\gamma+1}{b}, \quad q = c\sqrt{\mu^2-1}, \end{cases} \tag{3.2}$$

and μ is defined in (1.5).

Note that via (1.5) the first inequality (3.1) is equivalent to the condition $1+\mu > 0$ that follows from the first inequality (2.4).

Via [10] the conditions (3.1), (3.2) are sufficient conditions for stability in probability of the zero solution of the equation (1.4) and therefore for stability in probability of the positive equilibrium point x^* of the equation (1.3).

3.2. Zero equilibrium. For the zero equilibrium the equation (1.4) has the form

$$\begin{gathered} \dot{y}(t) = ay^{\gamma}(t-h)e^{-by(t-h)} - cy(t) + \sigma y(t)\dot{w}(t), \quad t \geq 0, \\ y(s) = \phi(s), \quad s \in [-h, 0]. \end{gathered} \tag{3.3}$$

Lemma 3.2. *If $\gamma \geq 1$ and*

$$c > K_0 a + p, \qquad p = \frac{1}{2}\sigma^2, \tag{3.4}$$

where K_0 is defined in (2.1), then the zero equilibrium of the equation (3.3) is exponentially mean square stable.

Proof. Note that via Lemma 2.2 by the condition (3.4) the equation (3.3) has the zero equilibrium only. Note also that the solution of the equation (3.3) can be represented [5] in the form

$$y(t) = \left[y(0) + a\int_0^t y^\gamma(s-h)e^{-by(s-h)+\eta(s)}ds\right]e^{-\eta(t)}, \quad \eta(t) = \left(c + \frac{\sigma^2}{2}\right)t - \sigma w(t).$$

So, for a positive initial function $y(s) = \phi(s)$, $s \in [-h, 0]$, the solution $y(t)$ of the equation (3.3) remains a positive one for all $t \geq 0$.

Let L be the generator [5, 10] of the equation (3.3). Using Lemma 2.1 and positiveness of the solution $y(t)$, for the functional $V_1(y(t)) = e^{\lambda t}y^2(t)$, $\lambda > 0$, we have

$$\begin{aligned}
LV_1(y(t)) &= e^{\lambda t}[\lambda y^2(t) + 2y(t)(ay^\gamma(t-h)e^{-by(t-h)} - cy(t)) + \sigma^2 y^2(t)] \\
&= e^{\lambda t}[(\lambda - 2c + \sigma^2)y^2(t) + 2ay(t)y^\gamma(t-h)e^{-by(t-h)}] \\
&\leq e^{\lambda t}[(\lambda - 2c + \sigma^2)y^2(t) + 2K_0 ay(t)y(t-h)] \\
&\leq e^{\lambda t}[(\lambda - 2c + \sigma^2 + K_0 a)y^2(t) + K_0 ay^2(t-h)].
\end{aligned}$$

Following the procedure of Lyapunov functionals construction [9, 10] consider the additional functional V_2 in the form

$$V_2(y_t) = K_0 a\int_{t-h}^t e^{\lambda(s+h)}y^2(s)ds.$$

Then $LV_2(y_t) = K_0 a[e^{\lambda(t+h)}y^2(t) - e^{\lambda t}y^2(t-h)]$ and for the functional $V = V_1 + V_2$ we obtain

$$LV(y_t) \leq e^{\lambda t}\left[-2c + \lambda + (e^{\lambda h} + 1)K_0 a + \sigma^2\right]y^2(t).$$

From (3.4) it follows that there exists small enough $\lambda > 0$ such that the expression in the square brackets is negative, i.e., $LV(y_t) \leq 0$. From this via Theorem 2.1 it follows that by the condition (3.4) the zero solution of the equation (3.3) is exponentially mean square stable. The proof is completed. □

4. Numerical simulations. Here the stability conditions (3.1), (3.2) and (3.4) are used for construction of stability regions for the positive and the zero equilibriums in the space of the parameters (a, c) for different values of the other parameters.

In Figure 4.1 one can see the region (red and green) of stability in probability for the positive equilibrium x^* by $\gamma = 3$, $b = 1$, $h = 0.02$, $p = 20$. Via Lemma 3.1 red and green parts of the stability region satisfy respectively the conditions

$$\frac{\gamma - 1}{b} < x^* < \frac{\gamma + 1}{b} \quad \text{and} \quad x^* \geq \frac{\gamma + 1}{b}.$$

The red part of the stability region is placed between two straight lines: 1) $c = K_1 a$ and 2) $c = K_0 a$, where K_0 is defined in (2.1) and

$$K_1 = \left(\frac{\gamma + 1}{eb}\right)^{\gamma - 1}e^{-2}.$$

Via Lemma 3.2 the region of exponential mean square stability (yellow) of the equation (3.3) zero equilibrium is placed above the straight line 3) $c = K_0 a + p$.

In Figure 4.2 one can see the similar picture for other values of the parameters: $\gamma = 2$, $b = 2$, $h = 0.02$, $p = 20$.

Consider now a behavior of the solutions of the equations (1.3) and (3.3) in the points $A(700, 300)$, $B(900, 200)$ and $C(600, 400)$ (see Figure 4.1) respectively.

The point $A(700, 300)$ is placed (Figure 4.1) inside of the stability region of the positive equilibrium of the equation (1.3). Solving the algebraic equation (1.2) by $\gamma = 3$, $b = 1$, $a = 700$, $c = 300$, we obtain two equilibriums: $x_1^* = 1.1805$ and $x_2^* = 3.1270$. Since $x_1^* < \frac{\gamma-1}{b} < x_2^*$ then via Lemma 3.1 x_1^* is unstable equilibrium

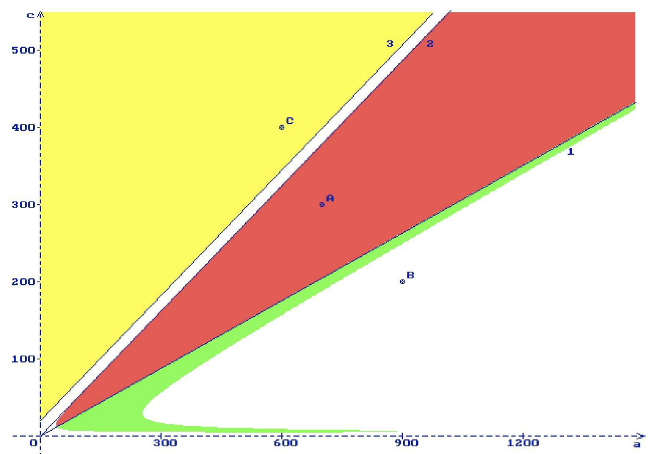

Figure 4.1

Figure 4.1. Stability regions for equation (1.3) (red and green) and equation (3.3) (yellow),$\gamma = 3$, $b = 1$, $h = 0.02$, $p = 20$

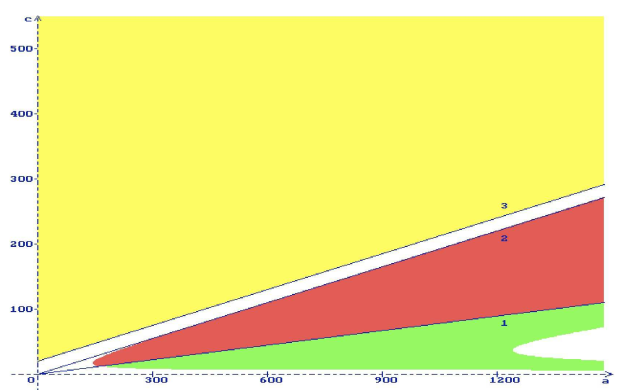

Figure 4.2

Figure 4.2. Stability regions for equation (1.3) (red and green) and equation (3.3) (yellow),$\gamma = 2$, $b = 2$, $h = 0.02$, $p = 20$

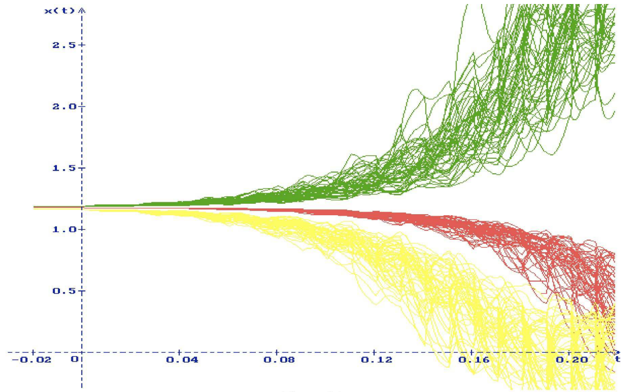

Figure 4.3

Figure 4.3. Trajectories of solution of equation (1.3) in unstable equilibrium for different initial functions: $x_0 = 1.19$ (green), $x_0 = 1.1805$ (red), $x_0 = 1.17$ (yellow), $A(700, 300)$, $\gamma = 3$, $b = 1$, $h = 0.02$, $p = 20$, $x_1^* = 1.1805$

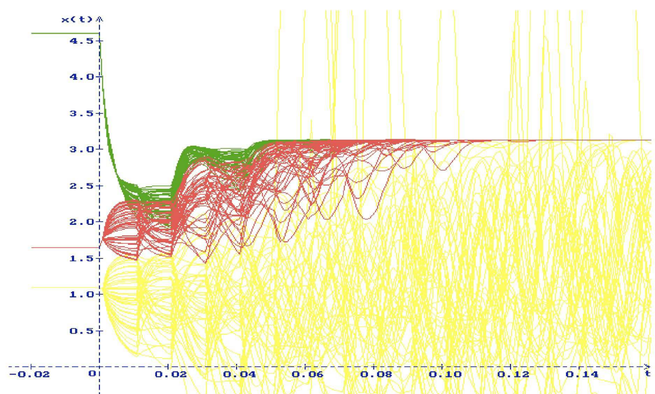

Figure 4.4

Figure 4.4. Trajectories of solution of equation (1.3) in stable equilibrium for different initial functions: $x_0 = 4.6$ (green), $x_0 = 1.65$ (red), $x_0 = 1.1$ (yellow), $A(700, 300)$, $\gamma = 3$, $b = 1$, $h = 0.02$, $p = 20$, $x_2^* = 3.1270$

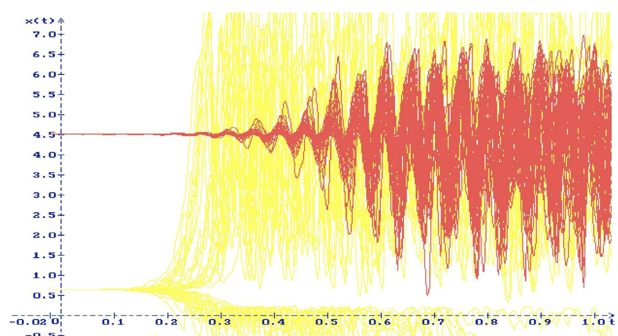

Figure 4.5

Figure 4.5. Trajectories of solution of equation (1.3) in unstable equilibrium for different initial functions: $x_0 = 0.6536$ (yellow) and $x_0 = 4.5215$ (red), $B(900, 200)$, $\gamma = 3$, $b = 1$, $h = 0.02$, $p = 20$, $x_1^* = 0.6527$ and $x_2^* = 4.5215$

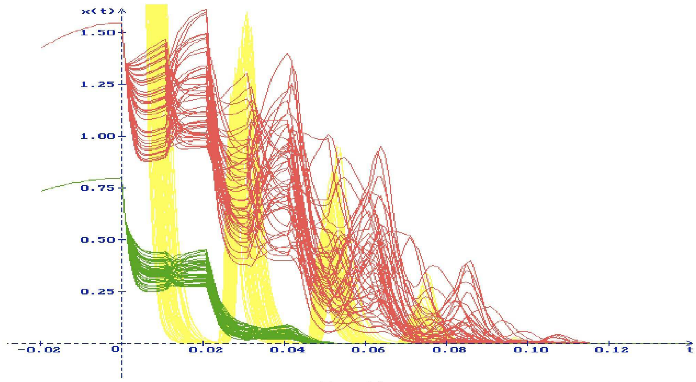

Figure 4.6

Figure 4.6. Trajectories of solution of equation (3.3) in stable zero equilibrium for different initial functions: $b_0 = 0.8$ (green), $b_0 = 1.55$ (red), $b_0 = 100$ (yellow), $C(600, 400)$, $\gamma = 3$, $b = 1$, $h = 0.02$, $p = 20$, $x^* = 0$

and x_2^* is stable in probability. In Figure 4.3 50 green, 50 red and 50 yellow trajectories of the equation (1.3) solution are shown in the point $A(700, 300)$ by $h = 0.02$, $p = 20$ and different initial function $x(s) = x_0$, $s \in [-h, 0]$: $x_0 = 1.19$ (green), $x_0 = 1.1805$ (red) and $x_0 = 1.17$. One can see that all trajectories beginning close enough to the unstable equilibrium $x^* = 1.1805$ do not converge to this equilibrium. In Figure 4.4 50 green, 50 red and 50 yellow trajectories of the equation (1.3) solution are shown in the stable point $A(700, 300)$. One can see that all green ($x_0 = 4.6$) and all red ($x_0 = 1.65$) trajectories converge to the stable in probability positive equilibrium $x_2^* = 3.1270$. However, yellow trajectories that begin far enough ($x_0 = 1.1$) from the equilibrium x_2^* do not converge to this equilibrium and fill by itself whole space. This situation is a natural one for stability in probability (see Definition 2.1).

The point $B(900, 200)$ is placed (Figure 4.1) out of the stability region of the positive equilibrium of the equation (1.3). Solving the algebraic equation (1.2) by $\gamma = 3$, $b = 1$, $a = 900$, $c = 200$, we obtain two equilibriums: $x_1^* = 0.6527$ and $x_2^* = 4.5215$. In Figure 4.5 50 yellow and 50 red trajectories of the equation (1.3) are shown in the unstable point $B(900, 200)$ by the initial function $x(s) = x_0 = 0.6536$, $s \in [-h, 0]$ (yellow) and $x(s) = x_0 = 4.5215$, $s \in [-h, 0]$ (red), $h = 0.02$, $p = 20$. One can see that all trajectories beginning close enough to the unstable positive equilibriums $x_1^* = 0.6527$ and $x_2^* = 4.5215$ do not converge to these equilibriums and fill by itself whole space.

The point $C(600, 400)$ is placed (Figure 4.1) inside of the region of exponential mean square stability of the zero equilibrium of the equation (3.3). In Figure 4.6 50 green, 50 red and 50 yellow trajectories of the equation (3.3) solution are shown for $\gamma = 3$, $b = 1$, $h = 0.02$, $p = 20$ and the initial function $y(s) = b_0 \cos(20s)$, $s \in [-h, 0]$: $b_0 = 0.8$ (green), $b_0 = 1.55$ (red) and $b_0 = 100$ (yellow). One can see that in contrast to stability in probability all trajectories converge to the exponentially mean square stable zero equilibrium for each initial function, even if the initial function is placed far enough from the zero ($b_0 = 100$).

5. Conclusion. In the present paper, the nonlinear delay differential neoclassical growth model under stochastic perturbations is analyzed. Stability conditions for the zero and positive equilibriums of the considered model are obtained. Numerical simulations show a principal difference in solution behavior in the case of stable and unstable equilibriums and in the case of stable equilibriums for different types of stability: stability in probability and exponential mean square stability. The proposed research method can be applied to investigation of other stochastic nonlinear models with the order of nonlinearity higher than one.

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