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Brief paper

Delay-induced stability of vector second-order systems via simple Lyapunov functionals ☆

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ABSTRACT

It is well known that some important classes of systems (e.g. inverted pendulums, oscillators, double integrators) that cannot be stabilized by a static output-feedback, may be stabilized by inserting an artificial time-delay in the feedback. Static output-feedback controllers have advantages over observer-based controllers in the presence of uncertainties in the system matrices and/or uncertain input/output delays, where the observer-based design becomes complicated. The existing Lyapunov-based methods that may treat the case of stabilizing delays and that lead to stability conditions in terms of Linear Matrix Inequalities (LMIs) suffer from high-dimensionality of the resulting LMIs with a large number of decision variables. In this paper, we suggest simple Lyapunov functionals for vector second-order systems with stabilizing delays that lead to reduced-order LMIs with a small number of decision variables. Moreover, differently from the existing methods, we show that the presented LMIs are always feasible for small enough delays.

1. Introduction

It is well known that time-delay is, in many cases, a source of instability. However, for some systems, the presence of delay can have a stabilizing effect (Abdallah, Dorato, Benitez-Read, & Byrne, 1993; Niculescu & Michiels, 2004; Richard, 2003). Thus, the double integrator

$$\ddot{y}(t) = u(t), \quad y(t) \in \mathbb{R}$$

is not stabilizable by the non-delayed static output-feedback $u(t) = K_0 y(t)$. However, this system is stabilizable by the delayed static output-feedback

$$u(t) = K_1 y(t) + K_2 y(t-h), \quad h > 0$$

since it is stabilizable by $u(t) = \bar{K}_1 y(t) + \bar{K}_2 \dot{y}(t)$ and

$$\dot{y}(t) \approx \frac{y(t) - y(t-h)}{h}, \quad h > 0.$$

Simple static output-feedback control has advantages over observer-based control in the presence of uncertainties in the system matrices and/or uncertain input/output delays, where the observer-based design becomes complicated. The idea of using artificial delay became attractive and was applied e.g. for sliding mode control (Seuret, Edwards, Spurgeon, & Fridman, 2009) and for delay-induced consensus in multi-agent systems (Yu, Chen, Cao, & Ren, 2013).

Lyapunov-based methods for stabilizing delay may treat uncertain systems and analyze system performance (see e.g. Fridman & Tsodik, 2009, Fridman, 2014, Gu, Kharitonov, & Chen, 2003). The main Lyapunov method leading to LMI conditions that has been used till now for the case of stabilizing delay is discretized Lyapunov method (Gu, 1997; Gu et al., 2003). This method is based on the discretization of the general Lyapunov functional that corresponds to necessary and sufficient conditions for stability of systems with constant delay. In some cases, augmented Lyapunov functional of Seuret and Gouaisbaut (2013) (that may be also considered as a certain choice of general Lyapunov functional Fridman, 2014) is applicable to the case of stabilizing delay. Both methods (Gu et al., 2003; Seuret & Gouaisbaut, 2013) lead to high-order LMIs with a large number of decision variables.

A model transformation-based method for stabilization by using constant artificial delay of the scalar second-order system that models inverted pendulum was suggested in Borne, Kolmanovskii, and Shaikhet (2000). The latter method uses simple Lyapunov

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functionals for the analysis of the transformed neutral type systems. The stability conditions in Borne et al. (2000) are given in terms of inequalities on the system coefficients.

The objective of the present paper are simple LMI conditions for delay-induced stabilization of the vector second-order systems. Both cases of constant and time-varying stabilizing delays are considered. Two main approaches are suggested for the Lyapunov–Krasovskii stability analysis via simple Lyapunov functionals. The first one is a new method based on the application of Wirtinger's inequality that leads to the simplest LMI conditions. However, this method employs a Lyapunov functional depending on the state derivative that seems to be not applicable to the stochastic case. The second approach extends the model transformation-based approach of Borne et al. (2000) and Shaikhet (2013) to linear vector second-order systems via a comprehensive technique. The latter approach is applied to deterministic and stochastic systems. Differently from Gu et al. (2003) and Seuret and Gouaisbaut (2013) we explain that the new LMI conditions are always feasible for small enough values of the delay. The presented LMIs essentially reduce the number of decision variables comparatively to Gu et al. (2003) and Seuret and Gouaisbaut (2013) leading in examples to a slightly more conservative results.

The paper is organized as follows. After Problem Formulation in Section 2, we present two different Lyapunov methods for delay-induced stability in Sections 3 and 4, and give illustrating examples in Section 5.

Notation. Throughout the paper the superscript '′' stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite, I_n is the identity $n\times n$-matrix. The symmetric elements of the symmetric matrix is denoted by $*$. By $O(h)$ we denote a matrix/scalar function of $h\in\mathbb{R}_+$ such that $\lim_{h\to 0^+}\frac{|O(h)|}{h}=M$, where $M>0$ is constant.

2. Problem formulation and preliminaries

We consider stabilization of the vector second-order system

$$\ddot{y}(t)=A_1y(t)+A_2\dot{y}(t)+Bu(t-h_1) \tag{2.1}$$

by a static output-feedback of the form

$$u(t)=K_1y(t)+K_2y(t-h). \tag{2.2}$$

Here, $y(t)\in\mathbb{R}^n$ is the measurement, $u(t)\in\mathbb{R}^k(k\le n)$ is the control input, $A_1,A_2\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times k}$ are system matrices. $K_1,K_2\in\mathbb{R}^{k\times n}$ are controller gains, $h_1\ge 0$ is input delay and $h>0$ is a 'stabilizing delay'. In the case of scalar y and $A_2=0$ the stability of (2.1), (2.2) was analyzed in Borne et al. (2000) and Shaikhet (2013) via Lyapunov–Krasovskii method. The scalar system (2.1) with $B=1$ and $A_2=0$ may model a double integrator (for $A_1=0$), an inverted pendulum (for $A_1>0$) and a double oscillator (for $A_1<0$). In the vector case, (2.1) may present e.g. a model of inverted pendulum on the cart (see Example 3) or large-scale system of double integrators, inverted pendulums or oscillators.

Denoting

$$x(t)=col\{x_1(t),x_2(t)\}=col\{y(t),\dot{y}(t)\},\quad h_2=h_1+h,$$

we present the closed-loop system (2.1), (2.2) as

$$\dot{x}(t)=\begin{bmatrix}0 & I_n\\ A_1 & A_2\end{bmatrix}x(t)+\sum_{i=1}^{2}\begin{bmatrix}0\\ B\end{bmatrix}K_ix_1(t-h_i). \tag{2.3}$$

We assume that the delay $h>0$ is constant and known. For the delay h_1, we will consider two cases:

(1) h_1 is constant and known;
(2) $h_1=h_1(t)$ is piecewise-continuous in time and bounded

$$h_1(t)\in[0,h_{1M}],\qquad h_2(t)=h+h_1(t)\in[h,h+h_{1M}],$$

where h_{1M} (and, thus, $h_{2M}=h+h_{1M}$) is known.

Throughout the paper we assume

A1 The pair $\left(\begin{bmatrix}0 & I_n\\ A_1 & A_2\end{bmatrix},\begin{bmatrix}0\\ B\end{bmatrix}\right)$ is stabilizable.

In case (1) of the known h_1, under **A1** there exist $k\times n$ gains $\bar{K}_1$ and $h_2\bar{K}_2$ such that the following matrix is Hurwitz:

$$\bar{D}_1=\bar{D}_1(h_2)=\begin{bmatrix}0 & I_n\\ A_1 & A_2\end{bmatrix}+\begin{bmatrix}0\\ B\end{bmatrix}[\bar{K}_1\ h_2\bar{K}_2]. \tag{2.4}$$

Assuming $h_1=O(h_2^2)$ (meaning that the stabilizing delay h is larger than the input one h_1) and substituting

$$\begin{aligned}x_1(t)&=x_1(t-h_1)+O(h_1),\\ h_2x_2(t)&=x_1(t)-x_1(t-h_2)+O(h_2^2)\\ &=x_1(t-h_1)-x_1(t-h_2)+O(h_2^2)\end{aligned}$$

into $\bar{K}_1x_1(t)+h_2\bar{K}_2x_2(t)$, the system $\dot{x}(t)=\bar{D}_1x(t)$ can be presented as

$$\dot{x}(t)=\begin{bmatrix}0 & I_n\\ A_1 & A_2\end{bmatrix}x(t)+\sum_{i=1}^{2}\begin{bmatrix}0\\ B\end{bmatrix}K_ix_1(t-h_i)+O(h_2^2)$$

with

$$K_1=\bar{K}_1+\bar{K}_2,\quad K_2=-\bar{K}_2. \tag{2.5}$$

Hence, under **A1** the system (2.3) is asymptotically stable for small enough h_2. From (2.5), we have $\bar{K}_1=K_1+K_2$, $\bar{K}_2=-K_2$. Substituting the latter into $\bar{D}_1$, we arrive at the Hurwitz matrix

$$D_1=D_1(h_2)=\begin{bmatrix}0 & I_n\\ A_1+B(K_1+K_2) & A_2-h_2BK_2\end{bmatrix}. \tag{2.6}$$

In the present paper, for case (1) given $K_i,h_i(i=1,2)$ we assume that D_1 defined by (2.6) is Hurwitz. In case (2), K_1 and K_2 may be found from (2.5), where $\bar{K}_1$ and $\bar{K}_2$ are such that the matrix $\bar{D}_1(h)$ defined by (2.4) is Hurwitz. In both cases, we will derive sufficient stability conditions for the system (2.3) in terms of the low-order LMIs with a small number of decision variables. We will show that these LMIs are always feasible for small enough h_2.

We present below some useful lemmas.

Lemma 1 (*Wirtinger's Inequality Liu, Suplin, & Fridman, 2010*)**.** *Let $z(t):(a,b)\to\mathbb{R}^n$ be absolutely continuous with $\dot{z}\in L_2(a,b)$ and $z(a)=0$ or $z(b)=0$. Then for any $n\times n$ matrix $W>0$ the following inequality holds:*

$$\int_a^b z'(\xi)Wz(\xi)d\xi\le\frac{4(b-a)^2}{\pi^2}\int_a^b\dot{z}'(\xi)W\dot{z}(\xi)d\xi. \tag{2.7}$$

Lemma 2 (*Jensen's Inequality Solomon & Fridman, 2013*)**.** *Denote*

$$G=\int_a^b f(s)x(s)ds,$$

where $a\le b$, $f:[a,b]\to[0,\infty)$, $x(s)\in\mathbb{R}^n$ and the integration concerned is well defined. Then for any $n\times n$ matrix $R>0$ the following inequality holds:

$$G'RG\le\int_a^b f(\theta)d\theta\int_a^b f(s)x'(s)Rx(s)ds. \tag{2.8}$$

Lemma 3 (Melchor-Aguilar, Kharitonov, & Lozano, 2010). *Consider the integral equation*

$$z(t)=\sum_{i=1}^{m}\int_{-h_i}^{0}F_i(s)z(t+s)ds,\quad z(t)\in\mathbb{R}^n, \tag{2.9}$$

where $F_i(s)\in\mathbb{R}^{n\times n}$ is integrable. If there exists a continuous functional $V(\varphi)$ such that $V(z_t)$ is differentiable in $t\geq 0$ and the following conditions hold

$$\alpha_1\sum_{i=1}^{m}\int_{-h_i}^{0}|\varphi(s)|^2ds\leq V(\varphi)\leq\alpha_2\sum_{i=1}^{m}\int_{-h_i}^{0}|\varphi(s)|^2ds, \tag{2.10}$$

$$\frac{d}{dt}V(z_t)\leq-\beta\sum_{i=1}^{m}\int_{-h_i}^{0}|z(t+s)|^2ds, \tag{2.11}$$

with some positive constants $\alpha_1\leq\alpha_2$ and β, then (2.9) is exponentially stable.

3. Wirtinger's inequality-based approach

In this section we consider (2.3) in case 2 of time-varying delay $h_1=h_1(t)\in[0,h_{1M}]$ and $h_2=h_2(t)\in[h,h_{2M}]$. The term with stabilizing delay $x_1(t-h_2)$ can be presented as

$$x_1(t-h_2)=x_1(t)-h_2x_2(t)+\delta_2(t),\quad \delta_2(t)=h_2x_2(t)-[x_1(t)-x_1(t-h_2)]. \tag{3.1}$$

Since $\dot x_1(t)=x_2(t)$, the delayed term $x_1(t-h_1)$ can be presented as

$$x_1(t-h_1)=x_1(t)+\delta_1(t),\quad \delta_1(t)=-\int_{t-h_1}^{t}x_2(s)ds. \tag{3.2}$$

Substituting (3.1) and (3.2) into (2.3) we arrive at the following system

$$\dot x(t)=D_1(h_2)x(t)+\sum_{i=1}^{2}\begin{bmatrix}0\\B\end{bmatrix}K_i\delta_i(t). \tag{3.3}$$

Theorem 1. *Given $K_i\in\mathbb{R}^{k\times n}(i=1,2)$, $h_{1M}\geq 0$ and $h>0$ assume that the matrix $D_1(h)$ defined by (2.6), is Hurwitz. Let there exist positive definite $k\times k$-matrices W,R and $2n\times 2n$-matrix $P>0$ that satisfy the following LMIs*

$$\Xi(h)<0,\qquad \Xi(h_{2M})<0, \tag{3.4}$$

where $h_{2M}=h+h_{1M}$ and

$$\Xi(h_2)=\begin{bmatrix}X_1 & X & X & h_{2M}^2D_1'[0\ K_2]'W\\ * & -R & 0 & h_{2M}^2B'K_2'W\\ * & * & -\frac{\pi^2}{4}W & h_{2M}^2B'K_2'W\\ * & * & * & -W\end{bmatrix},$$

$$X_1=D_1'(h_2)P+PD_1(h_2)+\mathrm{diag}\left\{0,h_{1M}^2K_1'RK_1\right\},$$
$$X=P\cdot col\{0,B\}.$$

Then the system (2.3) is asymptotically stable for all $h_1(t)\in[0,h_{1M}]$, $h_2(t)\in[h,h_{2M}]$. Moreover, the LMIs (3.4) are always feasible for small enough h and $h_{1M}=O(h^2)$ provided $A_2=O(h)$.

In case of constant $h_1\equiv h_{1M}$, the system (2.3) is asymptotically stable if $\Xi(h_{2M})<0$.

Proof. Differentiating $V_1(x)=x'Px$, $P>0$, along (3.3) we have

$$\frac{d}{dt}V_1(x(t))=2x'(t)P\left[D_1(h_2)x(t)+\sum_{i=1}^{2}\begin{bmatrix}0\\B\end{bmatrix}K_i\delta_i(t)\right]. \tag{3.5}$$

In order to compensate δ_1 and δ_2-terms in (3.5) consider

$$V_2(x_t)=h_{1M}\int_{t-h_{1M}}^{t}(s-t+h_{1M})x_2'(s)\hat Rx_2(s)ds,\quad \hat R=K_1'RK_1, \tag{3.6}$$

and

$$V_3(x_t)=h_{2M}^3\int_{t-h_{2M}}^{t}(s-t+h_{2M})\dot x_2'(s)\hat W\dot x_2(s)ds,\quad \hat W=K_2'WK_2, \tag{3.7}$$

respectively. We have

$$\frac{d}{dt}V_2(x_t)\leq h_{1M}^2x_2'(t)\hat Rx_2(t)-h_{1M}\int_{t-h_1}^{t}x_2'(s)\hat Rx_2(s)ds,$$
$$\frac{d}{dt}V_3(\dot x_t)\leq h_{2M}^4\dot x_2'(t)\hat W\dot x_2(t)-h_{2M}^3\int_{t-h_2}^{t}\dot x_2'(s)\hat W\dot x_2(s)ds.$$

Then, by Jensen's inequality (2.8)

$$\frac{d}{dt}V_2(x_t)\leq h_{1M}^2x_2'(t)\hat Rx_2(t)-\delta_1'(t)\hat R\delta_1(t). \tag{3.8}$$

By Wirtinger's inequality (2.7)

$$-h_{2M}^3\int_{t-h_2}^{t}\dot x_2'(s)\hat W\dot x_2(s)ds\leq-\frac{\pi^2}{4}h_{2M}\int_{t-h_2}^{t}[x_2(t)-x_2(s)]'\hat W[x_2(t)-x_2(s)]ds.$$

Therefore

$$\frac{d}{dt}V_3(\dot x_t)\leq h_{2M}^4\dot x_2'(t)\hat W\dot x_2(t)-\frac{\pi^2}{4}h_{2M}\int_{t-h_2}^{t}[x_2(t)-x_2(s)]'\hat W[x_2(t)-x_2(s)]ds.$$

Applying further Jensen's inequality (2.8) to the last integral term and taking into account that

$$\int_{t-h_2}^{t}[x_2(t)-x_2(s)]ds=\delta_2(t)$$

we arrive at

$$\frac{d}{dt}V_3(\dot x_t)\leq h_{2M}^4\dot x_2'(t)\hat W\dot x_2(t)-\frac{\pi^2}{4}\delta_2'\hat W\delta_2. \tag{3.9}$$

Choose Lyapunov functional

$$V(x_t,\dot x_t)=V_1(x(t))+V_2(x_t)+V_3(\dot x_t).$$

Denote $\eta(t)=col\{x(t),K_1\delta_1(t),K_2\delta_2(t)\}$. Differentiating V along (3.3) and using (3.5)–(3.9) we find

$$\frac{d}{dt}V(x_t,\dot x_t)\leq\eta'(t)\begin{bmatrix}X_1 & X & X\\ * & -R & 0\\ * & * & -\frac{\pi^2}{4}W\end{bmatrix}\eta(t)+h_{2M}^4\dot x_2'(t)K_2'WK_2\dot x_2(t). \tag{3.10}$$

Substituting

$$K_2\dot x_2=[0\ K_2]D_1x(t)+\sum_{i=1}^{2}K_2BK_i\delta_i(t)$$

into (3.10) and applying Schur complement we conclude that $\frac{d}{dt}V \le -c|x(t)|^2$ for some $c > 0$ if $\Xi(h_2) < 0$. Since $D_1 = D_1(h_2)$ is affine in h_2, the matrix $\Xi(h_2)$ is affine in h_2. Therefore, the feasibility of the LMIs (3.4) yields the inequality $\Xi(h_2) < 0$ for all $h_2 = h_2(t) \in [h, h_{2M}]$. The latter inequality implies the asymptotic stability of (3.3) and thus of (2.3).

We prove next the feasibility of LMIs (3.4) for small enough $h_{1M} = O(h^2)$. Let P be a solution of the Lyapunov equation

$$D_1'(h)P + PD_1(h) = -hI_{2n}.$$

Then $P = O(1)$ and for $h_{1M} = O(h^2)$

$$D_1'(h_{2M})P + PD_1(h_{2M}) = -hI_{2n} + O(h^2).$$

Choose $R = h_{1M}^{-1}I_k$ and $W = h_{2M}^{-2}I_k$. Applying Schur complements to $\Xi(h) < 0$ and $\Xi(h_{2M}) < 0$, we arrive at

$$D_1'(h)P + PD_1(h) + O(h^2) < 0,$$

that holds for small h.

For constant delays $h_1 = h_{1M}$ and $h_2 = h_{2M}$ the result follows from (3.10). □

Remark 1. Note that the objective of this paper is simple LMI conditions for stability. It is clear that by adding additional terms to V and applying appropriate inequalities (e.g. the standard by now term $\int_{t-h_{1M}}^t x'(s)K_1'SK_1x(s)ds$ with $S > 0$ and the reciprocally convex approach of Park, Ko, & Jeong, 2011) one can derive less conservative conditions.

The method of Theorem 1 is not extendable to stochastic case because of V_3-term in V that depends on $\dot{x}_2$. To derive results applicable to the stochastic case, in the next section we consider a model transformation-based approach.

4. Model transformation-based approach

In this section, we start with the case of constant and known delays. Only in the end of section (in Section 4.4), we consider the case of time-varying delays.

4.1. Neutral type model transformation of the system

The idea of the transformation that we use below is to represent the system (2.3) in the form of a neutral type system without delays in the right-hand side of the equation. The term with stabilizing delay $x_1(t - h_2)$ can be presented as

$$x_1(t-h_i) = x_1(t) - h_i x_2(t) + \frac{d}{dt}G_i(x_{2t}), \tag{4.1}$$

where $i = 2$ and

$$G_i(x_{2t}) = \int_{t-h_i}^t (s - t + h_i)x_2(s)ds. \tag{4.2}$$

Indeed, since $\dot{x}_1(t) = x_2(t)$ we have

$$\frac{d}{dt}G_i(x_{2t}) = h_i x_2(t) - [x_1(t) - x_1(t-h_i)].$$

The term $x_1(t - h_1)$ can be presented either as

$$\begin{aligned} x_1(t-h_1) &= x_1(t) + \frac{d}{dt}G_1(x_{1t}), \\ G_1(x_{1t}) &= -\int_{t-h_1}^t x_1(s)ds, \end{aligned} \tag{4.3}$$

or as (4.1), (4.2), where $i = 1$.

Different ways of the presentation for $x_1(t-h_1)$ lead to different neutral type systems and to different (complementary) stability conditions. Using G_1 of (4.3) and G_2 of (4.2), we represent the system (2.3) in the form of a neutral type system

$$\dot{z}(t) = D_1 x(t), \quad x(t), z(t) \in \mathbb{R}^{2n}, \tag{4.4}$$

where D_1 is given by (2.6) and

$$\begin{aligned} x(t) &= col\{x_1(t), x_2(t)\}, \\ z(t) &= x(t) - col\{0, B\}\sum_{i=1}^{2} K_i G_i(x_{it}). \end{aligned} \tag{4.5}$$

Using G_1 and G_2 of (4.2) we represent the system (2.3) in the form

$$\dot{z}(t) = D_2 x(t), \quad x(t), z(t) \in \mathbb{R}^{2n}, \tag{4.6}$$

where

$$\begin{aligned} x(t) &= col\{x_1(t), x_2(t)\}, \\ z(t) &= x(t) - col\{0, B\}\sum_{i=1}^{2} K_i G_i(x_{2t}), \\ D_2 &= \begin{bmatrix} 0 & I_n \\ A_1 + BK_1 + BK_2 & A_2 - \sum_{i=1}^{2} h_i BK_i \end{bmatrix}. \end{aligned} \tag{4.7}$$

Note that if the matrix D_1 is Hurwitz, then for small enough $h_1 = O(h_2^2)$ the matrix D_2 is Hurwitz too. In Borne et al. (2000) and Shaikhet (2013) representation (4.6), (4.7) was used.

Remark 2. Differently from the so-called first model transformation (see e.g. p. 81 of Fridman, 2014), the neutral type one does not introduce additional dynamics. Indeed, it is easy to see that if $x(t)$ subject to continuous initial condition $x(s) = \phi(s),\ s \in [-h_2, 0]$ satisfies (2.3) for $t \ge 0$, then it satisfies the transformed neutral type systems and vise versa.

4.2. Stability of the integral equations

In order to use the Lyapunov–Krasovskii theorem for the stability of the neutral type systems (see e.g. Theorem 8.1 on p. 293 of Hale & Verduyn Lunel, 1993), we first derive conditions for the exponential stability of the corresponding integral equations

$$z(t) = 0. \tag{4.8}$$

We will start with (4.8), where $z(t)$ is defined by (4.5), i.e. with the following system:

$$x_1(t) = 0, \qquad x_2(t) - BK_1G_1(x_{1t}) - BK_2G_2(x_{2t}) = 0.$$

It is clear that the latter system is exponentially stable if the equation $x_2(t) = BK_2G_2(x_{2t})$ is exponential stable, i.e. if the following integral equation

$$x_2(t) = \int_{t-h_2}^t (s - t + h_2)BK_2x_2(s)ds \tag{4.9}$$

is exponentially stable.

Lemma 4. *Let there exists an $n \times n$-matrix $S_2 > 0$ such that the following LMI holds*

$$h_2^2 K_2' B' S_2 B K_2 - 4h_2^{-2} S_2 < 0. \tag{4.10}$$

Then the integral equation (4.9) (and, thus, the integral equation (4.8), where $z(t)$ is defined by (4.5)) is exponentially stable.

Proof. It is easy to see that the functional

$$V_S(x_{2t}) = \int_{t-h_2}^{t} (s-t+h_2+\alpha)^2 x_2'(s)S_2x_2(s)ds, \tag{4.11}$$

where $\alpha > 0$ and $0 < S_2 \in \mathbb{R}^{n\times n}$, satisfies the condition (2.10) of Lemma 3. Calculating the derivative of the functional (4.11) we obtain

$$\begin{aligned}\frac{d}{dt}V_S(x_{2t}) \le\ & (h_2+\alpha)^2 x_2'(t)S_2x_2(t)\\ & -2\int_{t-h_2}^{t}(s-t+h_2)x_2'(s)S_2x_2(s)ds\\ & -2\alpha\int_{t-h_2}^{t}x_2'(s)S_2x_2(s)ds.\end{aligned} \tag{4.12}$$

Via Jensen's inequality (2.8) we have

$$\begin{aligned}&-2\int_{t-h_2}^{t}(s-t+h_2)x_2'(s)S_2x_2(s)ds\\ &\le -4h_2^{-2}G_2'(x_{2t})S_2G_2(x_{2t}).\end{aligned} \tag{4.13}$$

From this and (4.12) it follows that

$$\begin{aligned}\frac{d}{dt}V_S(x_{2t}) &\le (h_2+\alpha)^2 x_2'(t)S_2x_2(t)\\ &-4h_2^{-2}G_2'(x_{2t})S_2G_2(x_{2t}) - \beta\int_{t-h_2}^{t}|x_2'(s)|^2ds\end{aligned} \tag{4.14}$$

where $\beta = 2\alpha\lambda_{\min}(S_2) > 0$. Here, $\lambda_{\min}(S_2)$ is the minimal eigenvalue of S_2.

Substituting into (4.14) $x_2(t) = BK_2G_2(x_{2t})$, we obtain

$$\begin{aligned}\frac{d}{dt}V_S(x_{2t}) \le\ & G_2'(x_{2t})\left[(h_2+\alpha)^2K_2'B'S_2BK_2 - 4h_2^{-2}S_2\right]\\ & \times G_2(x_{2t}) - \beta\int_{t-h_2}^{t}|x_2(s)|^2ds.\end{aligned}$$

So, if

$$(h_2+\alpha)^2K_2'B'S_2BK_2 - 4h_2^{-2}S_2 < 0 \tag{4.15}$$

then the functional (4.11) satisfies also the condition (2.11) of Lemma 3 and therefore (4.9) is exponentially stable. It is easy to see that if (4.10) holds then (4.15) holds with a small enough $\alpha > 0$. □

Remark 3. Note that the feasibility of the LMI (4.10) is equivalent to the fact that all the eigenvalues of $0.5h_2^2BK_2$ are inside of the unit circle (this LMI is Lyapunov inequality for the discrete-time system $x(k+1) = 0.5h_2^2BK_2x(k),\ k = 0, 1, \ldots$).

Consider next (4.8), where $z(t)$ is defined by (4.7), i.e. the following system:

$$x_1(t) = 0, \qquad x_2(t) - BK_1G_1(x_{2t}) - BK_2G_2(x_{2t}) = 0.$$

In this case, the stability of (4.8) is reduced to the stability of the integral equation with two delays

$$x_2(t) = \sum_{i=1}^{2}\int_{t-h_i}^{t}(s-t+h_i)BK_ix_2(s)ds. \tag{4.16}$$

We immediately arrive at the following result:

Lemma 5. *Let there exist some positive definite $n\times n$-matrices S_1, S_2 such that the following LMI holds*

$$\begin{aligned}&\bar{B}'\left(h_1^2S_1 + h_2^2S_2\right)\bar{B} - \bar{S} < 0,\\ &\bar{B} = (BK_1\ BK_2), \qquad \bar{S} = \operatorname{diag}\{4h_1^{-2}S_1, 4h_2^{-2}S_2\}.\end{aligned} \tag{4.17}$$

Then the integral equation (4.16) (and, thus, (4.8) with notation (4.7)) is exponentially stable.

4.3. Stability of (2.3): constant delays

Consider the neutral type system (4.4), (4.5), where the matrix D_1 is Hurwitz.

Theorem 2. *Given $K_i \in \mathbb{R}^{k\times n} (i = 1, 2)$ and constant known delays $h_1 \ge 0$ and $h_2 > 0$ such that D_1 defined by (2.6) is Hurwitz, let there exist positive definite matrices $S_2 \in \mathbb{R}^{n\times n}$, $R_1, R_2 \in \mathbb{R}^{k\times k}$ and $P \in \mathbb{R}^{2n\times 2n}$ that satisfy LMIs (4.10) and*

$$\Psi_1 = \begin{bmatrix}\Phi_1 & \Phi & \Phi\\ * & -R_1 & 0\\ * & * & -4R_2\end{bmatrix} < 0, \tag{4.18}$$

$$\begin{aligned}&\Phi = D_1'P\cdot col\{0, B\},\\ &\Phi_1 = D_1'P + PD_1 + \operatorname{diag}\left\{h_1^2K_1'R_1K_1,\ h_2^4K_2'R_2K_2\right\}.\end{aligned} \tag{4.19}$$

Then the system (2.3) is asymptotically stable. Moreover, the LMIs (4.10) and (4.18) are always feasible for small enough h_2 and $h_1 = O(h_2^2)$ provided $A_2 = O(h_2)$.

Proof. Via the condition (4.10) the integral equation (4.9) is exponentially stable. Differentiating $V_1(x_t) = z'(t)Pz(t), P > 0$, along (4.4), (4.5) and using notation (4.18), we have

$$\begin{aligned}\frac{d}{dt}V_1(x_t) &= 2\left(x(t) - \begin{bmatrix}0\\ B\end{bmatrix}\sum_{i=1}^{2}K_iG_i(x_{it})\right)' PD_1x(t)\\ &= 2x'(t)PD_1x(t) - 2\sum_{i=1}^{2}K_i'G_i'(x_{it})\Phi'x(t).\end{aligned} \tag{4.20}$$

In order to compensate G_i-terms in (4.20) consider

$$\begin{aligned}V_2(x_t) &= h_1\int_{t-h_1}^{t}(s-t+h_1)x_1'(s)\hat{R}_1x_1(s)ds,\\ &+ h_2^2\int_{t-h_2}^{t}(s-t+h_2)^2x_2'(s)\hat{R}_2x_2(s)ds,\\ &\hat{R}_i = K_i'R_iK_i,\ i = 1, 2.\end{aligned} \tag{4.21}$$

Taking into account the representations (4.3), (4.2) for $G_1(x_{1t})$, $G_2(x_{2t})$ and applying Jensen's inequality (2.8), similarly to (4.13) we find

$$\begin{aligned}\frac{d}{dt}V_2(x_t) \le\ & h_1^2x_1'(t)\hat{R}_1x_1(t) + h_2^4x_2'(t)\hat{R}_2x_2(t)\\ & - G_1'(x_{1t})\hat{R}_1G_1(x_{1t}) - 4G_2'(x_{2t})\hat{R}_2G_2(x_{2t}).\end{aligned} \tag{4.22}$$

Denote $\eta(t) = col\{x(t), -K_1G_1(t, x_{1t}), -K_2G_2(t, x_{2t})\}$. Then for the Lyapunov functional $V(x_t) = V_1(x_t) + V_2(x_t)$ from (4.18), (4.20) and (4.22), (4.18) we obtain

$$\begin{aligned}\frac{d}{dt}V(x_t) \le\ & 2x'(t)PD_1x(t) - 2\sum_{i=1}^{2}x'(t)\Phi'K_iG_i(x_{it})\\ & + h_1^2x_1'(t)\hat{R}_1x_1(t) + h_2^4x_2'(t)\hat{R}_2x_2(t)\\ & - G_1'(x_{1t})\hat{R}_1G_1(x_{1t}) - 4G_2'(x_{2t})\hat{R}_2G_2(x_{2t})\\ =\ & \eta'(t)\Psi_1\eta(t) \le -c|x(t)|^2\end{aligned}$$

for some $c > 0$. The latter inequality guarantees asymptotic stability of the neutral type system (4.4), (4.5) (and, thus, of (2.3)) with the asymptotically stable integral equation (Hale & Verduyn Lunel, 1993).

We prove next the feasibility of LMIs (4.10) and (4.18) for small enough h_2. It is easy to see that (4.10) is feasible with $S_2 = I_n$ for small enough h_2. Let P be a solution of the Lyapunov equation $D_1'P + PD_1 = -h_2 I_{2n}$ and $R_i = h_i^{-i} I_k (i = 1, 2)$. Applying Schur complements to (4.18), we arrive at $D_1'P + PD_1 + O(h_2^2) < 0$, that holds for small enough h_2 because $-h_2 I_{2n} + O(h_2^2) < 0$. □

Consider next the system (4.6), (4.7), where D_2 is Hurwitz. Modifying the arguments of Theorem 2 and using the Lyapunov functional V in the form

$$V(x_t) = z'(t)Pz(t) + \sum_{i=1}^{2} h_i^2 \int_{t-h_i}^{t} (s-t+h_i)^2 x_2'(s) K_i' R_i K_i x_2(s) ds,$$
$$0 < P \in \mathbb{R}^{2n\times 2n},\ 0 < R_i \in \mathbb{R}^{k\times k},\ i = 1, 2,$$

we arrive at the following theorem.

Theorem 3. *Given $K_i \in \mathbb{R}^{k\times n} (i = 1, 2)$ and constant known delays $h_1 \geq 0$ and $h_2 > 0$ such that D_2 defined by (4.7) is Hurwitz, let there exist positive definite matrices $S_1, S_2 \in \mathbb{R}^{n\times n}$, $R_1, R_2 \in \mathbb{R}^{k\times k}$ and $P \in \mathbb{R}^{2n\times 2n}$ that satisfy LMIs (4.17) and*

$$\Psi_2 = \begin{bmatrix} \Phi_2 & \Phi & \Phi \\ * & -4R_1 & 0 \\ * & * & -4R_2 \end{bmatrix} < 0, \tag{4.23}$$

$$\begin{aligned} \Phi &= D_2'P \cdot col\{0, B\}, \\ \Phi_2 &= D_2'P + PD_2 + \text{diag}\left\{0, \sum_{i=1}^{2} h_i^4 K_i' R_i K_i\right\}. \end{aligned} \tag{4.24}$$

Then the system (2.3) is asymptotically stable. Moreover, the LMIs (4.17) and (4.23) are always feasible for small enough h_2 and $h_1 = O(h_2^2)$ provided $A_2 = O(h_2)$.

Remark 4. As seen from Example 2, Theorems 2 and 3 lead to close and complementary results for $h_1 > 0$ (they are equivalent for $h_1 = 0$). However, Theorem 3 is based on a higher-order LMI (4.17) with an additional decision variable S_1 for the stability of the integral equation.

4.4. Stability: variable delays and stochastic perturbations

In this subsection, we consider time-varying and unknown $h_1(t) \in [0, h_{1M}]$ and $h_2(t) \in [h, h_{2M}]$, where $h_{2M} = h + h_{1M}$. Assume that the matrix $D_1(h)$ is Hurwitz. We present

$$x_1(t-h_1) = x_1(t) + \delta_1(t),$$
$$x_1(t-h_2) = x_1(t) - hx_2(t) + \frac{d}{dt} G_2(x_{2t}) + \delta_2(t),$$
$$\delta_1(t) = -\int_{t-h_1}^{t} x_2(s) ds, \qquad \delta_2(t) = -\int_{t-h_2}^{t-h} x_2(s) ds,$$
$$G_2(x_{2t}) = \int_{t-h}^{t} (s-t+h) x_2(s) ds.$$

Similarly to (4.4) the system (2.3) can be written as

$$\dot{z}(t) = D_1(h)x(t) + \sum_{i=1}^{2} \begin{bmatrix} 0 \\ B \end{bmatrix} K_i \delta_i(t), \tag{4.25}$$
$$x(t) = [x_1'(t)\ x_2'(t)]', \qquad z(t) = x(t) - [0\ B']' K_2 G_2(x_{2t}).$$

Denote

$$Y = col\{0, B\}' X,\ X = Pcol\{0, B\}, \qquad \Phi = D_1'(h) X. \tag{4.26}$$

Differentiating $V_1(x_t) = z'(t)Pz(t)$, $P > 0$, along (4.25), we have

$$\begin{aligned} \frac{d}{dt} V_1(x_t) &= 2\left[x(t) - \begin{bmatrix} 0 \\ B \end{bmatrix} K_2 G_2(x_{2t})\right]' P \\ &\quad \times \left[D_1(h)x(t) + \sum_{i=1}^{2} \begin{bmatrix} 0 \\ B \end{bmatrix} K_i \delta_i(t)\right] \\ &= 2x'(t)\left[PD_1(h)x(t) + X\sum_{i=1}^{2} K_i\delta_i(t)\right] \\ &\quad - 2G_2'(x_{2t})K_2'\left[\Phi' x(t) + Y\sum_{i=1}^{2} K_i \delta_i(t)\right]. \end{aligned} \tag{4.27}$$

In order to compensate G_2, δ_1, δ_2-terms in (4.27) consider

$$\begin{aligned} V_2(x_t) &= h^2 \int_{t-h}^{t} (s-t+h)^2 x_2'(s)\hat{Q} x_2(s) ds \\ &\quad + h_{1M} \int_{t-h_{1M}}^{t} (s-t+h_{1M}) x_2'(s) \hat{R}_1 x_2(s) ds \\ &\quad + h_{1M} \int_{t-h_{2M}}^{t-h} (s-t+h_{2M}) x_2'(s) \hat{R}_2 x_2(s) ds \\ &\quad + h_{1M}^2 \int_{t-h}^{t} x_2'(s) \hat{R}_2 x_2(s) ds \\ \hat{Q} &= K_2' Q K_2,\ \hat{R}_i = K_i' R_i K_i,\ i = 1, 2, \end{aligned} \tag{4.28}$$

where Q, R_1 and R_2 are positive definite $k \times k$ matrices.

Differentiating $V_2(x_t)$, we find

$$\begin{aligned} \frac{d}{dt} V_2(x_t) &= h^4 x_2'(t)\hat{Q}x_2(t) + h_{1M}^2 x_2'(t)(\hat{R}_1 + \hat{R}_2)x_2(t) \\ &\quad - 2h^2 \int_{t-h}^{t} (s-t+h) x_2'(s) \hat{Q} x_2(s) ds \\ &\quad - h_{1M} \int_{t-h_{1M}}^{t} x_2'(s)\hat{R}_1 x_2(s) ds \\ &\quad - h_{1M} \int_{t-h_{2M}}^{t-h} x_2'(s) \hat{R}_2 x_2(s) ds. \end{aligned}$$

Note that via (4.2) and Jensen's inequality (2.8) we have

$$2G_2'(x_{2t})\hat{Q}G_2(x_{2t}) \leq h^2 \int_{t-h}^{t} (s-t+h) x_2'(s)\hat{Q}x_2(s) ds,$$
$$\delta_1'(t)\hat{R}_1\delta_1(t) \leq h_{1M} \int_{t-h_{1M}}^{t} x_2'(s)\hat{R}_1 x_2(s) ds.$$

Similarly, using that $h_2 - h \leq h_{2M} - h = h_{1M}$, we obtain

$$\delta_2'(t)\hat{R}_2\delta_2(t) \leq h_{1M} \int_{t-h_{2M}}^{t-h} x_2'(s)\hat{R}_2 x_2(s) ds.$$

So,

$$\begin{aligned} &\frac{d}{dt} V_2(x_t) \leq x_2'(t)[h^4\hat{Q} + h_{1M}^2(\hat{R}_1 + \hat{R}_2)]x_2(t) \\ &\quad - 4G_2'(x_{2t})\hat{Q}G_2(x_{2t}) - \delta_1'(t)\hat{R}_1\delta_1(t) - \delta_2'(t)\hat{R}_2\delta_2(t). \end{aligned}$$

As a result for the Lyapunov functional $V(x_t) = V_1(x_t) + V_2(x_t)$, we have

$$\begin{aligned} \frac{d}{dt} V(x_t) &\leq 2x'(t)\left[PD_1(h)x(t) + X\sum_{i=1}^{2} K_i \delta_i(t)\right] \\ &\quad - 2G_2'(x_{2t})K_2'\left[\Phi' x(t) + Y\sum_{i=1}^{2} K_i\delta_i(t)\right] \end{aligned}$$

$$\begin{aligned}&+x_2'(t)[h^4\hat{Q}+h_{1M}^2(\hat{R}_1+\hat{R}_2)]x_2(t)\\&-4G_2'(x_{2t})\hat{Q}G_2(x_{2t})-\delta_1'(t)\hat{R}_1\delta_1(t)\\&-\delta_2'(t)\hat{R}_2\delta_2(t)=-\eta'(t)\Psi_3\eta(t),\end{aligned}\tag{4.29}$$

where $\eta(t)=col\{x(t),-K_2G_2(x_{2t}),K_1\delta_1(t),K_2\delta_2(t)\}$ and

$$\Psi_3=\begin{bmatrix}\Phi_3 & \Phi & X & X\\ * & -4Q & Y & Y\\ * & * & -R_1 & 0\\ * & * & * & -R_2\end{bmatrix},\tag{4.30}$$

$$\begin{aligned}\Phi_3 &= PD_1(h)+D_1'(h)P\\&+\operatorname{diag}\left\{0,\,h^4K_2'QK_2+h_{1M}^2\sum_{i=1}^{2}K_i'R_iK_i\right\}.\end{aligned}$$

Suppose now that the elements of A_1 and A_2 in (2.1) are under the influence of stochastic perturbations of the type of white noise. So, the Eq. (2.1) takes the form

$$\begin{aligned}\ddot{y}(t) &= (A_1+\sigma_1\dot{w}_1(t))\,y(t)\\&+(A_2+\sigma_2\dot{w}_2(t))\,\dot{y}(t)+Bu(t-h_1),\end{aligned}\tag{4.31}$$

where $\sigma_j\in\mathbb{R}^{n\times n\times q_j}(j=1,2)$ are constant matrices, $w_j(t)\in\mathbb{R}^{q_j}$ are mutually independent Wiener processes with independent components (Gikhman & Skorokhod, 1972; Shaikhet, 2013). In this case the closed-loop system (4.31), (2.2) similarly to (4.25) can be presented as Ito stochastic differential equation (Gikhman & Skorokhod, 1972)

$$\begin{aligned}dz(t)&=\left(D_1(h)x(t)+\sum_{i=1}^{2}\begin{bmatrix}0\\B\end{bmatrix}K_i\delta_i(t)\right)dt\\&+C(x(t))dw(t),\end{aligned}\tag{4.32}$$

where

$$\begin{aligned}C(x(t)) &= col\{0,\sigma(x(t))\}\in\mathbb{R}^{2n\times q},\\ \sigma(x(t)) &= \big((\sigma_1x_1(t))(\sigma_2x_2(t))\big)\in\mathbb{R}^{n\times q},\\ w(t) &= col\{w_1(t),w_2(t)\}\in\mathbb{R}^{q},\quad q=q_1+q_2.\end{aligned}\tag{4.33}$$

Let L be the generator (Gikhman & Skorokhod, 1972; Shaikhet, 2013) of the stochastic differential equation (4.32). Using the functional $V(x_t)$ constructed above and (4.33) similarly to (4.29), we obtain

$$\begin{aligned}LV(x_t) &\le \eta'(t)\Psi_3\eta(t)+\operatorname{Tr}[C'(x(t))PC(x(t))]\\&=\eta'(t)\Psi_3\eta(t)+\sum_{i=1}^{2}x_i'(t)T_ix_i(t),\\&\quad T_i=\operatorname{Tr}[\sigma_i'[0\;I_n]P[0\;I_n]'\sigma_i],\quad i=1,2.\end{aligned}$$

Summarizing we arrive at the following result:

Theorem 4. *Given $K_i\in\mathbb{R}^{k\times n}(i=1,2)$, $h_{1M}\ge 0$ and $h>0$ assume that the matrix $D_1(h)$ defined by (2.6) is Hurwitz. If there exist positive definite matrices $S_2\in\mathbb{R}^{n\times n}$, $Q,R_1,R_2\in\mathbb{R}^{k\times k}$ and $P\in\mathbb{R}^{2n\times 2n}$ that satisfy LMIs (4.10) and $\Psi_3+\operatorname{diag}\{T_1,T_2,0_{3k\times 3k}\}<0$, where Ψ_3 is defined by (4.26) and (4.30), then the closed-loop system (4.31), (2.2) with time-varying and unknown delays $h_1(t)\in[0,h_{1M}]$ and $h_2(t)=h+h_1(t)$ is asymptotically mean square stable.*

Remark 5. The LMIs of all the theorems in this paper are affine in A_1 and A_2. Therefore, in case of A_1 and A_2 from an uncertain time-varying polytope one have to solve these LMIs simultaneously for all the vertices of the polytope applying the same decision matrices.

Remark 6. The integral terms of Lyapunov functionals that have been used for the proof of the theorems coincide with the widely used by now double and triple integral terms (Fridman, 2014; Sun, Liu, & Chen, 2009). Indeed, the first term of V_2 given by (4.21) can be presented as a double integral term

$$\int_{-h_1}^{0}\int_{t+\theta}^{t}x_1'(s)\hat{R}_1x_1(s)dsd\theta.$$

The second term can be written as a triple integral term:

$$2\int_{-h_2}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t}x_2'(s)\hat{R}_2x_2(s)dsd\lambda d\theta.$$

Similarly integral terms in (3.6), (3.7), (4.11) and (4.28) can be presented as double or triple integral terms. Thus, all the presented results are based on simple Lyapunov functionals.

5. Examples

Example 1 (*Gu et al., 2003*)**.** Consider the system

$$\begin{aligned}&\dot{x}(t)=\begin{bmatrix}0 & 1\\-2 & 0.1\end{bmatrix}x(t)+\begin{bmatrix}0\\1\end{bmatrix}u(t),\\&u(t)=K_2x_1(t-h),\quad x(t)=col\{x_1(t),x_2(t)\}.\end{aligned}\tag{5.1}$$

Clearly, (5.1) cannot be stabilized by a non-delayed feedback $u(t)=K_2x_1(t)$ for any K_2 because the resulting matrix of the closed-loop system $\begin{bmatrix}0 & 1\\-2+K_2 & 0.1\end{bmatrix}$ is not Hurwitz. Choose as in Gu et al. (2003) $K_2=1$. The system (5.1) can be presented as (2.3) with $K_1=0$, $K_2=1$, $h_1=0$ and $n=k=1$. By the frequency domain analysis, (5.1) is asymptotically stable for the constant delay $h\in(0.1002,\,1.7178)$.

We apply here LMIs of Theorem 2 with $h_1=0$ and of Theorem 1 with $h_{1M}=0$. As shown in Table 1, the results of Theorems 1 and 2 are more conservative than the results obtained via discretized Lyapunov method with $N=1$ (Gu et al., 2003) and augmented Lyapunov functional (Seuret & Gouaisbaut, 2013), respectively. However, the number of scalar decision variables in LMIs of Theorem 2 (with $R_1=S_1=0$) is equal to $2n(2n+1)/2+n(n+1)/2+k(k+1)/2=2.5n^2+1.5n+0.5k^2+0.5k$, which is essentially smaller than by the other two methods. LMIs of Theorem 1 have minimal number of decision variables. The results of Borne et al. (2000) and Shaikhet (2013) are confined to the case of $A_2=0$ (here $A_2=0.1$), and thus are not applicable to this example.

Remark 7. Note that the results of Gu et al. (2003) and Seuret and Gouaisbaut (2013) are derived for general linear time-delay systems. When these results are applied to a particular system (2.3), where delay appears in x_1 only, there are some redundant decision variables. Thus, Lyapunov functional of Seuret and Gouaisbaut (2013) for (2.3) with $h_1=0$ and $n=k$ can be modified as follows

$$\begin{aligned}V_{aug}&=\begin{bmatrix}x(t)\\ \int_{t-h}^{t}x_1(s)ds\end{bmatrix}'\mathcal{P}\begin{bmatrix}x(t)\\ \int_{t-h}^{t}x_1(s)ds\end{bmatrix}\\&+\int_{t-h}^{t}x_1'(s)Sx_1(s)ds+h\int_{t-h}^{t}\int_{\theta}^{t}x_2'(\theta)Rx_2(\theta)d\theta ds\end{aligned}$$

with positive matrices $\mathcal{P}\in\mathbb{R}^{3n\times 3n}$ and $S,R\in\mathbb{R}^{n\times n}$. This leads to $5.5n^2+2.5n$ decision variables which is less than the one brought by a modified in a similar manner Lyapunov functional of Gu et al. (2003), but greater than $3n^2+2n$ and $2.5n^2+1.5n$ decision variables of Theorems 1 and 2, respectively. More important, differently from Gu et al. (2003) and Seuret and Gouaisbaut (2013),

Table 1
Example 1: stability interval $h \in [h_{\min}, h_{\max}]$.

Method	$h_{\min}$	$h_{\max}$	No vars
Gu et al. (2003) ($N = 1$)	0.1006	1.4272	$22n^2 + 5n$
Seuret and Gouaisbaut (2013)	0.1006	1.55	$12n^2 + 4n$
Theorem 2	0.106	1.409	$0.5*(5n^2+3n+k^2+k)$
Theorem 1	0.108	1.20	$2n^2 + n + 0.5(k^2 + k)$

Table 2
Example 2 ($\sigma^2 = 0$): maximum h_1 preserving the stability.

h_2	0.99	0.8	0.6	0.4	0.2
Shaikhet (2013)	–	0.11	0.16	0.14	0.086
Theorem 2	0.069	0.21	0.20	0.15	0.089
Theorem 3	0.099	0.20	0.18	0.14	0.086

the LMIs of Theorems 1 and 2 are always feasible for small enough h provided $D_1(h)$ of (2.6) is Hurwitz.

Example 2 (*Inverted Pendulum Borne et al., 2000; Shaikhet, 2013*). Consider the controlled inverted pendulum

$$\begin{aligned} &\ddot{y}(t) = [1 + \sigma \dot{w}(t)]y(t) + u(t - h_1), \quad y(t) \in \mathbb{R}, \\ &u(t) = K_1 y(t) + K_2 y(t - h), \end{aligned} \tag{5.2}$$

where $w(t)$ is the scalar standard Wiener process and either $\sigma^2 = 0$ (deterministic case) or $\sigma^2 = 0.2$ (stochastic case). It is easy to see that for $h_1 = 0$ and $\sigma = 0$ the system (5.2) is not stabilizable by $u(t) = Ky(t)$ for any K. Indeed, the characteristic polynomial $\lambda^2 - \lambda - K$ of the resulting closed-loop system has roots with positive real parts for any K. We further choose $K_1 = -4$, $K_2 = 2$.

In case 1 of constant delays, we apply LMIs of Theorems 2 and 3 to the closed-loop system for the values of $h_2 = h_1 + h$ as given in Table 2 and find the maximum values of h_1 that preserve the stability. As it is seen from Table 2, both theorems lead to close but complementary results. The results are favorably compared to those of Borne et al. (2000) and Shaikhet (2013).

Consider next case 2 of variable $h_1(t)$ and $h = 0.8$, $h_2(t) = h + h_1(t)$, where $\sigma = 0$. Applying Theorems 1 and 4 we find that the system is asymptotically stable for all $h_1(t) \in [0, 0.06]$ and $h_1(t) \in [0, 0.14]$, respectively. Note that Theorem 4 improves the results of Theorem 1, but this is on the account of the LMIs complexity. For $\sigma^2 = 0.2$, where Theorem 1 is not applicable, Theorem 4 guarantees asymptotic mean square stability for $h_1(t) \in [0, 0.13]$.

Example 3 (*Inverted Pendulum on the Cart Fridman, 2014*). Consider a model of the inverted pendulum on a cart

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{-mg}{M} & 0 & 0 \\ 0 & \dfrac{(M+m)g}{Ml} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{a}{M} \\ \dfrac{-a}{Ml} \end{bmatrix} u(t - h_1)$$

with $M = 3.9249$, $m = 0.2047$, $l = 0.2302$, $g = 9.81$, $a = 25.3$. In this model, x and θ represent cart position coordinate and pendulum angle from vertical, respectively. The system can be stabilized by a state-feedback

$$\begin{aligned} &u(t) = \bar{K}_1[x(t)\ \theta(t)]' + h_2\bar{K}_2[\dot{x}(t)\ \dot{\theta}(t)]', \\ &[\bar{K}_1\ h_2\bar{K}_2] = [5.825\ 24.941\ 5.883\ 5.140]. \end{aligned}$$

Assume now that the measurement is given by $y(t) = col\{x(t), \theta(t)\}$, and we are looking for a static output-feedback that stabilizes the system. As in the previous example, for $h_1 = 0$ the inverted pendulum on the cart is not stabilizable by a non-delayed static output-feedback. We choose $h_2 = 0.01$ and obtain from (2.5)

$$K_1 = [594.125\ 538.941], \qquad K_2 = [-588.3\ -514]. \tag{5.3}$$

Applying Theorems 2 and 3 we find that the feedback

$$u(t - h_1) = \sum_{i=1}^{2} K_i[x(t - h_i)\ \theta(t - h_i)]'$$

stabilizes the system for $h_1 \le 0.0075$ and $h_1 \le 0.0073$, respectively. Numerical simulations of the solutions of the closed-loop system confirm the asymptotic stability for all constant delays $h_1 \le 0.0093$, whereas for $h_1 = 0.0094$ the system has unbounded solutions.

Consider next case 2 of variable $h_1(t)$ and $h = 0.01$, $h_2(t) = h + h_1(t)$ and the gain (5.3). Applying Theorems 1 and 4, we find that the closed-loop system is asymptotically stable for $h_1(t) \in [0, 0.0019]$ and $h_1(t) \in [0, 0.0039]$, respectively.

6. Conclusions

In the present paper, simple LMI conditions for stabilization of vector second-order systems by using static output-feedback controllers with artificial delays have been provided. The controllers' gains may be found from the gains of the corresponding state-feedback controllers. The results are applicable to linear uncertain, either deterministic or stochastic, systems with uncertain time-varying delays, where observer-based design becomes complicated. The suggested controllers may be useful for networked-based control, for delay-induced consensus in multi-agent systems and for many other control problems. As all sufficient conditions, the presented results are conservative and can be further improved. This may be a topic for the future research.

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