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RICCATI EQUATIONS AND
STABILITY OF STOCHASTIC LINEAR SYSTEMS
WITH DISTRIBUTED DELAY

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Abstract

Many real processes can be modeled by stochastic differential equations with aftereffect [1-3]. Stability conditions for such systems can be obtained by construction of appropriate Lyapunov functionals using special procedure of Lyapunov functionals construction [4-15]. In this paper asymptotic mean square stability of stochastic linear differential equations with distributed delay is considered. Stability conditions are formulated in terms of existence of positive definite solutions of matrix Riccati equations. The method of different Riccati equations construction is proposed.

Keywords: Asymptotic mean square stability, stochastic linear equations, distributed delay, matrix Riccati equations.

1. Introduction. Consider the stochastic linear differential equation

$$\dot{x}(t) = Ax(t) + Cx(t)\dot{\xi}(t). \tag{1}$$

Here A and C are constant $(n*n)$-matrices, $x(t) \in \mathbf{R}^n$, $\xi(t)$ is a scalar Wiener process.

Denote $P > 0$ any symmetric positive definite matrix. Then an appropriate Liapunov function V for the equation (1) is a quadratic form $V = x'Px$, where the matrix P is a positive solution of the linear matrix equation [16]

$$A'P + PA + C'PC = -Q. \tag{2}$$

The necessary and sufficient conditions of asymptotic mean square stability of the system (1) can be formulated in terms of existence of a positive definite solution P of the matrix equation (2) for any positive definite matrix Q.

But for stochastic linear differential equations with delays, for example,

$$\dot{x}(t) = Ax(t) + \int_{t-h(t)}^{t} Bx(s)ds + \int_{t-\tau(t)}^{t} Cx(s)ds\dot{\xi}(t), \tag{3}$$

$$x_0(s) = \varphi(s), \quad s \leq 0.$$

this problem is more complicated.

Below we will obtain the conditions of asymptotic mean square stability for the equation (3).

Let $\{\Omega, \sigma, \mathbf{P}\}$ be the probability space, $\{f_t, t \geq 0\}$ be the family of σ-algebras, $f_t \in \sigma$, H be the space of f_0-adapted functions $\varphi(s) \in R^n$, $s \leq 0$, $\|\varphi\|_0^2 = sup_{s \leq 0}\mathbf{E}|\varphi(s)|^2$, $\mathbf{E}$ be the mathematical expectation, $\|B\|$ be arbitrary matrix norm of matrix B, L be the generator of the equation (3).

Definition 1. The zero solution of the equation (3) is called mean square stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\mathbf{E}|x(t)|^2 < \epsilon$ for all $t \geq 0$ if $\|\varphi\|_0^2 < \delta$. If, besides, $lim_{t\to\infty}\mathbf{E}|x(t)|^2 = 0$, then the zero solution of the equation (3) is called asymptotically mean square stable.

Theorem 1. Let there exists the functional $V(t,\varphi)$, which satisfies the conditions $\mathbf{E}V(0,\varphi) \leq c_1\|\varphi\|^2$, $\mathbf{E}V(t,x_t) \geq c_2\mathbf{E}|x(t)|^2$, $\mathbf{E}LV(t,x_t) \leq -c_3\mathbf{E}|x(t)|^2$, where $c_i > 0$, $i = 1,2,3$, $x_t = x(t+s)$, $s \leq 0$. Then the zero solution of the equation (3) is asymptotically mean square stable.

2. Lyapunov functionals construction. From Theorem 1 it follows that the construction of stability conditions for the equation (3) is reduced to the construction of appropriate Lyapunov functionals. Constructing different Lyapunov functionals we can obtain different stabillity conditions. Using the general method of Lyapunov functionals construction [4-15], we will construct two different Lyapunov functionals for the equation (3).

It is supposed that the delays $h(t)$ and $\tau(t)$ are functions satisfying the conditions:

$$0 \le h_0 \le h(t) \le h_1, \quad \hat{h} = h_1 - h_0, \quad 0 \le \tau(t) \le \tau_1, \tag{4}$$

2.1. First way. We will construct the Lyapunov functional V in the form $V = V_1 + V_2$, where $V_1 = x'(t)Px(t)$. Calculating LV_1, we get $LV_1 = x'(t)(A'P + PA)x(t) + I_0 + I_1$, where

$$I_0 = \int_{t-\tau(t)}^{t} \int_{t-\tau(t)}^{t} x(s)C'PCx(\theta)d\theta ds, \tag{5}$$

$$I_1 = \int_{t-h(t)}^{t} (x'(s)B'Px(t) + x'(t)PBx(s))ds,$$

Note that for arbitrary vectors a, b and any $R > 0$ we have

$$a'b + b'a = a'Ra + b'R^{-1}b - (Ra - b)'R^{-1}(Ra - b) \le a'Ra + b'R^{-1}b. \tag{6}$$

Using (6) for any $R_0 > 0$, $a = x(s)$, $b = C'PCx(\theta)$ and (4) we obtain

$$I_0 = \frac{1}{2} \int_{t-\tau(t)}^{t} \int_{t-\tau(t)}^{t} (x'(\theta)C'PCx(s) + x'(s)C'PCx(\theta))d\theta ds \le$$

$$\le \frac{\tau_1}{2} \int_{t-\tau_1}^{t} x'(s)(R_0 + C'PCR_0^{-1}C'PC)x(s)ds. \tag{7}$$

Using (6) for $R_1 > 0$, $a = x(t)$, $b = PBx(s)$ and (4) we have

$$I_1 \le h_1 x'(t)R_1 x(t) + \int_{t-h_1}^{t} x'(s)B'PR_1^{-1}PBx(s)ds.$$

Then

$$LV_1 \le x'(t)(A'P + PA + h_1 R_1)x(t) + \int_{t-h_1}^{t} x'(s)B'PR_1^{-1}PBx(s)ds+$$

$$+\frac{\tau_1}{2} \int_{t-\tau_1}^{t} x'(s)(R_0 + C'PCR_0^{-1}C'PC)x(s)ds.$$

Choosing now the functional V_2 in the form

$$V_2 = \int_{t-h_1}^{t} (s - t + h_1)x'(s)B'PR_1^{-1}PBx(s)ds+$$

$$+\frac{\tau_1}{2} \int_{t-\tau_1}^{t} (s - t + \tau_1)x'(s)(R_0 + C'PCR_0^{-1}C'PC)x(s)ds.$$

we have

$$LV_2 = h_1 x'(t)B'PR_1^{-1}PBx(t) - \int_{t-h_1}^{t} x'(s)B'PR_1^{-1}PBx(s)ds+$$

$$+\frac{\tau_1^2}{2}x'(t)(R_0 + C'PCR_0^{-1}C'PC)x(t) - \frac{\tau_1}{2} \int_{t-\tau_1}^{t} x'(s)(R_0 + C'PCR_0^{-1}C'PC)x(s)ds.$$

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As a result for $V = V_1 + V_2$ we have $LV \leq -x'(t)Qx(t)$, where

$$Q = -[A'P + PA + h_1(R_1 + B'PR_1^{-1}PB) + \frac{\tau_1^2}{2}(R_0 + C'PCR_0^{-1}C'PC)]. \tag{8}$$

Thus, it is proved

Theorem 2. Let for some symmetric matrices $R_0 > 0$, $R_1 > 0$ and $Q > 0$ there exists a positive definite solution P of the matrix Riccati equation (8). Then the zero solution of the equation (3) is asymptotically mean square stable.

Remark 1. Using the inequality (6) with other representations for a and b it is possible to get other matrix Riccati equations in Theorem 2. For example, using (6) for $R_0 > 0$, $a = Cx(s)$, $b = PCx(\theta)$ we obtain

$$I_0 \leq \frac{\tau_1}{2} \int_{t-\tau_1}^t x'(s)C'(R_0 + PR_0^{-1}P)Cx(s)ds.$$

Using the inequality (6) for $R_1 > 0$, $a = Px(t)$ and $b = Bx(s)$ we have

$$I_1 \leq h_1 x'(t)PR_1Px(t) + \int_{t-h_1}^t x'(s)B'R_1^{-1}Bx(s)ds.$$

Choosing in this case the functional V_2 in the form

$$V_2 = \int_{t-h_1}^t (s - t + h_1)x'(s)B'R_1^{-1}Bx(s)ds +$$

$$+ \frac{\tau_1}{2} \int_{t-\tau_1}^t (s - t + \tau_1)x'(s)C'(R_0 + PR_0^{-1}P)Cx(s)ds,$$

as a result for $V = V_1 + V_2$ we have $LV \leq -x'(t)Qx(t)$, where

$$Q = -[A'P + PA + h_1(PR_1P + B'R_1^{-1}B) + \frac{\tau_1^2}{2}C'(R_0 + PR_0^{-1}P)C]. \tag{9}$$

Example. In scalar case both of the equations (8) and (9) have a positive solution if and only if $A + h_1|B| + \frac{1}{2}\tau_1^2 C^2 < 0$.

2.2. Second way. Consider now another way of Lyapunov functional construction. Represent the equation (3) in the form of a stochastic differential equation of neutral type

$$\dot{y}(t) = (A + h_1 B)x(t) - \int_{t-h_1}^{t-h(t)} Bx(s)ds + \int_{t-\tau(t)}^t Cx(s)ds\dot{\xi}(t),$$

where $y(t) = x(t) + \int_{t-h_1}^t (s - t + h_1)Bx(s)ds$. We will construct the Lyapunov functional V in the form $V = V_1 + V_2$, where $V_1 = y'(t)Py(t)$. Calculating LV_1, we get $LV_1 = x'(t)[(A + h_1 B)'P + P(A + h_1 B)]x(t) + I_0 + I_1 + I_2 + I_3$, where I_0 is described by (5),

$$I_1 = \int_{t-h_1}^t (s - t + h_1)[x'(s)B'P(A + h_1 B)x(t) + x'(t)(A + h_1 B)'PBx(s)]ds,$$

$$I_2 = -\int_{t-h_1}^{t-h(t)} [x'(t)PBx(s) + x'(s)B'Px(t)]ds,$$

$$I_3 = -\int_{t-h_1}^t \int_{t-h_1}^{t-h(t)} (s - t + h_1)[x'(s)B'PBx(\theta) + x'(\theta)B'PBx(s)]d\theta ds.$$

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Using (6) and (4)we obtain

$$I_1 \leq \frac{1}{2}h_1^2 x'(t)R_1 x(t) + \int_{t-h_1}^{t} (s-t+h_1)x'(s)B'P(A+h_1B)'PBR_1^{-1}(A+h_1B)'PBx(s)ds,$$

$$I_2 \leq \int_{t-h_1}^{t-h(t)} [x'(t)R_2 x(t) + x'(s)B'PR_2^{-1}PBx'(s)]ds \leq$$

$$\leq \hat{h}x'(t)R_2 x(t) + \int_{t-h_1}^{t-h_0} x'(s)B'PR_2^{-1}PBx'(s)ds,$$

$$I_3 \leq \int_{t-h_1}^{t} \int_{t-h_1}^{t-h(t)} (s-t+h_1)[x'(\theta)R_3 x(\theta) + x'(s)B'PBR_3^{-1}B'PBx(s)]d\theta ds \leq$$

$$\leq \frac{1}{2}h_1^2 \int_{t-h_1}^{t-h_0} x'(\theta)R_3 x(\theta)d\theta + \hat{h}\int_{t-h_1}^{t} (s-t+h_1)x'(s)B'PBR_3^{-1}B'PBx(s)ds.$$

Then

$$LV_1 \leq x'(t)[(A+h_1B)'P + P(A+h_1B) + \frac{1}{2}h_1^2 R_1 + \hat{h}R_2]x(t)+$$

$$+ \int_{t-h_1}^{t} (s-t+h_1)x'(s)B'P(A+h_1B)R_1^{-1}(A+h_1B)'PBx(s)ds+$$

$$+ \int_{t-h_1}^{t-h_0} x'(s)[B'PR_2^{-1}PB + \frac{1}{2}h_1^2 R_3]x(s)ds+$$

$$+ \hat{h}\int_{t-h_1}^{t} (s-t+h_1)x'(s)B'PBR_3^{-1}B'PBx(s)ds+$$

$$+ \frac{\tau_1}{2}\int_{t-\tau_1}^{t} x'(s)(R_0 + C'PCR_0^{-1}C'PC)x(s)ds.$$

Choosing the functional V_2 in the form

$$V_2 = \frac{1}{2}\int_{t-h_1}^{t} (s-t+h_1)^2 x'(s)B'P(A+h_1B)R_1^{-1}(A+h_1B)'PBx(s)ds+$$

$$+ \int_{t-h_1}^{t-h_0} (s-t+h_1)x'(s)[B'PR_2^{-1}PB + \frac{1}{2}h_1^2 R_3]x(s)ds+$$

$$+ \hat{h}\int_{t-h_0}^{t} x'(s)[B'PR_2^{-1}PB + \frac{1}{2}h_1^2 R_3]x(s)ds+$$

$$+ \frac{\hat{h}}{2}\int_{t-h_1}^{t} (s-t+h_1)^2 x'(s)B'PBR_3^{-1}B'PBx(s)ds+$$

$$+ \frac{\tau_1}{2}\int_{t-\tau_1}^{t} (s-t+\tau_1)x'(s)(R_0 + C'PCR_0^{-1}C'PC)x(s)ds,$$

as a result for the functional $V = V_1 + V_2$ we obtain $LV \leq -x'(t)Qx(t)$, where

$$Q = -[(A+h_1B)'P + P(A+h_1B) + \frac{1}{2}h_1^2(R_1 + B'P(A+h_1B)R_1^{-1}(A+h_1B)'PB)+$$

$$+ \hat{h}(R_2 + B'PR_2^{-1}PB) + \frac{1}{2}h_1^2\hat{h}(R_3 + B'PBR_3^{-1}B'PB) + \frac{\tau_1^2}{2}(R_0 + C'PCR_0^{-1}C'PC)]. \qquad (10)$$

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Remark 2. Functional V_1 in this case is so-called degenerated functional (i.e. nonpositive definite). It means that direct application of Theorem 1 is impossible. The appropriate modification of Theorem 1 for degenerated functionals there is in [1]. Using this modification we obtain the following

Theorem 3. Let the inequality $h_1^2\|B\| < 2$ hold and for some symmetric matrices $R_i > 0$, $i = 0, 1, 2, 3$ and $Q > 0$ there exists a positive definite solution P of the matrix Riccati equation (10). Then the zero solution of the equation (3) is asymptotically mean square stable.

Remark 3. By analogy with Remark 1 instead of the equation (10) can be obtained other variants of Riccati equations.

Remark 4. Analogously results can be obtained for more general systems:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} \int_{t-h_i(t)}^{t} B_i x(s)ds + \sum_{i=1}^{k} \int_{t-\tau_i(t)}^{t} C_i x(s)ds\dot{\xi}(t),$$

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