Sensor Allocation Problems on the Real Line

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Consider a set of \( n \) closed finite intervals with non-negative endpoints, symbolizing sensors on the real line. The sensors are placed randomly in such a way that the distances between the consecutive centres are independent random variables with expectation inversely proportional to \( n \). Sensor’s length is equal among all the sensors, and overlaps between sensors are possible. In this work we address two fundamental sensor allocation problems. The interference problem tries to reallocate the sensors from their initial positions so as to eliminate overlaps. The coverage problem, on the other hand, allows overlaps, but tries to eliminate gaps (uncovered spaces) between the originally placed sensors. Both problems seek to minimize the total sensor movement while reaching their respective goals.

The paper investigates asymptotic behaviour of the optimal costs as the number of sensors increases to infinity. We propose a new unifying approach using tools from queueing theory, reflected random walks and weak convergence. We show that the results depend substantially on whether the sensor length is asymptotically greater, less or equal to the average distance between the original midpoints. The methodology is then used to address a more complicated, modified coverage problem, in which the overlaps between any two sensors can not exceed a certain parameter.

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1. Introduction

A large number \( n \) of sensors are distributed over the positive half-line. Each sensor is represented by a finite closed interval of the same length \( a^n \), and the distances between consecutive midpoints are i.i.d. random variables with expectation \( 1/n \). An arbitrary

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realization will possibly have some overlapping sensors, as well as the sensors with gaps between them. The coverage and interference problems aim to reallocate the sensors in an optimal way to either eliminate the overlaps (interference problem) or rather to remove the gaps, securing uninterrupted coverage starting from the origin (coverage problem). In both problems the optimality criteria is chosen to be minimization of the total movement of sensors.

In this paper we consider both problems in the asymptotical setting where the number of sensors \( n \) increases to infinity, and the length \( \sigma^n \) becomes infinitely small, roughly inversely proportional to \( n \). Our main goal at this stage is not to solve the coverage and interference problems, but to understand the orders of magnitude for optimal allocation costs and their dependence on the limiting behaviour of \( \sigma^n \). The results are of a particular importance in computer science, communications and networking; and later may be used as a starting point for finding an optimal solution. To the best of our knowledge, such asymptotical setting has yet to be fully studied. Some initial results appear in the conference proceedings ([9], coverage) and ([10], interference) treating the special cases when the i.i.d. distances between sensor midpoints are either uniform [9] or exponential [10]. Both papers study performance bounds for some allocation algorithms, mostly using straightforward computations. The methods had their limitations - not all choices of \( \sigma^n \) could be treated, and the bounds were mostly one-sided. Besides, no coherent connection was detected between the coverage and the interference problems.

In what follows we introduce a unifying methodology to address both problems simultaneously. The spectrum of all values for the sensor length \( \sigma^n \) is decomposed into three different categories, characterized by whether the length is asymptotically greater, less or equal to the average original distance of \( 1/n \) between sensors’ midpoints. The three categories will be referred to as overloaded, underloaded or critical, respectively; each producing a distinct two-sided estimate for either coverage or interference optimal costs. The terminology clearly suggests an analogy with the queueing theory and, indeed, quite a bit of it was used, especially for the interference problem. In particular, we use a so called potential outflow function (3.1) and related functionals. Our main findings are summarized in Theorem 2.3. It is shown that both costs behave quite the opposite in either underloaded and overloaded cases, while sharing the same order of magnitude in the critical case.

The resulting table may seem intuitive or even simple (we prefer elegant), yet the situation was far from being that clear before the 'queueing link' had been established. In particular, the orders \( O(1) \) were not suspected. And, while one can intuitively understand this in the interference context, proving \( O(1) \) bounds for the coverage required a step up from standard techniques. Specifically, a certain Skorokhod reflection map (6.3) had to be introduced (the result which may be of a separate interest in queueing theory as it can be used for analyzing busy periods in \( G/D/\infty \) models). In addition, to show the almost sure relation we could no longer rely on the methodology of asymptotical queueing analysis (very valuable to establish the tightness, for example). Instead, we had to bound the reflection map by a certain discrete-time, continuous-space Markov chain in the spirit of [13], making it possible to apply the classical 'Law of Large Numbers for Markov chains'.
The new methods are then exploited to study a more complicated, mixed problem (see (2.7) for the formulation, and Theorem 2.4). Once again, the O(1) estimates were the hardest to show, requiring a modified version (8.2) of the previously introduced Skorokhod map. After that, the map (8.2) had to be bounded from both sides by reflected random walks, amenable for analysis. Overall we believe that our methodology is robust, and can be useful to treat various related questions.

Related literature
Both coverage and interference problems have been studied extensively in deterministic and random settings. An unpretentious sample of relevant works is discussed below. The paper [6] deals with covering of a two-dimensional domain by sensors. A specific and more efficient type of coverage is called barrier coverage and concerns covering the perimeter of a two dimensional object. For the barrier coverage problem the authors of [11] propose efficient algorithms to determine, after deploying the sensors, whether a region is k-barrier covered. For the problem of coverage on a line, the authors of [5] consider the complexity of the (total) displacement problem on a line and propose deterministic algorithms; similarly, the same authors in [4] analyze the complexity of the maximum displacement problem on a line and propose deterministic algorithms.

Interference in a network may be caused by node transmission power and proximity and affects the overall communication connectivity. In [2] the authors propose connectivity-preserving and spanner constructions that are interference-minimal. In [14] they study the problem of minimizing the average interference while still maintaining desired network properties, such as connectivity, point-to-point connections, or multicast trees. In [17] the authors propose a scheme to reduce interference whereby a node grows its transmission power until it finds a neighbour node in every direction.

Notation
We use $L[a, b] = b - a$ for the length of the interval $[a, b]$ on the real line. For $x, y \in \mathbb{R}$, we use $x \land y = \min\{x, y\}$ and $x \lor y = \max\{x, y\}$. We also use $x^+ = x \lor 0$ and $x^- = -(x \land 0)$ for positive and negative parts of $x \in \mathbb{R}$, respectively. For a vector $a \in \mathbb{R}^n$ we use $||a|| = \sum_{k=1}^{n} |a_k|$. For a function $f(\cdot)$ on $\mathbb{R}$ we use $f(u-) = \operatorname{limit}_{t \to u} f(t)$ to denote a left limit of $f$ at point $u$.

For positive sequences $\{x_n, n \geq 1\}$ and $\{y_n, n \geq 1\}$ we say that $x_n \in \Theta(y_n)$ if there exist two constants $c_1 > 0$ and $c_2 > 0$, independent of $n$, such that $c_1 y_n \leq x_n \leq c_2 y_n$ for all $n$ large enough. In addition, we say $x_n \in O(y_n)$ (resp., $x_n \in \Omega(y_n)$), if there exists a constant $c > 0$ such that $x \leq c y_n$ (resp., $c y_n \leq x_n$) for $n$ large enough. In the case when $x_n$ is random (and $y_n$ is not) we will distinguish between having $\Theta(\cdot)$, $O(\cdot)$ and $\Omega(\cdot)$ relations in either expectation or almost surely.

2. Setting and the main results
A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given, supporting all the random variables and stochastic processes defined below. Expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. The model is parameterized by $n \in \mathbb{N}$, the number of sensors under consideration. The $n$ sensors are
represented by \( n \) finite closed intervals
\[
[\tau_i^n, \tau_i^n + \sigma^n], \quad 1 \leq i \leq n, \tag{2.1}
\]
where we use \( \tau_i^n \) for the left endpoint of \( i^{th} \) sensor; and \( \sigma^n \) for its length. All sensors have the same length \( \sigma^n \). We start with a basic assumption.

\[\text{Figure 1.} \quad \text{Sensors (lifted above each other to clarify the exposition). Superscripts are omitted}\]

**Assumption 2.1.** For a given \( n \in \mathbb{N} \) the left endpoints \( \{\tau_i^n\}_{i=1}^n \) satisfy
\[
\tau_i^n - \tau_{i-1}^n = \zeta_i/n, \quad 1 \leq i \leq n, \tag{2.2}
\]
(we let \( \tau_0^n = 0 \)), where \( \{\zeta_i\}_{i \geq 1} \) are i.i.d. positive, absolutely continuous random variables with \( \mathbb{E}(\zeta_1) = 1 \) and \( \mathbb{E}(\zeta_1^2) < \infty \).

### 2.1. Interference and Coverage problems

This paper is about allocating (or, better, reallocating) the sensors along the real half-line to achieve certain properties. A set \( \pi^n = \{\pi_i^n\}_{i=1}^n \in \mathbb{R}^n \) will be called a *displacement policy*, each \( \pi_i^n \in \mathbb{R} \) being a displacement of \( i^{th} \) sensor from its original location \([\tau_i^n, \tau_i^n + \sigma^n]\) to a new location \([\tau_i^n - \pi_i^n, \tau_i^n - \pi_i^n + \sigma^n]\). For a policy \( \pi^n \) we let \( ||\pi^n|| = \sum_{i=1}^n |\pi_i^n| \) represent the total displacement cost. We next introduce our main problems.

- **Problem 1.** (Interference): minimize the total displacement cost needed to eliminate overlaps between the sensors. That is, find
\[
C^n_{\text{int}} = \min_{\pi^n \in \Pi^n} ||\pi^n||, \tag{2.3}
\]
where \( \pi^n \in \Pi^n \), if
\[
(\tau_i^n - \pi_i^n, \tau_i^n - \pi_i^n + \sigma^n) \cap (\tau_j^n - \pi_j^n, \tau_j^n - \pi_j^n + \sigma^n) = \emptyset, \quad i \neq j.
\]

\[\text{Figure 2.} \quad \text{Placement of sensors, with and without the overlaps. The first sensor is shifted to the left (hence, } \pi_1 > 0), \text{ while the last sensor is moved to the right (} \pi_5 < 0), \text{ in order to eliminate the overlaps}\]
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● Problem 2. (Coverage): minimize the total displacement cost needed to eliminate gaps between the sensors, that is, find

\[ C_{\text{cov}}^n = \min_{\pi^n \in \Pi_2^n} \|\pi^n\|, \]  

where \( \pi^n \in \Pi_2^n \), if

\[ \bigcup_{i=1}^{n} [\tau_i^n - \pi_i^n, \tau_i^n - \pi_i^n + \sigma^n] = \left[ 0, \max_{1 \leq i \leq n} \{ \tau_i^n - \pi_i^n + \sigma^n \} \right]. \]

Figure 3. Placement of sensors, with and without the gaps. The 1st and the 4th sensors are shifted to the left (hence, \( \pi_1 > 0 \) and \( \pi_4 > 0 \)), while the 3rd sensor is shifted to the right (\( \pi_3 < 0 \)), to achieve a continuous coverage starting from point zero.

The main objective of the paper is to understand the large \( n \) asymptotics of the total movement costs. It appears, the results strongly depend on whether the sensor length \( \sigma^n \) is greater, less or asymptotically (as parameter \( n \uparrow \infty \)) equal to the average distance of \( 1/n \) between initially placed sensors.

Assumption 2.2. The sensor length \( \sigma^n \) is deterministic and satisfies

\[ \alpha_\sigma := \lim_{n \to \infty} \frac{\sigma^n}{1/n} = \lim_{n \to \infty} n\sigma^n < \infty. \]  

Definition 1. We say that the set of sensors is

* of type U (underloaded), if \( \alpha_\sigma < 1 \);
* of type O (overloaded), if \( \alpha_\sigma > 1 \);
* of type C (critical), if \( \alpha_\sigma = 1 \) and \( \lim_{n \to \infty} \sqrt{n}(n\sigma^n - 1) \in (-\infty, \infty) \).

Theorem 2.3. (Interference and Coverage). Let Assumptions 2.1 and 2.2 hold. The table below summarizes asymptotics as \( n \to \infty \) of optimal displacement cost for problems (2.3) and (2.4), for each type of sensor sets
Table 1. Asymptotics of the optimal displacement cost for different types of sensor sets.

<table>
<thead>
<tr>
<th>Regime</th>
<th>Interference $C_{int}^n$</th>
<th>Coverage $C_{cov}^n$</th>
<th>Result holds in</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>$O(1)$</td>
<td>$\Theta(n)$</td>
<td>Expectation / a.s.</td>
</tr>
<tr>
<td>C</td>
<td>$\Theta(n^{1/2})$</td>
<td>$\Theta(n^{1/2})$</td>
<td>Expectation</td>
</tr>
<tr>
<td>O</td>
<td>$\Theta(n)$</td>
<td>$O(1)$</td>
<td>Expectation / a.s.</td>
</tr>
</tbody>
</table>

Remark 1. The categorization of the limiting behaviour of $n\sigma^n$ (according to Definition 1) may seem biased towards the queueing theory standards, and not covering the whole spectrum of possibilities. In particular, what if $n\sigma^n - 1 \approx n^{-\beta}$ for some $\beta$ other than $1/2$? What if $\alpha_\sigma = \infty$ or, say, what if $\alpha_\sigma$ does not exist at all, etc.? We admit the charges. The form of exposition is mostly motivated by our desire to keep it clear and simple; the issues are partially addressed in Section 10.

Theorem 2.3 will be proved in several stages, over the next five sections. Section 3 deals with some necessary prerequisites. The interference problem will be treated in Sections 4 and 5: at first we show certain lower and upper bounds (Section 4), which will later be analyzed asymptotically (Section 5). Similarly, the coverage problem will be treated in Sections 6 and 7.

Meanwhile we propose a short intuitive explanation of $O(1)$ and $\Theta(n)$ estimates. Consider the following extreme (!) cases of over- and underloaded sensor sets.

![Figure 4](image)

*a) Overloaded case b) Underloaded case*

*Figure 4.* The sensor length being either greater (overloaded) or smaller (underloaded) than the average distance between the left endpoints.

The pictures suggest that almost no action is required to achieve an uninterrupted coverage in the overloaded case, or to eliminate the overlaps in the underloaded. This explains $O(1)$. On the other hand, a lot of movement would be needed to achieve an uninterrupted coverage in the underloaded case, or to eliminate the overlaps in the overloaded. This gives $\Theta(n)$, which is the maximal possible order due to assumed $\alpha_\sigma < \infty$ and the relation $1/n + 2/n + .. + n/n = \Theta(n)$. The order of $\Theta(n^{1/2})$ comes from diffusion approximations, for which the reader can wait until Section 5.
2.2. Mixed problem
We next introduce a certain modification of the coverage problem. Let \( r^n \) be a deterministic sequence of positive numbers with \( r^n \leq \sigma^n \), and assume the existence of the limit
\[
\alpha_r := \lim_{n \to \infty} nr^n < \infty. \tag{2.6}
\]

- **Problem 3. (Mixed):** minimize the total displacement cost needed to eliminate gaps between the sensors, with an additional requirement that no two sensors can overlap by more than \( r^n \). That is, we seek
\[
C_{mix}^n = \min_{\Pi_2 \cap \Pi_3} \|\pi^n\|, \tag{2.7}
\]
where \( \pi^n \in \Pi_3 \), if
\[
\mathcal{L}\left\{[r_i^n - \pi_i^n, r_i^n - \pi_i^n + \sigma^n] \cap [r_j^n - \pi_j^n, r_j^n - \pi_j^n + \sigma^n]\right\} \leq r^n, \quad i \neq j.
\]

For the mixed problem we will show that the coverage results of Theorem 2.3 remain the same for underloaded and critical sensor sets. On the other hand, when the sensor set is overloaded, the results will depend on the asymptotical behaviour of the difference \( \sigma^n - r^n \). We start with the definition.

**Definition 2.** Recall Definition 1. An overloaded \( O \) sensor set is said to be
* of type \( O^{(u)} \) if \( \alpha_\sigma - \alpha_r < 1 \);
* of type \( O^{(c)} \) if \( \alpha_\sigma - \alpha_r > 1 \);
* of type \( O^{(c)} \) if \( \alpha_\sigma - \alpha_r = 1 \) and \( \lim_{n \to \infty} \sqrt{n}(n\sigma^n - nr^n - 1) \in (-\infty, \infty) \).

**Theorem 2.4. (Mixed).** Let Assumptions 2.1 and 2.2 hold. The table below summarizes asymptotics of \( C_{mix}^n \) for each type of sensor sets.

<table>
<thead>
<tr>
<th>Regimes</th>
<th>( C_{mix}^n )</th>
<th>Result holds in</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U, O^{(c)} )</td>
<td>( \Theta(n) )</td>
<td>Expectation / a.s.</td>
</tr>
<tr>
<td>( C, O^{(c)} )</td>
<td>( \Theta(n^{1/2}) )</td>
<td>Expectation</td>
</tr>
<tr>
<td>( O^{(u)} )</td>
<td>( O(1) )</td>
<td>Expectation / a.s.</td>
</tr>
</tbody>
</table>
Recall that all five categories \( U \), \( C \), \( O^{(o)} \), \( O^{(n)} \) and \( O^{(c)} \) are disjoint. The above statements mean that \( \Theta(n) \) estimates, for example, hold true for both \( U \) and \( O^{(o)} \) sensor sets, and so on. The theorem will be treated in Sections 8 and 9. Once again, some bounds are introduced and then analyzed asymptotically. To some extent, the problem resembles the coverage problem. Further explanations are postponed until Section 8.

3. Preliminary analysis

3.1. Potential outflow. Connection to queueing theory

For a counting process \( N^n(t) := \sum_{i=1}^{n} 1_{[\tau^n_i \leq t]} \) for \( 0 \leq t \leq \tau^n \), we define a potential outflow as

\[
W^n(t) = \sigma^n N^n(t) - t, \quad 0 \leq t \leq \tau^n. \tag{3.1}
\]

The notion comes from the \( G/D/1 \) queueing theory, where the left endpoints \( \{\tau^n_i\} \) play the role of arrival epochs with Generally distributed inter-arrival times, and the sensor’s length \( \sigma^n \) stands for Deterministic service requirement. \( N^n \) is an arrival process. The process \( W^n \) will play a key role in construction and asymptotic analysis of various displacement policies. Definition (3.1) implies (pay attention to the left-hand limit):

\[
W^n(\tau^n_k -) = (k-1)\sigma^n - \tau^n_k, \quad 1 \leq k \leq n. \tag{3.2}
\]

3.2. Base coverage policy

Introduce a policy \( \pi^n = \{\pi^n_k\}_{k=1}^{n} \) satisfying

\[
\pi^n_k = -W^n(\tau^n_k -) = \tau^n_k - (k-1)\sigma^n, \quad 1 \leq k \leq n, \tag{3.3}
\]

which trivially places \( k \)-th sensor in the interval \([((k-1)\sigma^n, k\sigma^n]\) for each \( k \).

The base coverage policy is clearly admissible for each of the problems (2.3), (2.4) and (2.7), and can be used as an upper bound for either \( C_{int}^n \), \( C_{cov}^n \) or \( C_{mix}^n \)

\[
C_{int/cov/mix}^n \leq \sum_{k=1}^{n} |W^n(\tau^n_k -)|. \tag{3.4}
\]
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Figure 6. An implementation of the base-coverage policy: 1\textsuperscript{st} and 4\textsuperscript{th} sensors are shifted to the left, while 2\textsuperscript{nd}, 3\textsuperscript{rd} and 5\textsuperscript{th} are shifted to the right. The shift $\pi_k = \tau_k - (k-1)\sigma$.

Such an upper bound, however, cannot be regarded as a universal panacea. Consider, for example, an (extremely) underloaded situation from Fig. 4 (b). In this case it seems (and will be shown) too costly to apply base coverage policy for the interference problem (2.3), since most of the sensors will initially be placed without overlaps.

3.3. Ordered policies

Throughout the sequel, we will restrict the policy sets $\Pi_1^n$, $\Pi_2^n$, $\Pi_3^n$ to policies satisfying (3.5), allowing to contain only the ordered (that is, according to their initial order) displacement of sensors. The claim is justified in Lemma 3.1 below. Such a reduction of $\Pi$’s is indeed crucial (!) and will later be used to establish a connection to queueing theory. To keep it simple we will abuse the notation and retain the letters $\Pi_1^n$, $\Pi_2^n$, $\Pi_3^n$.

Lemma 3.1. An optimal displacement policy for either of the problems (2.3), (2.4) or (2.7) necessarily satisfies

$$\tau_i^n - \pi_i^n \leq \tau_j^n - \pi_j^n, \quad \text{for} \quad i < j.$$  

(3.5)

Proof. For a pair of adjacent sensors the claim follows from Fig. 7 below and the obvious relation $|O_1A| + |O_2B| \leq |O_1B| + |O_2A|$, stating that it is cheaper to have the leftmost shifted sensor (the one, with initial point $O_1$) to be the shifted version of the originally leftmost sensor (the one, with initial point $A$). A general case for non-neighbouring sensors is shown analogously, by sequentially interchanging the neighbours, (starting with making the originally first sensor to become the shifted first, and so on...), thus improving the policy with each iteration.

As trivial as it sounds, Lemma 3.1 is only true because sensors are of the same length.
4. Interference problem

4.1. Characterization of a policy, and a lower bound on performance

As a consequence of Lemma 3.1, we will only be considering policies \( \pi^n \in \Pi^n \) and satisfying (3.5). Such policies admit the following representation. Let \( \xi_i \geq 0, (i = 1, \ldots, n) \) be the length of the gap between consecutive (!) shifted sensors \( i - 1 \) and \( i \), as in Fig. 8 below. Note that the shifted sensors will necessarily be consecutive, as required by (3.5).

![Figure 8. Gaps between shifted sensors (interference problem).](image)

Clearly, given the initial allocation \( \left\{ \tau^n_1, \tau^n_2, \ldots \right\} \), the set \( \left\{ \xi_i \right\}_{i=1}^n \) uniquely defines the displacement policy \( \left\{ \pi^n_1, \pi^n_2, \ldots \right\} \) and vice versa, due to relation

\[
\sum_{i=1}^{k-1} \xi_i + (k - 1)\sigma^n + \xi_k = \pi^n_k - \pi^n_k, \quad k = 1, \ldots, n.
\] (4.1)

Using (3.2), the above implies

\[
\pi^n_k = -\left( W^n(\tau^n_k) + \sum_{i=1}^{k} \xi_i \right), \quad 1 \leq k \leq n.
\] (4.2)

As a result, due to positivity of \( \xi_i \)'s, we can deduce the following lower bound

\[
C^n_{int} = \sum_{k=1}^{n} W^n(\tau^n_k) + \sum_{i=1}^{k} \xi_i \geq \sum_{k=1}^{n} \left( W^n(\tau^n_k) \right)^+.
\] (4.3)

4.2. Right-shift policy and an upper bound on performance

RIGHT-SHIFT POLICY: the first sensor remains on its original location. After that, for \( k \geq 1 \), the \((k + 1)^{st}\) sensor remains on its original location if it does not overlap with already allocated (possibly shifted) \( k^{th} \) sensor; otherwise, the \((k + 1)^{st}\) sensor is shifted to the right by the minimum distance required for that overlap to be eliminated.
Figure 9. An implementation of the right-shift policy: 2nd, 3rd and 4th sensors are shifted to the right by \( w_2, w_3 \) and \( w_4 \) units, respectively. The first and the last sensors do not move. The corresponding policy \( \{ \pi_2, \pi_3, \pi_4 \} \) is then given as 
\[
\pi_1 = \pi_5 = 0, \quad \pi_i = -w_i, \quad \text{for } i = 2, 3, 4.
\]

The gaps \( \{ \xi \} \) between shifted sensors in this case can be recursively calculated as
\[
\xi_k - \sum_{i=1}^{k-1} (\sigma^n + \xi_i) - \tau^n_k, \quad 1 \leq k \leq n, \tag{4.4}
\]
since \( \sum_{i=1}^{k-1} (\sigma^n + \xi_i) \) is the right endpoint of the \( (k-1)^{st} \) shifted sensor.

Referring to the queueing model, described in Subsection 3.1, and its terminology, the gap’s length \( \xi_k \) in (4.4) corresponds to the idling time between serving \( (k-1)^{st} \) and \( k^{th} \) customer, in a standard \( G/D/1 \) queue under the First-Come-First-Serve discipline. Moreover, the quantity \( \sum_{i=1}^{k-1} (\sigma^n + \xi_i) \) now represents the departure time of the \( (k-1)^{st} \) customer after being served. Combining the latter observation with (4.1) and (4.4), we see that the absolute value of the \( k^{th} \) displacement
\[
|\pi^n_k| = -\pi^n_k = \left( \sum_{i=1}^{k-1} (\sigma^n + \xi_i) - \tau^n_k \right) +, \quad 1 \leq k \leq n. \tag{4.5}
\]
represents the waiting time of the \( k^{th} \) arriving customer (see Fig. 9). Therefore, the total displacement cost equals to the total waiting time among \( n \) customers and, as a result
\[
C^n_{\text{int}} \leq \sum_{k=1}^{n} w^n_k, \tag{4.6}
\]
where \( w^n_k \) stands for the waiting time of the \( k^{th} \) customer in the corresponding \( G/D/1 \) queue.

5. Asymptotic analysis of Interference Problem

Proposition 5.1. (Overloaded regime). For type O of sensor sets, as \( n \to \infty \)
\[
\mathbb{E}(C^n_{\text{int}}) = \Theta(n) \quad \text{and} \quad C^n_{\text{int}} = \Theta(n) \quad \text{a.s.} \tag{5.1}
\]

Proof. Combining (3.4) and (4.3), one gets
\[
\sum_{k=1}^{n} \left( W^n(\tau^n_k) \right) \leq C^n_{\text{int}} \leq \sum_{k=1}^{n} \left| W^n(\tau^n_k) \right|, \tag{5.2}
\]
Start with the lower bound. Since \( n\sigma^n \to \alpha_\sigma > 1 \) there must exist constants \( \delta > 0 \) and \( c > 0 \) so that for \( n \) large enough

\[
\sigma^n - (1 + \delta)/n \geq c/n. \tag{5.3}
\]

Indeed, by definition 1, one must have \( \sigma^n \geq (1 + c_1)/n \) for some \( c_1 > 0 \) and \( n \) large enough. The relation (5.3) then holds for \( \delta = c = c_1/2 \). Recall (3.2) and Assumption 2.1, and rewrite \( W^n(\tau_k^n -) \) as

\[
W^n(\tau_k^n -) = -\sigma^n + \frac{k}{n} \left( \sigma^n - \frac{\zeta_i}{n} \right) + \frac{k}{n} \left( \frac{1 + \delta}{n} - \frac{\zeta_i}{n} \right) \tag{5.4}
\]

\[
= -\sigma^n + \frac{k}{n} \left( \sigma^n - \frac{1 + \delta}{n} \right) + \sum_{i=1}^{k} \left( \frac{1 + \delta}{n} - \frac{\zeta_i}{n} \right). \tag{5.5}
\]

Since \( \mathbb{E}(\zeta_i) = 1 \) and \( \delta > 0 \), the expectation of the last term is trivially positive. By the Law of Large Numbers (LLN), the positivity would still hold almost surely for \( k \) large enough. Hence, using (5.3)-(5.5) and Definition 1, we almost surely have

\[
\sum_{k=1}^{n} W^n(\tau_k^n -) \geq -n\sigma^n + cn \geq -(1 + \alpha_\sigma) + cn \geq (1/2)cn,
\]

for \( n \) large enough. Since \( (W^n)^+ \geq W^n \), the lower bound in (5.2) together with the last inequality, imply \( C_{\text{int}}^n = \Omega(n) \) a.s. and in expectation.

For the upper bound we shall use (see (3.2)) a crude estimate \(|W^n(\tau_k^n -)| \leq (k - 1)\sigma^n + \tau_k^n \). By Definition 1, we have \( \sum_{k=1}^{n} (k - 1)\sigma^n = O(n) \) so we just need to establish a bound for \( \sum_{k=1}^{n} \tau_k^n \). We have (recall (2.2)),

\[
\sum_{k=1}^{n} \tau_k^n = \sum_{k=1}^{n} (k/n) + \sum_{k=1}^{n} (\tau_k^n - k/n) = \frac{n - 1}{2} + \frac{n}{k} \sum_{i=1}^{k} \left( \frac{1}{k} \sum_{i=1}^{k} (\zeta_i - 1) \right).\]

The expectation of the last term on the above is trivially zero. Now, fix an arbitrary \( \varepsilon > 0 \). By the law of large numbers, almost surely we have \( (1/k) \sum_{i=1}^{k} (\zeta_i - 1) \leq \varepsilon \) for all \( k \) large enough. Therefore,

\[
\sum_{k=1}^{n} \tau_k^n \leq \frac{n - 1}{2} + \sum_{k=1}^{n} \frac{k}{n} \left( \frac{1}{k} \sum_{i=1}^{k} (\zeta_i - 1) \right) \leq n(1 + \varepsilon) \tag{5.6}
\]

for all \( n \) large enough. Hence \( \sum_{k=1}^{n} \tau_k^n = O(n) \) both in expectation and almost surely. This shows (5.1) and concludes the proposition.

**Proposition 5.2.** (Critical regime). For type C of sensor sets, as \( n \to \infty \)

\[
\mathbb{E}(C_{\text{int}}^n) = \Theta(n^{1/2}) \tag{5.7}
\]
Proof. Once again, we analyze the bounds from (5.2). Rewrite
\[ \sum_{k=1}^{n} \left( W^n(\tau_k^n) - \right)^{+} = \int_{0}^{\tau_n^n} \left( W^n(t) - \right)^{+} dN^n(t) \] (5.8)
\[ = \int_{0}^{\tau_n^n} \left( W^n(t) - \right)^{+} d[N^n(t) - a^n(t)] + \int_{0}^{\tau_n^n} \left( W^n(t) - \right)^{+} da^n(t) \] (5.9)
where (see [7], p. 98) \( a^n \) is a compensator of \( N^n \), hence the integrator \( N^n(t) - a^n(t) \) in the first expression of (5.9) is a martingale. As a result, the process \( M(\cdot) = \int_{0}^{\tau_n^n} \left( W^n(t) - \right)^{+} d[N^n(t) - a^n(t)] \) is a martingale (see Thm. 6.5.8 in [12], p. 88 ) with expectation zero, hence by the optional sampling theorem (see Thm. 3.22 in [8], p. 19) (since \( \tau_n^n \) is a stopping time), we get that the first term in (5.9) has zero expectation. Regarding the second term of (5.9), we use Lemma A3.1 from Appendix A3 as
\[ \int_{0}^{\tau_n^n} \left( W^n(t) - \right)^{+} da^n(t) \geq c_1 n \int_{0}^{\tau_n^n} \left( W^n(t) - \right)^{+} dt - c_2. \] (5.10)
for some constants \( c_1, c_2 \geq 0 \). (Recall that in the exponential case we simply have \( a^n(t) = nt \)). Combining (5.8) - (5.10) and taking the expectations, we obtain
\[ \mathbb{E} \left[ \sum_{k=1}^{n} \left( W^n(\tau_k^n) - \right)^{+} \right] \geq c_1 n \mathbb{E} \left[ \int_{0}^{\tau_n^n} \left( W^n(t) - \right)^{+} dt \right] - c_2. \] (5.11)
Introduce \( B^n(t) := n^{-1/2}(N^n(t) - nt) \) and rewrite \( W^n(t) = \sigma^n N^n(t) - t \) as
\[ W^n(t) = n^{-1/2} \left[ (n\sigma^n)B^n(t) + n^{1/2}(n\sigma^n - 1)t \right] \] (5.12)
It is well known that \( B^n \) converges to a standard Brownian motion (see Thm. 5.11 in [3], p. 110). In addition, we have \( \mathbb{E}(\tau^n) = 1 \). Recall the convergence limits \( \alpha_{n} \) and \( c \) from (2.5) and Definition 1. Next, we combine (5.12) and (5.11) with the lower bound inequality in (5.2), and use the Fatou’s Lemma to interchange limit and expectation, to get \( \mathbb{E}(C^n_{int}) = \Omega(n^{1/2}) \).

On the other hand, we can use Lemma 2 from ([1], page p.1102), saying that
\[ \mathbb{E}(B^n(t)) \leq c_1(1 + t^{c_2}), \quad n \in \mathbb{N}, \quad t \in \mathbb{R} \] (5.13)
for some constants \( c_1 \) and \( c_2 \) independent of \( n \) and \( t \). Applied to both (5.12) and the upper bound in (5.2), this yields \( \mathbb{E}(C^n_{int}) = \mathcal{O}(n^{1/2}) \) and (5.7).

Proposition 5.3. (Underloaded regime). For type U of sensor sets, as \( n \to \infty \)
\[ \mathbb{E}(C^n_{int}) = \mathcal{O}(1) \quad \text{and} \quad C^n_{int} = \mathcal{O}(1) \quad \text{a.s.} \] (5.14)
Proof. We take the bound (4.6) with \( w_k^n \) - the waiting time of the \( k \)th customer in a \( G/D/1 \) queue. Recall (note the dependence on the parameter \( n \)) that the times
between arrivals are i.i.d. random variables, distributed as $\zeta/n$, with a deterministic service duration of $\sigma^n$. From Lindley’s relations (Section 4.20 in [16])

$$w_{k+1}^n = \left( w_k^n + \sigma^n - (\tau^n_{k+1} - \tau^n_k) \right)^+ = \left( w_k^n + \sigma^n - \frac{\zeta + 1}{n} \right)^+. \quad (5.15)$$

Let $\phi_k^n = nw_k^n$ and rewrite the bound (4.6) as

$$C_{\text{int}}^n \leq \frac{1}{n} \sum_{k=1}^n \phi_k^n. \quad (5.16)$$

From (5.15) we have

$$\phi_{k+1}^n = \left( \phi_k^n + (n\sigma^n - \zeta + 1) \right)^+, \quad k \geq 0. \quad (5.17)$$

Together with obvious $\phi_0^n = 0$, this defines a random walk on a positive half-plane, as introduced in Appendix A1, with the increment $x_k = n\sigma^n - \zeta_k$.

To eliminate the dependence in $n$, take some $\bar{\sigma}$ satisfying $n\sigma^n < \bar{\sigma} < 1$, (the existence of such $\bar{\sigma}$ for $n$ large enough is guaranteed by having $n\sigma^n \rightarrow \alpha_\sigma < 1$). Let $y_k = \bar{\sigma} - \zeta_k$, and define a sequence $\{\phi_k^y, k \geq 0\}$ as $\phi_0^y = 0$ and

$$\phi_{k+1}^y = \left( \phi_k^y + y_{k+1} \right)^+, \quad k \geq 0.$$  

By the choice of $\bar{\sigma}$, we must have $x_k^n \leq y_k$. Next, we use the comparison Lemma A1.2 to get $\phi_k^n \leq \phi_k^y$ for all $k \geq 0$. Applied to (5.16), this reads

$$C_{\text{int}}^n \leq \frac{1}{n} \sum_{k=1}^n \phi_k^y. \quad (5.18)$$

and we are ready to apply Lemma A1.1 to $\Phi := \{\phi_k^y, k \geq 0\}$, with i.i.d. increments $\{y_1, y_2, \ldots\}$, satisfying $\mathbb{E}(y_1) < 0$, to get (5.14), finalizing the proposition. $\square$

We point out the clear advantage of the queueing bound (4.6) over the upper workload bound (5.2) that would still be at least $O(\sqrt{n})$ (see (5.7)). At the same time, the lower bound in (5.2) will notice a change in the regime and become $O(1)$, since in the underloaded case $W^n$ tends to be negative.

### 6. Coverage problem

#### 6.1. One trivial lower bound on performance

Let $\pi^n \in \Pi^n_2$, (which, recall, now also implies (3.5)). If $\tau^n_k \geq (k - 1)\sigma^n$ for some $k \geq 1$, then the corresponding $\pi^n_k$ must satisfy $\pi^n_k \geq \tau^n_k - (k - 1)\sigma^n$. Indeed, if it was not the case then the $k^{th}$ sensor would have been placed at $\tau^n_k - \pi^n_k > (k - 1)\sigma^n$ leaving the gap of at least $(\tau^n_k - \pi^n_k) - (k - 1)\sigma^n > 0$, therefore contradicting the assumed $\pi^n \in \Pi^n_2$.

The above observations implies $|\pi^n_k| \geq (\tau^n_k - (k - 1)\sigma^n)^+ = (-W^n(\tau^n_k - ))^+ = (W^n(\tau^n_k - ))^-$. And we have a lower bound in terms of

$$C_{\text{cov}}^n \geq \sum_{k=1}^n \left( W^n(\tau^n_k - ) \right)^-. \quad (6.1)$$
6.2. Left-shift policy and an upper bound on performance

**Left-shift policy:** the first sensor is shifted left to the origin. After that, for \( k > 1 \): the \((k+1)^{st}\) sensor remains on its original location if it overlaps with already allocated (possibly shifted) \(k^{th}\) sensor; otherwise, the \((k+1)^{st}\) sensor is shifted left to close the gap with already allocated (possibly shifted) \(k^{th}\) sensor, without overlapping it.

\[
\pi_k = \left( L^n(\tau_k^n -) \right)^-, \quad k = 1, \ldots, n, \quad (6.2)
\]

where
\[
L^n(t) = W^n(t) - \sup_{s \in [0,t]} (W^n(s) - \sigma^n)^+ \quad (6.3)
\]

is a Skorokhod map (see Fig. 11 below), reflecting \(W^n\) to \((-\infty, \sigma^n]\).

Lemma 6.1 implies a following upper bound for problem (2.4)
\[
C_{cov}^n \leq \sum_{k=1}^{n} \left( L^n(\tau_k^n -) \right)^-. \quad (6.4)
\]

**Proof of Lemma 6.1.** For \( k = 2, \ldots, n \), define \( \eta_k \) to be the length of the overlap between the \((k-1)^{st}\) and \(k^{th}\) sensors, after the left-shift policy had been performed. Assume \( \eta_1 = 0 \).

\[
\eta_1 = 0 \quad \eta_2 \quad \eta_3 = 0 \quad \eta_4
\]

*Figure 12.* Overlaps between shifted sensors (coverage by the left-shift policy)
Figure 11. Realization of the left-shift policy, governed by the map $L$. The length $\sigma = 1$ and the original left endpoints are given as $\tau_1 = 0.5$, $\tau_2 = 0.8$, $\tau_3 = 1.2$, $\tau_4 = 4.4$, $\tau_5 = 4.7$ and $\tau_6 = 4.9$. The function $L$ is a Skorokhod reflection of the potential outflow $W$ to $(-\infty, 1]$. The left shifts are given as $\pi_k = L(\tau_k^-)$, implying $\pi_1 = L(0.5^-) = 0.5$, $\pi_2 = \pi_3 = 0$, (because $L(0.8^-) > 0$ and $L(1.2^-) > 0$), $\pi_4 = 2.2$, $\pi_5 = 1.5$ and $\pi_6 = 0.7$.

Easy to see that the quantities $\{\eta_k\}$ satisfy the following recursive relation

$$\eta_k = \left( \sum_{i=1}^{k-1} (\sigma^n_i - \eta_i) - \tau_k^n \right)^+ = \left( W^n(\tau_k^n) - \sum_{i=1}^{k-1} \eta_i \right)^+,$$

(6.5)

where we use the convention $\sum_{i=1}^{0} = 0$. Indeed, the relation (6.5) is true because $\sum_{i=1}^{k-1} (\sigma^n_i - \eta_i)$ is the coordinate of the right endpoint of $(k - 1)^{st}$ sensor after the left-shift allocation. Consequently, we have

$$\pi_k^n = \left( \tau_k^n - \sum_{i=1}^{k-1} (\sigma^n_i - \eta_i) \right)^+ = \left( W^n(\tau_k^n) - \sum_{i=1}^{k-1} \eta_i \right)^-. $$

(6.6)
From (6.3), since $W^n$ is decreasing on $[\tau_{k-1}^n, \tau_k^n)$, we have

$$L(\tau_k^n) = W^n(\tau_k^n) - \sup_{s \in [0, \tau_{k-1}^n]} (W^n(s) - \sigma^n)^+,$$

and our main goal (6.2) will follow from (6.6) once we are able to show that

$$\sum_{i=1}^{k-1} \eta_i = \sup_{s \in [0, \tau_{k-1}^n]} (W^n(s) - \sigma^n)^+. \quad (6.7)$$

Introduce $R(t) = \sup_{s \in [0, t]} (W^n(s) - \sigma^n)^+$. The function $R$ is non-decreasing, piecewise constant function, with jumps possible only in the points $\{\tau_k^n\}_{k=1}^n$:

$$R(t) = \sum_{k : \tau_k^n \leq t} \Delta R(\tau_k^n), \quad t \geq 0, \quad \Delta R(\tau_k^n) := R(\tau_k^n) - R(\tau_{k-1}^n)^+$$

where $\Delta R(\tau_k^n) := R(\tau_k^n) - R(\tau_{k-1}^n)$ is a jump of $R$ at point $\tau_k^n$. In what follows we show that

$$\eta_k = \Delta R(\tau_k^n), \quad 1 \leq k \leq n, \quad (6.9)$$

which will in turn imply (6.7) and conclude the lemma. For $k = 1$ the relation (6.9) holds by the definition, since $\sup_{s \in [0, \tau_1^n]} (W^n(s) - \sigma^n)^+ = 0$ (see (3.1)).

At the same time, the quantities $\{\Delta R(\tau_k^n)\}$ satisfy, for $k \geq 2$

$$\Delta R(\tau_k^n) = \sup_{s \in [0, \tau_k^n]} (W^n(s) - \sigma^n)^+ - \sup_{s \in [0, \tau_{k-1}^n]} (W^n(s) - \sigma^n)^+$$

$$= \left( W^n(\tau_k^n) - \sigma^n - \sup_{s \in [0, \tau_{k-1}^n]} (W^n(s) - \sigma^n)^+ \right)^+ \quad (6.10)$$

$$= \left( W^n(\tau_k^n) - \sum_{i=1}^{k-1} \Delta R(\tau_i^n) \right)^+, \quad (6.11)$$

which is the same recursive relation as (6.5) for quantities $\{\eta_i\}$. Together with $\eta_1 = \Delta R(\tau_1^n)$, the recursion implies (6.9), concluding the lemma.

7. Asymptotic analysis of Coverage Problem

**Proposition 7.1.** (Underloaded regime). For type U of sensor sets, as $n \to \infty$

$$\mathbb{E}(C_{cov}^n) = \Theta(n) \quad \text{and} \quad C_{cov}^n = \Theta(n) \quad \text{a.s.} \quad (7.1)$$
Proof. Here we use the lower bound (6.1) together with the upper bound from base coverage policy (3.4)

\[ \sum_{k=1}^{n} \left( W^n(\tau_k^n) \right)^- \leq C_{cov}^n \leq \sum_{k=1}^{n} |W^n(\tau_k^n)|. \]  

(7.2)

Start with the lower bound. We claim there exist constants \( \delta > 0 \) and \( c > 0 \) so that for \( n \) large enough

\[ (1-\delta)/n - \sigma^n \geq c/n. \]  

(7.3)

Indeed, since \( n\sigma^n \rightarrow a_\sigma < 1 \), one must have \( \sigma^n \leq (1 - c_1)/n \) for some \( c_1 > 0 \) and \( n \) large enough, and the relation (7.3) holds for \( \delta = c = c_1/2 \). Similar to (5.4) we have

\[ -W^n(\tau_k^n-) = \sigma^n + \sum_{i=1}^{k} \left( \frac{\zeta_i}{n} - \sigma^n \right) \]  

(7.4)

\[ = \sigma^n + \sum_{i=1}^{k} \left( 1 - \frac{\delta}{n} - \sigma^n \right) + \sum_{i=1}^{k} \left( \frac{\zeta_i}{n} - 1 - \frac{\delta}{n} \right). \]  

(7.5)

Since \( \mathbb{E}(\zeta_i) = 1 \) and \( \delta > 0 \), the expectation of the last term is trivially positive. Moreover, by the Law of Large Numbers, the positivity also holds almost surely for \( k \) and \( n \) large enough. As result, for such \( n \), combining (7.3)- (7.5), we have

\[ -\sum_{k=1}^{n} W^n(\tau_k^n-) \geq n \sigma^n + (c/2)(n - 1) \geq (1/3)cn \]  

(7.6)

both in expectation and almost surely. Hence, since \( (W^n)^- \geq -W^n \), we have a lower bound of \( \Omega(n) \) both in expectation and almost surely. As for the upper bound, we once again use the estimate \( |W^n(\tau_k^n-)| \leq (k-1)\sigma^n + \tau_k^n \). By Definition 1, we have \( \sum_{k=1}^{n} (k-1)\sigma^n = O(n) \). Combining the latter with (5.6), we get the \( O(n) \) estimate both almost surely and in expectation, concluding the proposition.

Proposition 7.2. (Critical Regime). For type C of sensor sets, as \( n \rightarrow \infty \)

\[ \mathbb{E}(C_{cov}^n) = \Theta(n^{1/2}) \]  

(7.7)

Proof. We use bounds (7.2) together with the similar analysis (5.7)-(5.13).

The upper bound in (7.2) will not notice the change from underloaded to the overloaded regime, still producing \( O(n) \). Therefore, we need a stronger estimate (6.4).

Proposition 7.3. (Overloaded Regime). For type O of sensor sets, as \( n \rightarrow \infty \)

\[ \mathbb{E}(C_{cov}^n) = O(1) \text{ and } C_{cov}^n = O(1) \text{ a.s.} \]  

(7.8)

Proof. Our job will be to show the boundedness (in the expectation and almost surely) of (6.4). It would be easier to operate with \( \tau_k^n \) instead of \( \tau_k^n- \), and the definition (6.3)
and Fig. 11 suggest an obvious inequality
\[
(L^n(\tau^+_k))^- \leq (L^n(\tau^+_k))^- + \sigma^n. \tag{7.9}
\]
Since \(n\sigma^n < \infty\), the boundedness of (6.4) will follow once we can show the boundedness of \(\sum_{k=1}^n (L^n(\tau^+_k))^-\). To simplify the notation, let \(L^n_k = L^n(\tau^+_k)\), \(W^n_k = W^n(\tau^+_k)\). Since \(W^n(\cdot)\) can only increase at points \(\{\tau^+_n\}\), we can rewrite (6.3) as
\[
L^n_k = W^n_k - \max_{r \in \{0, \ldots, k\}} (W^n_r - \sigma^n)^+, \quad k = 0, \ldots, n,
\]
where we used \(\tau^n_0 = 0\). Define
\[
M^n_k = W^n_k - \max_{r \in \{0, \ldots, k\}} W^n_r, \quad k = 0, \ldots, n.
\]
Since \(W^n_0 = 0\), we have
\[
\max_{r \in \{0, \ldots, k\}} W^n_r = \max_{r \in \{0, \ldots, k\}} (W^n_r)^+ \geq \max_{r \in \{0, \ldots, k\}} (W^n_r - \sigma^n)^+ \tag{7.10}
\]
yielding \(L^n_k \geq M^n_k\) and, in particular, due to non-positivity of \(M^n\),
\[
(L^n_k)^- \leq (M^n_k)^-. \tag{7.11}
\]
Let \(\psi^n_k = nM^n_k\). Together, the relations (6.2), (6.4), (7.9) and (7.11) imply
\[
C^n_{cov} \leq n\sigma^n + \frac{1}{n} \sum_{k=1}^n (\psi^n_k)^-. \tag{7.12}
\]
From the definition of \(M^n\), we have \(\psi^n_0 = 0\) and
\[
\psi^n_k = nW^n_k - \max_{r \in \{0, \ldots, k\}} nW^n_r, \quad k \geq 1, \tag{7.13}
\]
where, using (3.1) and Assumption 2.1,
\[
nW^n_k = \sum_{i=1}^k (n\sigma^n - n[\tau^n_i - \tau^n_{i-1}]) = \sum_{i=1}^k (n\sigma^n - \zeta_i). \tag{7.14}
\]
The relations (7.13) - (7.14) define \(\Psi^n = \{\psi^n_k, \quad k \geq 0\}\) - a random walk on the negative half-plane, as described in Appendix A2, with i.i.d. increments \(\{n\sigma^n - \zeta_k, \quad k \geq 1\}\).

To properly analyze (7.12) we first need to eliminate \(n\) - dependence in the increment \(n\sigma^n - \zeta_k\). Take some \(\hat{\sigma}\), satisfying \(1 < \hat{\sigma} < n\sigma^n\) (the existence of such \(\hat{\sigma}\) for \(n\) large enough is guaranteed by having \(n\sigma^n \sim \alpha\) \(> 1\)). Define \(z_k = \hat{\sigma} - \zeta_k\) and introduce \(\Psi^z = \{\psi^z_k, \quad k \geq 0\}\) defined as \(\psi^z_0 = 0\) and
\[
\psi^z_{k+1} = -(\psi^z_k + z_{k+1})^-, \quad k \geq 0.
\]
Once again, as described in Appendix A2, \(\Psi^z\) is a random walk on the negative half-plane, with i.i.d. increments \(\{z_k, \quad k \geq 1\}\). By the choice of \(\hat{\sigma}\), we must have \(z_k \leq n\sigma^n - \zeta_k\), hence one can apply a comparison result of Lemma A2.2 to get \((\Psi^n)^- \leq (\Psi^z)^-\). Applying the latter to (7.12), together with Lemma A2.1 (valid due to \(E(z_1) > 0\)), we get
\[
C^n_{cov} \leq n\sigma^n + \frac{1}{n} \sum_{k=1}^n (\psi^z_k)^- < \infty, \tag{7.15}
\]
in expectation and almost surely. This concludes the proposition. □

8. Mixed Problem

Introduce a new policy

Mixed policy (MP): the first sensor is shifted left to the origin. After that, for every \( k \geq 1 \): the \((k + 1)\text{st}\) sensor remains on its original location if it overlaps with already allocated \( k\text{th}\) sensor by no more than \( r^n \); in the case the overlap is greater than \( r^n \), the \((k + 1)\text{st}\) sensor shifts to the right to make the overlap equal \( r^n \). Lastly, if there is no overlap between the \((k + 1)\text{st}\) sensor and the already allocated \( k\text{th}\), the \((k + 1)\text{st}\) sensor shifts left until it touches the \( k\text{th}\) sensor, without overlapping it (see Fig. 13).

\[ \text{Lemma 8.1.} \quad \text{The MP policy is characterized by the displacement} \]

\[ \tau_k^n = \left( P^n(\tau_k^n -) \right)^- - \left( P^n(\tau_k^n -) - r^n \right)^+, \quad (8.1) \]

where \( P \) is a modified Skorokhod reflection map

\[ P^n(t) = W^n(t) - \sum_{k: \tau_k^n t} \left( P^n(\tau_k^n -) \right)^+ \wedge r^n. \quad (8.2) \]

Discussion of Lemma 8.1. The policy says that (recall Section 2.1), the \( k\text{th}\) sensor moves right if \( P^n(\tau_k^n -) > r^n \); moves left if \( P^n(\tau_k^n -) < 0 \); and remains in its original place if \( 0 \leq P^n(\tau_k^n -) < r^n \) (see Fig. 14).

Looking closer, we can see that (8.2) has quite a lot of similarities with (6.3). Indeed, from the definition (and Fig. 11) it is easy to see that \( L^n(t) = W^n(t) - \sum_{k: \tau_k^n t} (L(\tau_k^n -))^+ \). Therefore, the processes \( L^n \) and \( P^n \) would be identical if not for the \( r^n \) restriction.
Figure 14. Mixed mapping for sensors displacement. Here we have the same set of original sensors as in Fig. 11. Both functions \( P \) and \( L \) (reflected mapping from Fig. 11) behave identically up to the jump at the point \( 0.8 \), and this is due to the fact that \( P(0.8) = 0.2 < 0.3 \), in which case \( P \) can not jump above the level \( \sigma = 1 \). On the other hand, at the jump point \( 1.2 \) the function \( P \) does go above the level \( \sigma = 1 \), (since \( P(1.2) = 0.6 > 0.3 \) although its jump is now shortened by \( r = 0.3 \), and equals to \( 1 - 0.3 = 0.7 \).

Naturally, the MP policy produces the following upper bound on the optimal cost, which will be later used to obtain the estimates in (9.11).

\[
C^m_{\text{mix}} \leq \sum_{k=1}^{n} \left[ \left( P^n(\tau^n_{k} - \cdot) \right)^- + \left( P^n(\tau^n_{k} - \cdot) - r^n \right)^+ \right]. \tag{8.3}
\]

**Proof of Lemma 8.1:** similarly to the proof of Lemma 6.1 (see Fig. 12), let \( \{\hat{\eta}_k\}_{k=2}^n \) be the overlaps between \((k-1)^{st}\) and \(k^{th}\) sensors in the final placement according to the MP policy. Assume \( \hat{\eta}_1 = 0 \). One can show that the MP policy implies the following recursive relation

\[
\hat{\eta}_k = \left( \sum_{i=1}^{k-1} (\sigma^n - \hat{\eta}_i) - \tau^n_k \right)^+ \land r^n = \left( W^n(\tau^n_k - \cdot) - \sum_{i=1}^{k-1} \hat{\eta}_i \right)^+ \land r^n, \tag{8.4}
\]
where we use the convention $\sum_{i=1}^{0} = 0$. Again, the relation (8.4) is true because $\sum_{i=1}^{k-1}(\sigma^n - \tilde{\eta}_i)$ is the coordinate of the right endpoint of $(k - 1)^{st}$ sensor after the left-shift allocation. Consequently, we have

$$\pi^n_k = \left(\tau^n_k - \sum_{i=1}^{k-1}(\sigma^n - \tilde{\eta}_i)\right)^+ + \left(\sum_{i=1}^{k-1}(\sigma^n - \tilde{\eta}_i) - \tau^n_k - r^n\right)^+$$  \hspace{1cm} (8.5)$$

$$= \left(W^n(\tau^n_k) - \sum_{i=1}^{k-1}\tilde{\eta}_i\right)^- + \left(W^n(\tau^n_k) - \sum_{i=1}^{k-1}\tilde{\eta}_i - r^n\right)^+.$$  \hspace{1cm} (8.6)$$

From (8.2)

$$P^n(\tau^n_k) = W^n(\tau^n_k) - \sum_{i=1}^{k-1}\left(P^n(\tau^n_i)\right)^+ \wedge r^n.$$  \hspace{1cm} (8.7)$$

and the relation (8.1) will follow once we show that

$$\tilde{\eta}_k = \left(P^n(\tau^n_k)\right)^+ \wedge r^n, \hspace{1cm} 1 \leq k \leq n$$  \hspace{1cm} (8.8)$$

Since $P^n(\tau^n_k) < 0$, the relation (8.8) holds trivially for $k = 1$. Next, proceed by induction. For $k \geq 2$, assume that (8.8) holds for $1, 2, \ldots, k - 1$. Then from (8.4), we must have

$$\tilde{\eta}_k = \left(W^n(\tau^n_k) - \sum_{i=1}^{k-1}\tilde{\eta}_i\right)^+ \wedge r^n$$

$$= \left(W^n(\tau^n_k) - \sum_{i=1}^{k-1}\left(P^n(\tau^n_i)\right)^+ \wedge r^n\right)^+ \wedge r^n =$$

$$= \left(P^n(\tau^n_k)\right)^+ \wedge r^n,$$

as desired, with the last equality following from (8.7). The above finalizes the lemma. \Box

9. Asymptotic analysis of Mixed Problem

In this section we apply the previously developed methodology to study the Mixed Problem. First of all, due to the definitions of problems (2.4) and (2.7), one must have

$$C^n_{mix} \geq C^n_{cov}.$$  \hspace{1cm} (9.1)$$

We now proceed to asymptotic analysis. As before, different regimes need to be considered separately.
Proposition 9.1. (Underloaded regime). For type \textit{U} of sensor sets, as \( n \to \infty \)
\[
\mathbb{E}(C_{\text{mix}}^n) = \Theta(n). \tag{9.2}
\]

\textbf{Proof.} Due to (9.1), (7.2) and an upper bound (3.4) from the base coverage policy, we still have
\[
\sum_{k=1}^{n} \left( W^n(\tau^n_k) \right) - C_{\text{mix}}^n \leq \sum_{k=1}^{n} \left| W^n(\tau^n_k) \right| \tag{9.3}
\]
and the arguments from (7.2)-(7.6) for \( n\sigma^n \to \alpha \sigma < 1 \) imply (9.2) \( \Box \)

Proposition 9.2. (Critical regime). For type \textit{C} of sensor sets, as \( n \to \infty \)
\[
\mathbb{E}(C_{\text{mix}}^n) = \Theta(n^{3/2}) \tag{9.4}
\]

\textbf{Proof.} Yet again, we can use the bounds (9.3) together with the considerations from (5.8) - (5.7), to get (9.4) \( \Box \)

Note that in obtaining both estimates (9.2) and (9.4) we did not use any information about \( r^n \). It comes into play in the last, considerably more involved, asymptotic regime for this problem. Each of the three different sub-regimes will be treated separately.

Proposition 9.3. (\( O^{(\ddagger)} \) regime). For type \textit{O} of sensor sets, as \( n \to \infty \)
\[
\mathbb{E}(C_{\text{mix}}^n) = \Theta(n) \quad \text{and} \quad C_{\text{mix}}^n = \Theta(n) \quad \text{a.s.} \tag{9.5}
\]

\textbf{Proof.} We begin with the lower bound. Due to Lemma 3.1, we consider only \textit{ordered} policies, hence a policy in \( \Pi^n_2 \) must satisfy
\[
\tau^n_i - \pi^n_i + \sigma^n - r^n \leq \tau^n_{i+1} - \pi^n_{i+1}, \quad \text{for} \quad 1 \leq i \leq n - 1. \tag{9.6}
\]
The above condition can be interpreted as that, once "shortened" by \( r^n \) \textit{from the right}, the modified sensors no longer interfere. That is, by allocating original sensors to satisfy (9.6), we automatically allocate "shortened" sensors to avoid interference. Naturally, the relation does not work in another direction, since one still needs to achieve coverage (that is, a policy must satisfy the requirements of \( \Pi^n_2 \)), therefore
\[
C_{\text{mix}}^n \geq C_{\text{int}}^{n, \sigma^n - r^n}, \tag{9.7}
\]
with \( C_{\text{int}}^{n, \sigma^n - r^n} \) being the analog of \( C_{\text{int}}^n \) from (2.3) for the case when all sensors are shortened by \( r^n \) \textit{from the right}. Hence, applying to (9.7) the modified version of (4.3), we have
\[
C_{\text{mix}}^n \geq \sum_{k=1}^{n} \left( W_{\sigma^n - r^n}^n(\tau^n_k) \right)^+, \tag{9.8}
\]
with \( W_{\sigma^n - r^n}^n(t) \) being the analog of (3.1)-(3.2), accommodating the shortened length of \( \sigma^n - r^n \), that is \( W_{\sigma^n - r^n}^n(t) = (\sigma^n - r^n)N^n(t) - t \). For the upper bound we follow the
previous discussion and use the base coverage (3.4) but for sensors shortened by $r^n$ from the right (obviously, such base coverage satisfies the requirements of the mixed problem). Combined with (9.8), we now have

$$
\sum_{k=1}^{n} \left( W_{\sigma^n-r^n}(\tau_k^n) \right) \leq C_{\text{mix}}^{n} \leq \sum_{k=1}^{n} \left| W_{\sigma^n-r^n}(\tau_k^n) \right|,
$$

(9.9)

and the assertion (9.5) follows from (5.2)-(5.6).

\[\square\]

**Proposition 9.4.** \((\text{O}^{(e)})\) regime. For type \(\text{O}^{(e)}\) of sensor sets, as \(n \to \infty\)

$$
\mathbb{E}(C_{\text{mix}}^{n}) = \Theta(n^{1/2})
$$

(9.10)

**Proof.** We use the bounds from (9.9) and proceed analogously to (5.7)-(5.13), replacing \(\sigma^n \) by \(\sigma^n - r^n\).

\[\square\]

**Proposition 9.5.** \((\text{O}^{(u)})\) regime. For type \(\text{O}^{(u)}\) of sensor sets, as \(n \to \infty\)

$$
\mathbb{E}(C_{\text{mix}}^{n}) = O(1) \quad \text{and} \quad C_{\text{mix}}^{n} = O(1) \quad \text{a.s.}
$$

(9.11)

**Proof.** Let \(\{\pi\}\) be the MP policy from (8.1). The statement of the proposition will follow once we show that

$$
\sup_{n} \|\pi^n\| < \infty, \quad \text{a.s.} \quad \sup_{n} \mathbb{E}(\|\pi^n\|) < \infty
$$

(9.12)

Define

$$
P_k^n := P_1^n(\tau_k^n - \zeta_{k+1}/n), \quad 1 \leq k \leq n.
$$

(9.13)

From (8.2), (3.2) and Assumption 2.1 we have \(P_1^n = -\zeta_1/n < 0\) and

$$
P_{k+1}^n = P_k^n + Z_{k+1}^n - (P_k^n)^+ \wedge r^n, \quad 1 \leq k \leq n-1.
$$

(9.14)

**Lemma 9.6.** For the sequence \(\{P_k^n\}\) from (9.14) and a random walk \(\{X_k^n, \ k \geq 1\}\), defined as \(X_1^n = P_1^n\) and \(X_{k+1}^n = -(X_k^n + Z_{k+1}^n)\), we have

$$
(P_k^n)^- \leq (X_k^n)^-, \quad 1 \leq k \leq n.
$$

(9.15)

**Proof of Lemma 9.6.** Consider an auxiliary sequence \(\{M_k^n, \ k \geq 1\}\), defined recursively as \(M_1^n = P_1^n\) and \(M_{k+1}^n = M_k^n + Z_{k+1}^n - (M_k^n)^+\). Comparing \(M_k^n\) with (9.14) it is easy to see that \(M_k^n \leq P_k^n\) for all \(k \geq 1\). Indeed, the relation holds for \(k = 1\). Assume now it holds for \(k\) and use the induction. If \(M_k^n \leq P_k^n \leq 0\) then \(M_{k+1}^n = M_k^n + Z_{k+1}^n \leq P_k^n + Z_{k+1}^n = P_{k+1}^n\). If \(M_k^n \leq 0 \leq P_k^n\) then \(M_{k+1}^n = M_k^n + Z_{k+1}^n \leq Z_{k+1}^n\), while \(P_{k+1}^n = P_k^n + Z_{k+1}^n - (P_k^n)^+ \wedge r^n \geq Z_{k+1}^n\) again implying \(M_{k+1}^n \leq P_{k+1}^n\). Lastly, if \(0 \leq M_k^n \leq P_k^n\), we get \(M_{k+1}^n = Z_{k+1}^n\) and \(P_{k+1}^n + Z_{k+1}^n - (P_k^n)^+ \wedge r^n \geq Z_{k+1}^n\) since \(P_k^n - (P_k^n)^+ \wedge r^n \geq 0\). Therefore, we get \(M_k^n \leq P_k^n\) for all \(k \geq 1\).

The next step would be to show that \(X_k^n \leq M_k^n\). Once again, the argument will follow by induction. Indeed, the relation holds for \(k = 1\). Assume now it holds for \(k\). As before,
Lemma 9.7. Recall (9.13). Let \( U^n_k = Z^n_k - r^n \) for \( 1 \leq k \leq n \) and define a random walk \( \{ Y^n_k, k \geq 1 \} \), as follows: \( Y^n_0 = 0 \) and \( Y^n_{k+1} = (Y^n_k + U^n_{k+1})^+ \). Then
\[
(P^n_k - r^n)^+ \leq Y^n_k, \quad 1 \leq k \leq n. \tag{9.16}
\]

Proof of Lemma 9.7. Since \( Y^n_k \geq 0 \), it would be enough to show \( P^n_k - r^n \leq Y^n_k \) for all \( k \geq 1 \). We will use the induction argument. The relation clearly holds for \( k = 1 \). Now, assume that \( P^n_k - r^n \leq Y^n_k \) holds for some \( k \geq 1 \), and show that \( P^n_{k+1} - r^n \leq Y^n_{k+1} \).

Case 1: \( P^n_k - r^n \leq Y^n_k = 0 \). Then \( Y^n_{k+1} = (U^n_{k+1})^+ \). At the same time
\[
P^n_{k+1} - r^n = \begin{cases}
Z^n_{k+1} - r^n, & \text{if } 0 \leq P^n_k \leq r^n, \\
(P^n_k + Z^n_{k+1}) - r^n, & \text{if } P^n_k < 0,
\end{cases} \tag{9.17}
\]
and we have \( P^n_{k+1} - r^n \leq (Z^n_{k+1} - r^n)^+ = (U^n_{k+1})^+ = Y^n_{k+1} \).

Case 2: \( P^n_k - r^n \leq 0 \). In this case the relation follows once again from (9.17) since \( P^n_{k+1} - r^n \leq (Z^n_{k+1} - r^n)^+ = (U^n_{k+1})^+ \leq (Y^n_k + U^n_{k+1})^+ = Y^n_{k+1} \).

Case 3: \( 0 \leq P^n_k - r^n \leq Y^n_k \). In this case (recall (9.14))
\[
P^n_{k+1} - r^n \leq Y^n_k + Z^n_{k+1} - r^n \tag{9.18}
\]
\[
\leq Y^n_k + Z^n_{k+1} - r^n \leq (Y^n_k + Z^n_{k+1} - r^n)^+ = (Y^n_k + U^n_{k+1})^+ = Y^n_{k+1}.
\]

This finalizes the proof of (9.16). □

We are now ready to finalize the statement (9.12). Namely, (8.3), (9.15) and (9.16) imply
\[
\|\pi\| = \sum_{k=1}^{n} |\pi_k^n| \leq \sum_{k=1}^{n} (X^n_k)^- + \sum_{k=1}^{n} Y^n_k = \frac{1}{n} \sum_{k=1}^{n} \left[ (\psi^n_k)^- + \phi^n_k \right], \tag{9.19}
\]
where \( \psi^n_k = n X^n_k \) and \( \phi^n_k = n Y^n_k \) satisfy \( \psi^n_1 = -\zeta_1, \phi^n_1 = 0 \), and for \( k \geq 2 \)
\[
\psi^n_{k+1} = -\psi^n_k + n (\sigma^n - \zeta_1)^-, \quad \phi^n_{k+1} = (\phi^n_k + n (\sigma^n - r^n) - \zeta_{k+1})^+.
\]

Take some \( \hat{\sigma} \), satisfying \( 1 < \hat{\sigma} < n\sigma^n \) (the existence of such \( \hat{\sigma} \) for \( n \) large enough is guaranteed by having \( n\sigma^n \to \alpha_\sigma > 1 \)). Define \( z_k = \hat{\sigma} - \zeta_k \) and introduce \( \{ \psi^n_k, k \geq 1 \} \) as \( \psi^n_1 = -\zeta_1 \), and recursion as defined in (A2.4). By such a choice of \( \hat{\sigma} \), we must have \( z_k \leq n\sigma^n - \zeta_k \), hence one can apply the comparison result of Lemma A2.2 to get \( (\psi^n_k)^- \leq (\psi^n_k)^- \).
Similarly, take some $\bar{\sigma}$, satisfying $n(\sigma^n - r^n) < \bar{\sigma} < 1$ (the existence of such $\bar{\sigma}$ for $n$ large enough is guaranteed by having $n(\sigma^n - r^n) \to \alpha_{\sigma} - \alpha_{r} < 1$). Define $y_k = \bar{\sigma} - \zeta_k$ and introduce $\{\phi_k^y, k \geq 1\}$ as defined in (A1.4). By such a choice of $\bar{\sigma}$, we must have $y_k \geq n(\sigma^n - r^n) - \zeta_k$, hence one can apply the comparison result of Lemma A1.2 to get $\phi_k^y \leq \phi_k^y$.

Combining the above estimates with (9.19), we get

$$C_{mix}^n \leq \frac{1}{n} \sum_{k=1}^{n} \left[ (\psi_k^x)^{-} + \phi_k^y \right].$$

Now, apply Lemma A1.1 for $\{\phi_k^y, k \geq 1\}$ and Lemma A2.1 for $\{\psi_k^x, k \geq 1\}$ in order to get (9.12).

10. Concluding remarks

We have presented a new queueing approach to analyze the asymptotical behaviour of sensor allocation problems. The above method is robust and can be naturally expanded in various directions. Just to name a few, one can introduce new allocation costs, omit the assumption about equal sensor length; or, maybe try to find an asymptotically optical solution - in either of the regimes. Besides, one can further explore the reflection mappings (6.3), (8.2) and their applicability in related generalizations.

Regarding Remark 1, there is no principal difference as for whether or not the limit of $n\sigma^n$ exists. In fact, the results of Theorem 2.3 will stay for the underloaded regime $\mathbf{U}$ characterized by $0 \leq \liminf n\sigma^n \leq \limsup n\sigma^n < 1$, or overloaded regime $\mathbf{O}$ characterized by $1 < \liminf n\sigma^n$. For the critical $\mathbf{C}$ case when $n\sigma^n \to 1$, one can introduce a lower intermediate regime $n\sigma^n \approx 1 - \delta n^{-\beta}$ for $\delta > 0$ and $\beta \in (0, 1/2)$, bridging the critical regime (corresponding to $\beta = 1/2$) down to the underloaded ($\beta = 0$); as well as the upper intermediate regime $n\sigma^n \approx 1 + \delta n^{-\beta}$ for $\delta > 0$ and $\beta \in (0, 1/2)$, connecting the critical ($\beta = 1/2$) and overloaded ($\beta = 0$) regimes. The estimates for the interference problem can be easily adjusted to include $\Theta(n^{\beta})$ for the lower intermediate, and $\Theta(n^{1-\beta})$ for the upper intermediate regimes. A similar situation, (but with interchanged $\Theta(n^{\beta})$ and $\Theta(n^{1-\beta})$) we believe holds for the coverage problem, though we have not verified all the technicalities. In addition, a super-overloaded regime $n\sigma^n \approx \gamma_n$ for some sequence $\gamma_n \uparrow \infty$ will imply the estimate of $\Theta(n^{\gamma_n})$ for the interference problem; the coverage problem will remain $O(1)$.

References


Appendices

A1. Random walk on a positive half-plane

Assume we have a sequence of numbers \( \{x_k, k \geq 1\} \) and define a new sequence \( \{\phi_k, k \geq 0\} \) as follows

\[
\phi_0 = 0, \quad \phi_{k+1} = (\phi_k + x_{k+1})^+, \quad n \geq 0.
\]

(A1.1)

By induction (Lemma B1, p.197, [15]), in that case \( \phi_k \) satisfies for \( k \geq 1 \)

\[
\phi_k = s_k - \min_{0 \leq r \leq k} s_r, \quad \text{where} \quad s_0 = 0, \quad s_k = x_1 + \ldots + x_k.
\]

(A1.2)
In fact, both representations (A1.1) and (A1.2) are equivalent.

**Lemma A1.1.** Assume \( \{x_1, \ldots \} \) are i.i.d. random variables with \( \mathbb{E}(x_1) < 0 \) and \( \mathbb{E}(x_1^2) < \infty \). Then the random process \( \Phi = \{\phi_n, n \geq 0\} \) satisfies
\[
\sup_n \left( \frac{1}{n} \sum_{k=1}^{n} \phi_k \right) < \infty, \quad \text{a.s.} \quad \sup_n \mathbb{E} \left( \frac{1}{n} \sum_{k=1}^{n} \phi_k \right) < \infty \quad \text{(A1.3)}
\]

**Proof.** The first statement would follow from Theorem 17.0.1 of ([13], p. 422) showing the result for a positive Harris recurrent chain with finite invariant probability that has at least one moment. Let us make sure that all the conditions hold. The recurrence of \( \Phi \) has been established in Prop. 8.5.1. of ([13], p. 193), the Harris recurrence (see the definition in [13], p. 199) follows from \( \psi \)-irreducibility of \( \Phi \) (see [13], Prop. 4.3.1 and the last paragraph of p. 87), and the Harris recurrence of the petite set / atom at \( \{0\} \) ([13], Prop. 9.1.7 on p. 205), since, with probability one, the chain visits \( \{0\} \) infinitely often. The latter claim, though very intuitive, follows from the boundedness of the expected busy period -time between successive visits to \( \{0\} \), which can be found in ([15], Thm. 7 on p. 27, using their notation \( \mathbb{E}(N) < \infty \) if \( \alpha < 0 \)). This last reference also implies the existence of a finite invariant measure (see [13], Thm. 10.2.1.(iii), p. 234). The properly normalized invariant measure then becomes a stationary probability measure. Finally, the existence of moments for the stationary probability is guaranteed by ([13], Prop 14.4.1, p. 352).

For the second statement in (A1.3), since \( \phi_0 = 0 \), we can almost surely (path-wise) bound \( \Phi \) from above by stationary \( \Phi^\infty = \{\phi_k^\infty, k \geq 0\} \), such that \( \phi_0^\infty \) is already stationary distributed. As a result, for each \( k \geq 1 \) we have \( \mathbb{E}(\phi_k) \leq \mathbb{E}(\phi_k^\infty) < \infty \) (since the stationary distribution has at least one moment), and the second bound of (A1.3) follows.

We end this section with a small comparison lemma.

**Lemma A1.2.** (comparison) Assume we are given two sets of numbers \( \{x_k, k \geq 1\} \) and \( \{y_k, k \geq 1\} \) satisfying \( x_k \leq y_k \) for all \( k \geq 1 \). Let \( \Phi^x = \{\phi_k^x, n \geq 0\} \) be defined as in (A1.1) and \( \Phi^y = \{\phi_k^y, k \geq 0\} \) be defined as
\[
\phi_0^y = 0, \quad \phi_{k+1}^y = (\phi_k^x + y_k + 1)^+, \quad k \geq 0.
\]

Then \( \Phi^x \leq \Phi^y \), that is \( \phi_k^x \leq \phi_k^y \) for all \( k \geq 0 \).

**Proof.** For \( k = 0 \) the relation holds by the definition. After that the statement follows from the induction argument, because having \( \phi_{k-1}^x \leq \phi_{k-1}^y \) for a certain \( k \geq 1 \) will necessarily imply \( \phi_{k-1}^x + x_k \leq \phi_{k-1}^y + y_k \), which immediately yields \( \phi_k^x = (\phi_{k-1}^x + x_k)^+ \leq (\phi_{k-1}^y + y_k)^+ = \phi_k^y \).
A2. Random walk on a negative half-plane

Alternatively, for \( \{x_k, k \geq 1\} \) we can define a sequence \( \Psi = \{\psi_k, k \geq 0\} \) as follows

\[
\psi_0 = 0, \quad \psi_{k+1} = - (\psi_k + x_{k+1})^-, \quad k \geq 0. \tag{A2.1}
\]

Note that \( \psi_k \leq 0 \) for all \( k \geq 0 \). Once again, the above representation is equivalent to (see (A1.2))

\[
\psi_k = s_k - \max_{0 \leq r \leq k} s_r, \quad k \geq 0. \tag{A2.2}
\]

Lemma A2.1. Assume \( \{x_1, \ldots\} \) are i.i.d. random variables with \( \mathbb{E}(x_1) > 0 \) and \( \mathbb{E}(x_1^2) < \infty \). Then the relations (A1.3) hold for \( \Phi = -\Psi = (\Psi)^- \).

Proof. First of all, note that \( -\Psi = (\Psi)^- \) due to non-positivity of \( \Psi \). Now, let \( \Phi = -\Psi \). From (A2.2),

\[
\phi_k = -\psi_k = -s_k + \max_{0 \leq r \leq k} s_r = \bar{s}_k - \min_{0 \leq r \leq k} \{\bar{s}_r\},
\]

where \( \bar{s}_k = -s_k = (-x_1) + \ldots + (-x_k) \). Relation (A2.3) defines a random walk on the positive half-plane, as described in Appendix A1, with the i.i.d. increments \( \{-x_i, i = 1, \ldots\} \). Since \( \mathbb{E}(-x_1) < 0 \), the statement of the lemma follows from Lemma A1.1.  \( \square \)

Lemma A2.2. (comparison) For two sets \( \{x_k, k \geq 1\} \) and \( \{z_k, k \geq 1\} \) assume \( z_k \leq x_k \) for all \( k \geq 1 \). Let \( \Psi^x = \{\psi_k^x, n \geq 0\} \) be defined as in (A2.1) and \( \Psi^z = \{\psi_k^z, n \geq 0\} \) be defined as

\[
\psi_0^x = 0, \quad \psi_{k+1}^x = - (\psi_k^x + z_{k+1})^-, \quad k \geq 0. \tag{A2.4}
\]

Then \( \Psi^x \leq \Psi^z \) or \( (\Psi^x)^- \leq (\Psi^z)^- \).

Proof. For \( k = 0 \) the relation holds by the definition. Next, from having \( \psi_{k-1}^z \leq \psi_{k-1}^x \) for a certain \( k \geq 1 \) we get \( \psi_{k-1}^z + z_k \leq \psi_{k-1}^x + x_k \). This implies \( \psi_k^z = - (\psi_{k-1}^x + z_k)^- \leq - (\psi_{k-1}^x + x_k)^- = \psi_k^x \), concluding the lemma.  \( \square \)

A3. Compensators

Lemma A3.1. Let \( a^n \) be a compensator of \( N^n(t) = \sum_{i=1}^n 1_{\{\tau_i^t \leq t\}} \) and let \( W^n = \sigma^n N^n(t) - t \). There exist constants \( c_1, c_2 \geq 0 \) such that for any \( T > 0 \)

\[
\int_0^T \left(W^n(t-)\right)^+ da^n(t) \geq c_1 n \int_0^T \left(W^n(t-)\right)^+ dt - c_2, \quad \text{almost surely.} \tag{A3.1}
\]

Proof. The compensator must satisfy

\[
\frac{da^n(t)}{dt} = n \frac{f_\zeta(nt - n\tau_{N^n(t-))})}{1 - F_\zeta(nt - n\tau_{N^n(t-))})} = n h_\zeta(nt - n\tau_{N^n(t-))}), \tag{A3.2}
\]

\[
\frac{da^n(t)}{dt} = n \frac{f_\zeta(nt - n\tau_{N^n(t-))})}{1 - F_\zeta(nt - n\tau_{N^n(t-))})} = n h_\zeta(nt - n\tau_{N^n(t-))}), \tag{A3.2}
\]
where $h_\zeta$ is a hazard rate of random variable $\zeta$ (in the exponential case we have $h_\zeta(t) = 1$).
We claim that there exist constants $c_1$ and $c_2$ such that for any constant $T > 0$
\[
 n \int_0^T \left( W^n(t-) \right)^+ h_\zeta(nt - n\tau_{N^n(t-)}^n) dt \geq c_1 n \int_0^T \left( W^n(t-) \right)^+ dt - c_2. \tag{A3.3}
\]

Indeed, the inequality (A3.3) immediately holds if the hazard rate $h_\zeta$ is bounded from below by some constant $c_1$. Otherwise, if $h_\zeta(0) = 0$, we can decrease the integral in the l.h.s. of (A3.3) by eliminating the neighbourhoods where the argument inside $h_\zeta(\cdot)$ is close to zero, hence leaving only the area (denote it by $D$) on which $h_\zeta(\cdot) \geq c_1$, for some $c_1$ small enough:
\[
n \int_0^T \left( W^n(t-) \right)^+ h_\zeta(nt - n\tau_{N^n(t-)}^n) dt \geq n \int_0^T \left( W^n(t-) \right)^+ h_\zeta(nt - n\tau_{N^n(t-)}^n) \mathbb{I}_D dt \tag{A3.4}
\]
\[
\geq n c_1 \int_0^T \left( W^n(t-) \right)^+ \mathbb{I}_D dt. \tag{A3.5}
\]

At the same time, the area $D$ would consist of $n$ intervals each of the length $\Theta(1/n)$, and due to the "triangular" shape of $W^n$ in the neighbourhoods of jump points, the loss of restricting $[0, T]$ to $D$ would be $n \times \Theta(1/n^2) = \Theta(1/n)$. Therefore,
\[
n c_1 \int_0^T \left( W^n(t-) \right)^+ \left( 1 - \mathbb{I}_D \right) dt \leq c_2
\]
for some constant $c_2$. The latter, together with (A3.4)-(A3.5), implies (A3.3). \qed