The Regularity Lemma and Applications

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Arithmetic progressions

An arithmetic sequence is a set of integers of the form

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then some set C_i contains an arithmetic progression of length k.

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An argument using a probabilistic technique (the Lovász Local Lemma) gives that

$$n(k, r) > \left(\frac{r^k}{erk}\right)(1 + o(1)).$$

Plain language

Our version of van der Waerden says:

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Which color?

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If we color a set with r colors, then one of those colors will contain $1/r$ of the set.

That set is a very likely candidate. Is that the one?

Dense subsets

More generally, we can ask the following:

> If we have a subset of $\{1, 2, \ldots, n\}$ with positive density, does it have an arithmetic progression of length k?

Dense subsets

Let $A_n \subset \{1, 2, \ldots, n\}$. The family A_n is said to have positive upper density if there exists an ϵ such that $|A_n| > \epsilon n$.

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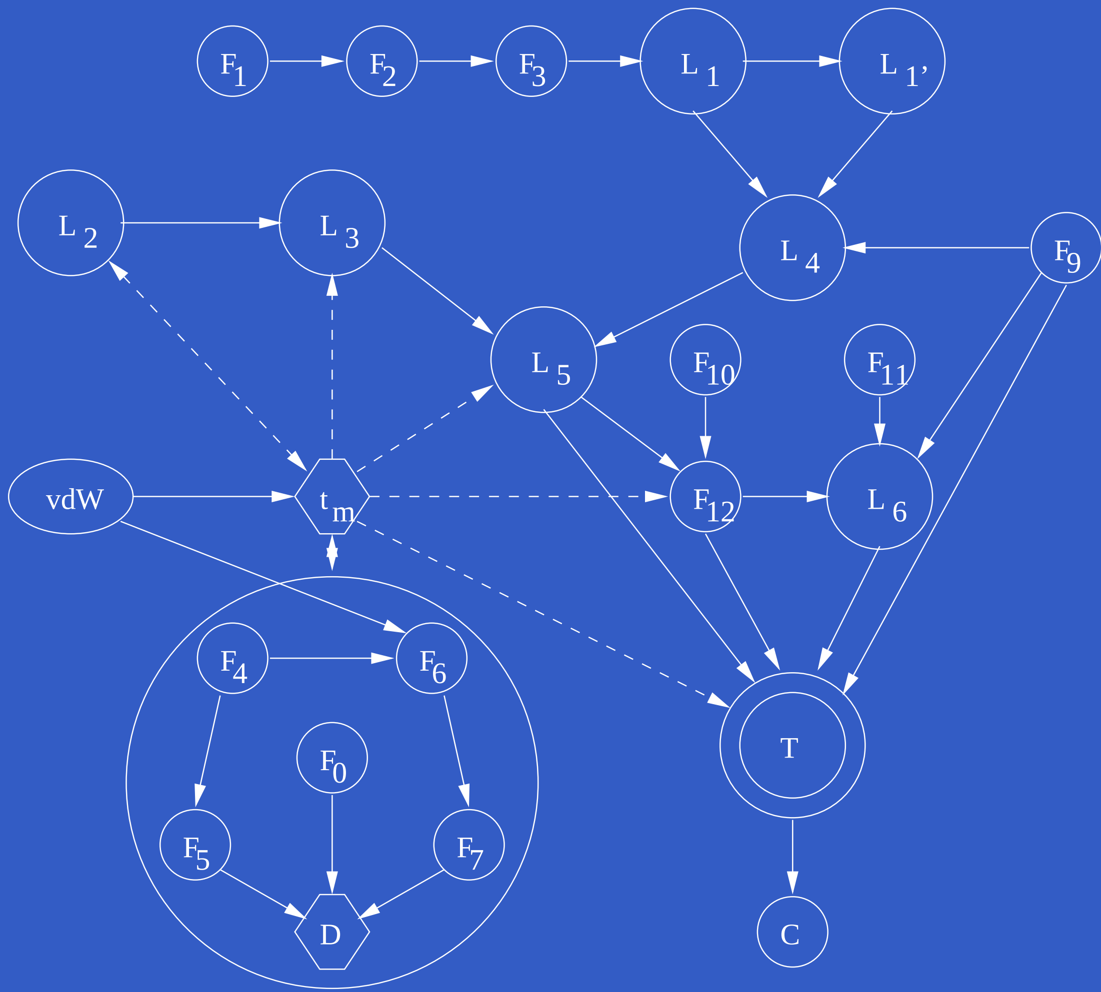
THEOREM [Szemerédi, 1975]
For every integer $k > 2$ and every $\epsilon > 0$, there exists a threshold $n_0 = n_0(k, \epsilon)$ such that if, for some $n \geq n_0$, $A \subset \{1, 2, \ldots, n\}$ and $|A| > \epsilon n$, then A must contain an arithmetic progression of k terms.

The proof of Szemerédi's theorem

The proof itself is quite long, complicated and ingenious.

For example, consider a diagram of the structure of the proof.

The proof of Szemerédi’s theorem



The Regularity Lemma

Buried in the proof of Szemerédi's theorem is a primitive version of the Regularity Lemma.

But the Regularity Lemma is not a statement about number theory, it's a statement about graphs.

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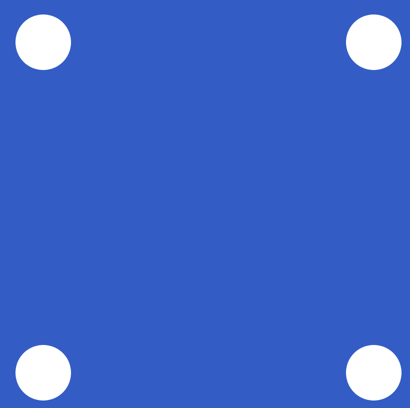

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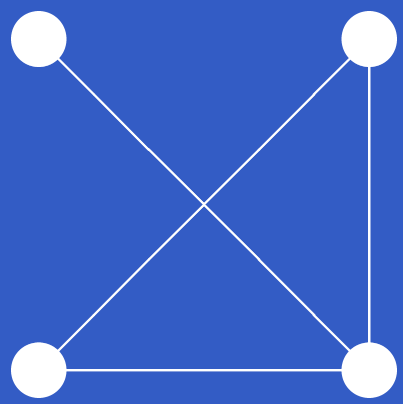

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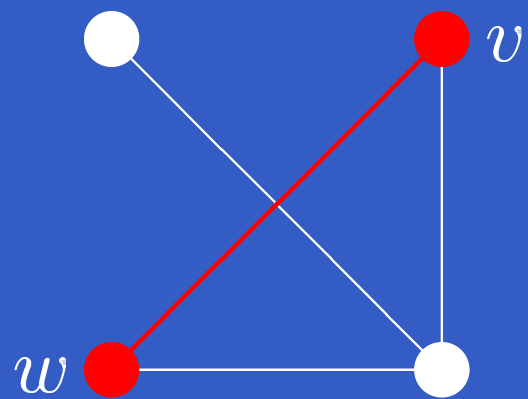

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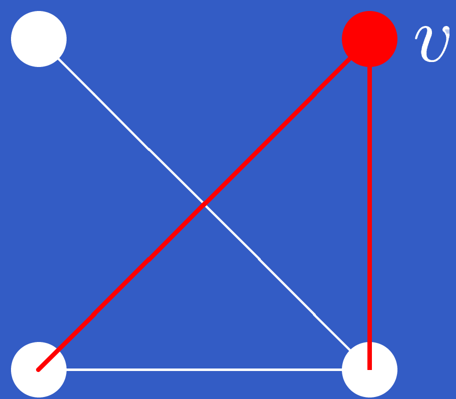

v is said to be adjacent to w if there is an edge between them. The degree of v, $\deg(v)$, is the number of edges touching v.

Bipartite

A graph G is bipartite if we can partition the vertex set into A and B so that all edges are **between** A and B:

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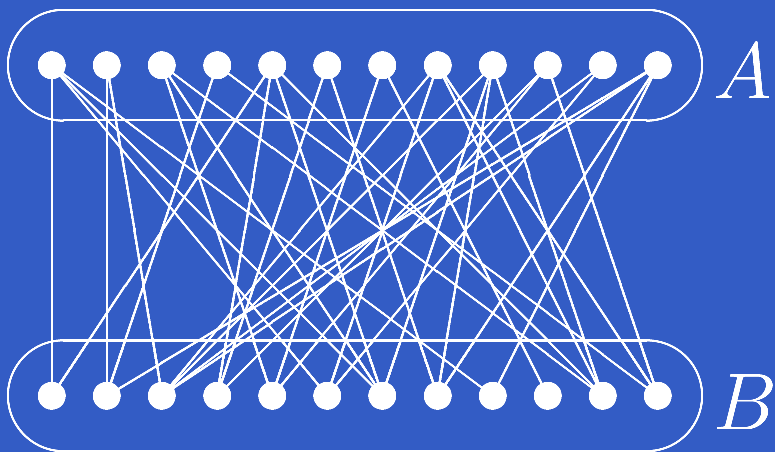

We denote such a graph as $G = (A, B; E)$.

Random bipartite graphs

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Binomial model

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I.e., take an $a \in A$ and $b \in B$ and flip a biased coin to see whether or not ab is an edge.

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The average density is

$$E[d(A, B)] = \frac{dn^2}{n^2} = d.$$

Subgraphs

Choose $A' \subseteq A$ and $B' \subseteq B$.
What is the probability that

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Easy to compute because $d(A', B')$ is a binomial random variable.

I.e., it is the average of coin flips.

Chernoff bound

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$$\Pr\left\{|d(A', B') - d| \geq \epsilon\right\} \leq 2\exp\left(-2\epsilon^2|A'||B'|\right)$$

Chernoff bound

In fact, if $|A'| > \epsilon n$ and $|B'| > \epsilon n$, then we have:

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This is ... tiny ...

Applies simultaneously to all such sets

In fact, the probability that ALL pairs of sets, each with size $> \epsilon n$, have density in

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approaches 1
as $n \to \infty$.

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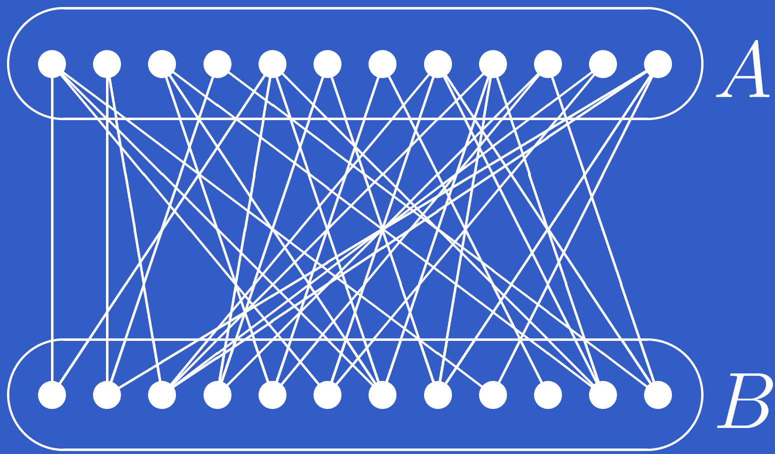
  we have $|d(A', B') - d| < \epsilon$.

So, ϵ-regular pairs mimic random pairs.

Random graphs

ϵ-regular pairs are tied inexorably to random pairs.



$$d(A, B) = \frac{1}{4}$$

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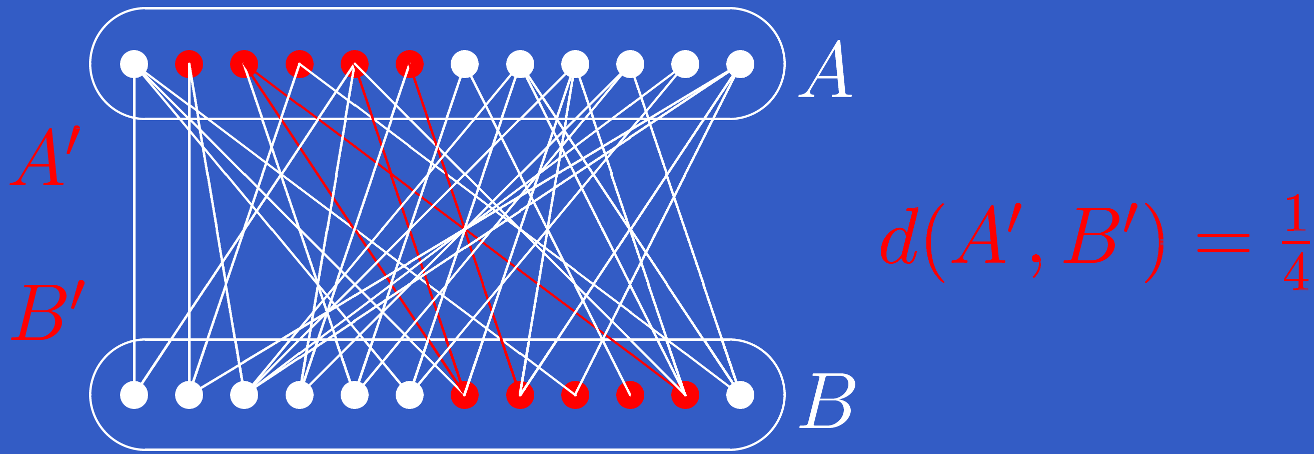

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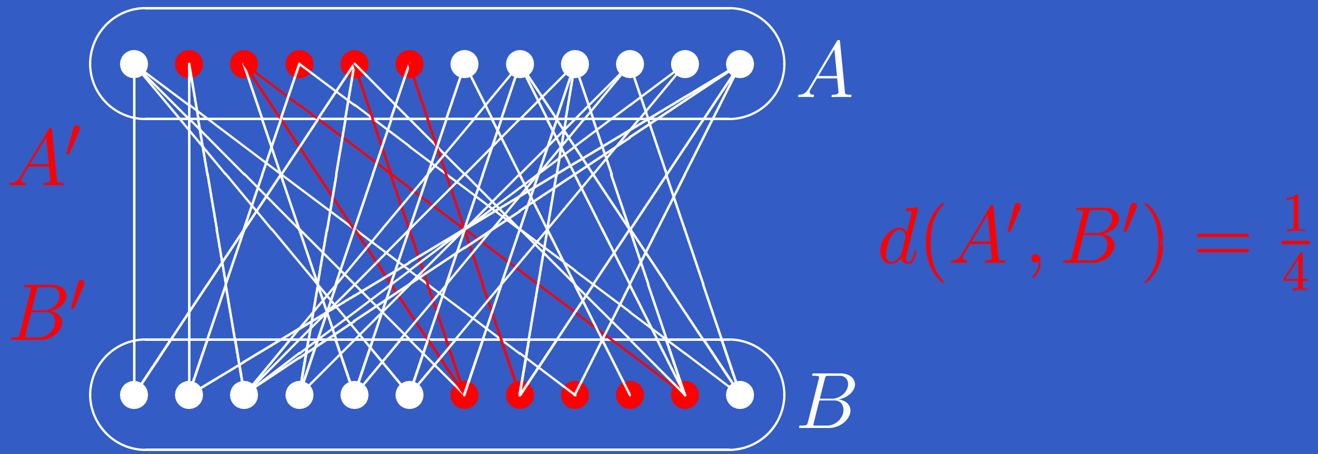

In fact, an ϵ-regular pair has a surprising number of properties that random pairs have.

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It's easier to see in a picture.

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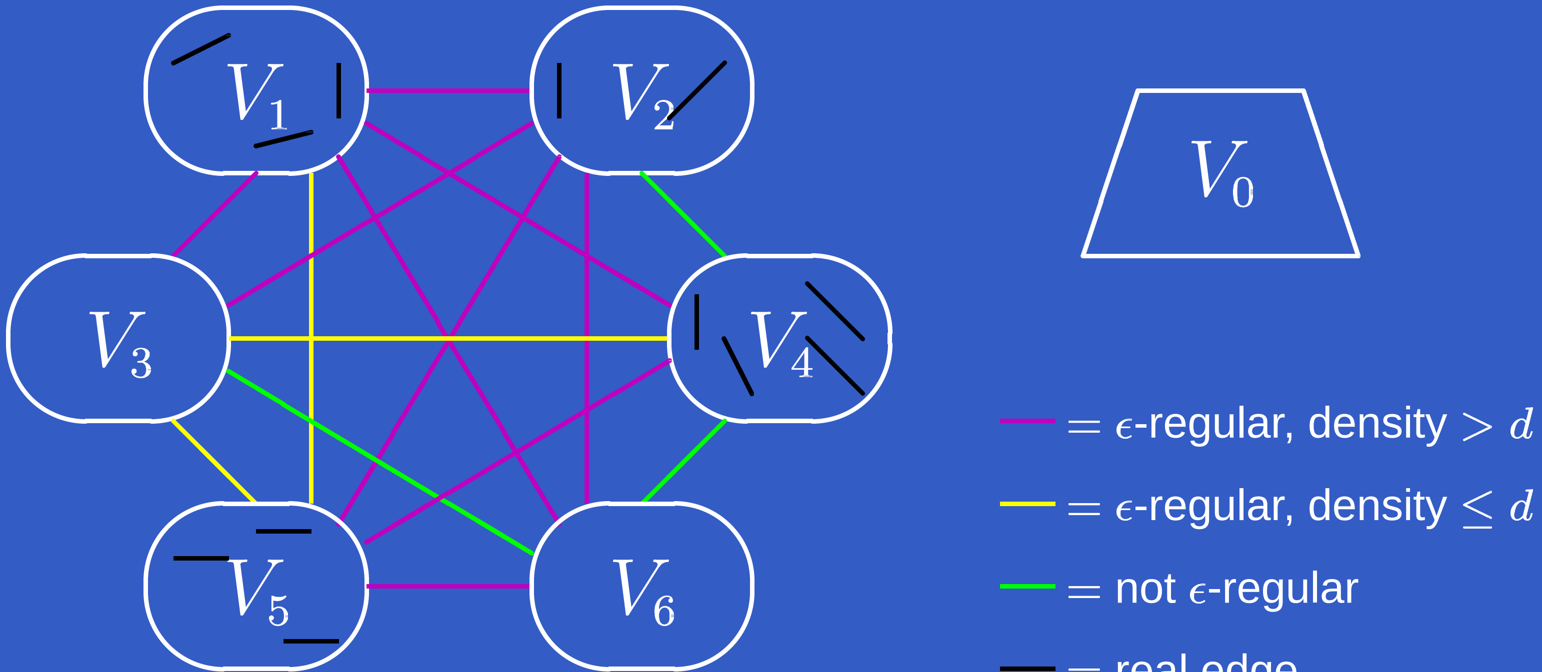

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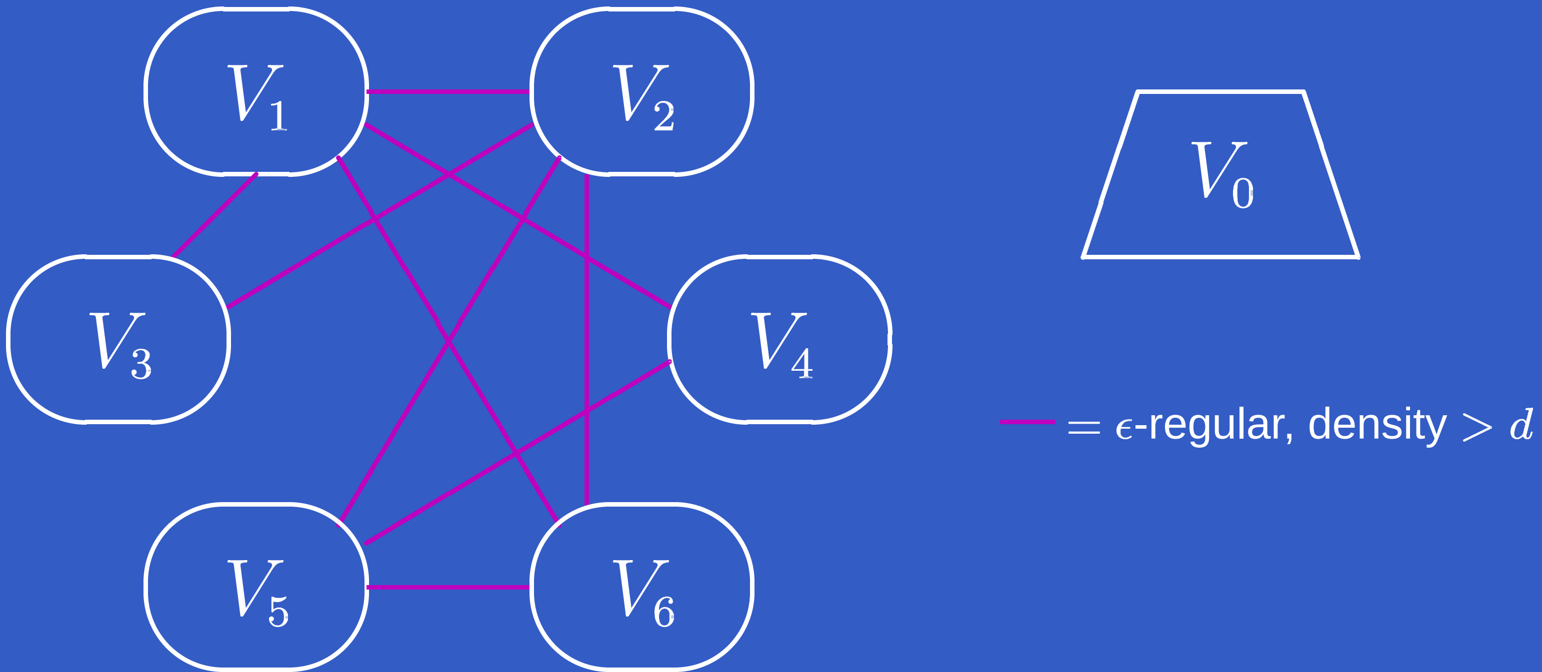

G' is very applicable.

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- no G' edges inside a V_i, $i = 1, \ldots, k$
- **all** pairs $G'(V_i, V_j)$, ($i, j \geq 1$) are ϵ-regular, each with a density **either** 0 **or** $> d$.

Geometry applications

One application is to look at distances in the plane.
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There are

$$\binom{n}{2} = \frac{n(n-1)}{2} \approx \frac{n^2}{2}$$

pairs of points.

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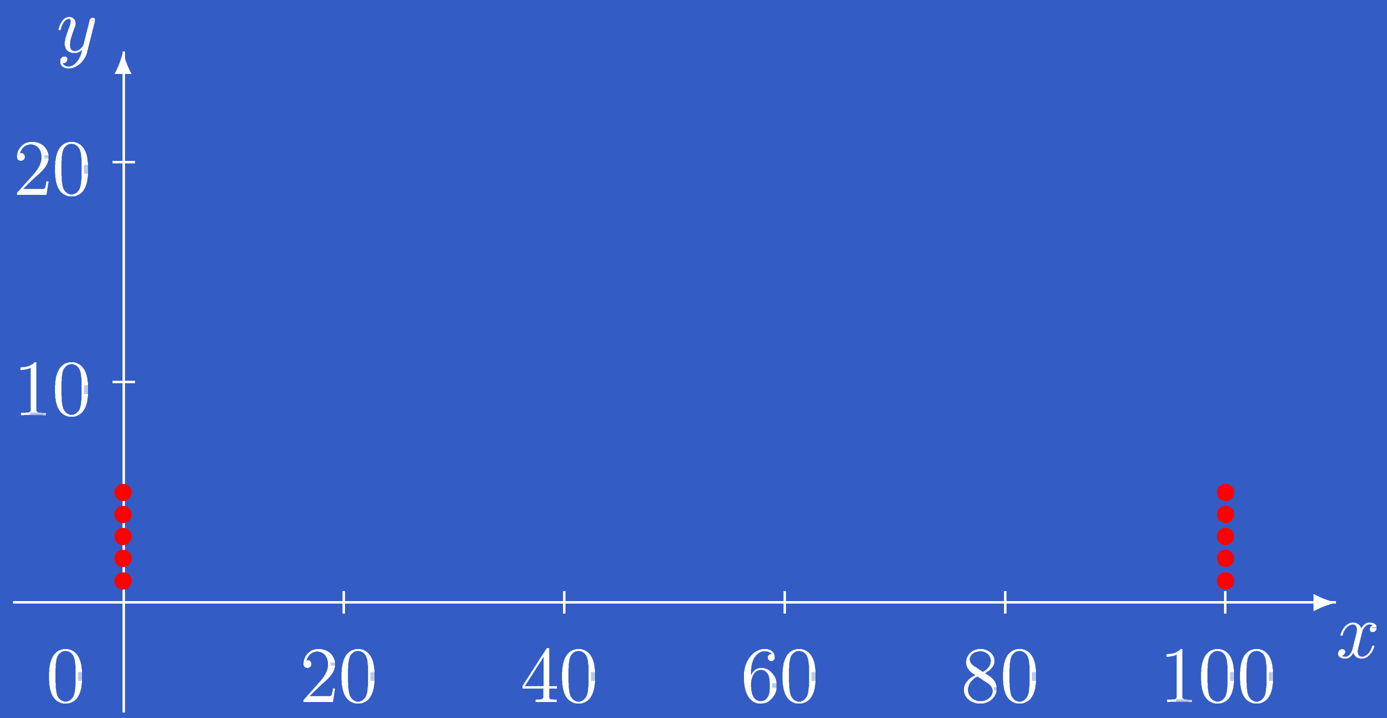
Even if $i = i'$, the distance is at least 1.

A picture of this family



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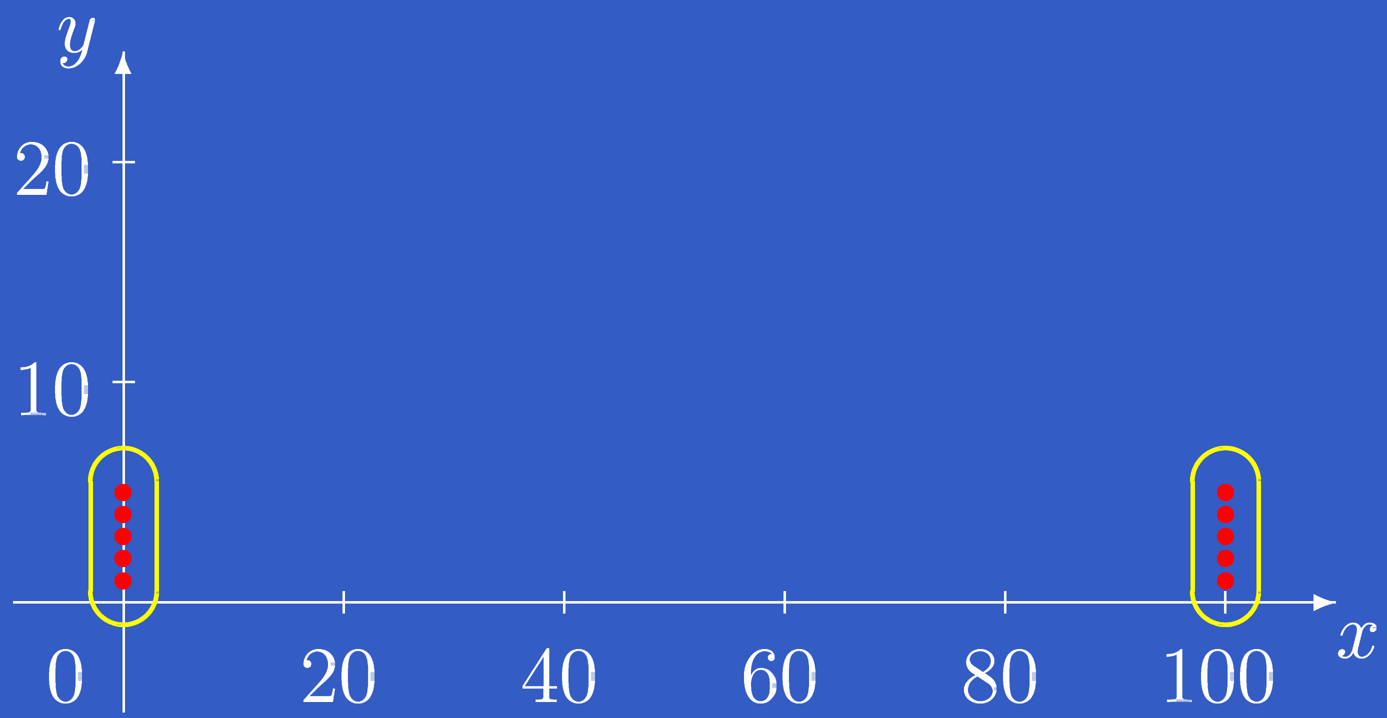

Distances are restricted

Thus, P has at least

$$\left(1-\frac{1}{2}\right)\frac{n^2}{2}=\frac{n^2}{4}$$

pairs with distances in

$$[n^2, n^2+1].$$

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There is a generalization.

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THEOREM [Erdős-Makai-Pach, 1993]
Given $k \in \mathbb{Z}^+$ and $\epsilon > 0$, there is a c and a positive integer n_0 satisfying the following condition:
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$$\|p_i - p_j\| \in \bigcup_{\ell=1}^{k} [t_\ell, t_\ell + c\sqrt{n}]$$

is at most $(1 - 1/(k+1) + \epsilon)\,(n^2/2)$

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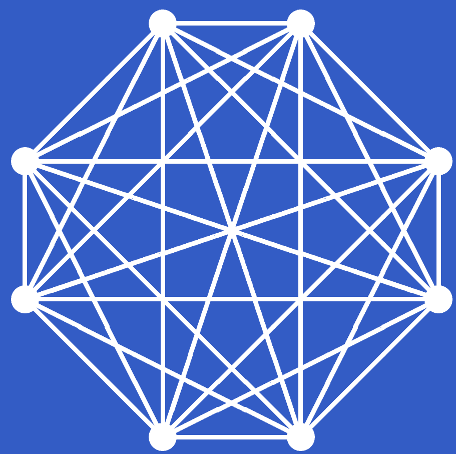

K_8

Theorem of Turán

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THEOREM [Turán, 1941]

- Let K_3 be the complete graph on 3 vertices.
- Let G_n, on n vertices, contain no K_3 as a subgraph.

Theorem of Turán

Turán's theorem is a classical result. Here is a special case.

THEOREM [Turán, 1941]

- Let K_3 be the complete graph on 3 vertices.
- Let G_n, on n vertices, contain no K_3 as a subgraph. Then,

$$|E(G_n)| \leq \frac{n^2}{4}$$

Consequence of Regularity Lemma

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$$|E(G_n)| > \left(\frac{1}{2} + \beta\right)\frac{n^2}{2},$$

then G_n contains at least $\left(\frac{\epsilon}{M(\epsilon)}n\right)^3$ copies of K_3.

Zero to many

There are graphs G_n such that

$$|E(G_n)| = \left(\frac{1}{2}\right)\frac{n^2}{2} - o(n^2)$$

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So, just a few edges takes us from zero copies of K_3 to a large number:

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Note that $M(\epsilon)$ is a constant.

Size of $M(\epsilon)$

Question: How big is this $M(\epsilon)$?

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$$\left\lceil \frac{10}{\epsilon^5} \right\rceil \left\{ 4^{4^{4^{\cdot^{\cdot^{\cdot^{4^{4^{(1/\epsilon)}}}}}}}} \leq M(\epsilon) \leq 5^{5^{5^{\cdot^{\cdot^{\cdot^{5^{5^{(1/\epsilon)}}}}}}}} \right\} \left\lceil \frac{10}{\epsilon^5} \right\rceil$$

Does it need to be so big?

Some kind of tower function is necessary.

THEOREM [Gowers, 1997] For any $\epsilon > 0$, there is a graph so that any application of the Regularity Lemma requires that the number of clusters is at least a number which is a tower of twos of height proportional to $\log(1/\epsilon)$.

Results are still satisfying

Despite the size requirement, there are still pretty results.

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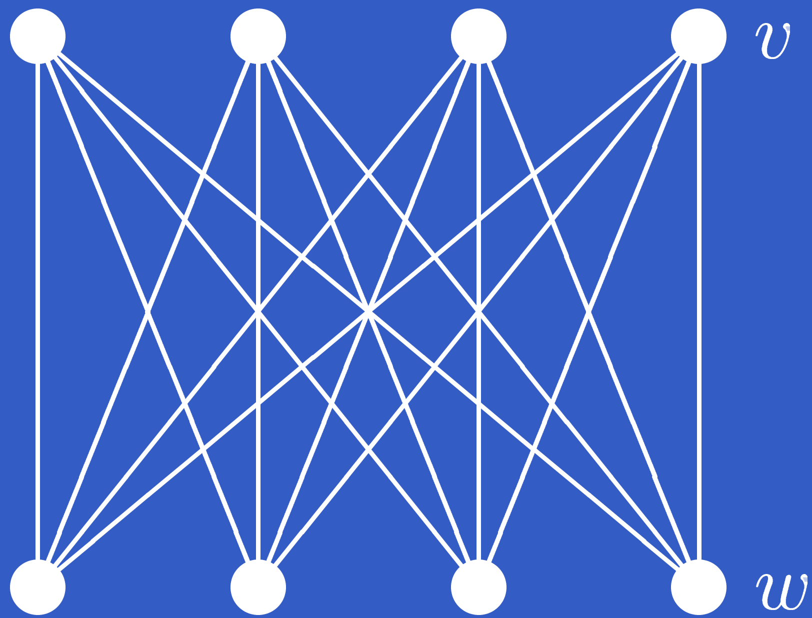
- A diameter 2 graph is one that has a path of length 2 between any pair of nonadjacent vertices.
- A minimal diameter 2 graph is a diameter 2 graph but $G \setminus e$ is **not**, $\forall e \in E(G)$.
- Any complete bipartite graph is minimal diameter 2.

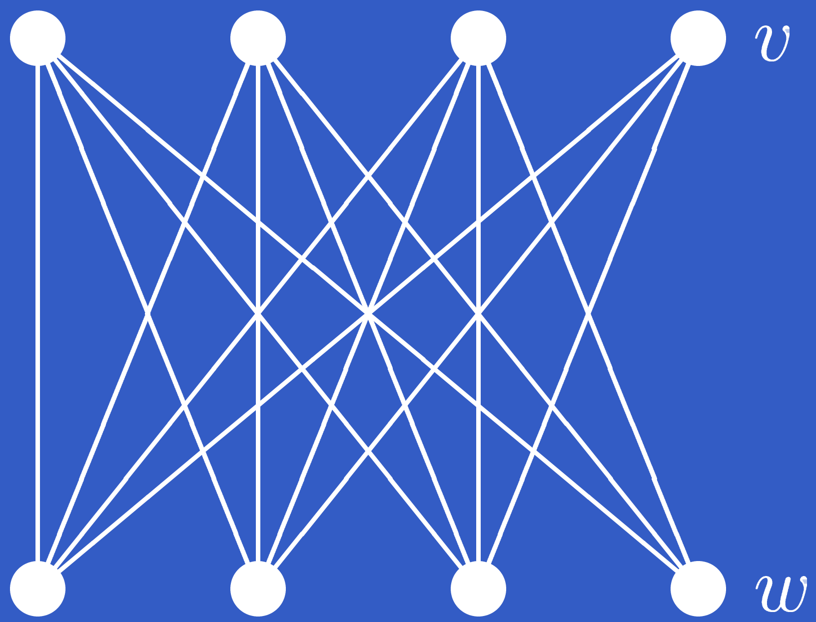
Complete bipartite graphs

Consider a complete bipartite graph $K_{4,4}$:



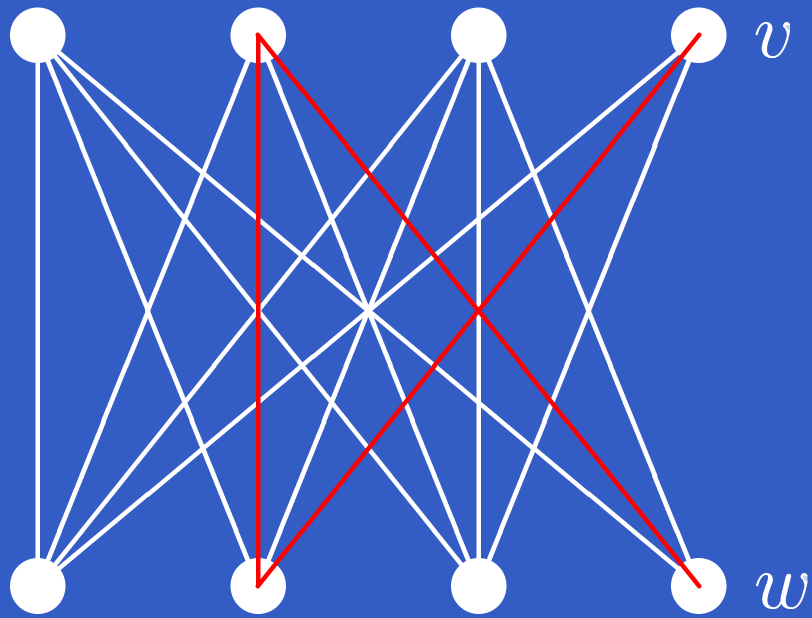
Complete bipartite graphs

Delete an edge vw.



Complete bipartite graphs

Distance between v and w is 3.



Minimal diameter 2 theorem

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There is a n_0 such that if $n \geq n_0$ and G_n is a minimal graph of diameter 2, then

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$$|E(G_n)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor$$

Furthermore, equality occurs if and only if

$$G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}.$$

Loebl's Conjecture

CONJECTURE [Loebl]

If G_n is a graph on n vertices, and at least $n/2$ vertices have degrees at least $n/2$, then G contains, as subgraphs, all trees with at most $n/2$ edges.

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THEOREM [Ajtai-Komlós-Szemerédi, 1994]
For every $\epsilon > 0$, there is a n_0 such that if G_n has at least $(1+\epsilon)n/2$ vertices of degree at least $(1+\epsilon)n/2$, then G_n contains, as subgraphs, all trees with at most $n/2$ edges.

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THEOREM [Zhao, 2003?]
There is a constant n_0 so that Loebl's Conjecture holds for $n \geq n_0$.

Hajnal-Szemerédi

There is a classical theorem, not proven by regularity, that gives a condition for which a graph can be covered by copies of K_p.

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THEOREM [Corrádi-Hajnal, 1963]
If $\delta(G_n) \geq (2/3)n$, then G_n contains $\lfloor n/3 \rfloor$ vertex-disjoint copies of K_3.

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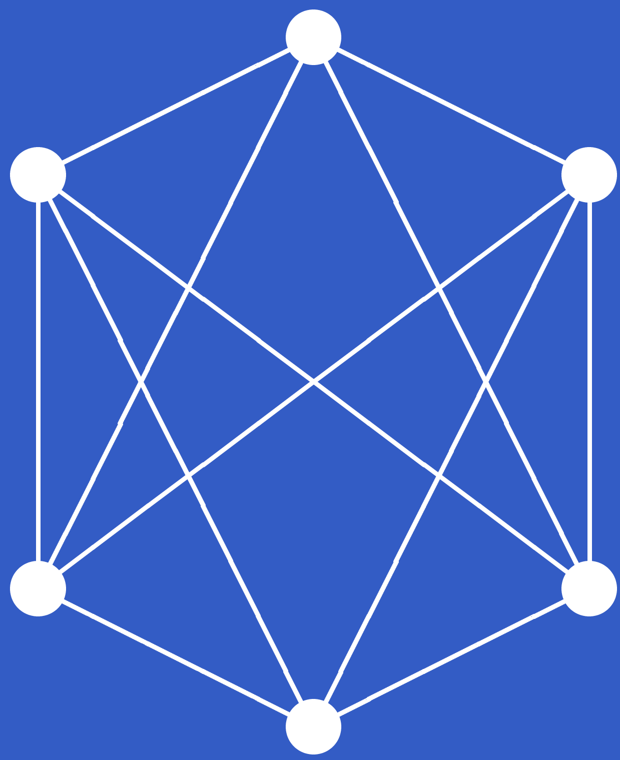
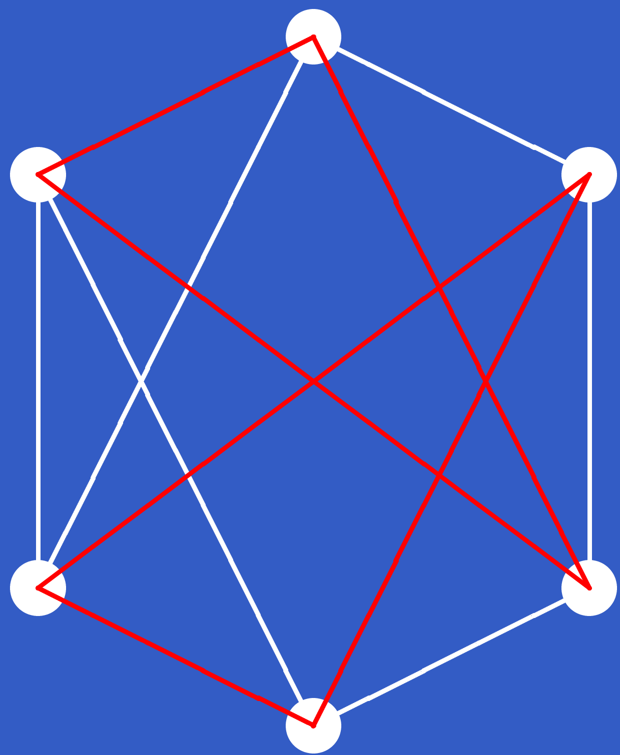
A small example

Here the minimum degree is
$4 = (2/3) \times 6$.



A small example

Hajnal-Szemerédi says that it can be covered by triangles (K_3's).



Tripartite version

The following conjecture is a natural extension of Corrádi-Hajnal, but not a consequence:

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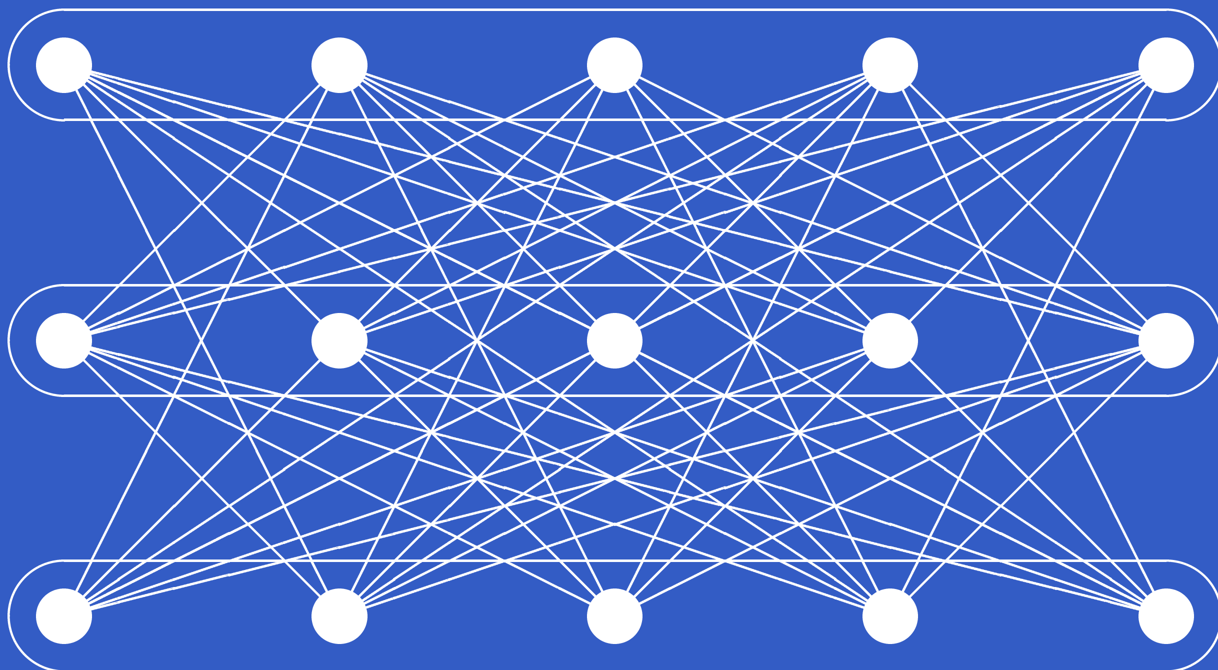
In fact, "$>$" can be replaced by "$\geq$", but there is one exceptional case.

An example

Each vertex is adjacent to at least $4 > (2/3) \times 5$ vertices in each of the other classes.



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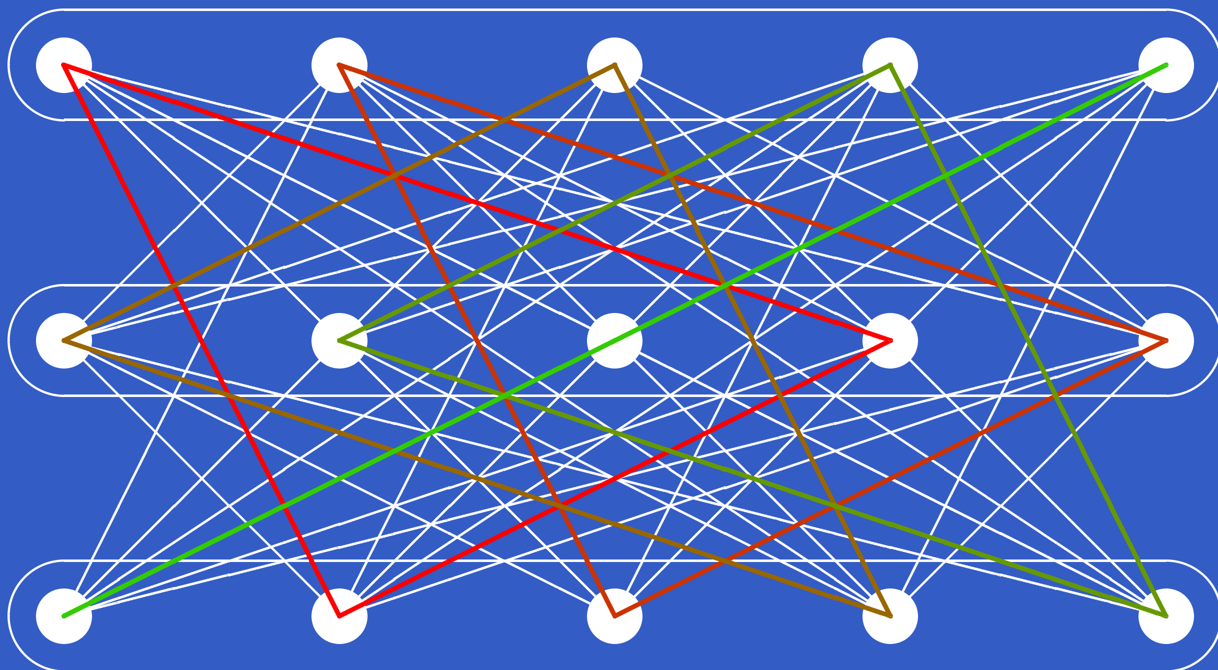

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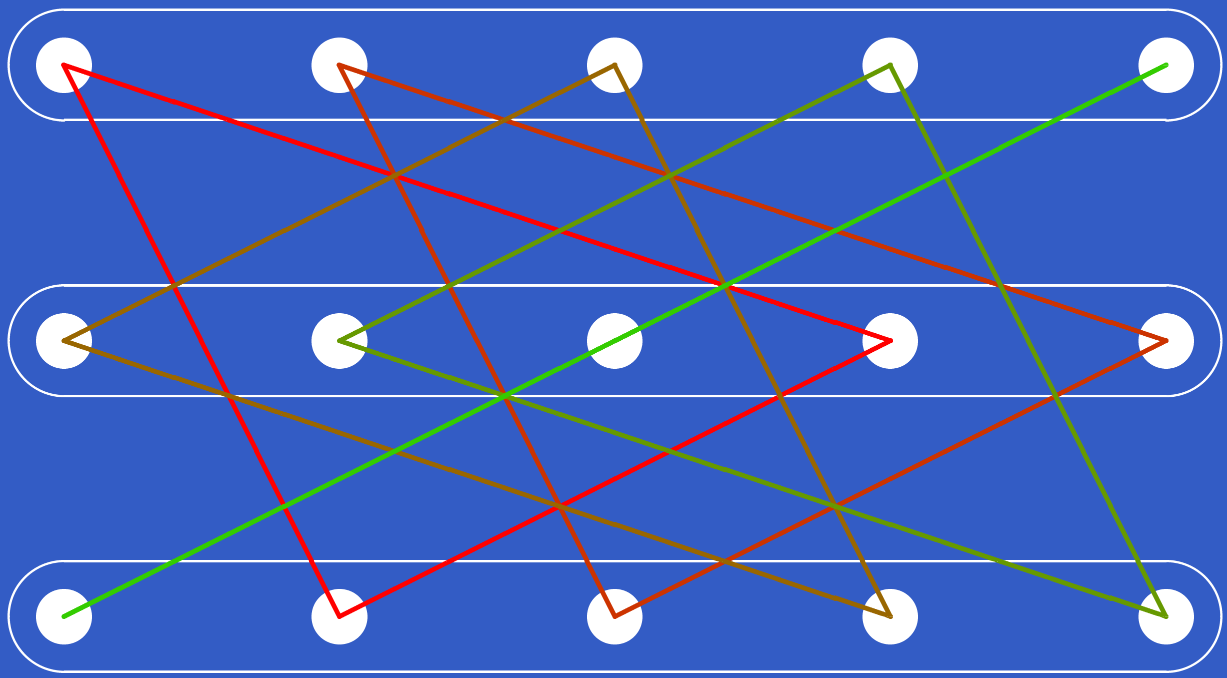
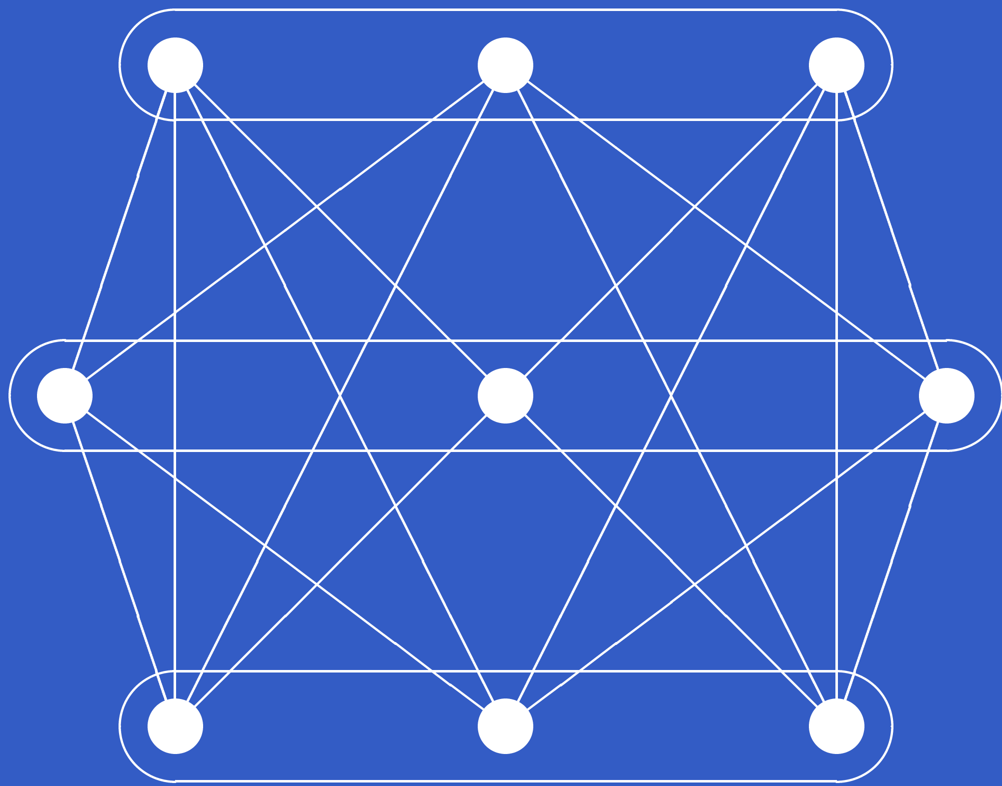


The result is tight

Consider the following example. Each vertex is adjacent to 2 in each other piece.



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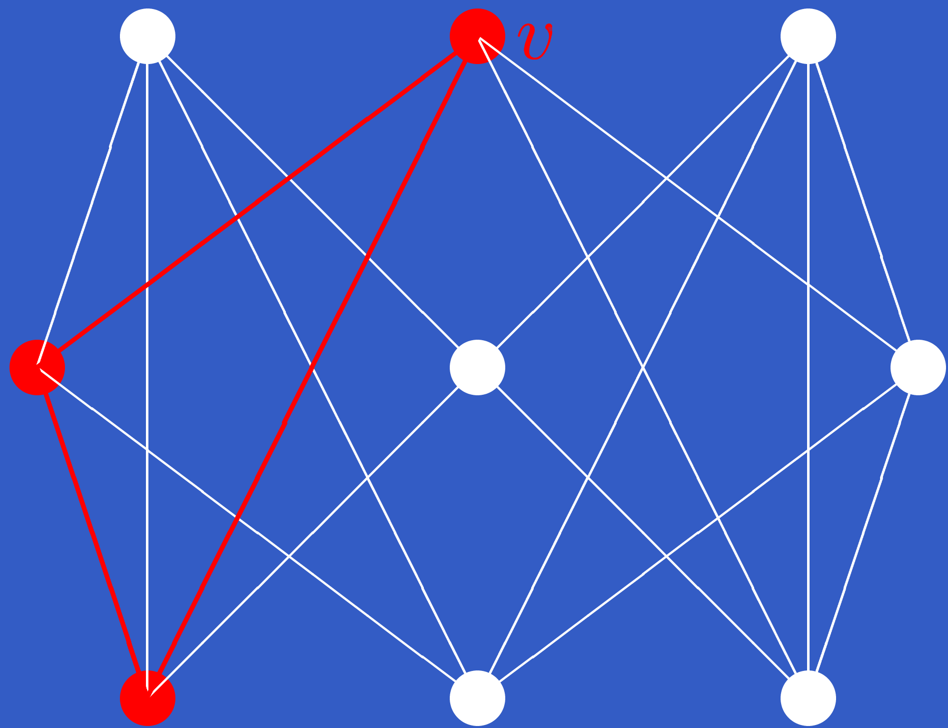
Without loss of generality, v must be in the red triangle.



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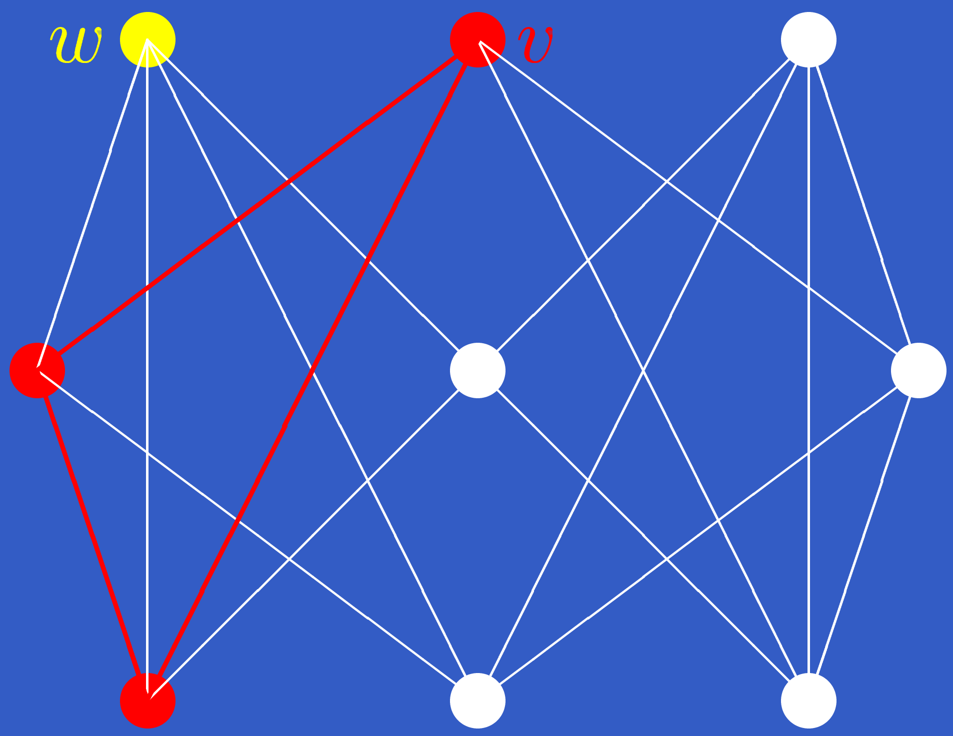
But then, w cannot be in any triangle.



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There is only one such graph (up to isomorphism) and we call it $\Gamma_3(N/3)$.

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Then either G has a subgraph which is N vertex-disjoint triangles, or $G = \Gamma_3(N/3)$ for $N/3$ an odd integer.

What's with N_0?

Yep, that N_0 is the $M(\epsilon)$ from the Regularity Lemma.

Quadripartite?

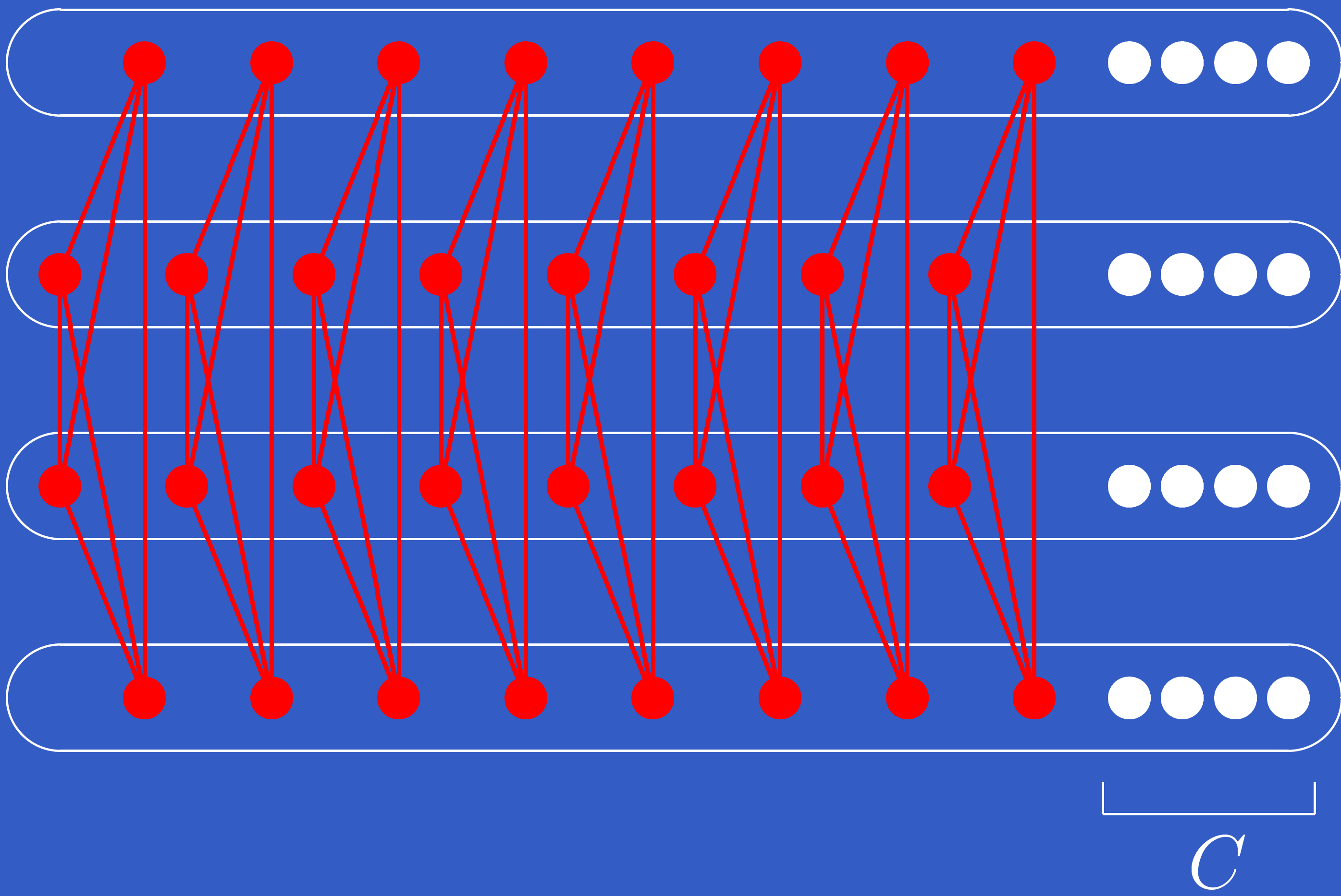
A partial result in the quadripartite case:

THEOREM [Fischer, 1999]
Let G be a quadripartite graph with

- N vertices in each part,
- each vertex adjacent to at least $(3/4)N$ vertices in each of the other three parts,

Then there is an absolute constant C such that G has a subgraph which is a family of $N - C$ vertex-disjoint K_4's.

Diagram of Fischer's result



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A final application

We will begin with a family of graphs:

$$\mathcal{H}(n,d) \stackrel{\text{def}}{=} \{H : \delta(H) \geq dn\}$$

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What is the diameter of the “random” graph G?

Many possibilities

Let's assume d is a small constant.

H could be a variety of possibilites, including

- a traditional random graph,
- an ϵ-regular pair,
- $\lfloor 1/d \rfloor$ disjoint cliques

Diameter 7

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- Let $m \to \infty$ as $n \to \infty$.

Then, $\operatorname{diam}(G) \leq 7$.

The algorithm

Construct $v_1, v_2, \ldots$ greedily such that

$$\mathrm{dist}(v_i, v_{i+1}) = 3.$$

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Since $|N(v_i)| \geq dn$, we stop in $\lfloor 1/d \rfloor$ steps.

Quick facts

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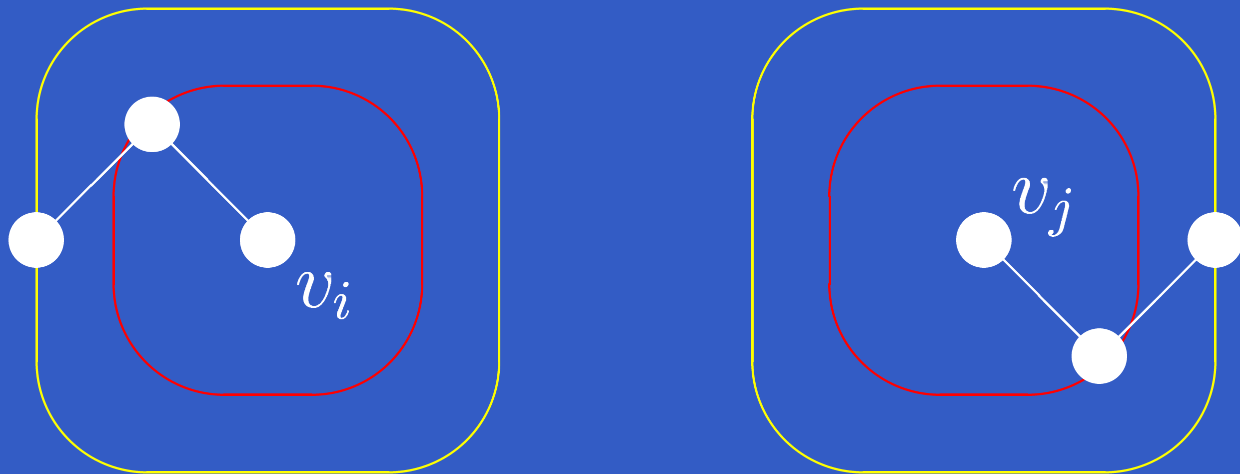
- There is an edge in any pair $(N(v_i), N(v_j))$ because $m(n) \to \infty$.
- So, for vertices u and w, the worst case is if they are in second neighborhoods of different v_i.

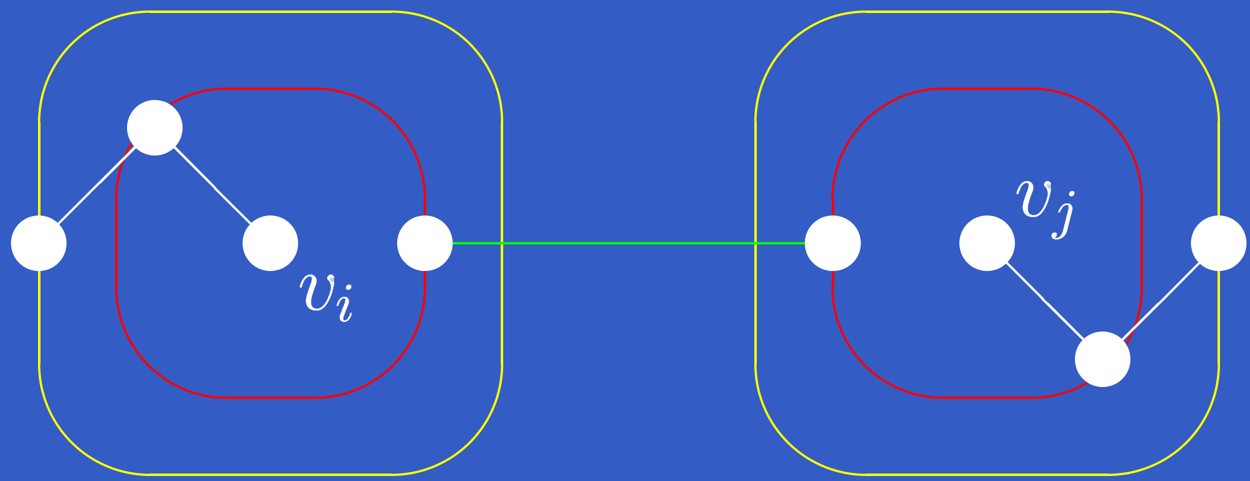
Diagram

First neighborhoods are red ovals.
Second neighborhoods are yellow ovals.



Diagram

A random edge is in $(N(v_i), N(v_j))$.



Diagram

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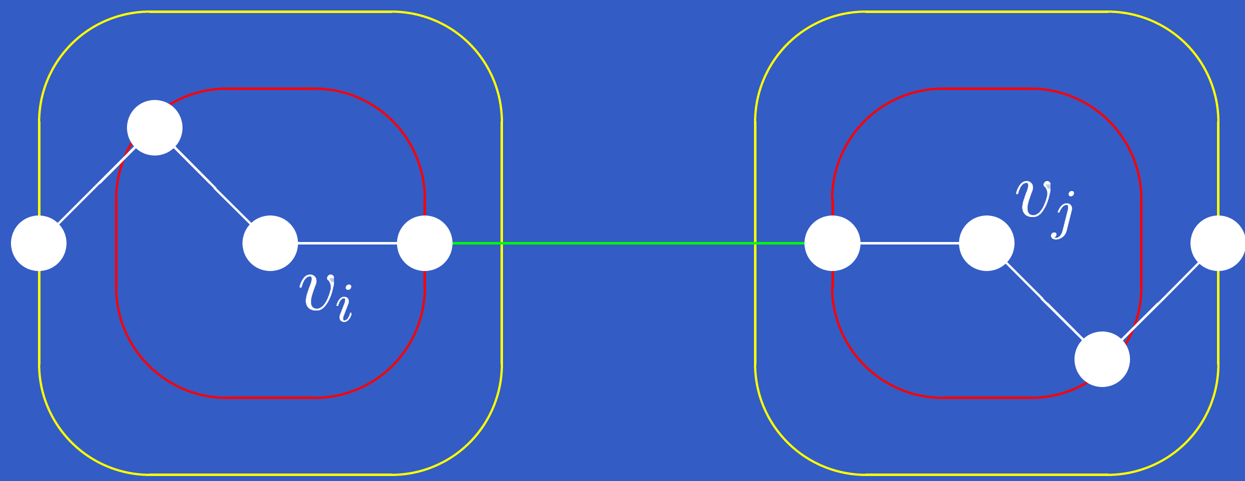
Voilá! A path of length 7.



Infinite edges

We can do better with Regularity:

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Small world problem

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He asked 160 families in Omaha, NE, to get a package to a stockbroker in Boston who had a home in Sharon, MA.

It had to be mailed or carried by hand from one acquaintance to the next, until it was delivered in person to the broker. Before the experiment began, Milgram asked his colleagues how many steps they thought it would take for the packages to make the trip.

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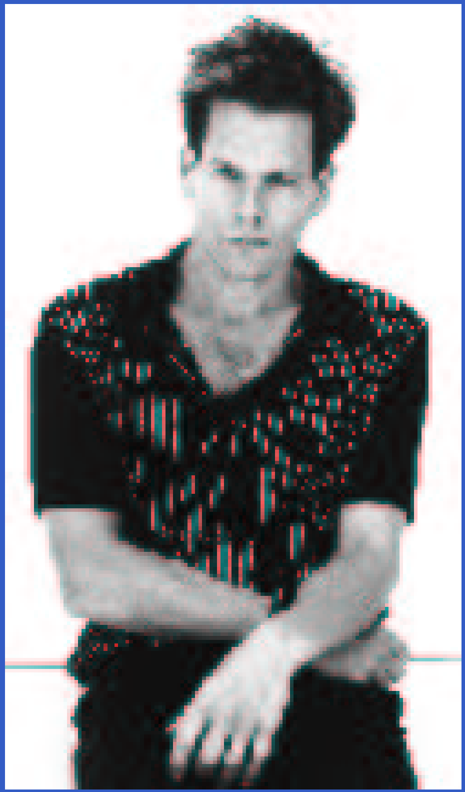
– the "six degrees of separation" that we’re familiar with today.

Popular culture

We've seen similar phenomena in the Kevin Bacon game.



Popular culture

Also in the Erdős number project.

Our answer

So, if familiarity grows proportionally with the size of the population and the amount of randomness in the system grows also, then

Our answer

So, if familiarity grows proportionally with the size of the population and the amount of randomness in the system grows also, then

It's FIVE degrees of separation, not six!

Thanks

Thank you for letting me talk today.

The file for this talk is available online at my website:
http://www.math.cmu.edu/~rymartin
These slides were created by the Prosper document preparation system.