ALEKSANDROV-BAKELMAN-PUCCI TYPE ESTIMATES FOR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this work we provide an Aleksandrov-Bakelman-Pucci type estimate for a certain class of fully nonlinear elliptic integro-differential equations, the proof of which relies on an appropriate generalization of the convex envelope to a nonlocal, fractional-order setting and on the use of Riesz potentials to interpret second derivatives as fractional order operators. This particular elliptic family under consideration is large enough to capture the second order theory as the order of the integro-differential equations tends to 2. Moreover, our estimate is uniform in the order of the equations, resulting in a genuine extension of the existing ABP estimate. This result also gives a new comparison theorem for viscosity solutions of such equations which only depends on the L^∞ and L^n norms of the right hand side, in contrast to previous comparison results which utilize the continuity of the right hand side for their conclusions. These results appear to be new even for the linear case of the relevant equations.

1. Introduction

We begin this work with a very basic question to motivate our results. Suppose that L is a uniformly elliptic operator (L could be a second order or integro-differential operator) and that u_k are appropriate weak solutions (read viscosity solutions) of the equations

$$\begin{cases} L(u_k, x) = f_k(x) & \text{in } B_1 \\ u = 0 & \text{on } \mathbb{R}^n \setminus B_1, \end{cases}$$

with the additional assumption that $0 \leq f_k \leq 1$. Then we pose the following question:

Question 1.1. *Under what conditions will it be true that* $|\{x : f_k(x) > 0\}| \to 0$ *as* $k \to \infty$ *also implies that* $\|u_k\|_{L^\infty} \to 0$*?*

(We note that the constant 0 function is a supersolution, and so always $u_k \leq 0$.)

In the case that L is a second order, uniformly elliptic operator,

$$L(u, x) = a_{ij}(x) u_{x_i x_j}(x),$$

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(for $\lambda \text{Id} \leq (a_{ij})_{i,j=1}^n \leq \Lambda \text{Id}$) the answer to the above question is indeed affirmative and is given by the celebrated Aleksandrov-Bakelman-Pucci (ABP) estimate which states that

$$-\inf_{B_1}\{u_k\} \leq \frac{C(n)}{\lambda}\|f_k\|_{L^n}$$

(where the equation is set in $B_1 \subset \mathbb{R}^n$).

The current understanding of this question is dramatically different, and not many results are known, when L is a uniformly elliptic integro-differential operator, namely

$$L(u,x) = \int_{\mathbb{R}^n} (u(x+y) - u(x) - Du(x)\cdot y \mathbb{1}_{|y|\leq 1}(y))K(x,y)dy,$$

$$\text{where } \lambda \leq K(x,y)|y|^{n+\sigma} \leq \Lambda \ \forall\, x, y \in \mathbb{R}^n.$$

In fact, using current results the only occasions in which something could be said about the possibility of $\|u_k\|_{L^\infty} \to 0$ would be when an explicit Green's function for the operator L in B_1 is known to exist. For many applications and also the possibility of treating nonlinear equations, this is an unsatisfactory answer to Question 1.1.

In this work, we present a new Aleksandrov-Bakelman-Pucci (ABP for short) type estimate for subsolutions and supersolutions of integro-differential equations for particular kernels $K(x,y)$. Namely, the kernels for which the result applies include (see Remark 1.4)

$$K(x,y) = (2-\sigma)\frac{y^T A(x) y}{|y|^{n+\sigma+2}} \quad \text{for} \quad \sigma \in (0,2),$$

where $A(x)$ satisfies the ellipticity condition $\lambda \text{Id} \leq A(x) \leq \Lambda \text{Id}\ \forall\, x$ (and is only measurable in x). Since these kernels are symmetric in y, we may rewrite $L(u,x)$ as

$$L(u,x) = (2-\sigma)\int_{\mathbb{R}^n} \delta u(x,y)\frac{y^T A(x) y}{|y|^{n+\sigma+2}}dy, \quad \sigma \in (0,2) \tag{1.1}$$

where we use the notation for second differences as

$$\delta u(x,y) := u(x+y) + u(x-y) - 2u(x).$$

Moreover, our result also covers fully nonlinear operators, such as

$$F(u,x) = \sup_\beta \inf_\alpha \{L_{\alpha\beta}(u,x)\} \tag{1.2}$$

where each $L_{\alpha\beta}$ is as in (1.1), with fixed σ. It gives new results for the Dirichlet problem:

$$\begin{cases} F(u,x) = f(x) & \text{in } D \\ u = g & \text{on } \mathbb{R}^n \setminus D, \end{cases} \tag{1.3}$$

where F is given by (1.2), g is continuous, and D is an open, bounded domain.

The ABP estimate is not only linked to convergence questions as explained above, but it also plays an important role in the regularity theory for solutions of equations related to (1.1)-(1.3). In particular, Theorem 1.2, below, implies both new comparison results for solutions of (1.3) and gives a new proof of the recent Hölder regularity results of [17, Section 12] for the operators which are in the class exemplified by (1.1).

The paradigm of ellipticity taken in [17, Section 3] (and [20, Chapter 2 and 5] for second order theory) is that F in (1.3) is considered elliptic if there are minimal and maximal operators, M^- and M^+, such that

$$M^-(u-v) \leq F(u,x) - F(v,x) \leq M^+(u-v).$$

In this work, we use a more restricted version of these minimal/maximal operators than those in [17, Section 3], which is reflected in the list of kernels covered as (1.1) being much smaller than those considered in [17]. Our extremal operators are defined as

$$M^-(u,x) = \inf_{\lambda \mathrm{Id} \leq A \leq \Lambda \mathrm{Id}} \left\{ (2-\sigma) \int_{\mathbb{R}^n} \delta u(x,y) \frac{y^T A y}{|y|^{n+\sigma+2}} dy \right\} \tag{1.4}$$

and

$$M^+(u,x) = \sup_{\lambda \mathrm{Id} \leq A \leq \Lambda \mathrm{Id}} \left\{ (2-\sigma) \int_{\mathbb{R}^n} \delta u(x,y) \frac{y^T A y}{|y|^{n+\sigma+2}} dy \right\}. \tag{1.5}$$

Supersolutions (respectively subsolutions) to (1.3) for these F in the ellipticity family are characterized by the fact that they are automatically supersolutions of a minimal (respectively maximal) equation. To this end, we assume throughout this note u to be a viscosity supersolution of a minimal equation with variable right hand side:

$$\begin{cases} M^-(u,x) \leq f(x) & \text{in } B_1 \\ u \geq 0 & \text{on } \mathbb{R}^n \setminus B_1, \end{cases} \tag{1.6}$$

and without loss of generality we assume that $f \geq 0$.

The main contribution of this work is to provide estimates on the infimum of u in terms of measure theoretic quantities of f, in particular for (1.6) the L^∞ and L^n norms of f. The main result of this work is:

Theorem 1.2. *Assume that $u \in L^\infty(\mathbb{R}^n) \bigcap LSC(\mathbb{R}^n)$ solves (1.6) in the viscosity sense and that $f \in C(\bar{B}_1)$. Then there exists a constant, $C(n)$, such that*

$$-\inf_{B_1}\{u\} \leq \frac{C(n)}{\lambda} (\|f\|_{L^\infty(K_u)})^{(2-\sigma)/2} (\|f\|_{L^n(K_u)})^{\sigma/2},$$

where $K_u \subset B_1$ is the coincidence set between u and a special envelope of u, made precise in Section 3.

A few immediate remarks related to Theorem 1.2 are in order:

Remark 1.3. For the definitions and basic properties of viscosity solutions of (1.3) and (1.6), the reader should consult the works: [3], [5], and [17, Sections 1-5]. We emphasize the *sign convention* for subsolutions and supersolutions in this work corresponds to that of [17] and [20]. A more detailed history and presentation pertaining to viscosity solutions of first and second order equations can be found [24].

Remark 1.4. We would like to at least make an attempt to give a concise explanation as to the need for the peculiar restriction that M^- is of the form (1.4) and hence why we can only treat kernels as in (1.1). Roughly speaking, our approach will take the function u, which solves a σ-order equation with $\sigma < 2$, and take its Riesz potential to invert the order by an amount $2 - \sigma$. Then to the potential, say P, of our original u, we can apply familiar second order techniques. In particular, we use the formula for the determinant of $D^2P(x)$ for those x with $D^2P(x) \geq 0$:

$$\left(\det(D^2P(x))\right)^{1/n} = \frac{1}{n}\inf\{\mathrm{Tr}(AD^2P(x)) : A \geq 0 \text{ and } \det(A) = 1\}.$$

When this computation is translated back to the original u, the derivatives on P can actually be transfered to derivatives on the *kernel* used to construct P, which results in a σ-order operator containing the term

$$L_A(u,x) = (2-\sigma)\int_{\mathbb{R}^n} \delta u(x,y)\frac{y^T A(x) y}{|y|^{n+\sigma+2}}dy.$$

All of these steps implemented carefully comprise Sections 3 - 5.

The analysis of fractional order integro-differential equations has gained much attention lately (see Section 2). It seems surprising, however, that despite so much interest, such estimates as those in Theorem 1.2 have not been obtained thus far. At least one need for an ABP type estimate for nonlocal equations was noted in [35, Remark 3.3] regarding the ability to prove homogenization for stationary ergodic families of integro-differential equations via the methods of [21]. This stochastic homogenization result will be presented in [34]. Other potential applications will be briefly discussed in Section 11. In [17, Section 8] an ABP type result was proved, but it involves the maximum of f over a *finite collection of cubes*, representing a Riemann Sum approximation to the usual ABP of second order theory. The result in [17, Section 8] was sufficient for the purposes of regularity theory (for very general kernels), nevertheless it has not been sufficient to answer Question 1.1, which is needed for the stochastic homogenization.

To prove Theorem 1.2 we introduce new machinery, specifically a nonlocal and fractional order replacement for the convex envelope and the Monge-Ampère operator of a Riesz potential of u, which can be expressed as a nonlinear integro-differential operator in u itself. This auxiliary operator makes up for current lack of a definitive analogue of the Monge-Ampère operator (which has both a divergence and a non-divergence structure), which has been and continues to be a significant obstacle to understanding the more geometric aspects of the regularity theory for nonlinear integro-differential equations.

We conclude this introduction with a brief outline of the sections of the paper. The two new tools mentioned in the previous paragraph, a generalization of the convex envelope and the auxiliary Monge Ampère operator, are explained in Sections 3 and 5 respectively. This new σ-order envelope solves a σ-order equation, just as in the second order setting, [20, Chapter 3], with the convex envelope and a second order equation. However, in order to gain access to familiar geometric arguments involving the usual convex envelope, we must push the order of the envelope and its equation up to 2. This is done by taking the Riesz potential of the envelope, and its properties are presented in Section 4, and in Section 5 we develop the Monge-Ampère operator of the potential as a σ-order operator acting on the envelope. In Section 6 we provide the final details of Theorem 1.2. In Sections 7, 8, 9, and 10 we present discussions involving respectively: the auxiliary operator of Monge-Ampère type, limits of these results as $\sigma \to 2$, other useful theorems related to Theorem 1.2 including comparison, and a proof of the L^ε estimates of elliptic regularity theory which nearly identically follows the classical one of [20, Lemmas 4.5 and 4.6], from which the Harnack inequality follows easily.

2. Background and Main Ideas

2.1. **Historical Background.** Analysis of integro-differential equations is by no means a new field. It is intimately linked, via their infinitesimal generators, with modeling involving Lévy or Lévy-Ito processes– which are much richer and more general than diffusion processes giving rise to second order non-divergence equations. A key feature intrinsic in modeling with Lévy-Ito processes and integro-differential equations is that they allow for *long range* interactions of various forms, and that the underlying stochastic processes can have jumps– in contrast to diffusion processes, which are continuous. There are *many* recent applications of these processes and their generators, and we list only a characteristic few: for particle systems and their hydrodynamic limit [27], for financial modeling [22] and [23], for optimal control (also related to financial modeling) [33], [39], and [38], for image processing [29].

There has been recently a growing interest in the analysis of integro-differential equations. The renewed interest seems not only due to the importance in modeling (Lévy processes have been around for a very long time) but to advances in both the probabilistic and partial differential equation analysis for treatment of equations related to (1.3) and (1.6). Here we list only a few references for recent advances, and suggest the interested reader to consult those references contained within, as a complete list would be impossible. On the analysis/PDE side, integro-differential equations were brought to the viscosity solutions framework in [39], [38], and later in [2], and the comparison theory for these equations was recently improved and re-presented in [5], with the Dirichlet problem being considered in [3]. Hölder regularity issues were considered in [37] for "non-divergence" form equations (cf. [37, Section 3.6] for a discussion of "divergence" versus "non-divergence" in this context), and a series of works making a uniform theory for both Hölder and higher regularity of more general versions of (1.3) was done in [15], [16], and [17]. Also by PDE methods, parabolic regularity was obtained for divergence form equations in [14]. Regularity for equations related to (1.3) was investigated by different, but still PDE, methods in [4]. Probabilistic analysis was used to investigate many integro-differential equations in both "divergence" and "non-divergence" forms to obtain various important results including Hölder regularity, Harnack inequalities, and other finer properties in [1], [6], [7], [8], [9], and [10].

2.2. **Another Simple Motivation For (1.6) and Theorem 1.2.** Let us now setup another very simple question related to that posed in the Introduction. It arises in the study of obstacle problems, and was utilized in [21] for homogenization (see [35] for the same use in a nonlocal setting). Consider the solution of an elliptic equation,

$$\begin{cases} F(v,x)=0 & \text{in } B_1 \\ v=0 & \text{on } \mathbb{R}^n\setminus B_1 \end{cases} \tag{2.1}$$

and the solution to the corresponding obstacle problem with the same operator, u, solving

$$u=\sup\{\phi : F(\phi,x)\geq 0 \text{ in } B_1 \text{ and } \phi\leq 0 \text{ in all of } \mathbb{R}^n\}. \tag{2.2}$$

A reasonable question is: what is the difference between u and v? Naturally, since u is a subsolution of the equation for v and they share the same data in $\mathbb{R}^n\setminus B_1$, then at least we can conclude $v\geq u$. But what about the reverse inequality? The defining feature of elliptic equations is that the maximal and minimal operators are used when investigating the difference of two solutions (cf. [17, Definition 3.1] and [20, Chapters 2 and 5]), namely

$$M^-(u-v,x) \leq F(u,x) - F(v,x).$$

Two of the key properties of the solutions to obstacle problems is that u is actually a *solution* to $F(u,x)=0$ whenever $u(x)\neq 0$, and that u inherits the equation from 0 when it does happen that $u(x)=0$, which is $F(u,x)\leq F(0,x)$ (here we mean the operator applied to the constant, 0, function). Therefore, the relevant inequalities are

$$M^-(u-v,x) \leq F(u,x) - F(v,x) \leq \mathbb{1}_{\{u=0\}}(x)F(0,x),$$

which by the boundedness of $F(0,x)\leq C$ becomes

$$\begin{cases} M^-(u-v,x) \leq C\mathbb{1}_{\{u=0\}}(x) & \text{in } B_1 \\ u-v=0 & \text{on } \mathbb{R}^n\setminus B_1. \end{cases} \tag{2.3}$$

After considering an f which is a continuous approximation from above of $\mathbb{1}_{\{u=0\}}(x)$, one can clearly see the question at hand and the motivation for this work: How can (1.6) be treated in a way which only depends upon *measure theoretic* properties of f and not *continuity* properties of f?

This leads us directly back to Question 1.1. The natural setting for the homogenization problem will involve a sequence of u_k and v_k as above with the additional information that $|\{u_k=0\}|\to 0$ as $k\to\infty$. Before this work was completed, there seemed to be very little results, in fact the authors found none, which could handle a situation in which the right hand side of (1.6) would converge in any sense other than in L^∞ (the relevant results in both [3] and [17] strongly require f to be continuous and converge to 0 in L^∞ to conclude similar statements). At least within the restricted class of operators presented here, Theorem 1.2 answers this question, which was not possible with previous known results.

2.3. **Theorem 1.2 As a Proof of Concept For Broader Results.** As hinted in the introduction, the full class of nonlinear nonlocal elliptic operators is much richer than simply those given in our definition of M^-. In [17], there is a more extremal version of M^- than the one appearing in (1.4); it is given as

$$M^-_{CS}(v,x) = \int_{\mathbb{R}^n}(2-\sigma)\left(\lambda\frac{(\delta v(x,y))^+}{|y|^{n+\sigma}} - \Lambda\frac{(\delta v(x,y))^-}{|y|^{n+\sigma}}\right)dy \tag{2.4}$$

$$= \inf_{\lambda\leq a(x,y)\leq\Lambda}\left\{(2-\sigma)\int_{\mathbb{R}^n}\delta v(x,y)\frac{a(x,y)}{|y|^{n+\sigma+2}}dy\right\}. \tag{2.5}$$

Hence the results of [17] correspond to a much larger family of equations than the one here. To go even further, as done in the linear case considered in [8], one could use this same definition with measures, $n(x,dy)$, "comparable" in a less restrictive sense to $|y|^{-n-\sigma}dy$ as opposed to only those measures with a density as above, $n(x,dy)=a(x,y)|y|^{-n-\sigma}dy$, for a uniformly bounded from below and above.

A reasonable guess for an ABP type theorem applying to (1.6) in these classes above would be

$$-\inf_{B_1}(u) \leq \frac{C}{\lambda}\|f\|_{L^p}, \tag{2.6}$$

for some $p > n$, depending on σ. At the time [17] and [35] were completed, it was not known whether or not such a result *should* be true for the general nonlocal ellipticity class or even a restricted class such as the one considered here. Now Theorem 1.2 indicates that at least some form of ABP type result holds for a restricted class of operators. This gives hope that ABP type estimates, such as those in Theorem 1.2, for the general class of nonlocal elliptic equations may still be true. Furthermore, it opens to door to answering the question of whether or not (2.6) is appropriate to expect for (1.6). The moral of the story is Theorem 1.2 indicates it is not that ABP type *results* are incompatible with the intrinsic properties of (1.6), but more importantly that the *existing machinery* is incompatible with (1.6). This illuminates one of the main difficulties in analysis of nonlocal equations: there is no known general framework to take the place of the very important Monge-Ampère operator and the gradient mapping of convex functions in the second order theory.

2.4. **Failure of The Convex Envelope.** For the convex envelope, say Γ, the information in the proof of the original second order ABP estimate is naturally encoded in the set $\{x : \det(D^2\Gamma(x)) > 0\}$. This matches very well with second order equations because $\det(D^2\Gamma) = 0$ whenever Γ and u do not coincide, and on the set where they do coincide, Γ inherits the supersolution property of u simply by comparison. In the nonlocal setting, the function $w = |x|^\alpha - 1$ will solve (1.6) for $\alpha > \sigma$ with a right hand side, f, which will still be bounded and continuous. However for $\sigma < \alpha < 1$, the *convex* envelope of w only coincides with w at one point, $x = 0$. Hence there would be a contradiction with the usual ABP estimate which would read:

$$-\inf_{B_1}\{w\} \leq C\|f\|_{L^n(\{w=\Gamma\})} = 0.$$

The problem here is exactly that such fractional order equations allow for much more drastic bending of supersolutions than is possible in a second order setting. Therefore, the convex envelope is not well suited for a measure theoretic estimate, such as Theorem 1.2. For purposes of studying the regularity of (1.3) with even more general operators than the F appearing in (1.2), the convex envelope was sufficient and strongly used in [17, Section 8]. In contrast to [17], in this work we must construct a different envelope which will be better suited to handle such a function as w, above. This will be the content of Section 3

2.5. **Main Ideas and Sketch of The Proof.** Here we give a brief sketch of how Theorem 1.2 is proved. The main ideas are the same as in the second order theory (cf. [20, Chapter 3] or [28, Chapter 9]), but the machinery and implementation are a bit more involved.

Everything starts with an appropriate envelope of u *from below*, which we will denote as Γ. It must be such that

$$\inf\{u\} = \inf\{\Gamma\}. \tag{2.7}$$

The first key feature is the existence of an operator, which we denote as $\mathcal{D}_\sigma$, such that

$$(-\inf\Gamma)^p \leq C\int_{B_3}(\mathcal{D}_\sigma(\Gamma, x))^p\,dx \tag{2.8}$$

for some p possibly depending on n and σ. At this point for the second order theory, we would have Γ as the convex envelope of u and $\mathcal{D}_\sigma$ would be $\det(D^2\Gamma)$, in which case the previous inequality is a consequence of *the geometry* of convex functions (what is known as

"Aleksandrov's estimate"). The second key feature is that the operator $\mathcal{D}_\sigma$ must satisfy *for a special class of* Γ,

$$\mathcal{D}_\sigma(\Gamma, x) \leq CM^-(\Gamma, x). \tag{2.9}$$

Finally, the third key feature is that

$$\mathcal{D}_\sigma(\Gamma, x) \leq 0 \text{ whenever } \Gamma(x) \neq u(x). \tag{2.10}$$

This is essential so that all the contribution of $\mathcal{D}_\sigma(\Gamma)$ can be ignored except for the contact set between u and Γ. This way, information about $M^-(u,x) \leq f(x)$ in the viscosity sense can be carried over to Γ via comparison, and the other values of $\mathcal{D}_\sigma(\Gamma)$ will not pollute the integral in the estimate (2.8). Essentially Γ acts as a test function on u, and the defining feature of viscosity solutions is that at those points where Γ touches u from below,

$$M^-(\Gamma, x) \leq f(x). \tag{2.11}$$

If all of (2.7), (2.8), (2.9), (2.10), and (2.11) can be satisfied (which is very nontrivial), then Theorem 1.2 can be proved.

It turns out to be quite difficult to simultaneously achieve all three of the key features, (2.8), (2.9), and (2.10). This delicate balance is what leads to Theorem 1.2 only being proved for a restricted class of operators, instead of the much more general class of [17].

2.6. **Notation.** We list here some notation which will be used throughout this work.

(1) The second difference operator: $\delta v(x,y) := v(x+y) + v(x-y) - 2v(x)$
(2) The $n-1$ dimensional sphere $S^{n-1} = \partial B_1 \subset \mathbb{R}^n$ and its surface area ω_n
(3) The complement of a set, $A^c = \mathbb{R}^n \setminus A$
(4) The following universal constant will appear often

$$A(n,\alpha) = \pi^{-\alpha-\frac{n}{2}} \frac{\Gamma(\frac{n+\alpha}{2})}{\Gamma(-\frac{\alpha}{2})}$$

(5) The inf-convolution of a function v, $v_\varepsilon(x) := \inf_y\{u(y) + (2\varepsilon)^{-1}|x-y|^2\}$
(6) The **one dimensional Fractional Laplacian in the direction** $\tau \in \mathbb{S}^{n-1}$,

$$-(-\Delta)_\tau^{\sigma/2} v(x) := A(n, 2-\sigma) \int_{\mathbb{R}} (\delta v(x, s\tau)) |s|^{-1-\sigma}\, ds$$

(7) v is $C^{1,1}$ from above at x (respectively from below) [17, Definition 2.1] if there exists a radius r, a vector p, and a constant M such that for all $|y| \leq r$

$$v(x+y) - v(x) - p\cdot x \leq M|y|^2$$

(respectively $v(x+y) - v(x) - p\cdot x \geq M|y|^2$)
(8) Contact Set between u and its envelope, Γ (defined in Definition 3.5),

$$K_u := \{x : u(x) = \Gamma(x)\}$$

(9) The convex envelope in B_R, v^{CE}, for a function $v \geq 0$ in $\mathbb{R}^n \setminus B_R$ is

$$v^{CE}(x) = \sup\{l(x) : l \text{ affine and } l \leq v \text{ in } B_R\}$$

(10) The extremal operators and ellipticity constants for the family governed by L of (1.1) are M^- and M^+ defined in (1.4) and (1.5).

3. The Fractional Envelope

This section is dedicated to constructing a new envelope for the supersolution, u, of (1.6) which will be essential to proving Theorem 1.2. The main idea is to imitate the most important features of the convex envelope as they pertain to the second order theory, cf. [20, Chapter 3] and [12, Proposition 2.12 and Appendix A] for the fully nonlinear version and [28, Chapter 9, Section 1] for the linear version. In fact, we will construct an envelope which is a fractional-order, nonlocal generalization of the convex envelope in the sense that the envelope constructed here will converge to the original convex envelope as $\sigma \to 2$. The goal is for the envelope to be in a class of functions for which M^- is comparable to a nonlocal version of the Monge-Ampère operator, and also to cause this nonlocal Monge-Ampère operator to vanish whenever u and its envelope do not touch. For the sake of explanation, let Γ be the convex envelope of u and $\mathcal{M}^-$ to be the second order minimal Pucci operator (cf. [20, Chapter 2, Section 2]). For Γ, the two requirements just mentioned above correspond to the two facts implied by convexity and the envelope property:

$$\prod_{k=1}^{n} e_k \leq \left(\sum_{k=1}^{n} e_k\right)^n \leq \left(\frac{1}{\lambda}\mathcal{M}^-(\Gamma)\right)^n \quad \text{for the eigenvalues, } e_k, \text{ of } D^2\Gamma$$

and

$$\det(D^2\Gamma(x)) = 0 \quad \text{if } u(x) \neq \Gamma(x).$$

The appropriate analogs to our new envelope appear subsequently as Lemmas 5.5 and 5.8.

In order to define the new envelope, we introduce a matrix-valued integro-differential operator, which will play the (auxiliary) role of an "integro-differential Hessian".

Definition 3.1. *Let v be a bounded function which is $C^{1,1}(x)$, then we define*

$$h_\sigma(v,x) := A(n, 2-\sigma)\int_{\mathbb{R}^n} \frac{y \otimes y}{|y|^{n+\sigma+2}}\delta v(x,y)dy \tag{3.1}$$

The use of this matrix is dictated by our approach based on Riesz potentials (see Section 4). By this we mean the following, if $\alpha = 2-\sigma$ and v is smooth enough then (see Lemma 5.1)

$$D^2(v * |x|^{-n-\alpha}) = c(n,\sigma)h_\sigma(v,x) + \left\{(-\Delta)^{\sigma/2}v(x)\right\}\tilde{c}(n,\sigma)\ \mathrm{Id}$$

Remark 3.2. It is immediate that if u touches v from above at x and they are both smooth enough then we have the matrix inequality $h_\sigma(u,x) \geq h_\sigma(v,x)$.

Remark 3.3. As $\sigma \to 2^-$ (and thus $\alpha \to 0$), the above identity gives us

$$h_\sigma(v,x) \to c_n D^2v(x) + \tilde{c}_n\Delta v(x)\mathrm{Id}$$

Remark 3.4. Moreover, the fully nonlinear integro-differential operators that our methods handle are exactly those that can be written in the form

$$I(u,x) = F(h_\sigma(u,x))$$

where F is a function defined on the space of symmetric matrices and which is elliptic in the usual sense.

Moving forward, the *"σ-order envelope"* of u is defined as the solution of an obstacle problem which is a generalization of the obstacle problem satisfied by the convex envelope:

Definition 3.5.

$$\Gamma_u^\sigma(x) = \sup\left\{v(x) : E_\sigma(v) \geq 0 \ \textit{in } B_3, \ \textit{and } v \leq u \mathbb{1}_{B_3} \ \textit{in } \mathbb{R}^n\right\}. \tag{3.2}$$

Here we define E_σ as a fractional order replacement for the smallest eigenvalue operator on the Hessian for second order equations:

$$E_\sigma(v,x) := \lambda_1\{h_\sigma(v,x)\} = \inf_{\tau \in S^{n-1}} \{(h_\sigma(v,x)\ \tau)\cdot\tau\}. \tag{3.3}$$

By remark 3.3, this does not recover the smallest eigenvalue of the Hessian of u in the limit $\sigma \to 2$, instead

$$\lim_{\sigma\to 2} E_\sigma(u,x) = C_n\left(\lambda_1(D^2u(x)) + \frac{1}{n+2}\Delta u(x)\right)$$

In particular, as $\sigma \to 2^-$ the envelope Γ_u^σ converges to the solution of the upper obstacle problem for the operator above with u as the upper obstacle. The solution to this problem lies above the convex envelope of u but it does not agree with it, however, the estimates we obtain are uniform in σ and converge to the old ABP as $\sigma \to 2$.

Remark 3.6. The usual convex envelope used in the second order theory has an analogous structure as the solution to an obstacle problem. The interested reader should consult [32] for an exposition.

Remark 3.7. All of the results of this section hold (with small modifications) if instead we use

$$E^*_\sigma(u,x) := \inf_{\tau\in S^{n-1}} \{-(-\Delta)^\sigma_\tau u(x)\}$$

Here $(-\Delta)^\sigma_\tau u$ is the one dimensional fractional Laplacian of u defined in Section 2.6. This is a much more drastic notion of envelope (such an envelope would be below the one defined using E_σ, and touch u on a much smaller domain, which can be problematic). In principle, it is perfectly tailored to handle much more general kernels:

$$K(y) = \frac{a(y)}{|y|^{n+\sigma}} \ \text{ where } \ a(ry) = a(y)\ \forall\, r \in \mathbb{R}.$$

However, it is not yet clear how one can go about proving L^∞ bounds for this envelope in terms of a "convenient" integral quantity (i.e. one that can be controlled by the $M^-(\Gamma)$). In contrast, the envelope we use admits integral bounds granted by its compatibility with the Riesz potential. Whether this argument work for the more "drastic" envelope is not clear and perhaps an entirely different approach is need, this question will be addressed in future work.

Remark 3.8. Some nonlinear one directional operators related to this alternative operator E^*_σ have also been considered in [11]. In particular, the one dimensional Fractional Laplacian in the direction of ∇u gives rise to a natural integro-differential analogue of the Infinity Laplacian.

For the remainder of this section, some properties of Γ_u^σ will be collected for later use. Also, we will dispense with the notation Γ_u^σ and instead simply use Γ except in special cases which will be appropriately noted.

In proving Theorem 1.2, it will be useful to be able to work with Γ as a classical solution to various inequalities to follow instead of only as a viscosity solution. The beauty of fractional order elliptic equations is that $C^{1,1}$ is already enough regularity to guarantee that a weak solution is classical, and this is much more convenient than in the second order setting. These results follow along the same lines as those in [17, Sections 3 and 4]. In anticipation of gaining the required regularity through an inf-convolution of u before the envelope is made, many of the statements in this section require u being $C^{1,1}$ from above. This regularization of u will be used in Section 6 and details of how Γ behaves with respect to the inf-convolution of u are presented at the end of this section.

Lemma 3.9 (Compact Support of Γ)**.** *$\Gamma = 0$ in $\mathbb{R}^n \setminus B_3$.*

Proof of Lemma 3.9. This is a result of the fact that the obstacle for Γ was truncated to be 0 outside of B_3, along with standard properties of solutions to obstacle problems in the viscosity sense. Basic arguments show that Γ can be attained as the limit of the penalized Dirichlet Problems:

$$\begin{cases} E_\sigma(v^\delta, x) = b(\dfrac{u - v^\delta}{\delta}) & \text{in } B_3 \\ v^\delta = 0 & \text{on } \mathbb{R}^n \setminus B_3, \end{cases} \tag{3.4}$$

with the function b chosen to be smooth, bounded, strictly monotone decreasing, and to satisfy

$$b(s) = C(u) \text{ for } s \leq 0$$
$$b(s) = 0 \text{ for } s \geq 0.$$

Here $C(u)$ is taken to be $C(u) = \sup_x\{E_\sigma(u, x)\}$. Further details can be found in [21, Section 2]. We note that there are some modifications required do to the fact that E_σ is only degenerate elliptic, but we do not discuss that here. □

Lemma 3.10 (Γ regularity from above)**.** *If u is $C^{1,1}$ from above, then Γ will be as well, with the same constant bounding the second differences.*

Proof of Lemma 3.10. This statement as it pertains to the operator, $((-\Delta)^{\sigma/2})$, was proved in [37, Proposition 3.10]. The result actually has nothing to do with the operator in particular (either E_σ or $(-\Delta)^{\sigma/2}$), but rather uses superadditivity of the operator, 1-homogeneity of the operator, constants evaluate to 0 in the operator, and also the fact that Γ is a supremum of functions which are below u. We provide some cursory details for the sake of completeness. Let $y \in \mathbb{R}^n$ be fixed. The $C^{1,1}$ estimate for u is recorded on the first line below, which can be translated into a bound for $\Gamma(x + y) + \Gamma(x - y)$ using the fact that $\Gamma \leq u$:

$$u(x + y) + u(x - y) \leq 2u(x) + C\,|y|^2$$
$$\Gamma(x + y) + \Gamma(x - y) \leq 2u(x) + C\,|y|^2$$
$$\frac{1}{2}\left(\Gamma(x + y) + \Gamma(x - y) - C\,|y|^2\right) \leq u(x)$$

The properties of the envelope and the operator E_σ allow us to carry this estimate over to a full $C^{1,1}$ from above estimate on Γ. Define the function v as

$$v(x) := \frac{1}{2}\left(\Gamma(x + y) + \Gamma(x - y) - C\,|y|^2\right)$$

Because E_σ is a superadditive and 1-homogeneous operator which evaluates to 0 on constants, we know that the function v will still be a subsolution of $E_\sigma v \geq 0$ which is also below u. Hence because Γ is the largest such subsolution,

$$v(x) \leq \Gamma(x)$$
$$\Gamma(x+y) + \Gamma(x-y) \leq 2\Gamma(x) + C\,|y|^2\,,$$

with the same constant, C, which was used for u.

□

The following fact is a straightforward consequence of the definition of M^- and E_σ, we state it as a lemma without proof.

Lemma 3.11 (Ordering of E_σ and M^-)**.** *For any $C^{1,1}(x)$ function, v,*

$$E_\sigma(v,x) \geq 0 \;\Rightarrow\; E_\sigma(v,x) \leq \frac{C(n)}{\lambda} M^-(v,x). \tag{3.5}$$

Lemma 3.12 (Almost Everywhere Equation For $E_\sigma(\Gamma)$)**.** *If u is $C^{1,1}$ from above, then for all x in B_3, $E_\sigma(\Gamma,x)$ is classically defined in the sense of absolutely convergent integrals with the bound*

$$E_\sigma(\Gamma,x) \geq 0, \tag{3.6}$$

and for almost every x satisfies

$$E_\sigma(\Gamma,x) \leq C(n)\mathbb{1}_{K_u}(x) M^-(u,x), \tag{3.7}$$

where $K_u = \{x : u(x) = \Gamma(x)\}$.

Proof of Lemma 3.12. First we will prove (3.6) holds classically for all x. Let $x_0 \in B_3$ be fixed. The fact that u and hence Γ by Lemma 3.10, are $C^{1,1}$ from above tells us that for some C large enough and r small enough, the function ϕ_r,

$$\phi_r(x) := \begin{cases} C\,|x-x_0|^2 + \Gamma(x_0) & \text{in } B_r(x_0) \\ \Gamma(x) & \text{on } \mathbb{R}^n \setminus B_r(x_0), \end{cases}$$

will be globally above Γ and touch at x_0. The function Γ is a viscosity subsolution of $E_\sigma(\Gamma) \geq 0$ because it is the supremum of classical (hence viscosity) subsolutions by definition. Thus for all r small enough, there is a global max of $\Gamma - \phi_r$ at x_0, and the function ϕ_r is a valid test function because it is $C^{1,1}$ in a neighborhood of x_0 (actually by [17, Lemma 4.3], punctually $C^{1,1}$ at x_0 is enough), we conclude that for all $e \in \mathbb{S}^{n-1}$:

$$A(n, 2-\sigma) \int_{\mathbb{R}^n} \frac{y_i y_j}{|y|^{n+\sigma+2}} \delta\phi_r(x_0,y) e_i e_j dy = (h_\sigma(\phi_r, y))_{ij} e_i e_j \geq 0.$$

Furthermore, we note that the sequence of functions (of the variable y), $\delta\phi_r(x_0,y)$, are bounded by above by

$$\delta\phi_r(x_0,y) \leq C\,|y|^2\, \mathbb{1}_{\{|y|<r_0\}}(t) + 4\|\Gamma\|_\infty \mathbb{1}_{\{|y|\geq r_0\}}(y),$$

for some $r_0 > 0$, and are decreasing to $\delta\Gamma(x_0,y)$. Therefore, by the Monotone Convergence Theorem we can conclude that $\delta\Gamma(x_0,y)\,|y|^{-n-\sigma}$is integrable and satisfies

$$\int_{\mathbb{R}^n} \frac{y_i y_j}{|y|^{n+\sigma+2}} \delta\phi_r(x_0, y) e_i e_j dy \geq 0.$$

Hence the infimum over e is also well defined and non-negative, which is exactly (3.6).

Now we turn to equation (3.7). We split the proof into two parts, both holding for almost every x:

$$E_\sigma(\Gamma, x) = 0 \text{ if } \Gamma(x) \neq u(x)$$

and

$$E_\sigma(\Gamma, x) \leq C(n) M^-(u, x) \text{ if } \Gamma(x) = u(x).$$

The proof of Lemma 3.9 implies that Γ is a viscosity solution of $E_\sigma(\Gamma, \cdot) = 0$ in the set $\{\Gamma \neq u\}$. Therefore, for any x such that Γ has a second order Taylor Expansion from below, we know that in fact Γ is punctually $C^{1,1}$ at x, by including the estimate from Lemma 3.10, and hence by [17, Lemma 4.3] the equation holds for Γ at x. The semiconcavity of Γ (Lemma 3.10) implies that Γ has a second order Taylor Expansion at almost every x (cf. [25, Chapter 6]). Thus (3.7) holds for the x such that $\Gamma(x) \neq u(x)$.

To prove the second part of (3.7), we note that at any x such that Γ has a second order order expansion and touches u from below, it follows that u is punctually $C^{1,1}$ as well. Thus both $E_\sigma(\Gamma, x)$ and $E_\sigma(u, x)$ are classically defined and ellipticity gives

$$E_\sigma(\Gamma, x) \leq E_\sigma(u, x).$$

Applying Lemma 3.11 gives the rest of (3.7).

□

Exactly as with Lemma 3.11, the following is a straightforward consequence of the definitions, and so we again omit the proof.

Lemma 3.13. *If u is $C^{1,1}$ from above, then for all $x \in B_3$, the operators on Γ are classically defined and satisfy for all $A \geq 0$:*

$$\int_{\mathbb{R}^n} \delta\Gamma(x, y) \frac{y^T A y}{|y|^{n+\sigma+2}} dy \geq 0.$$

Lemma 3.14. *If $u^\varepsilon \nearrow u$ as $\varepsilon \searrow 0$, then $\limsup K_\varepsilon \subset K$, where $K_\varepsilon = \{x : u_\varepsilon(x) = \Gamma_\varepsilon(x)\}$ and $K = \{x : u(x) = \Gamma(x)\}$.*

Proof of Lemma 3.14. The increasing property of u_ε implies for $\varepsilon_2 < \varepsilon_1$ that Γ_{ε_1} is an admissible subsolution below u_{ε_2}, and hence because Γ_{ε_2} is the largest such subsolution, $\Gamma_{\varepsilon_2} \geq \Gamma_{\varepsilon_1}$. Thus Γ_ε is also increasing, $\Gamma_\varepsilon \nearrow \Gamma$. Now suppose x is a point in the set $\limsup_{\varepsilon\to 0} K_\varepsilon$. We can extract a subsequence indexed by ε_k such that $u_{\varepsilon_k}(x) = \Gamma_{\varepsilon_k}(x)$.

It must be shown that $\Gamma(x) = u(x)$. We already know that $\Gamma(x) \leq u(x)$ by definition of Γ, so therefore we will show $\Gamma(x) - u(x) \geq 0$:

$$\begin{aligned} 0 &\geq \Gamma(x) - u(x) \\ &\geq \Gamma(x) - \Gamma_{\varepsilon_k}(x) + \Gamma_{\varepsilon_k}(x) - u_{\varepsilon_k}(x) + u_{\varepsilon_k}(x) - u(x) \\ &\geq 0 + 0 + u_{\varepsilon_k}(x) - u(x). \end{aligned}$$

Letting $k \to \infty$ ($\varepsilon_k \to 0$) yields the result. □

4. The Fractional-Order Potential

In the previous section we defined and presented some properties of an important envelope which will be used in the proof of Theorem 1.2. However, there are still many difficulties to be overcome in proving an ABP type estimate for (1.6), specifically one must relate the infimum of Γ to an integral of a quantity which is comparable to $M^-(\Gamma)$. In the second order setting (using a *convex* Γ), this is resolved by the very special geometry of the convex envelope which gives

$$-\inf\{\Gamma\} \leq C(n) \int_{B_3} \det(D^2\Gamma(x))dx.$$

A key feature in the proof is that we can transfer this argument to the Riesz potential of Γ, which will solve a second order equation. Moreover, the usual argument will result in looking at the gradient measure, $|\nabla P(B_3)|$, which is a nonlocal quantity of Γ. Inspiration for this choice came from the use of the Riesz potential in the setting of a nonlocal Porous Medium Equation treated in [18]. This method is also reminiscent of the regularity theory for the obstacle problem for the fractional Laplacian [37], which starts with the observation that the fractional Laplacian of the solution (of order σ) solved an equation of order $2-\sigma$.

As mentioned, we take our fractional-order potential to be the Riesz potential of the envelope (cf. [40, Chapter 5, Section 1] or [31, Ch. 1, Section 1]), which formally says

$$P = (-\Delta)^{-(2-\sigma)/2}\Gamma.$$

Precisely we mean for this to be the convolution with the Riesz kernel, $K_\alpha := A(n,\alpha)\left|\cdot\right|^{-n+\alpha}$,

$$P = \Gamma * K_\alpha, \tag{4.1}$$

for $\alpha = (2-\sigma)$, and the constant $A(n,\alpha)$ which is listed explicitly in section 2.6. Hence

$$(-\Delta)P = (-\Delta)^{\sigma/2}\Gamma \text{ and } (-\Delta)^{(2-\sigma)/2}P = \Gamma.$$

The whole motivation of using the potential, P, of order $(2-\sigma)/2$ is to make sure that the usual 2nd order Monge-Ampère operator, $\det(D^2P)$, becomes a σ-order operator when viewed as an operation on Γ. This way, there is hope that $\det(D^2P)$ will be comparable to $M^-\Gamma$, and familiar techniques from the proof of the second order ABP estimate can be used. Indeed, the form of $\det(D^2P)$ is investigated in Section 5 and will be shown to be a σ-order operator comparable to $M^-\Gamma$. However, some additional difficulties are introduced by using P instead of Γ itself. It will be necessary to compare the infimum of P to the infimum of Γ in a uniform fashion which does not depend on the continuity of Γ or u. This is not to be expected simply from the Riesz potential itself, but must be deduced from the equation satisfied by Γ. The relationship between P and Γ is the main result of this section, which is presented in Proposition 4.3.

To begin the results of this section, it will be helpful to remark upon the regularity of P pertaining to that of Γ. These results are presented in [36], and so we do not provide a proof. They all follow from either of the two equations involving P:

$$(-\Delta)^{(2-\sigma)/2}P = \Gamma$$

or

$$(-\Delta)P = (-\Delta)^{\sigma/2}\Gamma.$$

Lemma 4.1 (Proposition 2.9 of [36]). *Suppose that* Γ *is* $C^{1,1}$ *from above. The regularity on* P *inherited from* Γ *is*

i) *whenever* $(2-\sigma) \le 1$, $P \in C^{0,\alpha}$ *for any* $\alpha < (2-\sigma)$ *and*

$$\|P\|_{C^{0,\alpha}} \le C(\|P\|_\infty + \|\Gamma\|_\infty),$$

ii) *whenever* $(2-\sigma) > 1$, $P \in C^{1,\alpha}$ *for any* $\alpha < (2-\sigma) - 1$ *and*

$$\|P\|_{C^{1,\alpha}} \le C(\|P\|_\infty + \|\Gamma\|_\infty),$$

and

iii) $P \in C^{2,\gamma}$ *whenever* $\Gamma \in C^{1,1}(\mathbb{R}^n)$.

Now we proceed with the very interesting relationship between $\inf\{P\}$ and $\inf\{\Gamma\}$ which is imposed by the equation governing Γ.

Lemma 4.2. *Let* $0 < \sigma < 2$ *and* x_0 *be such that* $\Gamma(x_0) = \inf_{\mathbb{R}^n}\{\Gamma\}$. *Define the set*

$$A_{x_0} := \{y : \Gamma(x_0 + y) - \Gamma(x_0) \le f(x_0)\,|y|^\sigma\} \tag{4.2}$$

and rings with radius $r_k = \rho_0 2^{-k}$, $R_k = B_{r_k}(x_0) \setminus B_{r_{k+1}}$ *(where* ρ_0 *will be chosen later for the proof of Proposition 4.3). If the indices,* k^*, *indicate bad rings for which*

$$\left|A^c_{x_0} \bigcap R_k\right| \ge \frac{1}{2}\,|R_k|\,, \tag{4.3}$$

then the number of such indices satisfies

$$(\#k^*) \le \frac{2}{(2-\sigma)\lambda\omega_n}$$

Proof of Lemma 4.2. Because x_0 is a location of a global minimum of Γ which is also a contact point with u, we know that $M^-(\Gamma, x_0) \le f(x_0)$ by comparison with u. Furthermore, there are two things which result from $\delta\Gamma(x_0, y) \ge 0$: one, all of the operators appearing in the definition of M^- (equation (1.4)) are bounded below by the one corresponding to $\lambda\mathrm{Id}$, and two, we can neglect as much of the set of integration as we like in the evaluation of $M^-(\Gamma, x_0)$.

Therefore, evaluating (1.6) at x_0 and estimating as suggested above, we have

$$
\begin{aligned}
&f(x_0) \geq M^-(\Gamma, x_0) \\
&\geq \lambda \int_{\mathbb{R}^n} (2-\sigma)\delta\Gamma(x_0,y)\,|y|^{-n-\sigma}\,dy \\
&= \lambda \int_{\mathbb{R}^n} (2-\sigma)(\Gamma(x_0+y)-\Gamma(x_0))\,|y|^{-n-\sigma}\,dy + \lambda \int_{\mathbb{R}^n} (2-\sigma)(\Gamma(x_0-y)-\Gamma(x_0))\,|y|^{-n-\sigma}\,dy \\
&\geq \lambda \int_{A^c_{x_0}} (2-\sigma)(\Gamma(x_0+y)-\Gamma(x_0))\,|y|^{-n-\sigma}\,dy \\
&\geq \lambda \sum_{k^*} \int_{A^c_{x_0}\bigcap R_{k^*}} (2-\sigma)(\Gamma(x_0+y)-\Gamma(x_0))\,|y|^{-n-\sigma}\,dy \\
&\geq \lambda \sum_{k^*} \int_{A^c_{x_0}\bigcap R_{k^*}} (2-\sigma)f(x_0)\,|y|^{\sigma}\,|y|^{-n-\sigma}\,dy \\
&\geq \lambda f(x_0) \sum_{k^*} (2-\sigma) \left|A^c_{x_0}\bigcap R_{k^*}\right| r_{k^*+1}^{-n+\sigma-\sigma} \\
&\geq \lambda f(x_0) \sum_{k^*+1} (2-\sigma)\omega(n)(1-2^{-n}) \\
&\geq \lambda f(x_0)(2-\sigma)\frac{1}{2}\lambda\omega_n(\#k^*)
\end{aligned}
$$

Therefore, we conclude

$$(\#k^*) \leq \frac{2}{(2-\sigma)\lambda\omega_n}$$

□

Proposition 4.3. *Let x_0 be such that $\Gamma(x_0) = \inf_{\mathbb{R}^n}\{\Gamma\}$ and $\sigma \in (0,2)$. Then*

$$-\inf_{\mathbb{R}^n}\{P\} = -\inf_{B_3}\{P\} \geq C(n,\lambda)(-\Gamma(x_0))^{2/\sigma}\left(\frac{1}{2f(x_0)}\right)^{(2-\sigma)/\sigma}.$$

Proof of Proposition 4.3. Once we choose an appropriate ρ_0 from Lemma 4.2, this proposition will follow directly from the definition of P. We would like to restrict the beginning radius, ρ_0, so that whenever $y \in A_{x_0}\bigcap R_{\rho_0}$, y also satisfies (recall Γ is always negative)

$$\Gamma(x_0+y) \leq \frac{1}{2}\Gamma(x_0).$$

This is guaranteed if we let

$$\rho_0 = \left(\frac{-\Gamma(x_0)}{2f(x_0)}\right)^{1/\sigma},$$

which gives for $|y| \leq \rho_0$ and $y \in A_{x_0}$

$$\Gamma(x_0+y) - \Gamma(x_0) \leq f(x_0)\,|y|^{\sigma} \leq \frac{-1}{2}\Gamma(x_0).$$

Also in the estimates to follow, it will be very useful to observe that if $\{k_g\}$ are the collection of indices resulting in good rings satisfying

$$\left|A_{x_0}\bigcap R_k\right| \geq \frac{1}{2}\,|R_k|\,,$$

and $N(x_0)$ is the upper bound on the bad indices, $(\#k^*)$, where the collection $\{k^*\}$ are the ones satisfying (4.3), then

$$\sum_{k_g} r_{k_g}^{2-\sigma} \geq \sum_{k \geq N(x_0)} r_k^{2-\sigma}.$$

Recall also that

$$N(x_0) = \frac{2}{(2-\sigma)\lambda\omega_n}.$$

Now we have

$$\begin{aligned}
-P(x_0) &= A(n, 2-\sigma) \int_{\mathbb{R}^n} -\Gamma(x_0 + y) \, |y|^{-n+2-\sigma} \, dy \\
&\geq A(n, 2-\sigma) \sum_{k_g} \int_{A_{x_0} \bigcap R_{k_g}} -\frac{1}{2}\Gamma(x_0) \, |y|^{-n+2-\sigma} \, dy \\
&\geq A(n, 2-\sigma)(-\frac{1}{2}\Gamma(x_0)) \sum_{k_g} \left| A \bigcap R_{k_g} \right| r_{k_g+1}^{-n+2-\sigma} \\
&\geq A(n, 2-\sigma)(-\Gamma(x_0))C(n) \sum_{k_g} r_{k_g}^{2-\sigma} \\
&\geq A(n, 2-\sigma)(-\Gamma(x_0))C(n) \sum_{k \geq N(x_0)} r_k^{2-\sigma} \\
&= A(n, 2-\sigma)(-\Gamma(x_0))C(n)\rho_0^{2-\sigma} \sum_{k \geq N(x_0)} 2^{-k(2-\sigma)} \\
&= A(n, 2-\sigma)(-\Gamma(x_0))C(n)\rho_0^{2-\sigma} (2^{-(2-\sigma)})^{N(x_0)} \sum_{k \geq 0} (2^{-(2-\sigma)})^k \\
&\geq A(n, 2-\sigma)(-\Gamma(x_0))C(n)\rho_0^{2-\sigma} (2^{-2/(\lambda\omega_n)})(1 - 2^{-(2-\sigma)})^{-1}.
\end{aligned}$$

In the calculations above, the "constant", $C(n)$, was used multiple times as different numbers to absorb any extra constants which only depended on the dimension.

At this point we can collect the various information about ρ_0, $N(x_0)$. Observe that

$$\lim_{\sigma \to 2} A(n, 2-\sigma)(1 - 2^{(2-\sigma)})^{-1} = C(n),$$

which follows from the fact that $A(n, 2-\sigma) \sim 2-\sigma$ and $2^{2-\sigma} \sim 1$ as $\sigma \to 2$. We conclude

$$-P(x_0) \geq C(n) 2^{-2/(\lambda\omega_n)} (-\Gamma(x_0)) \rho_0^{(2-\sigma)}.$$

Recall that $\rho_0 = \left(\frac{-\Gamma(x_0)}{2f(x_0)}\right)^{1/\sigma}$ and that trivially $-P(x_0) \leq -\inf P$ so that

$$-\inf P \geq C(n) 2^{-2/(\lambda\omega_n)} (-\Gamma(x_0)) \left(\frac{-\Gamma(x_0)}{2f(x_0)}\right)^{(2-\sigma)/\sigma},$$

which proves the proposition. □

Remark 4.4. The proofs of Lemma 4.2 and Proposition 4.3 were in fact not special to the envelope property of Γ. They really only used the fact that $\delta\Gamma(x_0, y) \geq 0$ for all y and that the minimal equation inequality held at x_0. Therefore, the same results are also true for the function u itself, with the relevant potential now being $u * K_{(2-\sigma)}$ (as opposed to the one we used, $\Gamma * K_{(2-\sigma)}$). Moreover, the second difference sign, $\delta\Gamma(x_0, y)$ also holds for any x_0 which is in the coincidence set between u and its *convex* envelope taken in an appropriately sized ball (cf. [17, Section 8]).

There is another way of stating Proposition 4.3 which allows for a more intuitive interpretation as an interpolation result.

Corollary 4.5. *There exists $C(n,\lambda) > 0$ such that if σ, Γ and P are all as above, then:*

$$\|\Gamma\|_\infty \leq C(n)\|P\|_\infty^{2/\sigma}\ \|f\|_\infty^{(2-\sigma)/2}$$

Written in this form, the nature of Proposition 4.3 as an interpolation estimate becomes clear. To see why such an estimate is to be expected, we prove a closely related result.

Proposition 4.6. *Suppose σ, Γ and P are as before, then if P is $C^{1,1}(\mathbb{R}^n)$ we have*

$$\|\Gamma\|_\infty \leq C(n,\lambda)\|P\|_\infty^{2/\sigma}\ [P]_{C^{1,1}}^{(2-\sigma)/2}$$

Proof of Proposition 4.6. The proof is tailored after interpolation estimates for Hölder norms (for similar, but not identical statements, the interested reader should see [28, Chapter 6, Appendix]).

Since $\Gamma = (-\Delta)^{(2-\sigma)/2}P$ we always have (with $A = |A(n, -(2-\sigma))|$)

$$|\Gamma(x)| = A\left|\int_{\mathbb{R}^n}\frac{\delta P(x,y)}{|y|^{n+2-\sigma}}dy\right|$$

$$\leq A\left|\int_{B_\rho}\frac{\delta P(x,y)}{|y|^{n+2-\sigma}}dy\right| + A\left|\int_{B_\rho^c}\frac{\delta P(x,y)}{|y|^{n+2-\sigma}}dy\right| \quad \forall\rho > 0. \tag{4.4}$$

The second integral can be controlled as follows

$$\left|\int_{B_\rho^c}\frac{\delta P(x,y)}{|y|^{n+2-\sigma}}dy\right| \leq 2\|P\|_{L^\infty(\mathbb{R}^n)}\int_{B_\rho^c}\frac{1}{|y|^{n+2-\sigma}}dy = 2\omega_n\|P\|_{L^\infty(\mathbb{R}^n)}\int_\rho^{+\infty} r^{-3+\sigma}dr$$

$$= \|P\|_{L^\infty(\mathbb{R}^n)}\frac{2\omega_n}{2-\sigma}\rho^{-(2-\sigma)}. \tag{4.5}$$

For the first integral we have

$$\left|\int_{B_\rho}\frac{\delta P(x,y)}{|y|^{n+2-\sigma}}dy\right| \leq \left(\sup_{|y|\leq\rho}\frac{|\delta P(x,y)|}{|y|^2}\right)\int_{B_\rho}\frac{1}{|y|^{n-\sigma}}dy$$

$$= \left(\sup_{|y|\leq\rho}\frac{|\delta P(x,y)|}{|y|^2}\right)\frac{\omega_n}{\sigma}\rho^\sigma \leq [P]_{C^{1,1}(\mathbb{R}^n)}\frac{\omega_n}{\sigma}\rho^\sigma. \tag{4.6}$$

We may now plug inequalities (4.5) and (4.6) in (4.4) to obtain:

$$|\Gamma(x)| \leq A\omega_n\left(\frac{2}{2-\sigma}\|P\|_{L^\infty(\mathbb{R}^n)}\rho^{-(2-\sigma)} + \frac{C_n[P]_{C^{1,1}}(\mathbb{R}^n)}{\sigma}\rho^\sigma\right) \quad \forall\rho > 0. \tag{4.7}$$

Then we may pick the $\rho > 0$ that minimizes the right hand side of (4.7). Since we have a convex function of the parameter ρ we only need to get its critical point:

$$\frac{d}{d\rho}\left(\frac{a}{2-\sigma}\rho^{-(2-\sigma)} + \frac{b}{\sigma}\rho^\sigma\right) = 0 \Rightarrow a\rho^{-(3-\sigma)} = b\rho^{\sigma-1} \Rightarrow \rho = \sqrt{a/b}.$$

Finally, putting $\rho = \sqrt{a/b}$ gives us the minimum value (basically we used Young's inequality)

$$\frac{a}{2-\sigma}a^{-(2-\sigma)/2}b^{(2-\sigma)/2} + \frac{b}{\sigma}a^{\sigma/2}b^{-\sigma/2} = \frac{2}{(2-\sigma)\sigma}a^{\sigma/2}b^{(2-\sigma)/2}.$$

Where $a = 2\|P\|_{L^\infty(\mathbb{R}^n)}$ and $b = C_n[P]_{C^{1,1}(\mathbb{R}^n)}$, going back to (4.7) we obtain

$$|\Gamma(x)| \le \frac{2^{1+\sigma/2}\omega_n C_n^{(2-\sigma)/2} A(n,-(2-\sigma))}{(2-\sigma)\sigma}\|P\|_{L^\infty(\mathbb{R}^n)}^{\sigma/2}[P]_{C^{1,1}(\mathbb{R}^n)}^{(2-\sigma)/2}.$$

The proof is now done, as the term on the right hand side stays uniformly bounded for $\sigma \in (0,2)$, which is thanks to the behavior of $A(n,-(2-\sigma))$ as $\sigma \to 2$. □

Before we conclude this section, we must prove one last straightforward feature of the potential, P. We must be able to compare the values of P on the boundary of some ball, B_R, and make sure that the difference between the values on the boundary and the infimum of P are still comparable to the infimum itself.

Lemma 4.7 (Oscillation of The Potential)**.** *There exists a radius, R_σ, depending on σ and n such that*

$$-\inf_{B_{R_\sigma}}\{P\} + \inf_{\partial B_{R_\sigma}}\{P\} \ge \frac{1}{2}\left(-\inf_{B_{R_\sigma}}\{P\}\right).$$

Proof of Lemma 4.7. Let $x \in \partial B_R$ and x_0 be such that $-P(x_0) = -\inf(P)$. Lemma 3.9 tells us that $\Gamma = 0$ in $\mathbb{R}^n \setminus B_3$. Therefore we can estimate the integral

$$\begin{aligned}
-P(x) &= A(n,2-\sigma)\int_{\mathbb{R}^n} -\Gamma(x+y)\,|y|^{-n+2-\sigma}\,dy \\
&= A(n,2-\sigma)\int_{x+y\in B_3} -\Gamma(x+y)\,|y|^{-n+2-\sigma}\,dy \\
&= A(n,2-\sigma)\int_{w\in B_3} -\Gamma(x_0+w)\,|x_0-x+w|^{-n+2-\sigma}\,dw \\
&\le A(n,2-\sigma)\int_{w\in B_3} -\Gamma(x_0+w)\,|w|^{-n+2-\sigma}\left(\frac{R}{3}-1\right)^{-n+2-\sigma}dw \\
&= P(x_0)\left(\frac{R}{3}-1\right)^{-n+2-\sigma}.
\end{aligned}$$

Now, choosing R_σ so that

$$\left(\frac{R}{3}-1\right)^{-n+2-\sigma} \le \frac{1}{2}$$

gives the result. □

5. $\det(D^2P)$ as an Integro-Differential Operator on Γ

The point of using the potential, P, is that we can convert our fractional order equation into a 2nd order equation and appeal to known results. In this way we would like also to recognize the operation $\det(D^2P)$ as a σ-order operator on Γ, which one can hope will be comparable to M^- by below (proved in Lemma 5.5). This comparison would then allow us to bring together the geometric result involving $\inf(P)$ and the envelope property that $M^-(\Gamma) \le Cf$ on the set K_u. Fundamental to this pursuit is the formula for the determinant:

$$(\det(B))^{1/n} = \frac{1}{n}\inf\{\mathrm{Tr}(AB) : A \ge 0 \text{ and } \det(A) = 1\}, \quad \text{whenever } B \ge 0.$$

The interested reader should consult [13] and [26] for further discussion. The integro-differential nature of $\det(D^2P)$ acting on Γ is developed by making rigorous the formal computation of convolving Γ with the derivatives of the Riesz Kernel.

Lemma 5.1 (Derivatives of The Potential)**.** *Assume that Γ is $C^{1,1}(\mathbb{R}^n)$. Then the formula for derivatives of P in terms of an integro-differential operator on Γ is*

$$(P)_{x_ix_j} = \frac{(-n-\sigma+2)(-n-\sigma)}{2}\int_{\mathbb{R}^n} A(n, 2-\sigma)\delta\Gamma(x,y)\left[\frac{y_iy_j}{|y|^{n+\sigma+2}} - \frac{\delta_{ij}}{(n+\sigma)\,|y|^{n+\sigma}}\right]dy. \tag{5.1}$$

Proof of Lemma 5.1. We consider as a regularization of P, the convolution of Γ with a more regular Riesz Kernel, $C(\alpha)K_\alpha$. We take *initially* $\alpha > 2$ so that $(K_\alpha)_{x_ix_j}$ are integrable in B_1 and $C(\alpha) = (2-\alpha)/\sigma$ so that as the regularization parameter, α, approaches $2-\sigma$, we recover the original P. It will be handy to have the formula that for $y \ne 0$, the second derivatives of K_α are

$$(K_\alpha)_{y_iy_j}(y) = A(n,\alpha)\left((-n+\alpha-2)(-n+\alpha)y_iy_j\,|y|^{-n+\alpha-4} + (-n+\alpha)\,|y|^{-n+\alpha-2}\right), \tag{5.2}$$

and moreover, by reflection symmetry, the following integrals evaluate to 0:

$$\int_{B_1} y_iy_j\frac{1}{|y|^m}dS(y) = 0 \quad \text{as long as } i \ne j \tag{5.3}$$

Denote the regularization of P as $P_\alpha := C(\alpha)\Gamma * K_\alpha$. We first remark on the regularity of P and P_α. Because of the semigroup property of the convolutions with Riesz Kernels, we see that in fact

$$P_\alpha = (-\Delta)^{-(\alpha-(2-\sigma))/2}P,$$

and so P_α will have all the regularity of P, plus some more. Since we assume Γ is $C^{1,1}(\mathbb{R}^n)$, Lemma 4.1 implies that P, and hence also P_α, are uniformly in $C^{2,\gamma}$ with a norm that only depends on the $C^{1,1}(\mathbb{R}^n)$ estimate of Γ (because in this case $(-\Delta)^{\sigma/2}\Gamma$ is Hölder continuous). Therefore, at least along a subsequence– still denoted by α– we know that as $P_\alpha \to P$, then also we will have the derivatives $(P_\alpha)_{x_ix_j} \to P_{x_ix_j}$.

To conclude (5.1), we will compute the derivatives using (5.2) inside of the convolution. Then, just as in the computation of [31, p.45-47], we can analytically continue the formula through the value of $\alpha = 2$, down to $\alpha = (2-\sigma)$. It is in this analytic continuation argument where the constant, $C(\alpha)$, is crucial. We begin by adding and subtracting the value

$$\Gamma(x)C(\alpha)\int_{B_1}(K_\alpha)_{y_iy_j}(y)dy = \Gamma(x)C(\alpha)\int_{B_1}A(n,\alpha)(-n+\alpha)|y|^{-n+\alpha-2}dy$$

$$= \Gamma(x)C(\alpha)C(n,\sigma)A(n,2-\sigma)\frac{1}{\alpha-2}\omega(n).$$

The equality can be seen by (5.2) and (5.3), and the only quantities actually changing with α in an important way are the terms $C(\alpha)$ and $1/(\alpha-2)$. The choice of $C(\alpha)$ was exactly made to cancel $1/(\alpha-2)$ as $\alpha \to 2$. Our calculation now goes similarly to [31, p.45-47]:

$$\begin{aligned}
&2(P_\alpha)_{x_ix_j}(x) = 2C(\alpha)\Gamma * (K_\alpha)_{x_ix_j}(x) \\
&= C(\alpha)\int_{\mathbb{R}^n\setminus B_1} \left(\Gamma(x+y)+\Gamma(x-y)\right)(K_\alpha)_{y_iy_j}(y)dy \\
&\quad + C(\alpha)\int_{B_1} \left(\Gamma(x+y)+\Gamma(x-y)-2\Gamma(x)\right)(K_\alpha)_{y_iy_j}(y)dy \\
&\quad + 2\Gamma(x)C(\alpha)\int_{B_1}(K_\alpha)_{y_iy_j}(y)dy \\
&= C(\alpha)\int_{\mathbb{R}^n\setminus B_1} \left(\Gamma(x+y)+\Gamma(x-y)\right)(K_\alpha)_{y_iy_j}(y)dy \\
&\quad + C(\alpha)\int_{B_1} \left(\Gamma(x+y)+\Gamma(x-y)-2\Gamma(x)\right)(K_\alpha)_{y_iy_j}(y))dy \\
&\quad + 2\Gamma(x)C(n,\sigma)A(n,2-\sigma)C(\alpha)\frac{1}{\alpha-2}\omega(n).
\end{aligned}$$

At this point, we are safe to let $\alpha > 2$ pass through 2 to $\alpha < 2$, as $\alpha \searrow (2-\sigma)$. Moreover again using (5.2), (5.3), and the $C^{1,1}(\mathbb{R}^n)$ estimate of Γ for the convergence of integral over B_1, we recover for $\alpha < 2$ (not yet letting $\alpha \searrow (2-\sigma)$)

$$\begin{aligned}
&2(P_\alpha)_{x_ix_j}(x) \\
&= C(\alpha)\int_{\mathbb{R}^n\setminus B_1} \left(\Gamma(x+y)+\Gamma(x-y)\right)(K_\alpha)_{y_iy_j}(y)dy \\
&\quad + C(\alpha)\int_{B_1} \left(\Gamma(x+y)+\Gamma(x-y)-2\Gamma(x)\right)(K_\alpha)_{y_iy_j}(y)dy \\
&\quad - 2\Gamma(x)C(\alpha)\int_{\mathbb{R}^n\setminus B_1}(K_\alpha)_{y_iy_j}(y)dy.
\end{aligned}$$

Finally, we can combine the integrals and let $\alpha \to (2-\sigma)$ to recover

$$2(P)_{x_ix_j}(x) = \int_{\mathbb{R}^n} \left(\Gamma(x+y)+\Gamma(x-y)-2\Gamma(x)\right)(K_{(2-\sigma)})_{y_iy_j}(y)dy.$$

Dividing by 2 and using (5.2) gives the result.

□

The act of evaluating the determinant can be seen as introducing a new nonlocal operator which is a sort of generalization of the determinant of the Hessian to the fractional-order setting. However, there is much freedom in these generalizations, and such a fractional order determinant is complicated by the huge amount of freedom provided in the family of admissible kernels for the elliptic operators– much more than simply transforming by the quadratic form y^TAy. This topic will be expanded upon in Section 7. For now we stick to the development of such an operator only as it will be needed in the proof of Theorem 1.2.

Lemma 5.2 (Integro-Differential Form of $\mathrm{Tr}(AD^2P)$)**.** *Assume Γ is $C^{1,1}(\mathbb{R}^n)$. For any $A \geq 0$, $\mathrm{Tr}(AD^2P)$ defines an integro-differential operator acting on Γ, with a formula given as:*

$$\mathrm{Tr}(AD^2P)(x) = \frac{(-n-\sigma+2)(-n-\sigma)}{2}\int_{\mathbb{R}^n} A(n,2-\sigma)\delta\Gamma(x,y)\left[\frac{y^TAy}{|y|^{n+\sigma+2}} - \frac{\mathrm{Tr}(A)}{(n+\sigma)\,|y|^{n+\sigma}}\right]dy. \tag{5.4}$$

Proof of Lemma 5.2. This is a direct consequence of Lemma 5.1. □

Definition 5.3 (Integro-Differential Form of $\det(D^2P)$)**.** *The operator, $\det(D^2P)(x)$, defines a nonlinear integro-differential operator acting on $\Gamma \in C^{1,1}(x)$ via (5.4) as*

$$\mathcal{D}_\sigma(\Gamma,x) := \left(\mathcal{T}_{A(x)}(\Gamma,x)\right)^n,$$

where $\mathcal{T}_{A(x)}(\Gamma,x)$ is defined as the right hand side of the equality (5.4) with $A(x)$ being the cofactor matrix of $D^2P(x)$. When $\Gamma \in C^{1,1}(\mathbb{R}^n)$, this definition agrees with

$$\mathcal{D}_\sigma(\Gamma,x) = \left(\frac{1}{n}\inf\left\{\mathcal{T}_A(\Gamma,x) :\ A \geq 0 \text{ and } \det(A) = 1\right\}\right)^n = \det(D^2P),$$

at the points where $D^2P(x) \geq 0$.

Remark 5.4. It is worthwhile to remark on the peculiar form of formulas (5.1) and (5.4). Note that the term $y^TAy/|y|^2 - \mathrm{Tr}(A)/(n+\sigma)$ might take negative values for some of the matrices A being considered, so the kernels appearing in the formulas are not necessarily positive. In particular, $\mathcal{D}_\sigma(\Gamma,x) = \det(D^2P(x))$ is not an elliptic operator for a general Γ (not even if $E_\sigma(\Gamma) \geq 0$). Despite these issues, $\mathcal{D}_\sigma$ serves a crucial purpose in the proof of Theorem 1.2 as will be seen in Section 6. In particular, the first term in $\mathcal{D}_\sigma^{1/n}$ gives a much more convenient, (possibly degenerate) elliptic operator, this is introduced in Definition 5.6. We give further justification of (5.1) in Section 8.

Lemma 5.5 (Nonlocal Determinant - Minimal Operator Ordering)**.** *If Γ is $C^{1,1}(x)$ and solves $E_\sigma(\Gamma,x) \geq 0$, then*

$$D^2P(x) \geq 0 \Rightarrow \mathcal{D}_\sigma(\Gamma,x) \leq \frac{C(n)}{\lambda^n}(M^-(\Gamma,x))^n.$$

Proof of Lemma 5.5. This result is one of the main reasons for having a good definition of the envelope, Γ. Lemma 3.13 tells us in particular that $(-\Delta)^{\sigma/2}\Gamma(x) \leq 0$, and so we can ignore the second term in the formula of Lemma 5.2, which is

$$-\int_{\mathbb{R}^n} C(n)(2-\sigma)\delta\Gamma(x,y)\frac{\mathrm{Tr}(A)}{(n+\sigma)}\,|y|^{-n-\sigma}\,dy.$$

Thus thanks to the assumption of $D^2P(x) \geq 0$, we have

$$\begin{aligned}
(\mathcal{D}_\sigma(\Gamma,x))^{1/n} &\leq \inf_{\det(A)=1}\left\{\frac{1}{n}\mathcal{T}_A(\Gamma,x)\right\}\\
&\leq \inf_{\lambda\mathrm{Id}\leq B\leq\Lambda\mathrm{Id}}\left\{\frac{1}{n}\frac{1}{(\det(B))^{1/n}}\mathcal{T}_B(\Gamma,x)\right\}\\
&\leq \inf_{\lambda\mathrm{Id}\leq B\leq\Lambda\mathrm{Id}}\left\{\frac{1}{n}C(n)(2-\sigma)\frac{1}{\lambda}\int_{\mathbb{R}^n}\delta\Gamma(x,y)\frac{y^TBy}{|y|^{n+\sigma+2}}dy\right\}.
\end{aligned}$$

□

The proof of Lemma 5.5 indicates that it will be useful to introduce some further notation to deal with $\mathcal{D}_\sigma$ in the class of functions satisfying $E_\sigma \geq 0$. Namely that the negative term of $\mathcal{T}_A$ in many expressions can be ignored up to an inequality. Therefore, we give a definition of the operator without this term:

Definition 5.6. *Let v be $C^{1,1}(x)$ and such that $E_\sigma(v,x) \geq 0$, then define*

$$d_\sigma(v,x) := \inf_{\substack{\det(A)=1\\ A\geq 0}} \left\{ \frac{(-n-\sigma+2)(-n-\sigma)}{2} \int_{\mathbb{R}^n} A(n,2-\sigma)\delta v(x,y) \frac{y^T A y}{|y|^{n+\sigma+2}} dy \right\}.$$

As remarked in the proof of Lemma 5.5, we have an ordering of the operators $\mathcal{D}_\sigma$ and d_σ within the class of functions satisfying $E_\sigma \geq 0$ at those points where $D^2P \geq 0$. The details within the proof of Lemma 5.5 allow us to conclude the corollary,

Corollary 5.7. *If $\Gamma \in C^{1,1}(x)$ and $E_\sigma(\Gamma,\cdot) \geq 0$, then*

$$D^2P(x) \geq 0 \Rightarrow \mathcal{D}_\sigma(\Gamma,x) \leq d_\sigma(\Gamma,x) \leq \frac{C(n)}{\lambda^n}(M^-(\Gamma,x))^n.$$

The restriction to points x where $D^2P(x) \geq 0$ is not too severe, as later on we will reduce to those x for which P coincides with its convex envelope, and there we shall always have $D^2P \geq 0$. Very much related to this fact is the following crucial Lemma which will also guarantee that we will only need to test the equation at points where $u = \Gamma$.

Lemma 5.8 ($\{D^2P > 0\}$ is contained in the contact set)**.** *If Γ is $C^{1,1}$ from above and $\Gamma \neq 0$, then for almost every x_0 such that $\Gamma(x_0) \neq u(x_0)$, we can find some direction τ such that $P_{\tau\tau}(x) \leq 0$.*

Proof of Lemma 5.8. Lemma 3.12 tells us that at almost every such x_0, $E_\sigma(\Gamma,x_0) = 0$. Hence there exists a direction, such that

$$D^\sigma_{ij}(\Gamma,x_0)\tau_i\tau_j = 0$$

Hence we can plug τ into the formula (5.2) for $P_{\tau\tau}$ and get (recall $\tau_i\tau_j = |\tau| = 1$)

$$P_{\tau\tau} = D^\sigma_{ij}(\Gamma,x_0)\tau_i\tau_j - C_{n,\sigma}\int_{\mathbb{R}^n} \frac{\delta\Gamma(x,y)}{|y|^{n+\sigma}}dy = -C_{n,\sigma}\int_{\mathbb{R}^n} \frac{\delta\Gamma(x,y)}{|y|^{n+\sigma}}dy \leq 0$$

□

6. Proof of The Main Theorem

6.1. **An Approximation of Γ Appropriate For Theorem 1.2.** In proving Theorem 1.2, it will be necessary to start with $\Gamma \in C^{1,1}(\mathbb{R}^n)$ to have the necessary regularity that $\det(D^2P) = \mathcal{D}_\sigma(\Gamma,\cdot)$. Not just any approximation of Γ will work due to the delicate nature of the definitions of $\mathcal{D}_\sigma$ and d_σ. The issue which must be overcome here is the lack of $C^{1,1}$ control of Γ, which is inherent in the properties of solutions of fractional order equations. Specifically, we know that Γ always satisfies $E_\sigma(\Gamma,\cdot) \geq 0$, and even $E_\sigma(\Gamma,\cdot) = 0$ away from the contact set, but *a priori* there is no reason to be able to assert that Γ will have any $C^{1,1}$ regularity from below (to the contrary, $C^{1,1}$ from *above* is completely natural to assume as we have seen it is inherited from u which can be approximated in a fashion which gives such control.). This is in strong contrast to the second order setting with the convex envelope. Nonetheless, the whole point is that the behavior we expect should have no impact at all on the estimates we need. Specifically, it is likely that $\delta\Gamma(x,y)$ could become very negative for some $x \notin K_u$ with $|y|$ small and still maintain $E_\sigma(\Gamma,x)\geq 0$, which should not hurt in finding estimates of $\mathcal{D}_\sigma(\Gamma,\cdot)$ from above. These difficulties are overcome with a regularized version of Γ, which we note is *an envelope of* Γ:

Definition 6.1. *The regularization of* Γ *is defined for* $\eta > 0$

$$\Gamma^\eta := \sup\{v(x) : D^2 v(x) \geq -\eta \mathrm{Id},\ E_\sigma(v)\geq 0 \text{ in } B_3,\ \text{and } v\leq \Gamma \text{ in } \mathbb{R}^n\}. \tag{6.1}$$

Remark 6.2. This definition is far from being the most economical with regards to machinery. However, it is convenient to first have the original envelope, Γ, intact and then to produce a smoothing by applying a further envelope with a more regular family. This way we can clearly keep track of the original contact set, K_u, and to keep intact the nice properties of Γ away from the set K_u, which would be trickier to follow, but still equivalent, if we were to work with the regularized envelope of u directly.

We now prove various statements about Γ^η and Γ.

Lemma 6.3. $\Gamma^\eta \leq \Gamma$ *and* $\Gamma^\eta \nearrow \Gamma$ *as* $\eta\to\infty$.

Proof of Lemma 6.3. This follows simply by the fact that the sets over which the supremum for each η are taken are increasing as $\eta\to\infty$. □

Lemma 6.4. *If* Γ *is* $C^{1,1}$ *from above, then* $\Gamma^\eta\in C^{1,1}(\mathbb{R}^n)$.

Proof of Lemma 6.4. This fact follows from Lemma 3.10 plus the fact that the uniform estimate from below on $D^2v\geq -\eta$ in the definition of Γ^η is preserved by the supremum. □

Lemma 6.5. *If* $\Gamma^\eta(x)=\Gamma(x)$, *then*

$$d_\sigma(\Gamma^\eta,x)\leq d_\sigma(\Gamma,x).$$

Proof of Lemma 6.5. This follows simply from comparison because d_σ is degenerate elliptic and Γ^η touches Γ from below at such points. □

Lemma 6.6. *If* Γ *is* $C^{1,1}$ *from above, then*

$$\limsup_{\eta\to\infty}\int_{B_3} d_\sigma(\Gamma^\eta,x)dx \leq \int_{B_3} d_\sigma(\Gamma,x)dx$$

Proof of Lemma 6.6. Suppose that the global $C^{1,1}$ from above constant for Γ and Γ^η is C. We remark that due to Lemma 3.13 this implies

$$d_\sigma(\Gamma^\eta,x)\leq C(n)\int_{B_1} C\,|y|^2\,|y|^{-n-\sigma}\,dy + C(n)\int_{\mathbb{R}^n\setminus B_1} 4\|\Gamma^\eta\|_\infty\,|y|^{-n-\sigma}\,dy. \tag{6.2}$$

Thus we have

$$\begin{aligned}
&\int_{B_3} d_\sigma(\Gamma^\eta, x)dx \\
&= \int_{\Gamma^\eta=\Gamma} d_\sigma(\Gamma^\eta, x)dx + \int_{\Gamma^\eta\neq\Gamma} d_\sigma(\Gamma^\eta, x)dx \\
&\leq \int_{\Gamma^\eta=\Gamma} d_\sigma(\Gamma, x)dx + \int_{\Gamma^\eta\neq\Gamma} d_\sigma(\Gamma^\eta, x)dx.
\end{aligned}$$

Thanks to Lemma 6.3 and the boundedness from (6.2), we conclude by taking the lim sup in η and using dominated convergence for the integrals (note $\mathbb{1}_{\{\Gamma^\eta\neq\Gamma\}} \to 0$ pointwise). □

Lemma 6.7. *If Γ is $C^{1,1}$ from above and P^η is the Riesz potential of Γ^η, then*

$$\limsup_{\eta\to\infty} \int_{B_3\cap\{P^\eta=(P^\eta)^{CE}\}} \det(D^2P^\eta(x))dx \leq \int_{B_3\cap\{P=P^{CE}\}} d_\sigma(\Gamma, x)dx. \tag{6.3}$$

Proof of Lemma 6.7. Thanks to Lemma 6.4, the computation of Lemma 5.1, the fact that $D^2P^\eta \geq 0$ on $\{P^\eta = (P^\eta)^{CE}\}$, and definition 5.3 we see that $\det(D^2P^\eta(x)) = \mathcal{D}_\sigma(\Gamma^\eta, x)$ for a.e.x. Combining this with the order of $\mathcal{D}_\sigma$ and d_σ from Corollary 5.7 and Lemma 6.6, we have the result. □

Lemma 6.8. *Assume that Γ is $C^{1,1}$ from above. Then for almost every $x \in K_u$*

$$d_\sigma(\Gamma, x) \leq \frac{C(n)}{\lambda^n}(f(x))^n$$

Proof of Lemma 6.8. The proof of this Lemma is due to Corollary 5.7 and the fact that Γ is $C^{1,1}(x)$ for a.e.x due to its global $C^{1,1}$ from above regularity. □

Finally, there is one last fact we need which is nothing but a rephrasing of Lemma 5.8.

Corollary 6.9. *If Γ is $C^{1,1}(\mathbb{R}^n)$, then*

$$\{D^2P \geq 0\} \subset K_u. \tag{6.4}$$

6.2. **Proof of Theorem 1.2.** We are now in a position to prove the main theorem, Theorem 1.2. The main step left to prove is an estimate on $\inf(P)$ which involves a measure theoretic norm of f. This will come from the geometric set-up of the second order ABP estimate for P, thanks to the various results of section 5.

Proof of Theorem 1.2. The theorem will first be proved under the assumption that u is $C^{1,1}$ from above, and then we will remove this restriction at the end of the proof using the standard method of inf-convolution of u. Therefore, assume u is $C^{1,1}$ from above.

As mentioned in the lead-up to this proof, we will be applying the geometric set-up of second order ABP estimate to P, but a crucial factor to be determined is the domain in which the argument will be applied. We must work with an appropriately large radius according to Lemma 4.7 in order to make sure that the difference of the values of P on ∂B_{R_σ} and $\inf(P)$ are still comparable to $\inf_{\mathbb{R}^n}(P)$. Therefore we are working in B_{R_σ} from Lemma 4.7.

Before we can proceed, we must apply the regularization of definition 6.1 on Γ so that Γ and P have enough regularity that $\det(D^2P) = \mathcal{D}_\sigma(\Gamma, \cdot)$. To this end, we temporarily replace Γ by

Γ^η given in definition 6.1. Thus because Γ^η is $C^{1,1}(\mathbb{R}^n)$ (from Lemma 6.4), we know by Lemma 5.2 and Definition 5.3 that $\det(D^2P^\eta) = \mathcal{D}_\sigma(\Gamma^\eta, \cdot)$. Therefore, if $P^{\eta CE}$ is the convex envelope of P^η in B_{R_σ}, we know also that $P^{\eta CE} \in C^{1,1}(B_{R_\sigma})$ (cf. [20, Chapter 3]). The usual geometry of the convex envelope (cf. [20, Chapter 3]) then implies

$$-\inf_{B_{R_\sigma}}\{P^\eta\} \leq \left(-\inf_{\partial B_{R_\sigma}}\{P^\eta\}\right) + C(n)\left(\int_{\{P^\eta = P^{\eta CE}\}} \det(D^2P^{\eta CE}(x))dx\right)^{1/n} \tag{6.5}$$

$$\leq \left(-\inf_{\partial B_{R_\sigma}}\{P^\eta\}\right) + C(n)\left(\int_{\{P^\eta = P^{\eta CE}\}} \det(D^2P^{\eta}(x))dx\right)^{1/n} \tag{6.6}$$

$$= \left(-\inf_{\partial B_{R_\sigma}}\{P^\eta\}\right) + C(n)\left(\int_{\{P^\eta = P^{\eta CE}\}} d_\sigma(\Gamma^\eta, x)dx\right)^{1/n}. \tag{6.7}$$

We would now like to pass to the limit as $\eta \to \infty$ in the previous calculation. We note that the same proof of Lemma 3.14 applies to the sets $\{P^\eta = P^{\eta CE}\}$ seeing as the convex envelope, $P^{\eta CE}$, is defined in a similar fashion to Γ (cf. [17, Section 3] or [32]). Using also the fact that $\Gamma^\eta \nearrow \Gamma$ and also for $P^\eta \nearrow P$ with Lemma 6.7 we can conclude that

$$-\inf_{B_{R_\sigma}}\{P\} \leq \left(-\inf_{\partial B_{R_\sigma}}\{P\}\right) + C(n)\left(\int_{\{P = P^{CE}\}} d_\sigma(\Gamma, x)dx\right)^{1/n}.$$

Finally, using Lemmas 6.8, 5.8, and 4.7 we conclude

$$-\inf_{\mathbb{R}^n}\{P\} \leq \frac{C(n)}{\lambda}\left(\int_{K_u} f^n dx\right)^{1/n}. \tag{6.8}$$

In order to conclude the theorem in the case that u is $C^{1,1}$ from above we must now recall how $\inf(P)$ relates to $\inf(\Gamma)$. We can now combine the result of Proposition 4.3 with the previous calculation to obtain (with obvious abuse of the dimensional constant, $C(n)$):

$$C(n,\lambda)(-\Gamma(x_0))^{2/\sigma}\left(\frac{1}{2f(x_0)}\right)^{(2-\sigma)/\sigma} \leq \frac{C(n)}{\lambda}\left(\int_{K_u} f^n dx\right)^{1/n},$$

which after rearranging gives

$$-\inf\{\Gamma\} \leq \frac{C(n)}{\lambda}(\|f\|_{L^\infty(K_u)})^{(2-\sigma)/2}(\|f\|_{L^n(K_u)})^{\sigma/2}.$$

Now to finish the theorem for a generic u, we work with the inf-convolution of u (cf. [12, Appendix A] for the analogous argument in the second order setting),

$$u_\varepsilon(x) = \inf_{|x-y|\leq\sqrt{\varepsilon}\|u\|_\infty}\{u(y) + \frac{1}{\varepsilon}|x-y|^2\}.$$

The main properties we will use are that u_ε is $C^{1,1}$ from above in B_3, that u_ε is *increasing to* u, and that u_ε solves the equation (1.6) with f replaced by f^ε (see [12, Lemma A.3] and [17, Proposition 5.5]) given as

$$f^\varepsilon(x) = \sup_{|x-y|\leq\sqrt{\varepsilon}\|u\|_\infty}\{f(y)\}.$$

For Γ_ε, P_ε, and K_{u_ε} denoting all the corresponding operations using u_ε in place of u, (6.8) becomes

$$-\inf_{\mathbb{R}^n}\{P_\varepsilon\} \leq \frac{C(n)}{\lambda}\left(\int_{K_{u_\varepsilon}} (f^\varepsilon)^n dx\right)^{1/n}.$$

So long as $\limsup_\varepsilon f^\varepsilon(x) \leq f(x)$, which is given by continuity of f in this case, we can conclude the estimate by letting $\varepsilon \to 0$ and using Lemma 3.14. □

Remark 6.10. For future reference, we would like to collect a very important fact which is a cornerstone of the proof of Theorem 1.2. It is the relationship between $\inf(\Gamma)$ and the integral of $d_\sigma(\Gamma)$,

$$-\inf_{\mathbb{R}^n}(\Gamma) \leq \frac{C(n)}{\lambda}\left(\int_{K_u} d_\sigma(\Gamma, x)dx\right)^{1/n}, \tag{6.9}$$

which holds whenever $\Gamma \in C^{1,1}(\mathbb{R}^n)$.

7. Fractional Monge-Ampère Type Operators

The Monge-Ampère type operator $\mathcal{D}_\sigma(\Gamma, x)$ is crucial to our proof of Theorem 1.2, as it allows us to borrow the divergence structure of the standard Monge-Ampère which is essential for the classical ABP Theorem. One may take the point of view that $T_u(x) = \nabla P_u$ is a nonlocal gradient map given by u, and that $\mathcal{D}_\sigma$ is just the Jacobian of the map T_u. Then, part of the effort in our proof (Section 4) has been relating the infimum of Γ to the size of the image of the map $T_\Gamma(x)$.

We insist in referring to $\mathcal{D}_\sigma$ as a "Monge-Ampère type operator" and not just the "Monge-Ampère operator", as it is not clear that $\mathcal{D}_\sigma$ would be a definite, canonical analogue of this operator for integro-differential equations. Note also that as P_u is not necessarily convex the map ∇P_u might not be monotone, and in fact the operator $\mathcal{D}_\sigma(u)$ might fail to be elliptic even on the class of functions satisfying $E_\sigma \geq 0$ (for the potential P might not be convex). Another, perhaps more natural candidate for an "integro-differential Monge-Ampère" is given by the operator

$$\mathcal{D}^*_\sigma(u, x) := \left(\inf_{A\geq 0, \det A = 1} \int_{\mathbb{R}^n} \frac{C(n)(2-\sigma)}{|A^{-1}y|^{n+\sigma}}\delta u(x, y)dy\right)^{n_\sigma} \tag{7.1}$$

Where n_σ may be chosen to be $n_\sigma \equiv n$ or at least such that $n_\sigma \to n$ as $\sigma \to 2$. Unlike $\mathcal{D}_\sigma$, this second operator is affine invariant and it can be checked easily that it is well defined and (degenerate) elliptic in the class of functions which are subsolutions of $E_\sigma = 0$. The main disadvantage of $\mathcal{D}^*_\sigma$ with respect to $\mathcal{D}_\sigma$ is that it is hard to relate directly the size $\mathcal{D}^*_\sigma\Gamma$ to the infimum of Γ. This relationship could be quantified if $\mathcal{D}^*_\sigma$ had a connection to some sort of gradient map, which is (informally speaking) how the divergence structure of Monge-Ampère contributes to the classical ABP (recall $\mathcal{D}_\sigma = \det(DT_u)$). Nevertheless, they both converge as $\sigma \to 2$ to the Monge-Ampère operator (see respectively Section 8 below and also [17, Section 6]), and hence qualify as possible generalizations.

The operator $\mathcal{D}^*_\sigma$ as defined in (7.1) would be a natural operator to consider in the class of equations given by

$$L_A(u, x) = \int_{\mathbb{R}^n} \frac{C(n)(2-\sigma)}{\det(A)|A^{-1}y|^{n+\sigma}}\delta u(x, y)dy \quad \text{where } A^t = A,\ \lambda \mathrm{I} \leq A^2 \leq \Lambda \mathrm{I} \tag{7.2}$$

A posteriori, one can see the reasons for the proof of Theorem 1.2 being restricted to a much smaller class of equations than those appearing in [17]. There must be a balance to ensure each one of Lemmas 5.5, 5.8, and (6.9) hold. In principle, one could create a new notion of determinant altogether, such that those three meta-lemmas would still be valid for much richer families of kernels. The obstacle comes from the fact that considering a larger family of kernels makes the operator M^- more extremal, and thus the question of finding a "geometric" operator that is comparable M^- becomes much harder to tackle. In conclusion, all these issues underline the lack of geometric equations for integro-differential operators.

8. Important Limits As $\sigma \to 2$

In the series of works, [15], [16], and [17], all of the results were obtained in a fashion in which they are preserved as $\sigma \to 2$ and recover the corresponding results already proved for second order equations. This has led to a unified picture of the second order and the fractional order (i.e. nonlocal) theories. We adopt this view in the current work, and take this section to discuss explicitly how our result relates to the relevant second order theory.

8.1. **Recovery of A Second Order Envelope as $\sigma \to 2$.** One may expect that as $\sigma \to 2$ the envelopes Γ_σ should behave more and more like the convex envelope of u. This is almost the case at least whenever u is $C^{1,1}$ from above, the discrepancy arises only because of the behavior of E_σ as $\sigma \to 2$ (see Section 3).

Proposition 8.1. *Assume u is $C^{1,1}$ from above, then as $\sigma \to 2$ the envelopes Γ_σ converge uniformly to a function lying below u and above the convex envelope of u in B_3.*

Proof of Proposition 8.1. Here we will use the operator, E^*_σ, which was introduced in Remark 3.6. Let Γ_σ and Γ^*_σ be respectively the envelopes of u made using Definition 3.5 with respectively the operators E_σ and E^*_σ. We note here that because $E^*_\sigma\Gamma^*_\sigma \geq 0$ in B_3, then it also holds that $E_\sigma\Gamma^*_\sigma \geq 0$ in B_3 as well. Moreover, Γ^*_σ satisfies $\Gamma^*_\sigma \leq u$ by construction. Hence because Γ_σ is the supremum of such functions, we see that

$$\Gamma^*_\sigma \leq \Gamma_\sigma.$$

Due to the $C^{1,1}$ assumption on u we can use Lemma 3.10 to conclude that both families, $\{\Gamma_\sigma\}_\sigma$ and $\{\Gamma^*_\sigma\}_\sigma$ are uniformly $C^{1,1}$ from above. In particular there exists a constant $C > 0$ independent of σ such that

$$-(-\Delta)^{\sigma/2}\Gamma_\sigma \leq C \ \text{ and } \ -(-\Delta)^{\sigma/2}\Gamma^*_\sigma \leq C$$

Since we also have by construction both

$$-(-\Delta)^{\sigma/2}\Gamma_\sigma \geq 0 \ \text{ and } \ -(-\Delta)^{\sigma/2}\Gamma^*_\sigma \geq 0,$$

we conclude that both $\|(-\Delta)^{\sigma/2}\Gamma_\sigma\|_{L^\infty(B_3)} \leq C$ and $\|(-\Delta)^{\sigma/2}\Gamma^*_\sigma\|_{L^\infty(B_3)} \leq C$. Since each Γ_σ and Γ^*_σ are identically zero outside B_3 we may use (for instance) the Poisson kernel for $(-\Delta)^\sigma$ in [31] to conclude both Γ_σ and Γ^*_σ are Hölder continuous in $\mathbb{R}^n$ and uniformly in σ.

Therefore, from each sequence $\sigma_k \to 2$ we may select a subsequence that converges uniformly in $\mathbb{R}^n$ to some functions Γ_2 and Γ^*_2.

We note the convergence of $E_\sigma(v,x)$ as $\sigma \to 2$ to $C_n(\lambda_1(D^2v(x))+\frac{1}{n+2}\Delta v(x))$ (cf. Proposition 8.2) and $E^*_\sigma(v,x)$ to $\lambda_1(D^2v(x))$. Thus Γ_2 and Γ^*_2 are the unique solutions to their respective limiting obstacle problems, and hence all subsequential limits converge to the same function. But in the limit Γ_2^* is the convex envelope of u, which concludes the proposition. □

We proved the proposition assuming u is $C^{1,1}$ from above for the sake of simplicity, but as it is reasonable to expect the envelopes Γ_s to be somewhat regular regardless of u this convergence should hold under much more general circumstances.

8.2. **Derivatives of P Converge To Derivatives of Γ.** We now revisit the peculiar form of (5.1) and (5.4) mentioned in Remark 5.4. We give further justification of formula (5.1) by checking with a direct computation that when $\sigma \to 2^-$ it gives back the Hessian of Γ. This is to be expected since $P_u \to \Gamma_u^\sigma$ as $\sigma \to 2$ (u is fixed throughout this discussion).

Proposition 8.2. *Suppose Γ is a fixed function which has a second order Taylor expansion at x, then*

$$\lim_{\sigma \to 2^-} D^2 P(x) = D^2 \Gamma(x)$$

Proof of Proposition 8.2. The proof has two broad steps.

Step 1) We shall compute a formula for $\lim_{\sigma \to 2^-} P_{x_i x_j}$. To do this estimate the integral on the right hand side of (5.1) by breaking it in two parts. Fix $\epsilon > 0$, observe that

$$\left| (2-\sigma) \int_{B_\epsilon^C} \delta\Gamma(x,y) \left[\frac{y_i y_j}{|y|^{n+\sigma+2}} - \frac{\delta_{ij}}{(n+\sigma)\,|y|^{n+\sigma}} \right] dy \right|$$

$$\leq \|\Gamma\|_\infty (2-\sigma) C_n \int_{B_\epsilon^C} \frac{1}{|y|^{n+\sigma}} dy = \|\Gamma\|_\infty (2-\sigma) C_n \int_\epsilon^{+\infty} \frac{\omega_n}{t^{1+\sigma}} dt$$

The right hand side vanishes as $\sigma \to 2$ (for $\epsilon > 0$ fixed), thus

$$\lim_{\sigma \to 2^-} P_{xixj} = \frac{n(n+2)}{2} \lim_{\sigma \to 2^-} \int_{B_\epsilon(0)} A(n, 2-\sigma) \frac{\delta\Gamma(x,y)}{|y|^{n+\sigma}} \left[\frac{y_i y_j}{|y|^2} - \frac{\delta_{ij}}{(n+\sigma)} \right] dy \tag{8.1}$$

This holds for all $\epsilon > 0$ (which makes clear we are getting a local operator). To estimate the integral inside B_ϵ recall that Γ is $C^{1,1}(x)$, which means that $\nabla\Gamma(x)$ and $D^2\Gamma(x)$ exist in the sense that as $|y| \to 0$

$$\Gamma(x+y) = \Gamma(x) + \nabla\Gamma(x) \cdot y + \frac{1}{2} y^T (D^2\Gamma(x)) y + o(|y|^2)$$

In particular, setting $\hat{y} = y / |y|$

$$\frac{\delta\Gamma(x,y)}{|y|^2} = \hat{y}^T (D^2\Gamma(x)) \hat{y} + o(1) \ \ , \ |y| \to 0$$

Using this expansion in (8.1) we get further[1]

$$\lim_{\sigma \to 2^-} P_{x_i x_j} = \frac{n(n+2)}{2\omega_n} \lim_{\sigma \to 2^-} \left\{ (2-\sigma) \int_{B_\epsilon} \frac{\hat{y}^T (D^2\Gamma(x)) \hat{y}}{|y|^{n-2+\sigma}} \left[\hat{y}_i \hat{y}_j - \frac{\delta_{ij}}{n+\sigma} \right] dy \right\}$$

We may also use polar coordinates to see that

$$\int_{B_\epsilon} \frac{\hat{y}^T (D^2\Gamma(x)) \hat{y}}{|y|^{n-2+\sigma}} \left[\hat{y}_i \hat{y}_j - \frac{\delta_{ij}}{n+\sigma} \right] dy = \int_0^\epsilon \int_{S^{n-1}} e^T (D^2\Gamma(x)) e \left[e_i e_j - \frac{\delta_{ij}}{n+\sigma} \right] \frac{t^{n-1}}{t^{n-2+\sigma}} dS(e) dt$$

[1]also recall that $A(n, 2-\sigma)/(2-\sigma) \to \omega_n^{-1}$ as $\sigma \to 2^-$

$$= \frac{1}{2-\sigma}\epsilon^{2-\sigma}\int_{S^{n-1}} e^T(D^2\Gamma(x))e\left[e_ie_j - \frac{\delta_{ij}}{n+\sigma}\right]dS(e)$$

This gives us the formula

$$\lim_{\sigma\to 2^-} P_{x_ix_j} = \frac{n(n+2)}{2\omega_n}\int_{S^{n-1}}\left(e_ie_j - \frac{1}{n+2}\delta_{ij}\right)\, e^T(D^2\Gamma(x))e\; dS(e) \tag{8.2}$$

Step 2) All there is left to show is that the expression on the right in (8.2) always gives back the ij entry of $D^2\Gamma(x)$. By rotation invariance we may assume without loss of generality that $D^2\Gamma(x)$ is diagonal, in which case we have $e^T(D^2\Gamma(x))e = \sum_{l=1}^{n}\Gamma_{ll}(x)e_l^2$ for all e, in other words:

$$\lim_{\sigma\to 2^-} P_{x_ix_j} = \frac{n(n+2)}{2\omega_n}\sum_{l=1}^{n}\int_{S^{n-1}}\left(e_ie_j - \frac{\delta_{ij}}{n+2}\right)e_l^2\; dS(e)\Gamma_{x_lx_l}(x)$$

Note that by reflection symmetry the integral of $e_ie_je_l^2$ must be zero for all l when $i\neq j$, given that $D^2\Gamma(x)$ is diagonal this means $\lim_{\sigma\to 2^-} P_{x_ix_j} = 0 = \Gamma_{x_ix_j}(x)$ for $i\neq j$. We are left to consider the case $i=j$, to fix ideas we do it for $i=1$. Then

$$\lim_{\sigma\to 2^-} P_{x_1x_1} = \frac{n(n+2)}{2\omega_n}\sum_{l=1}^{n}\int_{S^{n-1}}\left(e_1^2 - \frac{1}{n+2}\right)e_l^2\; dS(e)\Gamma_{x_lx_l}(x)$$

Due to rotation invariance this can be rewritten as

$$\frac{n(n+2)}{2\omega_n}\left[\int_{S^{n-1}}\left(e_1^4 - \frac{e_1^2}{n+2}\right)\, dS(e)\Gamma_{x_1x_1} + \left(\int_{S^{n-1}}\left(e_1^2e_2^2 - \frac{e_2^2}{n+2}\right)\, dS(e)\right)\sum_{l=2}^{n}\Gamma_{x_lx_l}(x)\right]$$

One may compute explicitly the integrals above and get

$$\int_{S^{n-1}} e_i^2 dS(y) = \frac{\omega_n}{n},\quad \int_{S^{n-1}} e_i^2e_j^2 dS(y) = \begin{cases} \frac{\omega_n}{n(n+2)} & i\neq j\\ \\ \frac{3\omega_n}{n(n+2)} & i=j\end{cases}$$

Therefore

$$\int_{S^{n-1}}\left(e_1^4 - \frac{e_1^2}{n+2}\right)\, dS(e) = \frac{2\omega_n}{n(n+2)},\quad \int_{S^{n-1}}\left(e_1^2e_2^2 - \frac{e_2^2}{n+2}\right)\, dS(e) = 0$$

Plugging this in the last expression for $\lim_{\sigma\to 2^-} P_{x_1x_1}(x)$ we obtain

$$\lim_{\sigma\to 2^-} P_{x_1x_1}(x) = \frac{n(n+2)}{2\omega_n}\left(\frac{2\omega_n}{n(n+2)}\Gamma_{x_1x_1} + 0\right) = \Gamma_{x_1x_1}$$

This proves the proposition. □

Remark 8.3. The spherical integrals above are not hard to compute, they follow from counting the different terms appearing in the trivial identities:

$$\int_{S^{n-1}} e_1^2 + ... + e_n^2\; dS(e) = \int_{S^{n-1}}\left(e_1^2 + ... + e_n^2\right)^2 dS(e) = \int_{S^{n-1}} dS(e) = \omega_n$$

and from the relation $\int_{S^{n-1}} e_1^2e_2^2 dS(e) = \frac{1}{3}\int_{S^{n-1}} e_1^4 dS(e)$ which is a standard computation for $n=2$ and can be pushed for all n via induction (integrating along slices of the sphere and rescaling the lower dimensional formula in the inductive step).

9. Comparison Theorems Related to Theorem 1.2

In this section we collect various results that are either direct applications of Theorem 1.2 or straightforward modifications of its proof. Each of the results here are stated without proof, and we simply mention some of the modifications. First we mention in Theorem 9.1 the analog of Theorem 1.2 to more general domains than just B_1, and in Theorem 9.2 the applications to the comparison principle for (1.3).

Theorem 9.1. *Let equation (1.6) be set in a general bounded, connected, domain, D instead of in B_1. Then*

$$-\inf_{D}\{u\} \leq \frac{C(n)}{\lambda}\mathrm{diam}(D)(\|f\|_{L^\infty(K_u)})^{(2-\sigma)/2}(\|f\|_{L^n(K_u)})^{\sigma/2}.$$

We note that to modify the proof of Theorem 1.2 to incorporate D, one simply needs to modify the domain of truncation in the definition of Γ_u^σ, Definition 3.5, and also the selection of the radius, R_σ, from Lemma 4.7. In particular, if we define the set $D_3 := \{x : distance(x, D) \leq 3\}$, then we are concerned with the obstacle problem in D_3, using $u\mathbb{1}_{D_3}$ in Definition 3.5. Furthermore, the ball B_3 is no longer used in the proof of Lemma 4.7, but instead R_σ is chosen large enough to compensate for the size of D_3 instead of the size of B_3.

An immediate consequence of the fact that the difference between a subsolution and a supersolution with solve the minimal equation for M^- is that we get a comparison theorem for subsolutions and supersolutions of equations with the same operator, F, but different right hand sides.

Theorem 9.2. *Suppose that F is in the elliptic family for M^- and M^+. Let u and v be bounded and respectively usc subsolution and lsc supersolution of*

$$\begin{cases} F(u,x) \geq f(x) & \text{in } D \\ u = u_0 & \text{on } \mathbb{R}^n \setminus D \end{cases}$$

and

$$\begin{cases} F(v,x) \leq g(x) & \text{in } D \\ v = v_0 & \text{on } \mathbb{R}^n \setminus D. \end{cases}$$

then

$$\sup_D\{u-v\}$$
$$\leq \sup_{\mathbb{R}^n\setminus D}\{u_0 - v_0\} + \frac{C(n)}{\lambda}\mathrm{diam}(D)(\|(f-g)^-\|_{L^\infty(K_{u-v})})^{(2-\sigma)/2}(\|(f-g)^-\|_{L^n(K_{u-v})})^{\sigma/2}.$$

10. Special Cases of the Regularity Theory of Caffarelli and Silvestre

In this section we show how Theorem 1.2 can be used to prove the standard L^ϵ estimates for viscosity solutions of (1.6) (at least for a special family of operators). None of these results are new, as such estimates have already been proved in [17, Section 10] for very general nonlinear integro-differential equations. The L^ϵ estimate constitutes the backbone of the regularity theory for fully nonlinear elliptic equations; it is the key fact behind the Harnack inequality of Krylov-Safonov, the Evans-Krylov theorem and the respective Caffarelli-Silvestre theorems for nonlocal equations (again, see [20] and [17]). Our purpose in revisiting this part of the theory in our case is basically illustrating the uses of Theorem 1.2, in particular Theorem 1.2 allows us to do essentially the same proof of the L^ϵ bound for second order equations used in [20, Lemma 4.6] (compare to the different method needed in [17, Section 10]). Furthermore, the estimates obtained are uniform in the order of the equation, recovering the second order theory as $\sigma \to 2$, which was already done in [17]

We state without proof the following proposition regarding the construction of a special barrier function. It follows by an argument similar to that used in Lemma 9.1 of [17].

Proposition 10.1. *Given $0 < \lambda \leq \Lambda$ and $\sigma_0 \in (0,2)$ there exist constants $C_0, M > 0$ and a $C^{1,1}$ function $\eta(x) : \mathbb{R}^n \to \mathbb{R}$ such that*

(1) *supp $\eta \subset B_{2\sqrt{n}}(0)$*
(2) *$\eta \leq -2$ in Q_3 and $\|\eta\|_\infty \leq M$*
(3) *For every $\sigma > \sigma_0$ we have $M^-(\eta, x) \leq C_0 \xi$ everywhere where ξ is a continuous function with support inside $B_{1/4}(0)$ and such that $0 \leq \xi \leq 1$.*

With this special function in hand, Theorem 1.2 allows (just as in the second order theory) to control the *average size* of a supersolution in terms of its *value at a point.* This point to average estimate implies a weak L^ϵ estimate for supersolutions which is the key step in the proof of the Krylov-Safonov and Caffarelli-Silvestre regularity theorems.

Lemma 10.2. *Given n, λ, Λ and σ_0 such that $0 < \lambda \leq \Lambda$ and $\sigma_0 \in (0,2)$, one can find positive constants $M > 1, \mu < 1$ and δ_0 such that if u satisfies:*

(1) *$u \geq 0$ in $\mathbb{R}^n$*
(2) *$\inf_{Q_3} u \leq 1$*
(3) *$M^-(u, x) \leq f$ in $Q_{4\sqrt{n}}$ (for some $\sigma > \sigma_0$), and $\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \delta_0$, $\|f\|_{L^\infty(Q_{4\sqrt{n}})} \leq 1$.*

Then we have the bound

$$|\{x \in Q_1 : u(x) \leq M\}| \geq \mu |Q_1|. \tag{10.1}$$

Proof of Lemma 10.2. Consider the function $w = u + \eta$ (η as in Proposition 10.1), then w satisfies (in the viscosity sense)

$$\begin{cases} M^-(w, x) \leq f(x) + C_0 \xi & \text{in } B_{4\sqrt{n}} \\ w \geq 0 & \text{on } \mathbb{R}^n \setminus B_{4\sqrt{n}}. \end{cases} \tag{10.2}$$

Moreover $-\inf_{Q_3}\{w\} \geq 1$ since $u \leq 1$ somewhere in Q_3 and $\eta \leq -2$ everywhere in Q_3. In this situation, Theorem 1.2 (rescaled to the ball $B_{4\sqrt{n}}$) tell us that

$$1 \leq C\lambda^{-n} (1 + C_0)^{(2-\sigma)/2} (\|f + C_0 \xi\|_{L^n(K_w)})^{\sigma/2},$$

where we recall $K_\omega = \{x \in B_{4\sqrt{n}} : w(x) = \Gamma_w(x)\}$, then

$$C^{-2/\sigma} \lambda^{2n/\sigma} (1 + C_0)^{-(2-\sigma)/\sigma} \leq \|f + C_0 \xi\|_{L^n(K_w)}.$$

Hence

$$C^{-2/\sigma} \lambda^{2n/\sigma} (1 + C_0)^{-(2-\sigma)/\sigma} \leq \delta_0 + C_0 |K_w \cap B_{1/4}|^{1/n}. \tag{10.3}$$

One then sees that picking δ_0 universally small one gets for a universal $\mu \in (0,1)$ the lower bound

$$\mu |Q_1| \leq |K_w \cap B_{1/4}| \leq |K_w \cap Q_1|. \tag{10.4}$$

Now $x \in K_w$ implies in particular that $w \leq 0$ therefore $u \leq -\eta \leq M$, where $M = \sup|\eta|$. Since $K_w \cap Q_1 \subset \{x \in Q_1 : u \leq M\}$ this last inequality proves the lemma. □

As done in [20, Chapter 4, Section 4.2] one can use the Calderón-Zygmund decomposition and 10.2 to prove the weak L^ϵ estimates we previously mentioned.

Theorem 10.3 (weak L^ϵ estimate)**.** *Let $u \geq 0$ in $\mathbb{R}^n$ be a supersolution of $M^-(u,x) \leq f$ in B_1 (for $\sigma > \sigma_0$) and such that $u(0) \leq 1$. Suppose that*

$$\|f\|_\infty \leq 1, \;\; \|f\|_{L^n(B_2)} \leq \delta_0, \;\; \sigma \in (\sigma_0, 2).$$

Then there are universal constants C, δ_0 and ϵ (i.e. determined by n, σ_0, λ and Λ) such that for all $t > 0$ we have

$$\left|\{u > t\} \cap B_{1/2}\right| \leq Ct^{-\epsilon}. \tag{10.5}$$

With this lemma in hand, that viscosity solutions to (1.6) are Hölder continuous or that they satisfy a Harnack inequality follows by standard arguments that can be found in [20, Theorem 4.3 and Proposition 4.10] for second order equations and in [17, Theorem 12.1 and Theorem 11.1] for integro-differential equations.

11. Applications and Open Problems

Here we would like to make a few remarks about further research directions and open questions where we anticipate Theorem 1.2 could be useful, some of which are in obvious analogy to the second order theory. Accordingly, the discussion below is only suggestive but we include it with the hope of stimulating further work.

Stochastic homogenization. As pointed out in the introduction, Theorem 1.2 will play an important role in the homogenization of stationary ergodic families of equations within the ellipticity class governed by M^-, and this will be presented in [34].

"$W^{\sigma,p}$" estimates. These would be analogous to the $W^{2,p}$ theory of Caffarelli [19]. Such estimates would allow use of the regularity theory of [17, 16, 15] to obtain $W^{\sigma,p}$ regularity for viscosity solutions of (1.3) in terms of the L^p norm of the right hand side f. It is important to remark that such estimates are not yet available for any kind of fully nonlinear equation of fractional order.

More general equations. Comparison and regularity theory for more general equations where their ellipticity is considered on choices of L_A (with varying scopes of generality) such as:

$$L_A(v,x) = (2-\sigma)\int_{\mathbb{R}^n} \delta u(x,y) a(x,y) \, |y|^{-n-\sigma} \, dy \tag{11.1}$$

where $a(x,y)$ is homogeneous of degree 0 in y,

$$L_A(v,x) = (2-\sigma)\int_{\mathbb{R}^n} \delta u(x,y)(y^T A(x) y) n(dy) \tag{11.2}$$

where $n(dy)$ is comparable to $|y|^{-n-\sigma}\,dy$ only *in some subset* of $\mathbb{R}^n$ (indicated to the authors in [30]),

$$L_A(v,x) = (2-\sigma)\int_{\mathbb{R}^n} \delta u(x,y)\frac{a(x,y)}{|y|^{n+\sigma}}dy, \quad \lambda \leq a \leq \Lambda \tag{11.3}$$

which corresponds to the family in [17], and

$$L_A(v,x) = (2-\sigma)\int_{\mathbb{R}^n} \delta u(x,y) n(x,dy) \tag{11.4}$$

where $n(x,dy)$ (for each x) is again a measure comparable to $|y|^{-n-\sigma}\,dy$ only in some subsets of $\mathbb{R}^n$ (for example, along certain directions through the origin).

The "right" nonlocal Monge-Ampére equation. It is worthwhile to find out whether another notion of nonlocal determinant can be found which is both extremal and carries a geometric interpretation (or "divergence structure") such that it achieves the key features listed in Section 2.5–(2.8), (2.9), and (2.10)– for a more general family of equations, such as those considered in [17].

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