21-121 Calculus

Review Session for Exam 2

Cherry Wan
cwan@andrew.cmu.edu

Concept 1: Fundamental Theorem of Calculus

If f is continuous on [a,b], then the function g defined by

PART I

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is continuous on [a,b] and differentiable on (a,b), and g'(x) = f(x)

If f is continuous on [a,b], then

PART II

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that F' = f.

Exercises

Use Part I of FTC to find the derivative of the following functions

1.

$$f(x) := \int_x^{10} \tan(t)\,dt$$

$$f(x) = -\int_{10}^{x} \tan(t)\,dt$$

$$\frac{d}{dx} f(x) = -\frac{d}{dx}\left[\int_{10}^{x} \tan(t)\,dt\right]$$

$$\boxed{f'(x) = -\tan(x)}$$

2.

$$y = \int_1^{1-3x} \frac{u^3}{1+u^2}\,du$$

$$v = 1-3x$$

$$\frac{dv}{dx} = -3$$

$$y' = \frac{d}{dv}\int_1^v \frac{u^3}{1+u^2}\,du \;\frac{dv}{dx}$$

$$y' = \frac{v^3}{1+v^2}\cdot(-3)$$

$$\boxed{y' = \frac{-3(1-3x)^3}{1+(1-3x)^2}}$$

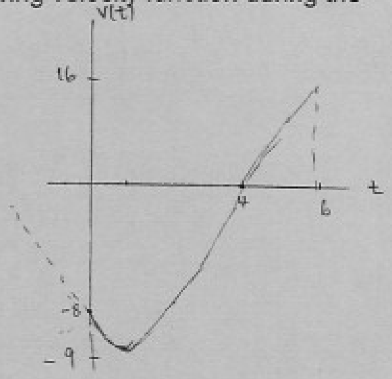
Concept 2: Net Change Theorem

~ It's basically Part II of FTC

Exercises

Note: distance is the total amount travelled
displacement is just the change from the beginning to the end

3. Find the distance and displacement travelled given the following velocity function during the given time interval



$v(t) := t^2 - 2t - 8 \qquad 0 \le t \le 6$

distance: $\int_0^6 |t^2-2t-8|\,dt$

displacement: $\int_0^6 t^2-2t-8\,dt$

displacement: $\int_0^6 t^2 - 2t - 8\,dt = \frac{1}{3}t^3 - t^2 - 8t\Big|_0^6$

$= \boxed{-12}$

distance $= -\int_0^4 t^2-2t-8\,dt + \int_4^6 t^2-2t-8\,dt$

$= -\left(\frac{1}{3}t^3-t^2-8t\Big|_0^4\right) + \left(\frac{1}{3}t^3-t^2-8t\Big|_4^6\right) = 26.67 + (-12+26.67) = \boxed{41.33}$

Concept 3: Integration by Substitution

Exercises

4. $$\int x^2\cdot\left(1+2\cdot x^3\right)^3 dx$$

$u = 1+2x^3 \quad du = 6x^2\,dx$

$x^2dx = \frac{1}{6}du$

$\frac{1}{6}\int u^3\,du = \frac{1}{6}\frac{1}{4}u^4 + C$

$= \boxed{\frac{1}{24}(1+2x^3)+C}$

5. $$\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$$

$u = \frac{1}{x} = x^{-1} \qquad du = -\frac{1}{x^2}dx$

$\frac{1}{x^2}dx = -du$

when $x = 1 \Rightarrow u = \frac{1}{1} = 1$

$x = 2 \Rightarrow u = \frac{1}{2}$

$\int_{1/2}^1 e^u\,du = e^u\Big|_{1/2}^1 = \boxed{e^1 - e^{1/2}}$

6. $\int \sec(x)\cdot\tan(x)\cdot\sqrt{1+\sec(x)}\,dx$

$u = 1+\sec(x) \quad du = \sec x \tan x\, dx$

$\int \sqrt{u}\,du = \int u^{1/2}\,du$

$= \frac{2}{3}u^{3/2} + C$

$= \boxed{\frac{2}{3}(1+\sec x)\sqrt{1+\sec(x)} + C}$

7. $\int_{e^1}^{e^4} \frac{1}{x\cdot\sqrt{\ln(x)}}\,dx$

when $x = e^1 \Rightarrow u = 1$

$x = e^4 \Rightarrow u = 4$

$u = \ln x \quad du = \frac{1}{x}\,dx$

$\int_1^4 \frac{1}{\sqrt{u}}\,du = \int_1^4 u^{-1/2}\,du = 2u^{1/2}\Big|_1^4$

$= 2\left[2-1\right] = \boxed{2}$

Concept 4: Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Note: you want to pick u so that it's derivative is simpler

Exercises

8. $\int (2x+3)\cdot e^x\,dx$

$u = 2x+3 \quad du = 2\,dx$

$dv = e^x dx \quad v = e^x$

$(2x+3)e^x - 2\int e^x\,dx$

$= \boxed{(2x+3)e^x - 2e^x + C}$

9. $\int \cos(\ln(x))\,dx$

$u = \cos(\ln x) \quad du = -\sin(\ln x)\cdot\frac{1}{x}\,dx$

$dv = dx \quad v = x$

$x\cos(\ln x) + \int x\sin(\ln x)\frac{1}{x}\,dx$

$= x\cos(\ln x) + \int \sin(\ln x)\,dx$

$u_1 = \sin(\ln x) \quad du_1 = \cos(\ln x)\frac{1}{x}\,dx$

$dv_1 = dx \quad v_1 = x$

$x\cos(\ln x) + \int \sin(\ln x)\,dx$

$= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx$

$2\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x)$

$\int \cos(\ln x)\,dx = \frac{1}{2}\left[x\cos(\ln x) + x\sin(\ln x)\right]$

Concept 5: Trigonometic Integrals

Rules

$$\boxed{\int \sin^{m}(x)\cdot\cos^{n}(x)\,dx}$$

If the power for cosine is odd, save one cosine factor and use $\cos^{2}(x) = 1 - \sin^{2}(x)$

Then substitute u = sin(x)

If the power of sine is odd, save one sine factor and use $\sin^{2}(x) = 1 - \cos^{2}(x)$

Then substitute u = cos(x)

If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^{2}(x) = \frac{1}{2}\cdot(1 - \cos(2x))$$

$$\sin x\cdot\cos x = \frac{1}{2}\cdot\sin(2x)$$

$$\cos^{2}(x) = \frac{1}{2}\cdot(1 + \cos(2x))$$

$$\boxed{\int \tan^{m}(x)\sec^{n}(x)\,dx}$$

If the power of secant is even, save a factor of $\sec^{2}(x)$ and use $\sec^{2}(x) = 1 + \tan^{2}(x)$

to express the remaining factors in terms of tan(x)

Then substitute u = tan(x)

If the power of tangent if odd, save a factor of sec(x)tan(x) and use $\tan^{2}(x) = \sec^{2}(x) - 1$

to express the remaining factors in terms of sec(x)

Then substitute u = sec(x)

$$\boxed{\int \tan(x)\,dx = \ln(|\sec(x)|) + C}$$

$$\boxed{\int \sec(x)\,dx = \ln(|\sec(x) + \tan(x)|) + C}$$

For the following cases on the left, considering using the properties on the right

$\int \sin(mx)\cdot\cos(nx)\,dx$ $\quad$ $\sin(A)\cdot\cos(B) = \frac{1}{2}\cdot(\sin(A-B) + \sin(A+B))$

$\int \sin(mx)\cdot\sin(nx)\,dx$ $\quad$ $\sin(A)\cdot\sin(B) = \frac{1}{2}\cdot(\cos(A-B) - \cos(A+B))$

$\int \cos(mx)\cdot\cos(nx)\,dx$ $\quad$ $\cos(A)\cdot\cos(B) = \frac{1}{2}\cdot(\cos(A-B) + \cos(A+B))$

Exercises

10. $\int \cos^5(x)\cdot\sin^4(x)\,dx$

$= \int \cos x \cdot \cos^4 x \cdot \sin^9 x \, dx$

$= \int \cos x \cdot (1-\sin^2 x)^2 \sin^4 x \, dx$

$= \int \cos x \cdot (1 - 2\sin^2 x + \sin^4 x) \sin^4 x \, dx$

$= \int \cos x \cdot (\sin^4 x - 2\sin^6 x + \sin^8 x) \, dx$

$u = \sin x \qquad du = \cos x \, dx$

$= \int u^4 - 2u^6 + u^8 \, du$

$= \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$

$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$

$= \boxed{\frac{1}{9}\sin^9 x - \frac{2}{7}\sin^7 x + \frac{1}{5}\sin^5 x + C}$

11. $\int \cos^7(x)\cdot\sin^9(x)\,dx$

$= \int \cos x \cdot \cos^6 x \cdot \sin^9 x \, dx$

$= \int \cos x \cdot (1-\sin^2 x)^3 \sin^9 x \, dx.$

$= \int \cos x \cdot (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x) \sin^9 x \, dx$

$= \int \cos x \cdot (\sin^9 x - 3\sin^{11} x + 3\sin^{13} x - \sin^{15} x) \, dx$

$u = \sin x \qquad du = \cos x \, dx$

$= \int u^9 - 3u^{11} + 3u^{13} - u^{15} \, du$

$= \frac{1}{10}u^{10} - \frac{1}{4}u^{12} + \frac{3}{14}u^{14} - \frac{1}{16}u^{16} + C$

$= \boxed{-\frac{1}{16}\sin^{16} x + \frac{3}{14}\sin^{14} x - \frac{1}{4}\sin^{12} x + \frac{1}{10}\sin^{10} x + C}$

12. $\displaystyle\int \cos^4(x)\cdot\sin^2(x)\,dx$

$= \frac{1}{8}\int (1+\cos(2x))^2\cdot(1-\cos(2x))\,dx$

$= \frac{1}{8}\int (1+2\cos(2x)+\cos^2(2x))(1-\cos(2x))\,dx$

$= \frac{1}{8}\int [1-\cos(2x)+2\cos(2x)-2\cos^2(2x)$
$+\cos^2(2x)-\cos^3(2x)]\,dx$

$= \frac{1}{8}\int 1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\,dx$

$= \frac{1}{8}x+\frac{1}{8}\int\cos(2x)dx-\frac{1}{8}\cdot\frac{1}{2}\int 1+\cos(4x)dx-\frac{1}{8}\int\cos^3(2x)dx$

$= \boxed{\frac{1}{8}x+\frac{1}{16}\sin(2x)-\frac{1}{16}x-\frac{1}{64}\sin(4x)-\frac{1}{16}\sin(2x)+\frac{1}{48}\sin^3(2x)+C}$

13. $\displaystyle\int \tan^5(x)\cdot\sec^4(x)\,dx$

$= \int \tan^5(x)(1+\tan^2(x))\sec^2x\,dx$

$= \int (\tan^5x+\tan^7x)\sec^2x\,dx$

$u = \tan x \quad du = \sec^2x\,dx$

$= \int (u^5+u^7)\,du$

$= \frac{1}{6}u^6+\frac{1}{8}u^8+C$

$\boxed{\frac{1}{6}\tan^6x+\frac{1}{8}\tan^7x+C}$

14. $\displaystyle\int \cos(7x)\cdot\cos(5x)\,dx$

$= \frac{1}{2}\int \cos(2x)+\cos(12x)\,dx$

$= \frac{1}{2}\int\cos(2x)\,dx+\frac{1}{2}\int\cos(12x)\,dx$

$= \boxed{\frac{1}{4}\sin(2x)+\frac{1}{24}\sin(12x)+C}$

15. $\displaystyle\int e^x\cdot\sin^7\left(e^x\right)dx$

$u = e^x \quad du = e^x\,dx$

$\int \sin^7u\,du$

$= \int \sin u\,(1-\cos^2u)^3\,du$

$= \int \sin u\,(1-3\cos^2u+3\cos^4u-\cos^6u)\,du$

$v = \cos u \quad dv = -\sin u\,du$

$= -\int 1-3v^2+3v^4-v^6\,dv$

$= -v+v^3-\frac{3}{5}v^5+\frac{1}{7}v^7+C$

$= \boxed{-\cos(e^x)+\cos^3(e^x)-\frac{3}{5}\cos^5(e^x)+\frac{1}{7}\cos^7(e^x)+C}$

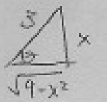
Concept 6: Trigonometric Substitution

Table of Trigonometric Substitutions

Expression	Substitution		Identity
$\sqrt{a^2-x^2}$	$x = a\cdot\sin(\theta)$	$\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$	$1-\sin^2(\theta)=\cos^2(\theta)$
$\sqrt{a^2+x^2}$	$x = a\cdot\tan(\theta)$	$\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$	$1+\tan^2(\theta)=\sec^2(\theta)$
$\sqrt{x^2-a^2}$	$x = a\cdot\sec(\theta)$	$0\le\theta\le\frac{\pi}{2}$ or $\pi\le\theta\le\frac{3\cdot\pi}{2}$	$\sec^2(\theta)-1=\tan^2(\theta)$

Exercises

16. $$\int x^3\sqrt{9-x^2}\,dx$$



$x = 3\sin\theta \qquad dx = 3\cos\theta\, d\theta$

$\int (3\sin\theta)^3\sqrt{9-9\sin^2\theta}\ 3\cos\theta\, d\theta$

$= 243\int \sin^3\theta\cdot\cos^2\theta\, d\theta$

$= 243\int \sin\theta\cdot(1-\cos^2\theta)\cos^2\theta\, d\theta$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$= 243\int \sin\theta\,(\cos^2\theta - \cos^4\theta)\, d\theta$

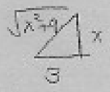
$= -243\int u^2 - u^4\, du = -\frac{243}{3}u^3 + \frac{243}{5}u^5 + c$

$= -81(\cos\theta)^3 + \frac{243}{5}\cos^5\theta + c$

$= -81\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + \frac{243}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5 + C$

$= \boxed{-3(9-x^2)\sqrt{9-x^2} + \frac{1}{5}(9-x)^2\sqrt{9-x^2} + C}$

17. $$\int \frac{x^3}{\sqrt{x^2+9}}\,dx$$



$x = 3\tan\theta \qquad dx = 3\sec^2\theta\, d\theta$

$\int \frac{27\tan^3\theta}{3\sec\theta}\cdot 3\sec^2\theta\, d\theta$

$= 27\int \tan^3\theta \sec\theta\, d\theta$

$= 27\int \tan\theta\sec\theta\,(\sec^2\theta - 1)\, d\theta \qquad u = \sec\theta$

$du = \sec\theta\tan\theta\, d\theta$

$= 27\int u^2 - 1\, du$

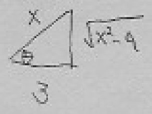
$= 27\left(\frac{1}{3}u^3 - u\right) + C = 9u^3 - 27u + C$

$= 9\sec^3\theta - 27\sec\theta + C$

$= 9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 27\left(\frac{\sqrt{x^2+9}}{3}\right) + C$

$= \boxed{\frac{1}{3}(x^2+9)\sqrt{x^2+9} - 9\sqrt{x^2+9} + C}$

18. $$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$$



$x = 3\sec\theta \qquad dx = 3\tan\theta\sec\theta\, d\theta$

$\int \frac{1}{9\sec^2\theta\cdot 3\cdot\tan\theta}\cdot 3\tan\theta\sec\theta\, d\theta$

$\sec\theta = \frac{x}{3}$

$\cos\theta = \frac{3}{x}$

$= \frac{1}{9}\int \cos\theta\, d\theta$

$= \frac{1}{9}\sin\theta + C \qquad = \boxed{\frac{1}{9}\left(\frac{\sqrt{x^2-9}}{x}\right) + C}$

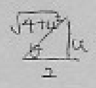
19. $$\int \frac{1}{\sqrt{t^2-6t+13}}\,dt$$

$\int \frac{1}{\sqrt{t^2-6t+9+4}}\, dt \qquad u = t-3$

$du = dt$

$= \int \frac{1}{\sqrt{(t-3)^2+4}}\, dt = \int \frac{1}{\sqrt{u^2+4}}\, du$

$u = 2\tan\theta \qquad du = 2\sec^2\theta\, d\theta$



$= \int \frac{1}{2\sec\theta}\cdot 2\sec^2\theta\, d\theta$

$= \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{\sqrt{4+u^2}}{2} + \frac{u}{2}\right| + C$

$= \boxed{\ln\left|\frac{\sqrt{4+(t-3)^2}}{2} + \frac{t-3}{2}\right| + C}$

Concept 7: Integration of Rational Functions

Note: Have to check if function is proper!

Four cases:

(1) The denominator Q(x) is a product of distinct linear factors
(2) Q(x) is a product of linear factors, some of which are repeated
(3) Q(x) containes irreducible quadratic factors, none of which is repeated
(4) Q(x) contains a repeated irreducible quadratic factor

Exercises

20. $$\int \frac{1}{x^2 - a^2}\,dx$$

$a \neq 0$

$$\frac{1}{(x+a)(x-a)} = \frac{B}{x+a} + \frac{C}{x-a}$$

$$= \frac{B(x-a)+C(x+a)}{(x+a)(x-a)}$$

$B+C=0$

$-Ba+Ca=1 \Rightarrow$ $B+C=0$ $B=-C$

$-B+C=\frac{1}{a}$

$2C=\frac{1}{a}$ $C=\frac{1}{2a}$ $B=-\frac{1}{2a}$

$$-\frac{1}{2a}\int\frac{1}{x+a}\,dx + \frac{1}{2a}\int\frac{1}{x-a}\,dx$$

$$= \boxed{-\frac{1}{2a}\ln|x+a| + \frac{1}{2a}\ln|x-a| + C}$$

21. $$\int \frac{1}{x^2 + a^2}\,dx$$

$$= \boxed{\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C}$$

$u = x-\frac{1}{2}$ $x = u+\frac{1}{2}$

$du = dx$ $-x+2 = -u-\frac{1}{2}+2$

$= -u+\frac{3}{2}$

$$(*) = \frac{1}{3}\ln|x+1| + \frac{1}{3}\int\frac{-u+\frac{3}{2}}{u^2+\frac{3}{4}}\,du$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int\frac{u}{u^2+\frac{3}{4}}\,du + \frac{1}{2}\int\frac{1}{u^2+\frac{3}{4}}\,du$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left|u^2+\frac{3}{4}\right| + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)+C$$

$$= \boxed{\frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left|(x-\frac{1}{2})^2+\frac{3}{4}\right| + \frac{\sqrt{3}}{4}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C}$$

22. $$\int \frac{x^3}{x^3 + 2x^2 + x}\,dx$$

$$\begin{array}{r} 1 \\ x^3+2x^2+x\,\overline{)\,x^3\qquad\qquad} \\ \underline{x^3+2x^2+x} \\ -2x^2-x \end{array}$$

$$= \int 1 + \frac{-2x^2-x}{x^3+2x^2+x}\,dx$$

$$= \int 1\,dx - \int\frac{2x^2+x}{x(x^2+2x+1)}\,dx$$

$$= x - \int\frac{2x^2+x}{x(x+1)^2}\,dx$$

$$\frac{2x^2+x}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} = \frac{A(x+1)^2+Bx(x+1)+Cx}{x(x+1)^2}$$

$$= \frac{A(x^2+2x+1)+Bx^2+Bx+Cx}{x(x+1)^2} = \frac{x^2(A+B)+x(2A+B+C)+(A)}{x(x+1)^2}$$

$$\begin{cases} A+B=2 \\ 2A+B+C=1 \\ A=0 \end{cases}$$

$$\int\frac{x^3}{x^3+2x^2+x}\,dx = x - \int\frac{2}{x+1}\,dx - \int\frac{1}{(x+1)^2}\,dx$$

$$= \boxed{x - 2\ln|x+1| + \frac{1}{x+1} + C}$$

$$\begin{cases} A=0 \\ B=2 \\ C=-1 \end{cases}$$

23. $$\int \frac{1}{x^3 + 1}\,dx$$

$$= \int\frac{1}{(x+1)(x^2-x+1)}\,dx$$

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$= \frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}$$

$$= \frac{A(x^2-x+1)+(Bx^2+Bx+Cx+C)}{(x+1)(x^2-x+1)}$$

$A+B=0$ $B=-A$

$-A+B+C=0$

$A+C=1$ $C=1-A$

$-A-A+1-A=0$

$3A=1$, $A=\frac{1}{3}$

$$\begin{cases} B=-\frac{1}{3} \\ C=\frac{2}{3} \end{cases}$$

$$\int\frac{1}{x^3+1}\,dx = \frac{1}{3}\int\frac{1}{x+1}\,dx + \frac{1}{3}\int\frac{-x+2}{x^2-x+1}\,dx$$

$$= \frac{1}{3}\ln|x+1| + \frac{1}{3}\int\frac{-x+2}{x^2-x+\frac{1}{4}+\frac{3}{4}}\,dx$$

$$= \frac{1}{3}\int\frac{-x+2}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx \quad (*)$$

24 $$\int \frac{\left(1-x+2x^2-x^3\right)}{x\cdot\left(x^2+1\right)^2}\,dx$$

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$= \frac{A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x}{x(x^2+1)^2}$$

$$= \frac{x^4(A+B)+x^3(C)+x^2(2A+B+D)+x(C+E)+A}{x(x^2+1)^2}$$

$A+B=0 \Rightarrow B=-1$

$C=-1$

$2A+B+D=2 \Rightarrow D=1$

$C+E=-1 \Rightarrow E=0$

$A=1$

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$$

$$= \int \frac{1}{x}\,dx \mp \int \frac{x+1}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \ln|x| - \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \boxed{\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}(x) - \frac{1}{2}\,\frac{1}{x^2+1} + C}$$

25 $$\int \frac{1}{x^2-2x}\,dx$$

$$= \int \frac{1}{x(x-2)}$$

$$\frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} = \frac{A(x-2)+Bx}{x(x-2)}$$

$A+B=0 \Rightarrow B=\frac{1}{2}$

$-2A=1 \Rightarrow A=-\frac{1}{2}$

$$\int \frac{1}{x^2-2x} = -\frac{1}{2}\int \frac{1\,dx}{x} + \frac{1}{2}\int \frac{1}{x-2}\,dx$$

$$= \boxed{-\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + C}$$

26 $$\int \frac{(2x+1)}{4x^2+12x-7}\,dx$$

$2x+1 = u-2$

$$= \int \frac{(2x+1)}{4x^2+12x+9-16}\,dx$$

$$= \int \frac{2x+1}{(2x+3)^2-16}\,dx$$

$u = 2x+3.$

$du = 2\,dx$

$$= \frac{1}{2}\int \frac{u-2}{u^2-16}\,du = \frac{1}{2}\int \frac{u-2}{(u+4)(u-4)}\,du.$$

$$\frac{u-2}{(u+4)(u-4)} = \frac{A}{u+4} + \frac{B}{u-4}$$

$$= \frac{Au-4A+Bu+4B}{u^2-16}$$

$A+B=1 \quad A=1-B$

$-4A+4B=-2$

$-4+4B+4B=2$

$8B=2 \quad B=\frac{1}{4}$

$A=\frac{3}{4}.$

$$= \frac{1}{2}\cdot\frac{3}{4}\int \frac{1}{u+4}\,du + \frac{1}{2}\,\frac{1}{4}\int \frac{1}{u-4}\,du$$

$$= \boxed{\frac{3}{8}\ln|2x+7| + \frac{1}{8}\ln|2x-1| + C}$$

27. $$\int \frac{\left(x^3-2x^2+x+1\right)}{x^4-3x^2-4}\,dx$$

$$= \int \frac{x^3-2x^2+x+1}{(x^2+1)(x^2-4)}\,dx = \int \frac{x^3-2x^2+x+1}{(x^2+1)(x+2)(x-2)}\,dx$$

$$\frac{x^3-2x^2+x+1}{(x^2+1)(x+2)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2} + \frac{D}{x-2}$$

$$= \frac{x^3(A+C+D)+x^2(B-2C+2D)+x(-4A+C+D)+(-4B-2C+2D)}{(x^2+1)(x+2)(x-2)}$$

$$\begin{cases} A=0 \\ B=-\frac{3}{5} \\ C=\frac{17}{20} \\ D=\frac{3}{20} \end{cases}$$

$$\int \frac{x^3-2x^2+x+1}{x^4-3x^2-4}\,dx$$

$$= -\frac{3}{5}\int \frac{1}{x^2+1}\,dx + \frac{17}{20}\int \frac{1}{x+2}\,dx + \frac{3}{20}\int \frac{1}{x-2}\,dx$$

$$= \boxed{\frac{3}{5}\tan^{-1}(x) + \frac{17}{20}\ln|x+2| + \frac{3}{20}\ln|x-2| + C}$$

Concept 8: Areas between Curves

The area between teh curves y = f(x) and y = g(x) and between x=a and x = b is

$$A = \int_a^b \left|f(x) - g(x)\right|\,dx$$

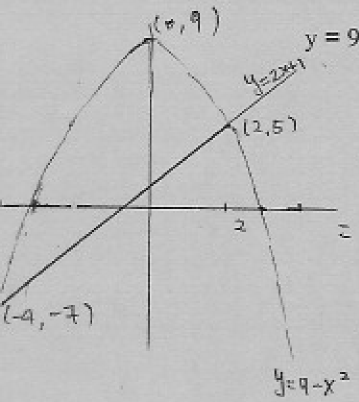
Exercises

Set up the area between the following curves

28.

$y = 2x + 1$

$y = 9 - x^2$



$$\int_{-4}^{2} |(9-x^2) - (2x+1)|\,dx$$

$$= \int_{-4}^{2} |-x^2-2x+8|\,dx$$

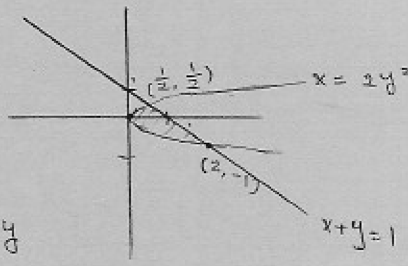
$$= \boxed{\int_{-4}^{2} (-x^2-2x+8)\,dx}$$

29.

$x = 1 - y$

$x + y = 1$

$x = 2y^2$



$$\int_{-1}^{1/2} |1-y-2y^2|\,dy$$

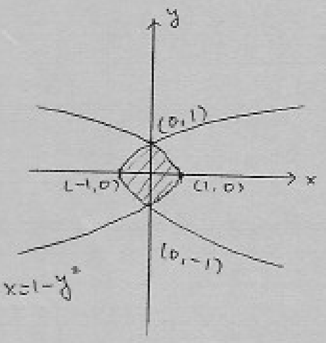
$$= \boxed{\int_{-1}^{1/2} (1-y-2y^2)\,dy}$$

30. $x = 1 - y^2$

$x = y^2 - 1$

$$\int_{-1}^{1} |1-y^2-y^2-1|\,dy$$

$$= \boxed{\int_{-1}^{1} (-2y^2)\,dy}$$



31. $y = x^3 - x$

$y = 3x$

$$= \int_{-2}^{0} x^3-x-3x\,dx +$$

$$\int_{0}^{2} 3x-x^3-x\,dx$$

$$= \boxed{\int_{-2}^{0} (x^3-4x)dx + \int_{0}^{2} (-x^3+2x)dx}$$