Adjustable Robust Optimization Models for Nonlinear Multi-Period Optimization

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March 1, 2005

Abstract

We study multi-period nonlinear optimization problems whose parameters are uncertain. We assume that uncertain parameters are revealed in stages and model them using the adjustable robust optimization approach. For problems with polytopic uncertainty, we show that quasi-convexity of the optimal value function of certain subproblems is sufficient for the reducibility of the resulting robust optimization problem to a single-level deterministic problem. We relate this sufficient condition to the quasi cone-convexity of the feasible set mapping for adjustable variables and present several examples and applications satisfying these conditions.

1 Introduction

Uncertainty is an inevitable feature of many decision-making environments. On a regular basis engineers, economists, investment professionals, and others need to make decisions to optimize a system with incomplete information and considerable uncertainty. Robust optimization (RO) is a term that is used to describe both modeling strategies and solution methods for optimization problems that are defined by uncertain inputs [3, 4]. The objective of robust optimization models and algorithms is to obtain solutions that are guaranteed to perform well (in terms of feasibility and near-optimality) for all, or at least most, possible realizations of the uncertain input parameters.

Standard robust optimization formulations assume that the uncertain parameters will not be observed until all the decision variables are determined and therefore do not allow for recourse actions that may be based on realized values of some of these parameters.

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This is not always the case for uncertain optimization problems. In particular, multi-period decision models involve uncertain parameters some of which are revealed during the decision process. Therefore, a subset of the decision variables can be chosen after these parameters are observed in a way to correct the sub-optimality of the decisions made with less information in earlier stages. *Adjustable robust optimization* (ARO) formulations model these decision environments, allowing recourse action. These models are related to the two-stage (or multi-stage) stochastic programming formulations with recourse.

ARO models were recently introduced in [6, 11] for uncertain linear programming problems. Consider, for example, the two-stage linear optimization problem given below whose first-stage decision variables ($\boldsymbol{x}^1$) need to be determined now, while the second-stage decision variables ($\boldsymbol{x}^2$) can be chosen after the uncertain parameters of the problem (A^1, A^2, and $\boldsymbol{b}$) are realized:

$$\min_{\boldsymbol{x}^1,\boldsymbol{x}^2} \{\boldsymbol{c}^\top \boldsymbol{x}^1 : A^1\boldsymbol{x}^1 + A^2\boldsymbol{x}^2 \leq \boldsymbol{b}\}. \tag{1}$$

Note that the second stage variables $\boldsymbol{x}^2$ do not appear in the objective function–this is what Ben-Tal *et al.* [6] call the "normalized" form of the problem. We can consider this simpler and convenient form without loss of generality, as discussed in [6, 11]. Let $\mathcal{U}$ denote the *uncertainty set* for parameters A^1, A^2, and $\boldsymbol{b}$, i.e., the set of all potentially realizable values of these uncertain parameters. The standard robust optimization formulation for this problem seeks to find vectors $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ that optimize the objective function and satisfy the constraints of the problem for all possible realizations of the constraint coefficients. In this formulation, both sets of variables must be chosen before the uncertain parameters can be observed and therefore cannot depend on these parameters. Consequently, the standard robust counterpart of this problem can be written as follows:

$$\min_{\boldsymbol{x}^1} \{\boldsymbol{c}^\top \boldsymbol{x}^1 : \exists \boldsymbol{x}^2 \ \forall (A^1, A^2, \boldsymbol{b}) \in \mathcal{U} : A^1\boldsymbol{x}^1 + A^2\boldsymbol{x}^2 \leq \boldsymbol{b}\}. \tag{2}$$

In contrast, the adjustable robust optimization formulation allows the choice of the second-period variables $\boldsymbol{x}^2$ to depend on the realized values of the uncertain parameters. As a result, the adjustable robust counterpart problem is given as follows:

$$\min_{\boldsymbol{x}^1} \{\boldsymbol{c}^\top \boldsymbol{x}^1 : \forall (A^1, A^2, \boldsymbol{b}) \in \mathcal{U},\ \exists \boldsymbol{x}^2 = \boldsymbol{x}^2(A^1, A^2, \boldsymbol{b}) : A^1\boldsymbol{x}^1 + A^2\boldsymbol{x}^2 \leq \boldsymbol{b}\}. \tag{3}$$

Clearly, the feasible set of the second problem is larger than that of the first problem in general and therefore the model is more flexible. ARO models can be especially useful when robust counterparts are unnecessarily conservative. The price to pay for this additional modeling flexibility appears to be the increased difficulty of the resulting ARO formulations. Even for problems where the robust counterpart is tractable, it can happen that the ARO formulation leads to an NP-hard problem; see, for example, Theorem 3.5 in [11]. One of the factors that contribute to the added difficulty in ARO models is the fact that the feasible set of the recourse actions (second-period decisions) depends on both the first-period decisions and the realization of the uncertain parameters. Consequently, the pioneering study of

Ben-Tal *et al.* [6] on this subject considers several simplifying assumptions either on the uncertainty set, or on the dependence structure of recourse actions to uncertain parameters.

Adjustable robust optimization models result from natural formulations of multi-stage decision problems with uncertain parameters and the development of efficient solution techniques for such problems represents the next frontier in robust optimization research. In this article, we contribute to this research by developing tractable ARO formulations for a class of multi-period optimization problems with nonlinear constraints and objective functions. After considering the simple case of finite uncertainty sets, we focus on polytopic uncertainty sets defined as a convex hull of a finite set of points. We investigate sufficient conditions under which the ARO problem reduces to a single deterministic optimization problem. In particular, we show that when the feasible sets of the second-period problem satisfy a certain quasi-convexity property such a reduction is possible. We provide examples exhibiting this property.

The rest of this article is organized as follows. In Section 2 we discuss adjustable robust optimization models for two-period optimization problems with finite and polytopic uncertainty sets and derive a sufficient condition for the tractability of these problems. In Section 3 we relate the quasi cone-convexity of the mapping that defines feasible sets for adjustable variables to the sufficient condition introduced in the previous section. In Section 4 we provide several low-dimensional examples of feasible set mappings that satisfy the quasi cone-convexity property. We discuss the application of the results we developed to problems in financial mathematics in Section 5.

2 Adjustable Robust Optimization Models

In this section we consider a two-period decision-making environment. We let $\boldsymbol{u}$ and $\boldsymbol{v}$ represent the first and second-period decision variables, respectively, and U and V represent their feasible sets. We let $\boldsymbol{p}$ denote a vector of parameters for the problem. The objective is to choose feasible vectors $\boldsymbol{u} \in U$ and $\boldsymbol{v} \in V$ such that the objective function, denoted by $f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p})$ is minimized:

$$\inf_{\boldsymbol{u} \in U} \inf_{\boldsymbol{v} \in V} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}). \tag{4}$$

When the vector $\boldsymbol{p}$ is known and the feasible set V for the second-period decision variables is independent of $\boldsymbol{u}$, the first-period decisions, this problem can be solved as a deterministic, single-period problem. We consider an environment where the parameter vector $\boldsymbol{p}$ is uncertain but is known to belong to an uncertainty set P. Throughout the paper, we will make the reasonable assumption that the uncertainty set P is closed and bounded. We assume that these parameters, possibly determined by events that take place between two periods, will be realized and observed after the first-period decision are made but before the second-period decisions need to be made. Furthermore, we assume that the feasible set V for the second-period decisions depends on the choice of $\boldsymbol{u}$ as well as the observed values of the parameters $\boldsymbol{p}$ and therefore, is denoted by $V(\boldsymbol{u}, \boldsymbol{p})$, or equivalently, by $V_{\boldsymbol{u}}(\boldsymbol{p})$ in the remainder of the article.

As we mentioned in the Introduction, an example of this framework with a linear objective function and linear constraints as in (1) is considered in [6, 11]. Problem (1) corresponds to the choices of $f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}) = \boldsymbol{c}^\top \boldsymbol{u}$, $U = \mathbb{R}^n$ for a vector $\boldsymbol{u}$ of dimension n, and $V(\boldsymbol{u}, \boldsymbol{p}) = \{\boldsymbol{v} : A_1(\boldsymbol{p})\boldsymbol{u} + A_2(\boldsymbol{p})\boldsymbol{v} \leq \boldsymbol{b}(\boldsymbol{p})\}$. For this problem, in addition to the standard robust counterpart (RC) problem (2), Ben-Tal *et al.* [6] introduce and study the so-called *adjustable robust counterpart* (ARC) problem given in (3). It is easy to see that the ARC is more flexible (has a larger feasible set) than the RC. Ben-Tal *et al.* argue that the ARC is also more difficult in general than the corresponding RC and give examples of problems whose robust counterparts are tractable while their ARC formulations are NP-hard. They also note two special cases: one where the ARC is equivalent to the RC, and therefore is easy when the RC is, and another where the ARC is a simple linear program. The first case arises when the uncertainty is assumed to be *constraint-wise*. The assumption of constraint-wise uncertainty is discussed in detail in [6, 11] and indicates that uncertain parameters appearing in a particular constraint of the problem do not appear in any of the remaining constraints. In fact, under the assumption of constraint-wise uncertainty, Guslitser shows that the ARC and RC are equivalent even for nonlinear convex programming problems [11]. The second case, namely the case where the ARC is a linear program arises if the matrix A^2 in (1) is certain and the uncertainty set for the matrix vector pair $(A^1, \boldsymbol{b})$ is given as the convex hull of a finite set. We will explore similar uncertainty sets below, but for nonlinear optimization problems.

2.1 Min-max-min Representation of the ARC Problem

For problem (4) with $V = V(\boldsymbol{u}, \boldsymbol{p})$, the adjustable robust counterpart problem is obtained as follows:

$$\inf_{\boldsymbol{u} \in U, t} \{t : \forall \boldsymbol{p} \in P \;\; \exists \boldsymbol{v} \in V(\boldsymbol{u}, \boldsymbol{p}) : f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}) \leq t\}. \tag{5}$$

We sometimes find it more convenient to work with the following representation of the ARC problem:

$$\inf_{\boldsymbol{u} \in U} \sup_{\boldsymbol{p} \in P} \inf_{\boldsymbol{v} \in V(\boldsymbol{u}, \boldsymbol{p})} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}). \tag{6}$$

Using the convention that $\inf_{\boldsymbol{v} \in V(\boldsymbol{u}, \boldsymbol{p})} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}) = \infty$ when $V(\boldsymbol{u}, \boldsymbol{p}) = \emptyset$ for some $\boldsymbol{u} \in U$ and $\boldsymbol{p} \in P$, the equivalence of problems (5) and (6) is shown in the following proposition:

Proposition 1. *The adjustable robust counterpart problem (5) and the min-max-min problem (6) are equivalent.*

Proof: As we discussed above, the ARC problem (5) was proposed in [6] where $\boldsymbol{u}$ is called a non-adjustable vector variable and $\boldsymbol{v}$ is called an adjustable vector variable. One of the following two cases must hold:

(a) there exists $\boldsymbol{u} \in U$ such that $V(\boldsymbol{u}, \boldsymbol{p}) \neq \emptyset$ for all $\boldsymbol{p} \in P$,

(b) for all $\boldsymbol{u} \in U$, there exists $\boldsymbol{p} \in P$ such that $V(\boldsymbol{u}, \boldsymbol{p}) = \emptyset$.

We'll show that problems (5) and (6) have identical optimal values in both cases.

For (a) we assume that there exists $\boldsymbol{u} \in U$ such that $V(\boldsymbol{u}, \boldsymbol{p}) \neq \emptyset$ for all $\boldsymbol{p} \in P$. Define the subset $U^{(a)}$ of U as

$$U^{(a)} \equiv \{\boldsymbol{u} \in U : V(\boldsymbol{u}, \boldsymbol{p}) \neq \emptyset, \ \forall \boldsymbol{p} \in P\}.$$

By our assumption, $U^{(a)}$ is nonempty. Next we show that (6) is equivalent to

$$\inf_{\boldsymbol{u} \in U^{(a)}} \sup_{\boldsymbol{p} \in P} \inf_{\boldsymbol{v} \in V(\boldsymbol{u}, \boldsymbol{p})} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}). \tag{7}$$

It is obvious that the optimal value of (6) is less than or equal to that of (7) because of $U^{(a)} \subseteq U$, so it is enough to show that the optimal solution $\boldsymbol{u}^* \in U$ of (6) must lie in $U^{(a)}$ for the equivalence of (6) and (7). Indeed, if we suppose that $\boldsymbol{u}^* \notin U^{(a)}$, there must exist $\boldsymbol{p} \in P$ such that $V(\boldsymbol{u}^*, \boldsymbol{p}) = \emptyset$ and optimal value (6) must be ∞. Recalling our assumption on the compactness of P, this contradicts the fact that $\sup_{\boldsymbol{p} \in P} \inf_{\boldsymbol{v} \in V(\boldsymbol{u}, \boldsymbol{p})} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p})$ is bounded above when $\boldsymbol{u} \in U^{(a)}$. Therefore $\boldsymbol{u}^* \in U^{(a)}$ and the equivalence of (6) and (7) is shown.
Next we show that (7) is equivalent to (5). To "normalize" the problem–this is the term used by Ben-Tal *et al.* [6] for problems with linear objective functions with no uncertainty–we introduce an artificial variable t to represent the objective function of (7) and impose the constraint $t \geq \inf_{\boldsymbol{v}(\boldsymbol{p}) \in V(\boldsymbol{u}, \boldsymbol{p})} f\left(\boldsymbol{u}, \boldsymbol{v}(\boldsymbol{p}), \boldsymbol{p}\right), \ \ \forall \boldsymbol{p} \in P, \forall \boldsymbol{u} \in U^{(a)}$. Then,

$$\begin{aligned}
& \inf_{\boldsymbol{u} \in U^{(a)}} \sup_{\boldsymbol{p} \in P} \inf_{\boldsymbol{v} \in V(\boldsymbol{u}, \boldsymbol{p})} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}) \\
\iff & \left| \begin{array}{ll} \inf\limits_{\boldsymbol{u} \in U^{(a)}} & t \\ \text{s.t.} & \inf\limits_{\boldsymbol{v}(\boldsymbol{p}) \in V(\boldsymbol{u}, \boldsymbol{p})} f\left(\boldsymbol{u}, \boldsymbol{v}(\boldsymbol{p}), \boldsymbol{p}\right) \leq t, \ \ \forall \boldsymbol{p} \in P \end{array} \right. \\
\iff & \inf_{\boldsymbol{u} \in U^{(a)}, t} \left\{t : \forall \boldsymbol{p} \in P \ \ \exists \boldsymbol{v} \in V\left(\boldsymbol{u}, \boldsymbol{p}\right) : f\left(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}\right) \leq t\right\}, \\
\iff & \inf_{\boldsymbol{u} \in U, t} \left\{t : \forall \boldsymbol{p} \in P \ \ \exists \boldsymbol{v} \in V\left(\boldsymbol{u}, \boldsymbol{p}\right) : f\left(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}\right) \leq t\right\},
\end{aligned}$$

and we find that (5) and (6) are equivalent.

In case (b), the ARC problem (5) has no feasible solutions and therefore the optimal value of (5) is ∞. Similarly, we observe that for all $\boldsymbol{u} \in U$, the optimal value of $\sup_{\boldsymbol{p} \in P} \inf_{\boldsymbol{v} \in V(\boldsymbol{u}, \boldsymbol{p})} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p})$ is also ∞. Therefore, both problems (5) and (6) attain the same optimal value ∞. □ ∎

In the next two subsections, we explore ARC problems for the cases of a finite uncertainty set and of a polytopic uncertainty set, i.e., a set given as the convex hull of a finite number of points.

2.2 Adjustable Robust Counterpart with Finite Uncertainty Sets

We first consider the case that P consists of a finite number of elements: $P = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k\}$. From problem (5), we see that for every $\boldsymbol{p} \in P$, there is a corresponding variable $\boldsymbol{v}$ satisfying constraints of (5). We introduce new variables $\boldsymbol{v}_i$ to represent the second-period decision variables corresponding to each element $\boldsymbol{p}_i, i \in \{1, \ldots, k\}$ of the uncertainty set and transform (5) into an equivalent single-level optimization problem.

$$
\begin{array}{rll}
\inf\limits_{\boldsymbol{u}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, t} & t & \\
\text{s.t.} & f(\boldsymbol{u}, \boldsymbol{v}_i, \boldsymbol{p}_i) \leq t, & (i = 1, \ldots, k) \\
& \boldsymbol{u} \in U, & \\
& \boldsymbol{v}_i \in V(\boldsymbol{u}, \boldsymbol{p}_i), & (i = 1, \ldots, k).
\end{array}
\tag{8}
$$

Despite the increase in the number of variables through duplication, this single-level, deterministic optimization problem is a tractable problem for many classes of functions f and sets $V(\boldsymbol{u}, \boldsymbol{p})$. As an example, we consider the following set up:

$$
\begin{array}{rcl}
f(\boldsymbol{u}, \boldsymbol{v}_i, \boldsymbol{p}_i) & \equiv & f_0(\boldsymbol{u}, \boldsymbol{v}_i, \boldsymbol{p}_i) \\
U & \equiv & \{\boldsymbol{u} : g_\ell(\boldsymbol{u}) \leq 0, \;\; \ell = 1, \ldots, m_1\}, \\
V(\boldsymbol{u}, \boldsymbol{p}_i) & \equiv & \{\boldsymbol{v}_i : f_\ell(\boldsymbol{u}, \boldsymbol{v}_i, \boldsymbol{p}_i) \leq 0, \;\; \ell = 1, \ldots, m_2\}
\end{array}
\tag{9}
$$

where

$$
\begin{array}{rl}
f_\ell(\boldsymbol{u}, \boldsymbol{v}_i, \boldsymbol{p}_i) = f_\ell(\boldsymbol{w}_i, \boldsymbol{p}_i) = & \boldsymbol{w}_i^\top Q_\ell(\boldsymbol{p}_i)\boldsymbol{w}_i + \boldsymbol{q}_\ell(\boldsymbol{p}_i)^\top \boldsymbol{w}_i + b_\ell(\boldsymbol{p}_i), \ell = 0, \ldots, m_2 \\
g_\ell(\boldsymbol{u}) = & \boldsymbol{u}^\top R_\ell \boldsymbol{u} + \boldsymbol{r}_\ell^\top \boldsymbol{u} + d_\ell, \ell = 1, \ldots, m_1 \\
\boldsymbol{w}_i = & (\boldsymbol{u}, \boldsymbol{v}_i)^\top, i = 1, \ldots, k.
\end{array}
$$

Above, we can use arbitrary functions $Q_\ell(\boldsymbol{p})$, $\boldsymbol{q}_\ell(\boldsymbol{p})$ and $b_\ell(\boldsymbol{p})$ of the uncertain parameter vector $\boldsymbol{p} \in P$ as long as the images of these functions are in the appropriate spaces. Using (9) and defining $Q_{i\ell} \equiv Q_\ell(\boldsymbol{p}_i), \boldsymbol{q}_{i\ell} \equiv \boldsymbol{q}_\ell(\boldsymbol{p}_i)$, and $b_{i\ell} \equiv b_\ell(\boldsymbol{p}_i)$ for all i, we rewrite problem (8) as follows:

$$
\begin{array}{rll}
\min\limits_{\boldsymbol{u}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, t} & t & \\
\text{s.t.} & \boldsymbol{w}_i^\top Q_{i0} \boldsymbol{w}_i + \boldsymbol{q}_{i0}^\top \boldsymbol{w}_i + b_{i0} \leq t, & (i = 1, \ldots, k) \\
& \boldsymbol{u}^\top R_\ell \boldsymbol{u} + \boldsymbol{r}_\ell^\top \boldsymbol{u} + d_\ell \leq 0, & (\ell = 1, \ldots, m_1) \\
& \boldsymbol{w}_i^\top Q_{i\ell} \boldsymbol{w}_i + \boldsymbol{q}_{i\ell}^\top \boldsymbol{w}_i + b_{i\ell} \leq 0, & (i = 1, \ldots, k, \ell = 1, \ldots, m_2).
\end{array}
\tag{10}
$$

This is a quadratically constrained optimization problem. If all the matrices $Q_{i\ell}$ as well as R_ℓ are positive-semi definite, then the feasible set is convex, the problem can be reformulated as a second-order cone programming problem as [5] shows, and can be solved easily using existing methods and software.

2.3 Adjustable Robust Counterpart with Polytopic Uncertainty Sets

In this subsection we consider uncertainty sets of the form $conv(P)$ where $P = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k\}$ and $conv(P)$ denotes the convex hull of P. Using this uncertainty set we consider the following adjustable robust optimization problem:

$$\inf_{\boldsymbol{u}\in U} \sup_{\boldsymbol{p}\in conv(P)} \inf_{\boldsymbol{v}\in V(\boldsymbol{u},\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}). \tag{11}$$

We are interested in characterizing tractable instances of this problem. In particular, we would like to identify conditions under which

$$\sup_{\boldsymbol{p}\in conv(P)} \inf_{\boldsymbol{v}\in V(\boldsymbol{u},\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}) = \max_{\boldsymbol{p}\in P} \inf_{\boldsymbol{v}\in V(\boldsymbol{u},\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}), \tag{12}$$

so that the ARC problem can be reduced to a single-level deterministic optimization problem as in the previous subsection. For this purpose, we first focus on the inner max-min problem in (11). Let us first define:

$$g_{\boldsymbol{u}}(\boldsymbol{p}) \equiv \inf_{\boldsymbol{v}\in V_{\boldsymbol{u}}(\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}). \tag{13}$$

Recall that $V_{\boldsymbol{u}}(\boldsymbol{p}) = V(\boldsymbol{u},\boldsymbol{p})$ with given $\boldsymbol{u}\in U$. Then, the inner max-min problem is:

$$\sup_{\boldsymbol{p}\in conv(P)} \inf_{\boldsymbol{v}\in V_{\boldsymbol{u}}(\boldsymbol{p})} f(\boldsymbol{u},\boldsymbol{v},\boldsymbol{p}) = \sup_{\boldsymbol{p}\in conv(P)} g_{\boldsymbol{u}}(\boldsymbol{p}).$$

A sufficient condition for (12) to hold is that with the given $\boldsymbol{u}\in U$, $g_{\boldsymbol{u}}(\boldsymbol{p})$ is a *quasi-convex* function in $\boldsymbol{p}\in conv(P)$, that is,

$$g_{\boldsymbol{u}}(\lambda\boldsymbol{p}_1 + (1-\lambda)\boldsymbol{p}_2) \le \max\{g_{\boldsymbol{u}}(\boldsymbol{p}_1), g_{\boldsymbol{u}}(\boldsymbol{p}_2)\}$$

holds for any $\boldsymbol{p}_1,\boldsymbol{p}_2\in conv(P)$ and $\lambda\in(0,1)$. Equivalently, $g_{\boldsymbol{u}}(\boldsymbol{p})$ is *quasi-convex* if all its level sets are convex sets. We state the following simple result without proof, which can be shown in, e.g., Corollary 2.14 of [17].

Proposition 2. *If $g_{\boldsymbol{u}}(\boldsymbol{p})$ defined in (13) is a quasi-convex function in $\boldsymbol{p}\in conv(P)$, then*

$$\max_{\boldsymbol{p}\in conv(P)} g_{\boldsymbol{u}}(\boldsymbol{p}) = \max_{\boldsymbol{p}\in P} g_{\boldsymbol{u}}(\boldsymbol{p}).$$

Therefore, when $g_{\boldsymbol{u}}(\boldsymbol{p})$ is quasi-convex, $conv(P)$ can be replaced by $P = \{\boldsymbol{p}_1,\dots,\boldsymbol{p}_k\}$ in (11), and the problem reduces to the single-level optimization problem (8) with finitely many constraints. In the next section, we will identify necessary and sufficient conditions on the sets $V_{\boldsymbol{u}}(\boldsymbol{p})$ that lead to quasi-convex $g_{\boldsymbol{u}}(\boldsymbol{p})$.

Remark 1. *In the remainder of the paper we consider a "normalized" version of problem (11), and assume that the objective function of the inner-most minimization problem is linear in $\boldsymbol{v}$ and is independent of the first period decision variables $\boldsymbol{u}$ and the uncertain parameters $\boldsymbol{p}$. This assumption can be made without loss of generality as indicated by the following* *simple transformation: For given* $\boldsymbol{u} \in U$ *and* $\boldsymbol{p} \in conv(P)$, $g_{\boldsymbol{u}}(\boldsymbol{p}) = \inf_{\boldsymbol{v} \in V_{\boldsymbol{u}}(\boldsymbol{p})} f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p})$ *is equivalent to:*

$$\begin{array}{ll} \inf_{\boldsymbol{v}, v_0} & v_0 \\ s.t. & f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}) \leq v_0 \\ & \boldsymbol{v} \in V_{\boldsymbol{u}}(\boldsymbol{p}). \end{array}$$

Defining $\tilde{V}_{\boldsymbol{u}}(\boldsymbol{p}) = \{\tilde{\boldsymbol{v}} = (\boldsymbol{v}, v_0) : \boldsymbol{v} \in V_{\boldsymbol{u}}(\boldsymbol{p}), f(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p}) \leq v_0\}$ *we observe that*

$$g_{\boldsymbol{u}}(\boldsymbol{p}) \equiv \inf_{\tilde{\boldsymbol{v}} \in \tilde{V}_{\boldsymbol{u}}(\boldsymbol{p})} \boldsymbol{c}^\top \tilde{\boldsymbol{v}}, \tag{14}$$

with $\boldsymbol{c} = [0 \ \ldots 0 \ 1]^\top$. *The normalized form of* $g_{\boldsymbol{u}}(\boldsymbol{p})$ *is useful in the succeeding discussion.*

3 Quasi-convex Maps and Functions

We argued in the previous section that the quasi-convexity of the function $g_{\boldsymbol{u}}(\boldsymbol{p}) = \inf_{\boldsymbol{v} \in V_{\boldsymbol{u}}(\boldsymbol{p})} \boldsymbol{c}^\top \boldsymbol{v}$ is a sufficient condition for the reducibility of the ARC problem (11) to a single-period optimization problem. Clearly, convexity properties of this function are related to the structure of the sets $V_{\boldsymbol{u}}(\boldsymbol{p})$ for $\boldsymbol{u} \in U$ and $\boldsymbol{p} \in P$. In what follows, we describe a necessary and sufficient condition on the set-valued mapping $V_{\boldsymbol{u}}(\boldsymbol{p})$ for $g_{\boldsymbol{u}}(\boldsymbol{p})$ to be a *quasi-convex* function in $\boldsymbol{p}$. We also consider explicit descriptions of the sets $V_{\boldsymbol{u}}(\boldsymbol{p})$ through constraints and investigate conditions on these constraint functions so that the sets $V_{\boldsymbol{u}}(\boldsymbol{p})$ satisfy the necessary and sufficient condition mentioned in the previous sentence.

3.1 Quasi-convex Maps $V_{\boldsymbol{u}}(\boldsymbol{p})$

For a given real topological vector space W, let 2^W denote its power set. Given $\boldsymbol{u} \in U$ and an appropriate choice of W, $V_{\boldsymbol{u}}(\boldsymbol{p})$ can be considered as a set-valued map $V_{\boldsymbol{u}} : conv(P) \to 2^W$. We also write $V_{\boldsymbol{u}} : conv(P) \rightsquigarrow W$. Let Q be a closed convex cone in W and define a relation $\leq_Q$ in W by the closed convex cone Q: for $\boldsymbol{v}_1, \boldsymbol{v}_2 \in W$, $\boldsymbol{v}_1 \leq_Q \boldsymbol{v}_2 \Leftrightarrow \boldsymbol{v}_2 - \boldsymbol{v}_1 \in Q$.

Definition 1. *A set-valued map* $V_{\boldsymbol{u}} : conv(P) \rightsquigarrow W$ *is said to be quasi Q-convex (see [2, 12]) if*

$$\begin{array}{l} \forall \boldsymbol{p}_1, \forall \boldsymbol{p}_2 \in conv(P), \forall \boldsymbol{v}_1 \in V_{\boldsymbol{u}}(\boldsymbol{p}_1), \forall \boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\boldsymbol{p}_2), \forall \alpha \in (0,1), \\ \quad \textit{if } \boldsymbol{w} \in W \textit{ satisfies } \boldsymbol{v}_1 \leq_Q \boldsymbol{w}, \boldsymbol{v}_2 \leq_Q \boldsymbol{w}, \\ \quad \textit{then } \exists \boldsymbol{v}' \in V_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2) \textit{ s.t. } \boldsymbol{v}' \leq_Q \boldsymbol{w}. \end{array} \tag{15}$$

Consider $Q = \{\boldsymbol{q} : \boldsymbol{c}^\top \boldsymbol{q} \geq 0\}$ defined using the coefficient vector $\boldsymbol{c}$ of the objective function.

Proposition 3. *Assume that* $V_{\boldsymbol{u}}(\boldsymbol{p})$ *is closed, bounded, and nonempty for any* $\boldsymbol{u} \in U$ *and* $\boldsymbol{p} \in P$. *Then,* $g_{\boldsymbol{u}}(\boldsymbol{p})$ *is a quasi-convex function in* $\boldsymbol{p}$ *if and only if the set-valued map* $V_{\boldsymbol{u}}(\boldsymbol{p})$ *is quasi Q-convex with* $Q = \{\boldsymbol{q} : \boldsymbol{c}^\top \boldsymbol{q} \geq 0\}$.

Proof: We observe that for every $\boldsymbol{u} \in U$ the quasi Q-convexity of the map $V_{\boldsymbol{u}}(\boldsymbol{p})$ is sufficient to guarantee that the function $g_{\boldsymbol{u}}(\boldsymbol{p})$ is quasi-convex function in $\boldsymbol{p}$. Indeed, for any $\boldsymbol{p}_1, \boldsymbol{p}_2 \in conv(P)$, choose

$$\begin{array}{l} \boldsymbol{v}_1 \in V_{\boldsymbol{u}}(\boldsymbol{p}_1) \text{ s.t. } g_{\boldsymbol{u}}(\boldsymbol{p}_1) = \boldsymbol{c}^\top \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\boldsymbol{p}_2) \text{ s.t. } g_{\boldsymbol{u}}(\boldsymbol{p}_2) = \boldsymbol{c}^\top \boldsymbol{v}_2. \end{array}$$

Such $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ exist since $V_{\boldsymbol{u}}(\boldsymbol{p}_i)$ are assumed to be closed and bounded. Define

$$\bar{\boldsymbol{w}} = \begin{cases} \boldsymbol{v}_1 & \text{if } \boldsymbol{c}^\top \boldsymbol{v}_1 > \boldsymbol{c}^\top \boldsymbol{v}_2 \\ \boldsymbol{v}_2 & \text{else,} \end{cases}$$

which indicates that $\boldsymbol{v}_1 \leq_Q \bar{\boldsymbol{w}}$ and $\boldsymbol{v}_2 \leq_Q \bar{\boldsymbol{w}}$. When $V_{\boldsymbol{u}}$ is quasi Q-convex, from (15), we have that for any $\alpha \in (0,1)$, there exists $\boldsymbol{v}' \in V_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)$ such that $\boldsymbol{v}' \leq_Q \bar{\boldsymbol{w}}$.

Then, using the above $\boldsymbol{v}'$ and $\bar{\boldsymbol{w}}$, we obtain

$$\begin{aligned} g_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2) &= \inf_{\boldsymbol{v} \in V_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)} \boldsymbol{c}^\top \boldsymbol{v} \\ &\leq \boldsymbol{c}^\top \boldsymbol{v}' \\ &\leq \boldsymbol{c}^\top \bar{\boldsymbol{w}} \\ &= \max\{\boldsymbol{c}^\top \boldsymbol{v}_1, \boldsymbol{c}^\top \boldsymbol{v}_2\} \\ &= \max\{g_{\boldsymbol{u}}(\boldsymbol{p}_1), g_{\boldsymbol{u}}(\boldsymbol{p}_2)\}. \end{aligned}$$

The second inequality follows from

$$\boldsymbol{v}' \leq_Q \bar{\boldsymbol{w}} \quad \Rightarrow \quad \bar{\boldsymbol{w}} - \boldsymbol{v}' \in Q \quad \Rightarrow \quad \boldsymbol{c}^\top (\bar{\boldsymbol{w}} - \boldsymbol{v}') \geq 0.$$

Therefore, $g_{\boldsymbol{u}}(\boldsymbol{p})$ is a quasi-convex function in $\boldsymbol{p}$.

Next, we show that (15) is also necessary for $g_{\boldsymbol{u}}(\boldsymbol{p})$ to be a quasi-convex function in $\boldsymbol{p}$. We suppose that (15) is not satisfied, and then show that $g_{\boldsymbol{u}}(\boldsymbol{p})$ cannot be a quasi-convex function in $\boldsymbol{p}$.

If (15) is not satisfied, there must exist $\bar{\boldsymbol{p}}_1, \bar{\boldsymbol{p}}_2 \in conv(P), \bar{\boldsymbol{v}}_1 \in V_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_1), \bar{\boldsymbol{v}}_2 \in V_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_2), \bar{\alpha} \in (0,1)$ such that

$$\begin{array}{c} \text{for some } \bar{\boldsymbol{w}} \in W \text{ s.t. } \bar{\boldsymbol{v}}_1 \leq_Q \bar{\boldsymbol{w}}, \bar{\boldsymbol{v}}_2 \leq_Q \bar{\boldsymbol{w}}, \\ \boldsymbol{v}' >_Q \bar{\boldsymbol{w}}, \forall \boldsymbol{v}' \in V_{\boldsymbol{u}}(\bar{\alpha}\bar{\boldsymbol{p}}_1 + (1-\bar{\alpha})\bar{\boldsymbol{p}}_2). \end{array}$$

From the definition (14) of $g_{\boldsymbol{u}}(\boldsymbol{p})$,

$$\begin{array}{l} g_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_1) \leq \boldsymbol{c}^\top \bar{\boldsymbol{v}}_1 \leq \boldsymbol{c}^\top \bar{\boldsymbol{w}} \\ g_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_2) \leq \boldsymbol{c}^\top \bar{\boldsymbol{v}}_2 \leq \boldsymbol{c}^\top \bar{\boldsymbol{w}}. \end{array} \tag{16}$$

Since $\forall \boldsymbol{v}' \in V_{\boldsymbol{u}}(\bar{\alpha}\bar{\boldsymbol{p}}_1 + (1-\bar{\alpha})\bar{\boldsymbol{p}}_2)$ satisfies $\boldsymbol{v}' >_Q \bar{\boldsymbol{w}}$ and $V_{\boldsymbol{u}}(\bar{\alpha}\bar{\boldsymbol{p}}_1 + (1-\bar{\alpha})\bar{\boldsymbol{p}}_2)$ is compact,

$$g_{\boldsymbol{u}}(\bar{\alpha}\bar{\boldsymbol{p}}_1 + (1-\bar{\alpha})\bar{\boldsymbol{p}}_2) > \boldsymbol{c}^\top \bar{\boldsymbol{w}}. \tag{17}$$

The above inequalities (16) and (17) show that

$$\max\{g_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_1), g_{\boldsymbol{u}}(\bar{\boldsymbol{p}}_2)\} < g_{\boldsymbol{u}}(\bar{\alpha}\bar{\boldsymbol{p}}_1 + (1-\bar{\alpha})\bar{\boldsymbol{p}}_2)$$

and we see that the condition of quasi-convex function:

$$g_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2) \leq \max\{g_{\boldsymbol{u}}(\boldsymbol{p}_1), g_{\boldsymbol{u}}(\boldsymbol{p}_2)\}, \quad \forall \boldsymbol{p}_1, \boldsymbol{p}_2 \in conv(P), \forall \alpha \in (0,1)$$

is violated at $\bar{\boldsymbol{p}}_1, \bar{\boldsymbol{p}}_2 \in conv(P)$ and $\bar{\alpha}$. Thus, if $g_{\boldsymbol{u}}(\boldsymbol{p})$ is a quasi-convex function in $\boldsymbol{p}$, $V_{\boldsymbol{u}}(\boldsymbol{p})$ satisfies the condition of quasi Q-convex set-valued map (15). □ ∎

3.2 Functional Description of $V_{\boldsymbol{u}}(\boldsymbol{p})$

In this subsection, we focus on the case where the sets $V_{\boldsymbol{u}}(\boldsymbol{p})$ are described explicitly using constraints and obtain sufficient conditions for quasi Q-convexity of the mapping $V_{\boldsymbol{u}}$ in Propositions 4 and 5.

For an arbitrary closed convex cone K, we consider a vector-valued function $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ that satisfies

$$V_{\boldsymbol{u}}(\boldsymbol{p}) = \{\boldsymbol{v} \mid F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p}) \leq_K \boldsymbol{0}\}.$$

We now investigate conditions on functions $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ that guarantee $V_{\boldsymbol{u}}(\boldsymbol{p})$ to be a quasi Q-convex set-valued map. Not surprisingly, we observe that quasi K-convexity of $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ is sufficient for this purpose. We first define this property [16]:

Definition 2. *$F : D \to W$ is a quasi K-convex vector-valued function in $\boldsymbol{d}$ if $\forall \boldsymbol{d}_1, \boldsymbol{d}_2 \in D$ and $\forall \alpha \in [0,1]$,*

$$F(\alpha \boldsymbol{d}_1 + (1-\alpha)\boldsymbol{d}_2) \leq_K \boldsymbol{z} \text{ holds}$$
$$\text{for any } \boldsymbol{z} \text{ satisfying } F(\boldsymbol{d}_1) \leq_K \boldsymbol{z}, \;\; F(\boldsymbol{d}_2) \leq_K \boldsymbol{z}.$$

Proposition 4. *Consider vector-valued functions $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p}) : V \times conv(P) \to W$ such that $V_{\boldsymbol{u}}(\boldsymbol{p}) = \{\boldsymbol{v} \mid F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p}) \leq_K \boldsymbol{0}\}$ for a given cone K. If $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ is quasi K-convex in $(\boldsymbol{v}, \boldsymbol{p})$ for all $\boldsymbol{u} \in U$, then $V_{\boldsymbol{u}}(\boldsymbol{p}) = \{\boldsymbol{v} \mid F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p}) \leq_K \boldsymbol{0}\}$ is a quasi Q-convex set-valued map for any closed convex cone Q.*

Proof: By definition, for any $\boldsymbol{v}_1 \in V_{\boldsymbol{u}}(\boldsymbol{p}_1)$ and $\boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\boldsymbol{p}_2)$,

$$F_{\boldsymbol{u}}(\boldsymbol{v}_1, \boldsymbol{p}_1) \leq_K \boldsymbol{0}, \text{ and } F_{\boldsymbol{u}}(\boldsymbol{v}_2, \boldsymbol{p}_2) \leq_K \boldsymbol{0}$$

holds, and under the assumption that $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ is quasi K-convex, we obtain

$$F_{\boldsymbol{u}}(\alpha \boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2, \alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2) \leq_K \boldsymbol{0}, \quad \forall \alpha \in [0,1],$$

which implies $\alpha \boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)$, $\forall \alpha \in [0,1]$.

Now, if $\boldsymbol{w} \in V$ satisfies $\boldsymbol{v}_1 \leq_Q \boldsymbol{w}$ and $\boldsymbol{v}_2 \leq_Q \boldsymbol{w}$, $\alpha \boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2 \leq_Q \boldsymbol{w}$ holds for every $\alpha \in [0,1]$, since $\boldsymbol{w} - \boldsymbol{v}_1 \in Q$, $\boldsymbol{w} - \boldsymbol{v}_2 \in Q$, and the convexity of Q indicates

$\alpha(\boldsymbol{w}-\boldsymbol{v}_1)+(1-\alpha)(\boldsymbol{w}-\boldsymbol{v}_2)=\boldsymbol{w}-\{\alpha\boldsymbol{v}_1+(1-\alpha)\boldsymbol{v}_2\}\in Q$. Therefore, $V_{\boldsymbol{u}}(\boldsymbol{p})$ satisfies the condition (15) of quasi Q-convexity. □

We stress that the cones K and Q in the proposition above need not coincide. Next, we consider an even more specific form for $V_{\boldsymbol{u}}(\boldsymbol{p})$ by defining $F_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})=(f^1_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p}),\ldots,f^m_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p}))^\top$ where each $f^i_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$ is a real-valued function and $K=R^m_+$. If $f^i_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$, $i=1,\ldots,m$, are quasi-convex functions in $(\boldsymbol{v},\boldsymbol{p})$, then $F_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$ is a quasi R^m_+-convex vector-valued function. Indeed, for any $\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{v}_1,\boldsymbol{v}_2$, and arbitrary $\boldsymbol{z}=(z_1,\ldots,z_m)^\top$ such that $f^i_{\boldsymbol{u}}(\boldsymbol{v}_1,\boldsymbol{p}_1)\le z_i$ and $f^i_{\boldsymbol{u}}(\boldsymbol{v}_2,\boldsymbol{p}_2)\le z_i$ $(i=1,\ldots,m)$, we have

$$\begin{aligned}f^i_{\boldsymbol{u}}(\alpha\boldsymbol{v}_1+(1-\alpha)\boldsymbol{v}_2,\alpha\boldsymbol{p}_1+(1-\alpha)\boldsymbol{p}_2)\le\max\{f^i_{\boldsymbol{u}}(\boldsymbol{v}_1,\boldsymbol{p}_1),f^i_{\boldsymbol{u}}(\boldsymbol{v}_2,\boldsymbol{p}_2)\}\le z_i\\ \forall\alpha\in[0,1],\quad i=1,\ldots,m,\end{aligned}$$

which shows the quasi R^m_+-convexity of $F_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$. Therefore, $V_{\boldsymbol{u}}(\boldsymbol{p})$ is a quasi Q-convex set-valued map by the proposition.

In fact, when $V_{\boldsymbol{u}}(\boldsymbol{p})=\{\boldsymbol{v}\mid f^i_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})\le 0, i=1,\ldots,m\}$ with quasi-convex functions $f^i_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$, $i=1,\ldots,m$, the mapping $V_{\boldsymbol{u}}(\boldsymbol{p})$ satisfies the stronger *Q-convexity* property. The Q-convex set valued map is defined in [2, 12]:

$$\begin{array}{l}\forall\boldsymbol{p}_1,\forall\boldsymbol{p}_2\in conv(P),\forall\boldsymbol{v}_1\in V_{\boldsymbol{u}}(\boldsymbol{p}_1),\forall\boldsymbol{v}_2\in V_{\boldsymbol{u}}(\boldsymbol{p}_2),\text{ and }\alpha\in(0,1),\\ \exists\boldsymbol{w}\in V_{\boldsymbol{u}}(\alpha\boldsymbol{p}_1+(1-\alpha)\boldsymbol{p}_2)\text{ s.t. }\boldsymbol{w}\le_Q\alpha\boldsymbol{v}_1+(1-\alpha)\boldsymbol{v}_2.\end{array}\tag{18}$$

We end this section by presenting the following proposition.

Proposition 5. *Consider the problem:*

$$g_{\boldsymbol{u}}(\boldsymbol{p})=\inf_{\boldsymbol{v}\in V_{\boldsymbol{u}}(\boldsymbol{p})}f_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p}),$$

If the objective function $f_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$ is quasi-convex in $(\boldsymbol{v},\boldsymbol{p})$ and $F_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$ is a quasi K-convex vector-valued function for some convex cone K, then $g_{\boldsymbol{u}}(\boldsymbol{p})$ is a quasi-convex function in $\boldsymbol{p}$.

Proof: We consider the problem

$$g_{\boldsymbol{u}}(\boldsymbol{p})=\inf_{v\in V_{\boldsymbol{u}}(\boldsymbol{p})}f_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})=\inf_{(\boldsymbol{v},v_0)\in\widetilde{V}_{\boldsymbol{u}}(\boldsymbol{p})}v_0,$$

where

$$\begin{aligned}\widetilde{V}_{\boldsymbol{u}}(\boldsymbol{p})\equiv&\ \{(\boldsymbol{v},v_0):f_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})\le v_0,\ \ \boldsymbol{v}\in V_{\boldsymbol{u}}(\boldsymbol{p})\}\\ =&\ \{(\boldsymbol{v},v_0):f_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})\le v_0,\ \ F_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})\le_K\boldsymbol{0}\}.\end{aligned}$$

For any $\boldsymbol{p}_1,\boldsymbol{p}_2\in conv(P)$, $(\boldsymbol{v}_1,v_{01})\in\widetilde{V}_{\boldsymbol{u}}(\boldsymbol{p}_1)$ and $(\boldsymbol{v}_2,v_{02})\in\widetilde{V}_{\boldsymbol{u}}(\boldsymbol{p}_2)$,

$$\begin{array}{l}f_{\boldsymbol{u}}(\boldsymbol{v}_1,\boldsymbol{p}_1)\le v_{01},\ \ f_{\boldsymbol{u}}(\boldsymbol{v}_2,\boldsymbol{p}_2)\le v_{02}\\ F_{\boldsymbol{u}}(\boldsymbol{v}_1,\boldsymbol{p}_1)\le_K\boldsymbol{0},\ \ F_{\boldsymbol{u}}(\boldsymbol{v}_2,\boldsymbol{p}_2)\le_K\boldsymbol{0}\end{array}$$

hold and the quasi K-convexity of $F_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{p})$ shows that $F_{\boldsymbol{u}}(\boldsymbol{v}',\boldsymbol{p}')\le_K\boldsymbol{0}$, where $\boldsymbol{v}'=\alpha\boldsymbol{v}_1+(1-\alpha)\boldsymbol{v}_2$ and $\boldsymbol{p}'=\alpha\boldsymbol{p}_1+(1-\alpha)\boldsymbol{p}_2$, for any $\alpha\in(0,1)$. Also, from the quasi-convexity of $f_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$, we see that $f_{\boldsymbol{u}}(\boldsymbol{v}', \boldsymbol{p}') \leq \max\{f_{\boldsymbol{u}}(\boldsymbol{v}_1, \boldsymbol{p}_1), f_{\boldsymbol{u}}(\boldsymbol{v}_2, \boldsymbol{p}_2)\} \leq \max\{v_{01}, v_{02}\} \equiv v_0'$. Therefore, $(\boldsymbol{v}', v_0') = \alpha(\boldsymbol{v}_1, v_0') + (1-\alpha)(\boldsymbol{v}_2, v_0')$ is in $\widetilde{V}_{\boldsymbol{u}}(\boldsymbol{p}')$, and if $(\boldsymbol{w}, w_0)$ satisfies $(\boldsymbol{v}_1, v_{01}) \leq_Q (\boldsymbol{w}, w_0)$ and $(\boldsymbol{v}_2, v_{02}) \leq_Q (\boldsymbol{w}, w_0)$ with $Q = \{(\boldsymbol{q}, q_0) : q_0 \geq 0\}$, then $(\boldsymbol{v}', v_0') \leq_Q (\boldsymbol{w}, w_0)$ holds. We have shown that $\widetilde{V}_{\boldsymbol{u}}(\boldsymbol{p})$ is a quasi Q-convex set-valued map, and from Proposition 3, $g_{\boldsymbol{u}}(\boldsymbol{p})$ is a quasi-convex function in $\boldsymbol{p}$. □ ∎

4 Examples with Quasi-convex Mappings

We investigate the condition (15) for quasi Q-convexity of set-valued mappings and two stronger (more restrictive) variants of this condition by studying three examples. The first one is an example of a Q-convex mapping, the second one is that of a naturally quasi Q-convex mapping, and the last one is that of a quasi Q-convex mapping. The Q-convexity condition was described above in (18). Before we present the examples, we define the naturally quasi Q-convexity condition [2, 12]:

Definition 3. *A set valued mapping $M : D \rightsquigarrow W$ is said to be naturally quasi Q-convex if $\forall \boldsymbol{d}_1, \boldsymbol{d}_2 \in D$,*

$$\begin{array}{l} \forall \alpha \in (0,1), \forall \boldsymbol{m}_1 \in M(\boldsymbol{d}_1), \forall \boldsymbol{m}_2 \in M(\boldsymbol{d}_2) \\ \exists \boldsymbol{w} \in M(\alpha \boldsymbol{d}_1 + (1-\alpha)\boldsymbol{d}_2) \text{ and } \exists \beta \in [0,1] \text{ s.t.} \\ \boldsymbol{w} \leq_Q \beta \boldsymbol{m}_1 + (1-\beta)\boldsymbol{m}_2. \end{array} \tag{19}$$

It is known that every convex set-valued map is also naturally quasi-convex, and every naturally quasi-convex set-valued map is also quasi-convex.

In the examples we describe below, the optimal solution of $\sup\limits_{\boldsymbol{p} \in conv(P)} \inf\limits_{\boldsymbol{v} \in V_{\boldsymbol{u}}(\boldsymbol{p})} \boldsymbol{c}^\top \boldsymbol{v}$ is obtained for $\boldsymbol{p}_i \in P$ for some i and we can ignore the constraints induced from the interior points of $conv(P)$. However, Examples 2 and 3 do not satisfy the sufficient conditions of Propositions 4 and 5 for quasi Q-convex set valued maps $V_{\boldsymbol{u}}(\boldsymbol{p})$. Thus, these examples indicate that conditions given in Propositions 4 and 5 are not necessary for quasi Q-convexity of the mapping $V_{\boldsymbol{u}}$ and more general problems can be reduced to the single-level optimization problem (8).

Example 1 (Q-convex $V_u(\boldsymbol{p})$) : *Consider the ARC problem described below:*

$$\min_{u \in U} \max_{\boldsymbol{p} \in conv(P)} \min_{\boldsymbol{v} = (v_1, v_2) \in V(\boldsymbol{p})} (-v_1 - u v_2) \tag{20}$$

with $P = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ and

$$V(\boldsymbol{p}) = \{(v_1, v_2) | \ (v_1 - p_1)^2 + (v_2 - p_2)^2 \leq 1, \ \ \boldsymbol{v} \geq \boldsymbol{0}\}.$$

Note that $conv(P) = \{\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \mid \boldsymbol{p} \geq \boldsymbol{0}, p_1 + p_2 = 1\}$. We dropped the subscript u from $V_u(\boldsymbol{p})$ since this set does not depend on u. The inner max-min problem in (20) is equivalent *to:*

$$\min_{u\in U, v_0} \left\{ v_0 |\ \forall \boldsymbol{p} \in conv(P)\ \exists \boldsymbol{v} : \begin{array}{l} v_0 \geq -v_1 - uv_2, \\ (v_1 - p_1)^2 + (v_2 - p_2)^2 \leq 1 \\ \boldsymbol{v} \geq \boldsymbol{0}. \end{array} \right\}$$

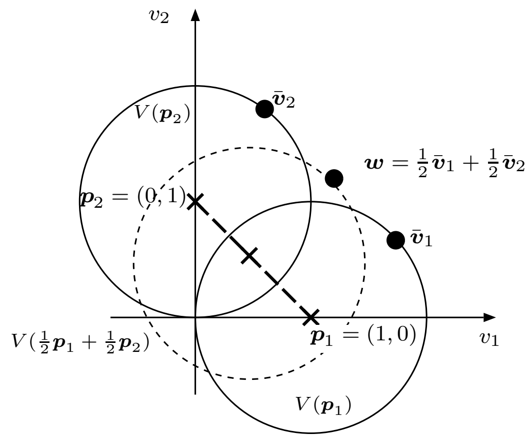

Figure 1: Feasible sets and optimal solutions (with $u = 1$) in Example 1

We observe that the mapping $V(\boldsymbol{p})$ is Q-convex for every cone Q containing zero. Indeed, for any $\boldsymbol{p}_1, \boldsymbol{p}_2 \in conv(P)$, $\alpha \in (0,1)$, $\bar{\boldsymbol{v}}_1 \in V(\boldsymbol{p}_1)$ and $\bar{\boldsymbol{v}}_2 \in V(\boldsymbol{p}_2)$, we can construct the inner point between $\bar{\boldsymbol{v}}_1$ and $\bar{\boldsymbol{v}}_2$: $\alpha\bar{\boldsymbol{v}}_1 + (1-\alpha)\bar{\boldsymbol{v}}_2$ which lies in the set $V(\alpha\boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)$, since $\alpha V(\boldsymbol{p}_1) + (1-\alpha)V(\boldsymbol{p}_2) = V(\alpha\boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)$. Now, the set-valued map V satisfies the condition of Q-convexity (18) with $\boldsymbol{w}$ chosen as $\boldsymbol{w} = \alpha\bar{\boldsymbol{v}}_1 + (1-\alpha)\bar{\boldsymbol{v}}_2$, since $\alpha\bar{\boldsymbol{v}}_1 + (1-\alpha)\bar{\boldsymbol{v}}_2 - \boldsymbol{w} = \boldsymbol{0} \in Q$.

Since every convex set-valued map is also quasi-convex [12], the set-valued map V is quasi Q-convex and therefore, $g_u(\boldsymbol{p})$ becomes a quasi-convex function in $\boldsymbol{p}$, $conv(P)$ in (20) can be replaced by P and this problem reduces to the single-level optimization problem with finitely many constraints.

Indeed, Figure 1 shows that it is sufficient to focus on the extreme cases $V(\boldsymbol{p}_1)$ and $V(\boldsymbol{p}_2)$, since the objective function is linear and an optimal solution is attained in some scenario $V(\boldsymbol{p}_1)$ or $V(\boldsymbol{p}_2)$.

Remark 2. *Note that this example contains a constraint in which the coefficients of adjustable vector variable $\boldsymbol{v}$ are affected by uncertainty:*

$$(v_1 - p_1)^2 + (v_2 - p_2)^2 \leq 1 \quad \Leftrightarrow \quad v_1^2 + v_2^2 - 2p_1v_1 - 2p_2v_2 + p_1^2 + p_2^2 \leq 1.$$

But $g_u(\boldsymbol{p})$ is quasi-convex and $conv(P)$ can be replaced by the finite set P in the ARC formulation. It is noted in [6] that when the constraint coefficients of the adjustable variables $\boldsymbol{v}$ are affected by uncertainty, the resulting ARC can be computationally intractable. For *example, this case is excluded in Theorem 2.2 of [6]. The example above shows a special case where the resulting ARC problem is still tractable.*

Example 2 (Naturally quasi Q-convex $V_u(p)$) : *We focus on the inner problem* $g_u(p) = \min_{\boldsymbol{v}\in V_u(p)} \boldsymbol{c}^\top \boldsymbol{v}$ *with a fixed* $u \in U$*, where*

$$\begin{aligned} &V_u(p) = \{(v_1, v_2)|\ \ u \le pv_1 \le 2u,\ u \le pv_2 \le 2u\}, \\ &P = \{\tfrac{1}{2}, 1\}, \quad conv(P) = [\tfrac{1}{2}, 1]. \end{aligned}$$

We'll show that the set-valued map $V_u(p)$ satisfies the condition of naturally quasi Q-convexity defined above. Indeed, for any $p_1, p_2 \in conv(P)$, $\alpha \in (0,1)$, $\bar{\boldsymbol{v}}_1 \in V_u(p_1)$ and $\bar{\boldsymbol{v}}_2 \in V_u(p_2)$, we can construct $\boldsymbol{w} = \beta\bar{\boldsymbol{v}}_1 + (1-\beta)\bar{\boldsymbol{v}}_2 \in V(\alpha p_1 + (1-\alpha)p_2)$ by computing $\beta \in [0,1]$ from

$$\frac{1}{\alpha p_1 + (1-\alpha)p_2} = \frac{\beta}{p_1} + \frac{1-\beta}{p_2}.$$

Therefore, the set-valued map V_u of this example satisfies the condition (19) of naturally quasi Q-convexity[1] *whenever we take any Q which includes $\mathbf{0}$. Assuming $u = 1$, Figure 2 shows, in the case of $\alpha = 1/2$, the inner point of $\bar{\boldsymbol{v}}_1$ and $\bar{\boldsymbol{v}}_2$: $\boldsymbol{w} = \beta\bar{\boldsymbol{v}}_1 + (1-\beta)\bar{\boldsymbol{v}}_2 \in V_u(\alpha p_1 + (1-\alpha)p_2)$ with $\beta = 1/3$.*

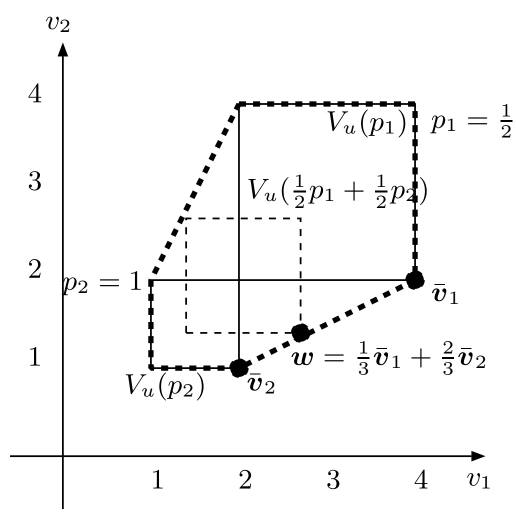

Figure 2: Feasible sets and optimal solutions (with $u = 1$) in Example 2

It is shown in [12] that every naturally quasi-convex set-valued map is also quasi-convex, and we find that the set-valued map V of this example is quasi Q-convex. However, we note that some constraint functions defining $V_u(p)$ are not quasi-convex in $(\boldsymbol{v}, \boldsymbol{p})$, and $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ do not satisfy the sufficient conditions for the quasi Q-convexity of set-valued map V_u.

[1] If an appropriate objective function $\bar{\boldsymbol{c}}^\top \boldsymbol{v}$ is given (for example, $\bar{\boldsymbol{c}} = \bar{\boldsymbol{v}}_1 - \bar{\boldsymbol{v}}_2$ and therefore, $Q' = \{\boldsymbol{q} : (\bar{\boldsymbol{v}}_1 - \bar{\boldsymbol{v}}_2)^\top \boldsymbol{q} \ge 0\}$), the set-valued map V_u might be Q'-convex.

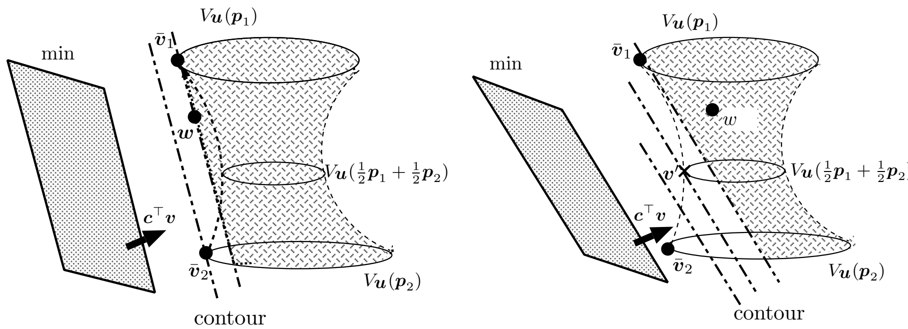

Figure 3: $V_{\boldsymbol{u}}(\boldsymbol{p})$ of the right figure is quasi Q-convex, but that of left figure is not.

Example 3 (Quasi Q-convex $V_{\boldsymbol{u}}(\boldsymbol{p})$) : *We now give a geometric example of a mapping $V_{\boldsymbol{u}}(\boldsymbol{p})$ that is quasi Q-convex for some convex cones Q but not for others. We define the convex cone Q depending on the linear objective function $\mathbf{c}^\top \boldsymbol{v}$ as follows: $Q = \{\boldsymbol{q} : \boldsymbol{c}^\top \boldsymbol{q} \geq 0\}$. In our previous examples and discussion, the cone Q in the definition of quasi Q-convex functions and mappings was largely irrelevant. Note however that while the mapping on the left in Figure 3 does not satisfy the condition (15), the mapping on the right satisfies this condition. In these figures, each horizontal cross-section of the hourglass shaped three-dimensional set corresponds to the image of the mapping $V_{\boldsymbol{u}}(\boldsymbol{p})$ with different values of $\boldsymbol{p}$. The figure on the left violates (15) with the points $\bar{\boldsymbol{v}}_1$, $\bar{\boldsymbol{v}}_2$, $\boldsymbol{w}$ and $\alpha = \frac{1}{2}$ forming the counter-example. In the figure on the right, we keep the mapping constant. By changing $\mathbf{c}$ and the orientation of the plane defining the cone Q, we see that the mapping is quasi Q-convex with respect to the tilted cone Q. For the example on the right, $g_{\boldsymbol{u}}(\boldsymbol{p}) \equiv \min_{\boldsymbol{v} \in V_{\boldsymbol{u}}(\boldsymbol{p})} \boldsymbol{c}^\top \boldsymbol{v}$ is quasi-convex in $\boldsymbol{p}$. Therefore, we can focus on the extreme scenario-cases $V_{\boldsymbol{u}}(\boldsymbol{p}_1)$ and $V_{\boldsymbol{u}}(\boldsymbol{p}_2)$, and conv$(P)$ in (ARC) can be replaced as the set of finite points P.*

Although in this example $F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p})$ of $V_{\boldsymbol{u}}(\boldsymbol{p}) = \{\boldsymbol{v} \mid F_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{p}) \leq_K \mathbf{0}\}$ is not a quasi K-convex vector-valued function (if quasi K-convex, $\alpha \boldsymbol{v}_1 + (1-\alpha)\boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\alpha \boldsymbol{p}_1 + (1-\alpha)\boldsymbol{p}_2)$ holds for $\boldsymbol{v}_1 \in V_{\boldsymbol{u}}(\boldsymbol{p}_1)$, $\boldsymbol{v}_2 \in V_{\boldsymbol{u}}(\boldsymbol{p}_2)$ and $\forall \alpha \in [0,1]$, which is clearly not the case), the set-valued map $V_{\boldsymbol{u}}(\boldsymbol{p})$ of the right figure satisfies the condition of quasi Q-convexity and $V_{\boldsymbol{u}}(\boldsymbol{p})$ is shown as a quasi Q-convex set-valued map.

5 Applications

While the quasi cone-convexity conditions we considered above may be difficult to verify in general settings, there are several application problems where one encounters these structures. Most natural examples come from the two-period formulation of optimization problems with quasi-convex objective functions where the feasible sets of the second-period variables depend "nicely" on the first period variables.

Quasi-convex (or quasi-concave) objective functions are quite common in applications.

A canonical example from economics is the utility maximization problem where the quasi-concavity of the utility function is a typical requirement. Since uncertainty and multi-period planning/decision making are natural occurrences and extensions for utility maximization, resulting problems can be addressed through the adjustable robust optimization approach we outlined in the previous sections.

Another rich class of quasi-convex objective functions appear in fractional programming problems [8, 15]. It is easy to verify that the fractional objective function $f(\boldsymbol{x}) \equiv p(\boldsymbol{x})/q(\boldsymbol{x})$ is quasi-convex on a convex domain S if p is convex and q is concave and positive on that domain [7]. Such functions arise frequently in the measurement of the efficiency of a system, e.g., with output to input ratios or reward to risk ratios. In what follows, we discuss applications that can be seen as particular instances of the fractional programming framework.

5.1 Robust Profit Opportunities

We consider an investment environment with n risky securities. Let v_i^1 denote the (random) time 1 value of \$1 invested in security i at time 0 and let x_i^0 denote the dollars invested in security i at time 0. We also define $r_i^1 = v_i^1 - 1$ to be the return on a dollar for security i. Letting $\boldsymbol{e}$ denote the vector of ones of appropriate size, we see that the initial value of the portfolio formed at time 0 is $\boldsymbol{e}^\top \boldsymbol{x}^0$ where $\boldsymbol{x}^0 = [x_1^0, \ \ldots, \ x_n^0]^\top$. At time 1 which is the end of the initial investment period this portfolio will have value $(\boldsymbol{v}^1)^\top \boldsymbol{x}^0$. One can treat $\boldsymbol{v}^1 = [v_1^1, \ \ldots, \ v_n^1]^\top$ as a random vector, and denote its expected value by $\bar{\boldsymbol{v}}^1$ and its $n \times n$ (symmetric, positive semidefinite) matrix of variance/covariances by Q^1. Similarly define $\boldsymbol{r}^1$ and $\bar{\boldsymbol{r}}^1$, its expected value vector.

The return from this investment is $(\boldsymbol{r}^1)^\top \boldsymbol{x}^0$ and the expected return is $(\bar{\boldsymbol{r}}^1)^\top \boldsymbol{x}^0$ with variance $(\boldsymbol{x}^0)^\top Q^1 \boldsymbol{x}^0$. For a given risk tolerance parameter $\theta \geq 0$, the quantity

$$f_\theta(\boldsymbol{x}^0) \equiv (\bar{\boldsymbol{r}}^1)^\top \boldsymbol{x}^0 - \theta \sqrt{(\boldsymbol{x}^0)^\top Q^1 \boldsymbol{x}^0}$$

is the *risk-adjusted expected return* from portfolio $\boldsymbol{x}^0$. In essence, f_θ is a utility function and the returns are penalized based on the risk taken to achieve them and the penalty rises with θ, the risk tolerance parameter. If we let $\mathcal{X}^0$ denote the feasible set of values for portfolio $\boldsymbol{x}^0$, the problem of maximizing $f_\theta(\boldsymbol{x}^0)$ subject to $\boldsymbol{x}^0 \in \mathcal{X}^0$ is one of the variants of the classical Markowitz mean-variance optimization problem.

Here, we consider an alternative problem. Instead of fixing the parameter θ, we make it a variable and look for the largest possible θ for which the risk-adjusted expected return is above a threshold value t, typically zero. The formulation we consider is:

$$\sup_{\theta, \boldsymbol{x}^0} \theta, \ \text{s.t.} \ (\bar{\boldsymbol{r}}^1)^\top \boldsymbol{x}^0 - \theta \sqrt{(\boldsymbol{x}^0)^\top Q^1 \boldsymbol{x}^0} \geq t, \ \ \boldsymbol{x}^0 \in \mathcal{X}^0. \tag{21}$$

The motivation for this alternative formulation comes from the models for *robust profit opportunities* (RPOs) developed in [14]. A RPO is defined as a portfolio that has a negative initial investment (i.e., positive initial cash-flow) and nonnegative future risk-adjusted value. A nonnegative risk-adjusted value makes future negative cash-flows unlikely. Formally, a portfolio vector x^0 is an RPO if it satisfies

$$(\bar{\boldsymbol{v}}^1)^\top \boldsymbol{x}^0 - \theta\sqrt{(\boldsymbol{x}^0)^\top Q^1 \boldsymbol{x}^0} \geq 0, \;\; \boldsymbol{e}^\top \boldsymbol{x}^0 < 0, \tag{22}$$

for a positive constant θ. It is argued in [14] that RPOs represent next-best alternatives to *arbitrage opportunities*, which are often assumed not to exist or persist in financial markets. Solving a problem similar to (21) with constraints (22), one finds maximum-θ RPOs.

While satisfying the inequality of the problem (21) (or the problem with constraints (22)) does not guarantee that the actual return will be above the threshold t (non-negative for (22)), maximization of θ is intended to maximize the likelihood of that event. It is easy to see that maximizing θ is actually equivalent to maximizing the probability that the random return vector $\boldsymbol{r}^1$ will satisfy the inequality $(\boldsymbol{r}^1)^\top \boldsymbol{x}^0 \geq t$ when the joint return distributions are normal. This argument is also related to the 3σ concepts in engineering–if the mean minus three standard deviations of a random variable is above a threshold, the random variable will "almost always" be above that threshold.

Next, we consider problem (21) in a two-period investment setting. For this purpose, let v_i^2 denote the (random) time 2 value of \$1 invested in security i at time 1, and let x_i^1 denote the dollars invested in security i at time 1. Let $\boldsymbol{r}^2$, $\bar{\boldsymbol{r}}^2$, Q^2, etc. be defined analogously to the earlier definitions. In this two-period investment setting, the investor will form a portfolio at time 0 that she will hold until time 1. At time 1 she will be able to rebalance her portfolio in a self-financing manner and hold this new portfolio until time 2. In the absence of transaction costs, the self-financing constraint is represented as follows:

$$\boldsymbol{e}^\top \boldsymbol{x}^1 = (\boldsymbol{v}^1)^\top \boldsymbol{x}^0, \tag{23}$$

i.e., the initial value of the portfolio constructed at time 1 must equal to value of the time 0 portfolio at time 1. Proportional transaction costs can be handled easily using additional linear variables. Fixed transaction costs however would require the use of binary variables. For simplicity, we do not consider either possibility here.

Since we do not know $\boldsymbol{v}^1$ beforehand, (23) is an uncertain (random) constraint. Let $\mathcal{X}^1(\boldsymbol{x}^0, \boldsymbol{v}^1)$ denote the feasible set of portfolios for time 1 defined by additional constraints that may or may not depend on $\boldsymbol{x}^0$ and $\boldsymbol{v}^1$. We focus on the self-financing constraint and treat it separately from the others. Given a particular value for $\boldsymbol{v}^1$ and the choice $\boldsymbol{x}^0$ for the time 0 portfolio, the problem we want to solve at time 1 is an analogue of (21) with the additional self-financing constraint:

$$\sup_{\theta, \boldsymbol{x}^1} \theta, \text{ s.t. } (\bar{\boldsymbol{r}}^2)^\top \boldsymbol{x}^1 - \theta\sqrt{(\boldsymbol{x}^1)^\top Q^2 \boldsymbol{x}^1} \geq t, \;\; \boldsymbol{e}^\top \boldsymbol{x}^1 = (\boldsymbol{v}^1)^\top \boldsymbol{x}^0, \;\; \boldsymbol{x}^1 \in \mathcal{X}^1(\boldsymbol{x}^0, \boldsymbol{v}^1). \tag{24}$$

For the two-period problem, one must decide how to approach the uncertain constraint $\boldsymbol{e}^\top \boldsymbol{x}^1 = (\boldsymbol{v}^1)^\top \boldsymbol{x}^0$. Here, we use the ARO approach and define an uncertainty set $\mathcal{U}$ that contains all possible values of the uncertain vector $\boldsymbol{v}^1$. We further assume that $\mathcal{U}$ is a polytopic set and is given as $\mathcal{U} = conv\{\boldsymbol{v}_1^1, \ldots, \boldsymbol{v}_k^1\}$. Note that ellipsoidal uncertainty sets commonly seen in robust modeling of financial problems [10] can be approximated by uniform sampling from the boundary of the ellipsoid and using the corresponding inscribed polytope.

Then, the two-period adjustable robust optimization model for this problem can be written as

$$\sup_{\boldsymbol{x}^0:\boldsymbol{x}^0\in\mathcal{X}^0}\ \inf_{\boldsymbol{v}^1\in\mathcal{U}}\ \sup_{\theta,\boldsymbol{x}^1\in\mathcal{X}^1(\boldsymbol{x}^0,\boldsymbol{v}^1)} \theta \text{ s.t. } \boldsymbol{e}^\top\boldsymbol{x}^1 = (\boldsymbol{v}^1)^\top\boldsymbol{x}^0,\ \ (\bar{\boldsymbol{r}}^2)^\top\boldsymbol{x}^1 - \theta\sqrt{(\boldsymbol{x}^1)^\top Q^2\boldsymbol{x}^1} \geq t. \tag{25}$$

It is reasonable to assume that Q^2 is nonsingular (otherwise there are redundant or riskless assets) and that $\mathbf{0}\notin\mathcal{X}^1$ (not doing anything is not an option). Then, $(\boldsymbol{x}^1)^\top Q^2\boldsymbol{x}^1$ is positive for all $\boldsymbol{x}^1\in\mathcal{X}^1$ and we can rewrite the problem as

$$\sup_{\boldsymbol{x}^0:\boldsymbol{x}^0\in\mathcal{X}^0}\ \inf_{\boldsymbol{v}^1\in\mathcal{U}}\ \sup_{\theta,\boldsymbol{x}^1\in\mathcal{X}^1(\boldsymbol{x}^0,\boldsymbol{v}^1)} \theta \text{ s.t. } \boldsymbol{e}^\top\boldsymbol{x}^1 = (\boldsymbol{v}^1)^\top\boldsymbol{x}^0,\ \ \theta - \frac{(\bar{\boldsymbol{r}}^2)^\top\boldsymbol{x}^1 - t}{\sqrt{(\boldsymbol{x}^1)^\top Q^2\boldsymbol{x}^1}} \leq 0. \tag{26}$$

Now we can eliminate the variable θ and obtain the following equivalent formulation:

$$\sup_{\boldsymbol{x}^0:\boldsymbol{x}^0\in\mathcal{X}^0}\ \inf_{\boldsymbol{v}^1\in\mathcal{U}}\ \sup_{\boldsymbol{x}^1\in\mathcal{X}^1(\boldsymbol{x}^0,\boldsymbol{v}^1)} \frac{(\bar{\boldsymbol{r}}^2)^\top\boldsymbol{x}^1 - t}{\sqrt{(\boldsymbol{x}^1)^\top Q^2\boldsymbol{x}^1}} \text{ s.t. } \boldsymbol{e}^\top\boldsymbol{x}^1 = (\boldsymbol{v}^1)^\top\boldsymbol{x}^0. \tag{27}$$

This is an ARO formulation for a fractional programming problem. If $\mathcal{X}^1(\boldsymbol{x}^0,\boldsymbol{v}^1)$ is a quasi-convex mapping (trivially satisfied when $\mathcal{X}^1$ is a fixed convex set independent of $\boldsymbol{x}^0$ and $\boldsymbol{v}^1$ and the only restriction on $\boldsymbol{x}^1$ from $\boldsymbol{x}^0$ and $\boldsymbol{v}^1$ is through the self-financing constraint) the only condition we need to verify to apply Proposition 5 is the quasi-concavity of the objective function in (27). Note that we need quasi-concavity rather than quasi-convexity since the inner problem is a maximization problem.

A solution to (27) is meaningful in the sense of a robust profit opportunity only when the optimal objective value of this problem is nonnegative. If we assume that this is the case, it is sufficient to verify the quasi-concavity of the objective function on the set $S = \{\boldsymbol{x} : (\bar{\boldsymbol{r}}^2)^\top\boldsymbol{x} \geq t\}$ since points in the complement of this set yield negative and therefore suboptimal objective values. To simplify the verification let $h(\boldsymbol{x}) \equiv \sqrt{\boldsymbol{x}^\top Q^2\boldsymbol{x}}$ and $g(\boldsymbol{x}) \equiv \frac{(\bar{\boldsymbol{r}}^2)^\top\boldsymbol{x}-t}{h(\boldsymbol{x})}$. Note that, since Q^2 is positive definite, h defines a norm and therefore is a convex function [7]. Given $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\lambda\in[0,1]$, define $\boldsymbol{x}_\lambda = \lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2$. Also, let $\tilde{\lambda} = \frac{\lambda h(\boldsymbol{x}_1)}{\lambda h(\boldsymbol{x}_1)+(1-\lambda)h(\boldsymbol{x}_2)}$ which is a number between 0 and 1. Then,

$$\begin{aligned}
g(\boldsymbol{x}_\lambda) &= \frac{(\bar{\boldsymbol{r}}^2)^\top\boldsymbol{x}_\lambda - t}{h(\boldsymbol{x}_\lambda)} \\
&\geq \frac{(\bar{\boldsymbol{r}}^2)^\top\boldsymbol{x}_\lambda - t}{\lambda h(\boldsymbol{x}_1) + (1-\lambda)h(\boldsymbol{x}_2)} \\
&= \tilde{\lambda} g(\boldsymbol{x}_1) + (1-\tilde{\lambda})g(\boldsymbol{x}_2) \\
&\geq \min\{g(\boldsymbol{x}_1), g(\boldsymbol{x}_2)\}
\end{aligned}$$

establishing the quasi-concavity of g on S (recall that g is quasi-concave if and only if $-g$ is quasi-convex and $-g$ is quasi-convex if and only if $-g(\boldsymbol{x}_\lambda) \le \max\{-g(\boldsymbol{x}_1), -g(\boldsymbol{x}_2)\}$ for every $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\lambda \in [0,1]$). The first inequality above follows from the convexity of h and the second one holds since a convex combination of two numbers is no less than the minimum of the two.

Note that g is not convex in general. Therefore, the problem (25) represents an interesting application of the ARO models developed here but not available in earlier works. Using Proposition 5 and the discussion in Section 2, we can write problem (27) as a single-level deterministic problem:

$$
\begin{array}{rl}
\sup\limits_{\boldsymbol{x}^0, \boldsymbol{x}_1^1, \ldots, \boldsymbol{x}_k^1, s} & s \\
\text{s.t.} & \frac{(\bar{\boldsymbol{r}}^2)^\top \boldsymbol{x}_j^1 - t}{\sqrt{(\boldsymbol{x}_j^1)^\top \boldsymbol{Q}^2 \boldsymbol{x}_j^1}} \ge s, \quad j = 1, \ldots, k \\
& \boldsymbol{e}^\top \boldsymbol{x}_j^1 = (\boldsymbol{v}_j^1)^\top \boldsymbol{x}^0, \quad j = 1, \ldots, k, \\
& \boldsymbol{x}^0 \in \mathcal{X}^0, \ \ \boldsymbol{x}_j^1 \in \mathcal{X}^1(\boldsymbol{x}^0, \boldsymbol{v}_j^1), \quad j = 1, \ldots, k
\end{array}
$$

which is also transformed to

$$
\sup_{\boldsymbol{x}} \inf_{j=1,\ldots,k} \frac{f_j(\boldsymbol{x})}{g_j(\boldsymbol{x})} \text{ s.t. } \boldsymbol{x} \in \mathcal{X}, \tag{28}
$$

where

$$
\begin{array}{l}
\boldsymbol{x} \equiv (\boldsymbol{x}^0, \boldsymbol{x}_1^1, \ldots, \boldsymbol{x}_k^1), \\
f_j(\boldsymbol{x}) = (\bar{\boldsymbol{r}}^2)^\top \boldsymbol{x}_j^1 - t, \ \ g_j(\boldsymbol{x}) = \sqrt{(\boldsymbol{x}_j^1)^\top \boldsymbol{Q}^2 \boldsymbol{x}_j^1}, \\
\mathcal{X} \equiv \left\{ \boldsymbol{x} : \begin{array}{l} \boldsymbol{e}^\top \boldsymbol{x}_j^1 = (\boldsymbol{v}_j^1)^\top \boldsymbol{x}^0, \quad j = 1, \ldots, k \\ \boldsymbol{x}^0 \in \mathcal{X}^0, \ \ \boldsymbol{x}_j^1 \in \mathcal{X}^1(\boldsymbol{x}^0, \boldsymbol{v}_j^1), \quad j = 1, \ldots, k \end{array} \right\}.
\end{array}
$$

Problem (28) is known as *max-min fractional program*, and there are several kinds of solution methods for solving this problem such as Dinkelbach's algorithm [8] and the interior-point algorithms proposed by Freund *et al.* [9] and Nemirovski [13].

In [14], the two-period maximum-θ RPO problem (25) is solved for the special case of $X^1 \equiv \mathbb{R}^n$ and $X^0 = \{\boldsymbol{x} : \boldsymbol{e}^\top \boldsymbol{x} < 0\}$. The techniques used in [14] rely on the homogeneity of the objective function as well as the constraints defining sets X^0 and X^1. Since we do not require such assumptions on the constraint sets, our approach here applies to more general RPO problems.

5.2 Maximum Sharpe Ratio Problem

If the investment environment contains a riskless security in addition to the risky securities we already considered above, it is possible to construct zero-investment portfolios by purchasing the portfolio $\boldsymbol{x}$ after borrowing $\boldsymbol{e}^\top \boldsymbol{x}$ dollars at the riskless rate r_f. If $\boldsymbol{e}^\top \boldsymbol{x} < 0$, this would correspond to lending $-\boldsymbol{e}^\top \boldsymbol{x}$ dollars. A well-studied problem in portfolio selection is that of finding a zero-investment portfolio that has the highest expected return to standard deviation ratio. This scale invariant quantity is called the Sharpe ratio and can be written as

$$\max_{\boldsymbol{x}} \frac{\bar{\boldsymbol{r}}^\top \boldsymbol{x} - r_f(\boldsymbol{e}^\top \boldsymbol{x})}{\sqrt{\boldsymbol{x}^\top Q \boldsymbol{x}}} \tag{29}$$

where we used the same notation as in the previous subsection (without time superscripts) and r_f represents the risk-free return rate. This function has the same structure as $g(\boldsymbol{x})$ in the previous subsection and hence is quasi-concave. Indeed, there is an equivalence between the maximum Sharpe ratio and the maximum-θ RPO problem provided that riskless securities are available, see [14]. For the problem of maximizing the final period Sharpe ratio in a two-period or multi-period framework, one can develop an analogous formulation to (27). Therefore, the two-period formulation of the maximum Sharpe ratio problem is another important example of the special nonlinear structures we considered in Sections 2 and 3 that lead to tractable formulations.

ACKNOWLEDGMENT

This work is partially supported by the National Science Foundation under grants CCR-9875559 and DMS-0139911, and Grant-in-Aid for Scientific Research from the Ministry of Education, Sports, Science and Culture of Japan under Grant No.16710110.

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