12) Distributions:

During the course, we used several times the so called **delta function** δ_x in two contexts:

i) as a density,

ii) as the Laplacian of the fundamental solution of the Laplace equation.

In both cases, what we did doesn't have any sense! Indeed, when we saw the delta function as a density, we saw δ_0 as representing the density of a particle of mass 1 placed at $x=0$; of course, the density of such an object is

$$\begin{cases} +\infty & x=0, \\ 0 & x\neq 0. \end{cases}$$

On the other hand, when we used the delta function to describe $\Delta\Gamma$, we said that

$$\Delta\Gamma = \begin{cases} 1 & \text{at } x=0, \\ 0 & \text{at } x\neq 0. \end{cases}$$

But the left-hand side doesn't have any meaning at $x=0$, since Γ is <u>not</u> defined at $x=0$!

We would like to give a meaning to all the above writing. To do so, let us start from the concept of density. Density is an average of some quantity over a set. An this is the only quantity that has a physical meaning, because it's something that can be measured. The density at a point has no meaning at all!

In particular, if u is the quantity that we want to understand [let's say, u is the temperature], the only things that we can measures are

$$(1) \qquad \frac{1}{Vol(S)} \int_S u \, dx,$$

for any set $S \subseteq \mathbb{R}^3$. It is possible to show that the Knowledge of (1) for all sets $S \subseteq \mathbb{R}^3$ is enough to **recover** u [up to a set that is "small", in the sense that, it is possible to Know u in the whole $\mathbb{R}^3$ up to a set $N \subseteq \mathbb{R}^3$ s.t.

$$\frac{1}{Vol(S)} \int_S u \, dx = \frac{1}{Vol(S \setminus N)} \int_{S \setminus N} u \, dx,$$

for all $S \subseteq \mathbb{R}^3$.] [This is called the

Now, we notice that (1) is one way to averaging, namely is an average where all points matter the same. But it is also possible to averaging in different ways, where different points have different weights in the average. Namely, a "way of averaging" is a function $\varphi \in C_c^\infty(\mathbb{R}^N)$ s.t. $\int_{\mathbb{R}^N} \varphi(x) dx = 1$.

Think at φ as a probability. The requirement that φ is regular, is because, physically, we can do that, **Moreover**, the compact support is to "localize" the averaging proceedure. So, we know how to measure: $\boxed{\int \varphi(x) u(x) dx}$ for every

So, the <u>fundamental change of point of view</u> is the following:

instead of considering our objects u as fuction, i.e., a "machine" whose input is a <u>point</u> and whose output is a number, we want to see our object u as a "machine" whose input is a <u>way of averaging</u> and whose outpus is a number.

These new objects are called <u>distributions</u> and were introduced by Schwartz in the 1940s. The idea wasn't completely new, but the way he applied them to analysis was revolutionary, and won him the Field medal [the "Nobel prize of math"]

To be more precise, a **distribution** is a linear $\rightsquigarrow$ technical and continuous functional $T: C_c^\infty(\mathbb{R}^N) \longrightarrow \mathbb{R}$.

Because if we change a little the way of averaging, we want the average to change just a little

we extend its physical meaning of average, by taking:

$$\tilde{\varphi}(x) := \frac{\varphi(x)}{\int_{\mathbb{R}^N} \varphi\,dx} \Rightarrow \int_{\mathbb{R}^N} \tilde{\varphi}\,dx = 1$$

The space of distributions is denoted by $\mathcal{D}'(\mathbb{R}^N)$.

- <u>Notice:</u> a function $u: \mathbb{R}^N \to \mathbb{R}$ can be seen as the distribution $T_u: C_c^\infty(\mathbb{R}^N) \to \mathbb{R}$ defined as:

$$\langle T_u, \varphi \rangle := \int_{\mathbb{R}^N} \varphi(x) u(x)\,dx .$$

[Read as: if we know the pointwise value of a]

- **Interlude** : is the space $C_c^\infty(\mathbb{R}^N)$ not empty?

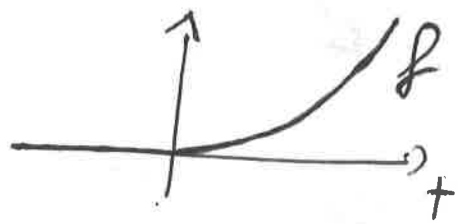
The space $C_c^\infty(\mathbb{R}^N)$ is called the space of test functions. We have to make sure that it is not empty, otherwise everything we said before has no meaning.
We will show that it is not empty in two ways: by providing an explicit example of a function belonging to it, and by providing a general method to construct functions belonging to it.

i) An explicit example:

- Let us consider the function:

$$f(t) := \begin{cases} e^{-1/t} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then, $f \in C^\infty(\mathbb{R})$.



- Fix a number $R > 0$, and consider the function:

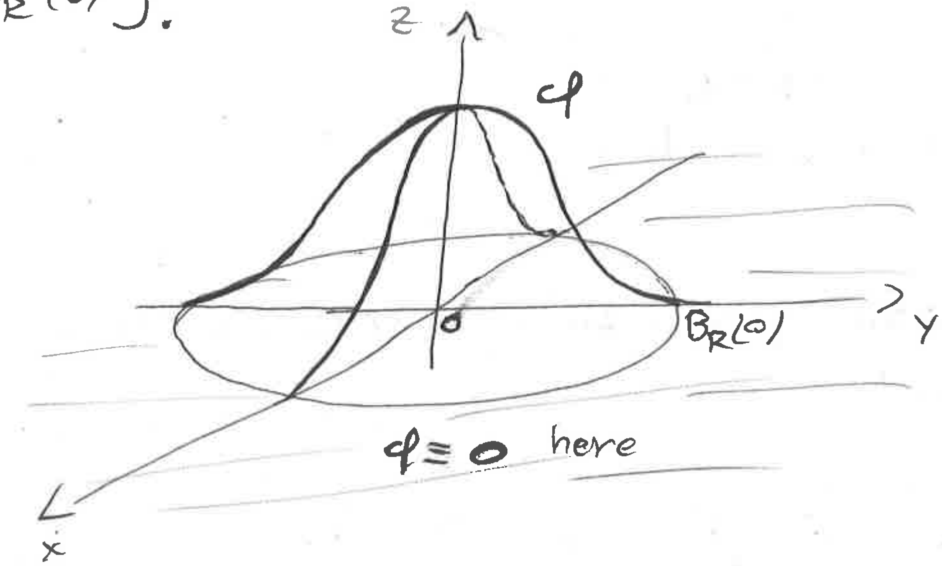
$$g(t) := f(R+t) f(R-t).$$

Then $g \in C^\infty(\mathbb{R})$ and $g \equiv 0$ outside $[-R, R]$.

- Consider the function $\varphi : \mathbb{R}^N \to \mathbb{R}$ defined as

$$\varphi(x) := g(|x|).$$

Then $\varphi \in C_c^\infty(\mathbb{R}^N)$ [In particular, $\varphi \equiv 0$ outside $B_R(0)$].



ii) <u>Convolution of functions</u>:

The idea of averaging is the key ingredient for using convolution to approximate generic functions with regular ones.
Let us be more precise:

- **<u>Def.</u>**: a function $f : \mathbb{R}^N \to \mathbb{R}$ is said to belong to the space $L^1(\mathbb{R}^N)$ [L stands for Lebesgue] if

$$\int_{\mathbb{R}^N} |f(x)|\,dx < +\infty.$$

- <u>Remark</u>: functions in $L^1(\mathbb{R}^N)$ can fail to be continuous; e.g., $f(x) := \begin{cases} 0 & |x| < 1 \\ 1 & 1 \le |x| \le 2 \end{cases}$

• **<u>Def</u>**: let $f \in L^1(\mathbb{R}^N)$ and let $g : \mathbb{R}^N \to \mathbb{R}$ be bounded. We define the <u>convolution (product)</u> of f and g, $f * g$, as

$$(f*g)(x) := \int_{\mathbb{R}^N} f(y)\, g(y-x)\, dy.$$

The above operation can be used to regularize a function.

• **<u>Def</u>**: let $\rho \in C_c^\infty(\mathbb{R}^N)$, be s.t.

• $\rho \equiv 0$ out of $B_1(0)$

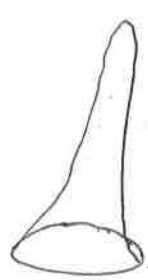
• $\displaystyle\int_{B_1(0)} \rho(x)\, dx = 1$

For every $\varepsilon > 0$, we define

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^N} \rho\left(\frac{x}{\varepsilon}\right).$$

The family $(\rho_\varepsilon)_{\varepsilon > 0}$ are called **<u>mollifiers</u>**.

• <u>Remark</u>: •



$B_\varepsilon(0)$

• $\displaystyle\int_{B_\varepsilon(0)} \rho_\varepsilon(x)\, dx = 1$

• $\rho_\varepsilon \equiv 0$ out of $B_\varepsilon(0)$

• ρ_ε is an average with range B_ε

The following holds:

- **Thm.:** Let $f \in L^1(\mathbb{R}^N)$. Then:

  i) $\forall \varepsilon > 0$, $f * \rho_\varepsilon \in C^\infty(\mathbb{R}^N)$

  ii) $$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} |(f * \rho_\varepsilon)(x) - f(x)|\, dx = 0$$

→ • i) says that $f * \rho_\varepsilon$ regularizes f

  • ii) says that, as $\varepsilon \to 0$, we recover, in some sense, f.

→ Notice that if f has compact support, then also $f * \rho_\varepsilon$ does.
In particular, $f * \rho_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ if $f \in L^1(\mathbb{R}^N)$ has compact support.

- **Notation:** $C_c^\infty(\mathbb{R}^N) =: \mathcal{D}(\mathbb{R}^N)$

- <u>Convergence of distributions</u>:

We now define the notion of convergence for distributions, that is the equivalent of pointwise convergence for functions, that we recall here:

- **Def**: let $(f_n)_n$ be functions in $\mathbb{R}$.
  We say that $f_n \to f$ <u>pointwise</u> if
  $$f_n(x) \to f(x) \qquad \forall x \in \mathbb{R}^N.$$

- **Def**: let $(T_n)_n \subseteq \mathcal{D}'(\mathbb{R}^N)$.
  We say that $T_n \xrightarrow{\mathcal{D}'} T$ if
  $$\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \quad \forall \varphi \in \underbrace{C_c^\infty(\mathbb{R}^N)}_{= \mathcal{D}(\mathbb{R}^N)}$$

- <u>The delta function</u>:

Le us define the distribution:
$$\langle \delta_0, \varphi \rangle := \varphi(0).$$

Then δ_0 is clearly linear and continuous, and thus it is a distribution. Of course δ_0 is <u>not</u> the distribution induced by a function, since:
$$\int_{\mathbb{R}^N} u(x)\varphi(x) = \varphi(0) \implies u \equiv 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}$$
$$\Downarrow$$
$$\int_{\mathbb{R}^N} u(x)\varphi(x)\,dx = 0 \quad \text{↯}.$$

It holds that:

- **Thm:**
  Let $f \in L^1(\mathbb{R})$ and define
  $$f_n(x) := n\, f(nx).$$
  Then $$f_n \xrightarrow{\mathcal{D}'} \delta_0.$$

We will apply the previous result to justify the assertion we made about the solution of the heat eq. in 1D:

let
$$u(x,t) := \int_{\mathbb{R}} g(y)\, \Gamma_D(x-y,t)\, dy.$$
Then, if $g \in C_c^\infty(\mathbb{R})$, we have that
$$(\circ) \quad u(x,t) \xrightarrow[t\to 0^+]{} g(x) \quad \forall x \in \mathbb{R}.$$

Indeed, let:
$$f(y) := \frac{1}{\sqrt{4\pi D}}\, e^{-\frac{y^2}{4D}}$$

and consider the function: $\frac{1}{\sqrt{t}} f\left(\frac{y}{\sqrt{t}}\right) = \Gamma_D(y,t)$

By the previous thm, we get that $\Gamma_D(\cdot,t) \xrightarrow[t\to 0^+]{\mathcal{D}'} \delta_0$.
By applying a translation, we also get
$$\Gamma_D(\cdot - x, t) \xrightarrow[t\to 0^+]{\mathcal{D}'} \delta_x \text{ and } \langle \delta_x, \varphi \rangle = \varphi(x)$$
$$\Downarrow$$
this is exactly $(\circ)$

- Derivative of a distribution:

In order to define the derivative of a distribution, we take inspiration from the case of a distribution generated by a function u of class C^1:

$$\int_{\mathbb{R}} u'(x)\varphi(x)dx = \underbrace{u(x)\varphi(x)\Big|_{-\infty}^{+\infty}}_{\substack{=0 \text{ for} \\ \varphi\in C_c^\infty(\mathbb{R})}} - \int_{\mathbb{R}} u(x)\varphi'(x)\,dx.$$

Thus, the following definition comes naturally:

- Def.: let $T\in\mathcal{D}'(\mathbb{R}^N)$. We define the partial derivative of T w.r.t. x_i as the distribution $\partial_{x_i}T$ defined as:

$$<\partial_{x_i}T,\varphi> := -<T,\partial_{x_i}\varphi> \quad \forall\varphi\in\mathcal{D}(\mathbb{R}^N).$$

- We now want to justify the claim:

$$-\Delta\Gamma=\delta_0,$$

where Γ is the fundamental solution for the Laplacian. By linearity, we have that:

$$<-\Delta\Gamma,\varphi> = <\Gamma,-\Delta\varphi>,$$

and we would like to prove that

$$<\Gamma,-\Delta\varphi> = \varphi(0) \qquad \forall\varphi\in\mathcal{D}(\mathbb{R}^N).$$

We have that:

$$<\Gamma, -\Delta\varphi> = \int_{\mathbb{R}^N} \Gamma(x)(-\Delta\varphi(x))\,dx$$ $\rightsquigarrow$ notice that Γ is <u>not</u> defined at $x=0$!

$$= \lim_{\varepsilon\to 0} \underbrace{\int_{\mathbb{R}^N\setminus B_\varepsilon(0)} \Gamma(x)(-\Delta\varphi(x))\,dx}$$

$\downarrow$ by Greens second identity

$$= \int_{\mathbb{R}^N\setminus B_\varepsilon(0)} \underbrace{(-\Delta\Gamma(x))}_{=0}\varphi(x)\,dx + \int_{\partial B_\varepsilon(0)} \left[-(\partial_\nu\varphi)\Gamma + \varphi(\partial_\nu\Gamma)\right]$$

So, let us consider:

$$\int_{\partial B_\varepsilon(0)} (\partial_\nu\varphi)\Gamma = \int_{\partial B_\varepsilon(0)} (\nabla\varphi\cdot\nu)\Gamma.$$

Since $\varphi\in C_c^\infty(\mathbb{R}^N)$, $\underbrace{\|\nabla\varphi\| \text{ is bounded}}_{\|\nabla\varphi\|\le C}$, and hence:

$\Gamma\geq 0$!

$$\left|\int_{\partial B_\varepsilon(0)} (\nabla\varphi\cdot\nu)\Gamma\right| \le \int_{\partial B_\varepsilon(0)} |\nabla\varphi\cdot\nu|\,\Gamma$$

$$\le \int_{\partial B_\varepsilon(0)} \|\nabla\varphi\|\,\Gamma$$

$|v\cdot w|\le\|v\|\,\|w\|$ Cauchy-Schwarz inequality

$$\le C\int_{\partial B_\varepsilon(0)} \Gamma$$

It is possible to show that

$$\int \Gamma \to 0 \quad \text{as } \varepsilon\to 0.$$

For the other term, we have that [assume $N \geq 3$]

$$\int_{\partial B_\varepsilon(0)} \varphi \, \partial_\nu \Gamma = \int_{\partial B_\varepsilon(0)} \varphi \, \nabla \Gamma \cdot \nu$$


$B_\varepsilon(0)$

$\nu = -\frac{x}{\|x\|}$

$$= \int_{\partial B_\varepsilon(0)} \varphi \left(-\frac{\partial}{\partial \|x\|} \left(\frac{1}{\omega_N N (N-2) \|x\|^{N-2}} \right) \right)$$

$$= \int_{\partial B_\varepsilon(0)} \varphi \, \frac{1}{N \omega_N \|x\|^{N-1}}$$

$x \in \partial B_\varepsilon(0) \Rightarrow \|x\| = \varepsilon$

$$= \frac{1}{\underbrace{N \omega_N \varepsilon^{N-1}}_{\text{Area of } \partial B_\varepsilon(0)}} \int_{\partial B_\varepsilon(0)} \varphi$$

$\leadsto$ this is the average of φ on $\partial B_\varepsilon(0)$. Since φ is continuous, it converges to $\varphi(0)$.