## SUPERSTABILITY AND CATEGORICITY IN ABSTRACT ELEMENTARY CLASSES

by

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#### Abstract

We<sup>1</sup> study the model theory of abstract elementary classes (AECs). They are a family of concrete categories closed under directed colimits where all the morphisms are monomorphisms, containing in particular the category of models of an  $\mathbb{L}_{\lambda,\omega}(Q)$ theory (with the natural notion of elementary embedding). This framework was identified by Shelah in 1977, when he proposed the far-reaching program of adapting his classification theory (originally developed for first-order logic) to AECs and hence to all "reasonable" infinitary logics. This thesis develops an analog of Shelah's first-order superstability theory to AECs. This involves studying forking-like notions of independence in this general framework, giving criteria for when they exist, and linking their properties to the stability spectrum and the behavior of chains of saturated models. We usually assume that the AEC has amalgamation, and also often that it is tame, a locality property of orbital (Galois) types introduced by Grossberg and VanDieren. It is conjectured that these properties should follow from categoricity.

We solve several open problems using the technology developed in this thesis. Shelah's eventual categoricity conjecture is the statement that an AEC categorical in *some* high-enough cardinal should be categorical in *all* high-enough cardinals. It is the main open question of the field. We apply the superstability theory to show that the conjecture holds in universal classes (a special kind of AECs: classes closed under isomorphisms, substructures, and union of  $\subseteq$ -increasing chains). Previous approximations were in a stronger set theory than ZFC and always assumed categoricity in a successor cardinal. Here, we eliminate the successor hypothesis and work in ZFC.

We also show that if an AEC **K** with amalgamation and no maximal models is categorical in  $\lambda > LS(\mathbf{K})$ , then the model of cardinality  $\lambda$  is saturated (in the sense of orbital types). This answers a question asked by both Baldwin and Shelah.

In several cases, the arguments developed for the superstability theory are useful also in case the AEC is stable but not superstable. This thesis develops the theory in this case as well, proving for tame AECs the equivalence between stability and no order property, as well as an eventual characterization of the stability spectrum under the singular cardinal hypothesis.

Independence relations (like Shelah's notion of a good frame) are *the* central pillar of the (super)stability theory developed here. They are a deep generalization of linear independence of vector spaces and algebraic independence in fields, so we expect that they will have many other applications, both to the abstract theory and to concrete (algebraic or perhaps analytic) examples.

Another contribution of this thesis is the definition of a *quasiminimal AEC*: it is an AEC with countable Löwenheim-Skolem-Tarski number which has a prime model, is closed under intersections, and has a unique generic type over every countable model. We show that quasiminimal AECs are exactly the quasiminimal pregeometry classes that Zilber used to study pseudo-exponential fields, motivated

<sup>&</sup>lt;sup>1</sup>The results of this thesis were written up in 22 separate papers submitted for publication in refereed journals, 16 of which have already been accepted, and 11 of the accepted ones have already appeared in print. While a majority of the papers are single author, some were written with collaborators (Boney, Grossberg, Kolesnikov, Lieberman, Rosický, Shelah, and VanDieren). Details and credits appear in Section 1.6 and at the start of every chapter.

#### ABSTRACT

by Schanuel's conjecture. In particular, an unbounded quasimimimal AEC is categorical in every uncountable cardinal. Along the way, we give new conditions under which a homogeneous closure operator has exchange and conclude that the exchange axiom is redundant in Zilber's definition of a quasiminimal pregeometry class.

We also study a more general notion than AECs:  $\mu$ -AECs, which are only required to be closed under  $\mu$ -directed (rather than  $\aleph_0$ -directed) colimits. We generalize some basic arguments from the theory of AECs and show that  $\mu$ -AECs are exactly the accessible categories whose morphisms are monomorphisms (this is joint work with Will Boney, Rami Grossberg, Michael Lieberman, and Jiří Rosický).

Finally, the thesis contains a chapter on simple first-order theories. We present a new proof of the existence of Morley sequences in such theories which avoids using the Erdős-Rado theorem and instead uses only Ramsey's theorem and compactness. The proof shows that the basic theory of forking in simple theories can be developed using only principles from "ordinary mathematics", answering a question of Grossberg, Iovino and Lessmann, as well as a question of Baldwin.

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## CHAPTER 1

## Introduction

### 1.1. Overview of this introduction

In Section 1.2.1, we give some background on the fields and problems that this thesis is concerned about. In Section 1.3, we present the main results of this thesis, as related to the problems introduced earlier. In Section 1.4, we discuss some other results that are peripheral to the core theme of the thesis. In Section 1.5, we make an attempt to sketch the proof of the main result of this thesis: Shelah's eventual categoricity conjecture for universal classes. Note that little background is assumed there and so we often have to be vague. However we hope that this short discussion can help give a flavor of what the thesis is about. In Section 1.6, we give a very short description of the chapters of this thesis, in the order in which they appear.

#### 1.2. Background

**1.2.1. Model theory and classification theory.** The topic of this thesis is model theory (a branch of mathematical logic), and more precisely classification theory.

The main concept of model theory is that of a structure (or model): it is a (potentially uncountably infinite) tuple  $M = (|M|, R_1, R_2, \ldots, f_1, f_2, \ldots)$  consisting of a universe |M|, relations  $R_1, R_2, \ldots$ , and functions  $f_1, f_2, \ldots$  which are allowed to be *n*-ary for any finite *n*. For example, a group can be seen as a structure  $G = (|G|, \cdot, x \mapsto x^{-1}, e)$ , where  $\cdot$  is a binary operation,  $x \mapsto x^{-1}$  is a 1-ary function, and *e* is a 0-ary function (i.e. a constant) standing for the identity element. A linear order is a structure I = (|I|, <) where < is a binary relation (at that point we do not specify what axioms the functions and relations must satisfy). The *vocabulary* (also called *similarity type*) of a structure describes uniquely what type of operation a structure has (e.g. the vocabulary of linear order will say that it consists of only a single binary relation).

Broadly speaking, model theory abstractly studies *classes* of structures (usually in the same vocabulary) of a particular form. Classically, model theorists have studied the class of models of a fixed first-order theory T. A first-order theory is a collection of first-order formulas, and a first-order formula is a statement like " $\forall x \forall y \exists z : x \cdot y = z \land y \cdot x = z$ ", where importantly everything inside the formula is finitary (for example only finitely many conjunctions are allowed) and we only allow quantification over objects of the universe, not subsets. Several interesting classes of mathematical objects are classes of models of a first-order theory (for example the classes of abelian groups, algebraically closed fields, or vector spaces over a fixed field). However several are not: consider locally finite groups (we have to say "every finitely generated subgroup is finite", and it turns out it is impossible to encode the notion "is finite" in first-order logic) or non-Archimedean fields (we have to say "there is an element that is above 1 and above 1+1 and above 1+1+1 and...", an infinite conjunction).

Roughly speaking, the goal of *classification theory* (for a certain class of structures) is to find *dividing lines*: if a class of structures is on the good side of the line, then one can analyze and understand its structure well. If it is on the bad side, then one can prove it behaves so wildly that there is little hope of understanding it. Classification theory originated in the seventies for classes of structures axiomatized by a first-order theory, the reference being Saharon Shelah's book [She90]. An example of a dividing line in that context is stability: let us say that a first-order theory T is stable if does not have the order property. T has the order property if there exists a model M of T, a first-order formula  $\phi(\bar{x}, \bar{y})$  with  $\ell(\bar{x}) = \ell(\bar{y}) = n$ and a sequence  $\langle \bar{a}_i : i < \omega \rangle$  (with  $\bar{a}_i$  an *n*-tuple in M for each  $i < \omega$ ) such that  $M \models \phi[\bar{a}_i, \bar{a}_i]$  if and only if i < j. In other words,  $\phi(\bar{x}, \bar{y})$  defines a linear order inside M. Shelah has shown that if a theory is stable then it admits a quite wellbehaved notion of independence (forking), and a local notion of dimension. If a theory has the order property, then it can define infinite linear orders which makes its analysis much harder. For example, if T has the order property then for all  $\lambda > |T|$ , T has  $2^{\lambda}$  non-isomorphic models of size  $\lambda$ . Examples of stable theories include the theory of algebraically closed fields or the theory of differentially closed fields of a fixed characteristic.

1.2.2. Shelah's main gap theorem. Since even among the stable theories, there are some that have  $2^{\lambda}$ -many non-isomorphic models in every size  $\lambda > |T|$ , Shelah isolated several more dividing lines within the stable theories (superstable, DOP, OTOP, and deep) and showed each time that (when the theory is countable) a theory on the good side of the line was nice and a theory on the bad side had "many" non-isomorphic models. Shelah concluded that if a theory falls on the good side of each of the lines, then its models can be completely analyzed and described by invariants. In particular, the theory will have "few" non-isomorphic models. Shelah called this result the main theorem of his book [She90, Chapter XII] (see [She85a] for an exposition):

FACT 1.2.1 (Shelah's main gap theorem). Let T be a countable first-order theory. Then exactly one of the following is true:

- (1) For any uncountable cardinal  $\lambda$ , T has  $2^{\lambda}$  many non-isomorphic models of size  $\lambda$ .
- (2) For any ordinal  $\alpha$ , T has at most<sup>1</sup>  $\beth_{\omega_1}(|\alpha|)$  many non-isomorphic models of size  $\aleph_{\alpha}$ . Moreover every model of T can be decomposed into a tree of countable submodels, each "as free as possible from the others".

The methods developed to prove Fact 1.2.1 have had a large impact on algebra, number theory, and geometry, see for example [**Bou99**]. Note that the main gap theorem can be extended to some non-elementary classes [**GH89**, **GL05**, **HS01**]. In fact, it is a major open question whether the main gap generalizes to AECs or even to *uncountable* first-order theories [**She00**, 4.7].

<sup>&</sup>lt;sup>1</sup>For an ordinal  $\gamma$  and a cardinal  $\mu$ , the cardinal  $\beth_{\gamma}(\mu)$  is the iteration of the operation  $\lambda \mapsto 2^{\lambda} \gamma$ -many times, starting with  $\mu$ . Intuitively, it is a tower of exponentials of height  $\gamma$  where the cardinal on the top is  $\mu$ .  $\beth_{\gamma}$  denotes  $\beth_{\gamma}(\aleph_0)$ .

#### 1.2. BACKGROUND

**1.2.3.** Abstract elementary classes. This thesis is about moving beyond the first-order context and looking at what can be said about classes of structures that are *not* axiomatized by a first-order theory. Although this has not yet fully materialized, it is expected that generalizing the first-order tools will also lead to applications to algebra, number theory, and geometry. See for example the work of Zilber on pseudo-exponentiation and Schanuel's conjecture [Zil05a], connected to Chapter 21 of this thesis.

We have mentioned the examples of locally finite groups and non-Archimedean fields: they fall into a framework for which it seems some model-theoretic analysis is possible: *abstract elementary classes* (AECs). Roughly speaking, an AEC is a concrete category (whose objects are structures) satisfying several axioms (for example, morphisms must be injective homomorphisms and the class must be closed under directed colimits). It generalizes the notion of a class axiomatized by a first-order theory, and also encompasses many non first-order logics such as  $\mathbb{L}_{\infty,\omega}$  (i.e. disjunctions and conjunctions of arbitrary, possibly infinite, length are allowed). For completeness, we give the definition [She87a]:

DEFINITION 1.2.2. An abstract elementary class (AEC) is a pair  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  where:

- (1) K is a class of structures in a fixed vocabulary  $\tau = \tau(\mathbf{K})$ .
- (2)  $\leq_{\mathbf{K}}$  is a partial order on K.
- (3)  $K \text{ and } \leq_{\mathbf{K}} \text{ are closed under isomorphisms: if } M \in K \text{ and } f : M \cong N, \text{ then } N \in K, \text{ and if } M, N \in K, M \leq_{\mathbf{K}} N, \text{ and } g : N \cong N', \text{ then } f[M] \leq_{\mathbf{K}} N'.$
- (4) If  $M \leq_{\mathbf{K}} N$ , then M is a  $\tau(\mathbf{K})$ -substructure of N (i.e. the way the functions and relations of M are defined agree with the way they are defined in N).
- (5) (Coherence axiom) If  $M_0, M_1, M_2 \in K$ ,  $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ , and  $M_0 \leq_{\mathbf{K}} M_2$ , then  $M_0 \leq_{\mathbf{K}} M_1$ .
- (6) (Löwenheim-Skolem-Tarski axiom) There exists a cardinal  $\mu \geq |\tau(\mathbf{K})| + \aleph_0$ such that whenever  $M \in K$  and  $A \subseteq |M|$ , there exists  $M_0 \in K$  with  $M_0 \leq_{\mathbf{K}} M, A \subseteq |M_0|$ , and<sup>2</sup>  $||M_0|| \leq |A| + \mu$ . We write LS(**K**) (the *Löwenheim-Skolem-Tarski number of* **K**) for the least such  $\mu$ .
- (7) (Tarski-Vaught chain axioms) Let I be a (non-empty) directed partial order (i.e. a partial order where every finite subset has an upper bound). Let  $\langle M_i : i \in I \rangle$  be increasing in **K** (i.e. for all  $i \leq j$  both in  $I, M_i \leq_{\mathbf{K}} M_j$ ). Let  $M := \bigcup_{i \in I} M_i$  (we define the relations and functions of M naturally). Then:
  - (a)  $M \in K$ .
  - (b)  $M_i \leq_{\mathbf{K}} M$  for all  $i \in I$ .
  - (c) If  $N \in K$  is such that  $M_i \leq_{\mathbf{K}} N$  for all  $i \in I$ , then also  $M \leq_{\mathbf{K}} N$ .

An example of an AEC familiar to logicians is  $\mathbf{K} := (\text{Mod}(T), \preceq)$ , for T a first-order theory and  $\preceq$  standing for elementary substructure, or even  $\mathbf{K}^* := (\text{Mod}(\psi), \preceq_{\Phi})$ , for  $\psi$  an  $\mathbb{L}_{\omega_1,\omega}$ -sentence and  $\Phi$  a countable fragment containing  $\psi$  (here Mod(T) and  $\text{Mod}(\psi)$  denote the class of models of T and  $\psi$  respectively). We have that  $\text{LS}(\mathbf{K}) = |\tau(T)| + \aleph_0$  and  $\text{LS}(\mathbf{K}^*) = \aleph_0$ . More algebraic examples include the class of all locally finite groups ordered by "being a subgroup" or, less

<sup>&</sup>lt;sup>2</sup>Given a structure  $M_0$ , we write  $|M_0|$  for its universe and  $||M_0||$  for the *cardinality* of the universe.

trivially, the class of all abelian groups ordered by "being a *pure* subgroup". Both have Löwenheim-Skolem-Tarski number  $\aleph_0$ .

There is a downside to working with AECs instead of only looking at classes of models of a first-order theory: the compactness theorem is no longer available and thus there are fewer ways of constructing new objects. However, in addition to having great expressive power and capturing several classical examples, one of the great advantage of AECs is that they exhibit a lot of closure: starting from an AEC, one can look at the AEC of models above a certain size, at a sub-AEC of saturated models, or even at a sub-AEC of models omitting a certain type. These closure properties are used crucially throughout the development of the theory, including in the categoricity transfers mentioned later.

Further reasons for working in the framework of AECs include:

- (1) It leads to interesting mathematics.
- (2) It generalizes the first-order context without tying us to another particular logic.
- (3) It leads to further insight about first-order model theory (for example the main gap was in fact proven using methods from non-first-order model theory [She83a, She83b]).
- (4) It closely relates to the framework of *accessible categories* (see Chapter 13). This suggests a strong potential for applications to category theory and algebra.

AECs were introduced by Shelah in the mid seventies [She87a], and the reader is encouraged to consult the introduction to Shelah's two-volume book [She09a, She09b] for more motivation on studying them. Other recommended beginning references are Grossberg's survey [Gro02] and Baldwin's book [Bal09].

**1.2.4.** Shelah's eventual categoricity conjecture. A long-term goal of the classification theory for AECs is to prove an analog of the main gap, however this seems out of reach at present. Shelah has suggested the following easier test question<sup>3</sup>:

CONJECTURE 1.2.3 (Shelah's eventual categoricity conjecture, N.4.2 in [She09a]). If an AEC is categorical in *some* high-enough cardinal, then it is categorical in *all* high-enough cardinals.

More precisely, there exists a map  $\mu \mapsto \lambda(\mu)$  such that any AEC **K** categorical in some  $\lambda \geq \lambda(\text{LS}(\mathbf{K}))$  is categorical in all  $\lambda' \geq \lambda(\text{LS}(\mathbf{K}))$ .

Here, we say that a class of structures (or a formula or theory) is *categorical* in  $\lambda$  if it has exactly one (up to isomorphism) model of cardinality  $\lambda$ . Shelah's eventual categoricity conjecture is inspired by the following classical result which started modern classification theory [**Mor65**]:

FACT 1.2.4 (Morley, 1965). If a countable first-order theory is categorical in *some* uncountable cardinal, then it is categorical in *all* uncountable cardinals.

Note that both the class of algebraically closed fields of characteristic zero and the class of vector spaces over  $\mathbb{Q}$  are categorical in all uncountable cardinals, and the reasons for this are "simple" and "uniform" (i.e. there is a notion of basis and any

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<sup>&</sup>lt;sup>3</sup>The order of presentation that we follow is not the historical one. In particular Conjecture 1.2.7 came much before 1.2.3; see the introduction of Chapter 8 for the history of the conjecture.

#### 1.2. BACKGROUND

two bases for the same structure have the same cardinality). Morley's theorem and Shelah's eventual categoricity conjecture state that such a phenomenon (namely having simple uniform reasons for categoricity) should hold of any "reasonable" (i.e. first-order axiomatizable or at least AEC) class of objects. Further, "small" objects can have pathological behavior (for example the rationals are the unique countable dense linear order without endpoints but there are many such orders in every uncountable size) so (at least at first) we should only care about the "eventual" behavior of the class.

We now give some examples illustrating difficulties in generalizing Morley's theorem to non-elementary classes of objects. The first example is logically quite simple (it is the reduct of an elementary class), but the set of cardinals in which it is categorical has "gaps".

EXAMPLE 1.2.5 (Silver's example). Let  $\tau$  be the vocabulary containing one unary predicate. Let K be the class of  $\tau$ -structures  $M = (|M|, P^M)$  such that:

(1) 
$$||M||$$
 is infinite.

(2) 
$$||M| |P^{M}|| = ||M||.$$

(3) 
$$2^{|P^{M}|} \ge ||M||$$

Then K is categorical in  $\lambda$  if and only if  $\lambda$  is a strong limit cardinal. Note however that K (ordered with, say, the  $\tau$ -substructure relation<sup>4</sup>) is not an AEC.

The next example shows that for the class of models of an  $\mathbb{L}_{\omega_1,\omega}$  sentence, categoricity in a "small" uncountable cardinal will not imply categoricity in all uncountable cardinals.

EXAMPLE 1.2.6 (Morley's example). For any  $\alpha < \omega_1$ , the structure ( $\langle V_\beta : \beta < \alpha \rangle, \in$ ) can be coded by an  $\mathbb{L}_{\omega_1,\omega}$  sentence  $\psi_{\alpha}$ . Such a sentence has models only up to cardinality  $\beth_{\alpha}$  (This is known to be optimal: if an  $\mathbb{L}_{\omega_1,\omega}$ -sentence has a model of size  $\beth_{\omega_1}$ , then it has arbitrarily large models).

Now let  $\phi_{\alpha}$  be the "disjoint" disjunction of  $\psi_{\alpha}$  with the sentence " $\exists x : x = x$ " (or any other totally categorical sentence). Then  $\phi_{\alpha}$  is categorical exactly in the cardinals  $\lambda > \beth_{\alpha}$ . This shows that  $\lambda(\aleph_0)$  in Conjecture 1.2.3 should be at least  $\beth_{\omega_1}$ . In fact, by a similar argument (working with AECs rather than  $\mathbb{L}_{\omega_1,\omega}$ ),  $\lambda(\mu) \ge \beth_{(2^{\mu})^+}$ .

Despite numerous approximations (e.g. [She83a, She83b, MS90, SK96, She99, She01a, She09a, She09b, GV06c, GV06a, Bon14b], and this thesis), Conjecture 1.2.3 is still open. Shelah stated a version for classes of models axiomatized by an  $\mathbb{L}_{\omega_1,\omega}$ -sentence already in the summer of 1976, more than forty years ago. The conjecture appears as open problem D.(3a) in [She90]. It says that the "high-enough" threshold given by Example 1.2.6 is optimal:

CONJECTURE 1.2.7 (see p. 218 of [She83a]). Let  $\psi$  be an  $\mathbb{L}_{\omega_1,\omega}$  sentence (in a countable vocabulary). If  $\psi$  is categorical in some  $\lambda \geq \beth_{\omega_1}$ , then  $\psi$  is categorical in all  $\lambda' \geq \beth_{\omega_1}$ .

Versions of Conjecture 1.2.3 are known in much easier frameworks, closer to first-order (homogeneous model theory [She70], simple finitary AECs [HK11], and

<sup>&</sup>lt;sup>4</sup>In general, it is known that any AEC **K** is closed under  $\mathbb{L}_{\infty,LS(\mathbf{K})^+}$ -elementary equivalence but it is easy to see (using the back and forth characterization of elementary equivalence) that Silver's example is not. Thus no ordering of Silver's example can make it into an AEC.

quasiminimal pregeometries [Zil05a]). We emphasize that the conjecture is a test question: no direct applications of a positive answer are known but on the other hand it is believed that the methods developed to answer it will turn out to be important.

**1.2.5.** Independence. We now outline the central concept and tool of modern classification theory, the theory of independence. A stable independence relation on an AEC **K** is a 4-ary relation  $M_1 \underset{M_0}{\downarrow} M_2$  which roughly can be thought of as saying " $M_3$  is a stable amalgam of  $M_1$  and  $M_2$  over  $M_0$ " or " $M_1$  is independent from  $M_2$  over  $M_0$  in  $M_3$ " or even " $M_3$  extends the free product of  $M_1$  and  $M_2$  over  $M_0$ ".

More precisely, we require that  $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$ ,  $\ell = 1, 2, \downarrow$  is invariant under isomorphisms and monotonic in natural ways, and  $\downarrow$  satisfies four natural properties:

- Existence: If  $M_0 \leq_{\mathbf{K}} M_\ell$ ,  $\ell = 1, 2$ , there exists  $M_3 \in \mathbf{K}$  and  $f_\ell : M_\ell \xrightarrow{M_0}$  $M_3$  such that  $f_1[M_1] \bigcup_{M_0}^{M_3} f_2[M_2]$ .

- Uniqueness: If  $M_1 \stackrel{M_3}{\downarrow} M_2$  and  $M_1 \stackrel{M'_3}{\downarrow} M_2$ , then there exists  $N \in \mathbf{K}$  with  $M_3 \leq_{\mathbf{K}} N$  and  $f: M'_3 \stackrel{M_3}{\longrightarrow} N$ . Symmetry:  $M_1 \stackrel{M_3}{\downarrow} M_2$  if and only if  $M_2 \stackrel{M_3}{\downarrow} M_1$ . Local character: If  $N_1 \leq_{\mathbf{K}} N_3$ ,  $M_2 \leq_{\mathbf{K}} N_3$ , there exists  $M_0, M_1, M_3 \in \mathbf{K}$  with  $N_3 \leq_{\mathbf{K}} M_3$ ,  $N_1 \leq_{\mathbf{K}} M_1$ ,  $M_1 \stackrel{M_3}{\downarrow} M_2$ , and  $||M_0|| \leq ||N_1||$ .

For example, let **K** be the class of vector spaces over  $\mathbb{Q}$  and define  $M_1 \stackrel{M_3}{\downarrow} M_2$ 

to hold if and only if  $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$ ,  $\ell = 1, 2$  and  $M_1 \cap M_2 = M_0$ . Then  $\downarrow$ is a stable independence notion (uniqueness follows from basic properties of linear independence). One can define a similar notion in classes of algebraically closed fields. One can localize the definition to allow sets  $A_1, A_2$  instead of just elements of the class  $M_1, M_2$ , and it then becomes clear that the notion of stable amalgamation generalizes linear independence in vector spaces and algebraic independence in fields..

The term "stable" is inspired from the first-order theory, where Shelah showed that a theory is stable if and only if its class of models admits a stable independence relation (he called the relation  $forking^5$ ). This is the central notion of Shelah's book on first-order classification theory [She90].

An additional property that  $\downarrow$  satisfies in vector spaces (because of the finite character of the span) is:

<sup>&</sup>lt;sup>5</sup>One can define forking as follows:  $A \stackrel{M}{\underset{M_0}{\downarrow}} B$  if and only if for every finite sequences  $\bar{a}$  and  $\bar{b}$  of

elements of A and  $|M_0| \cup B$  respectively, there exists a sequence  $\bar{a}'$  of elements of  $M_0$  such that for any first-order formula  $\phi(\bar{x},\bar{y})$ , if  $\phi[\bar{a},\bar{b}]$  holds in M, then  $\phi[\bar{a}',\bar{b}]$  holds in M. One can check that this agrees with the definition given for vector spaces.

#### 1.2. BACKGROUND

• Finite local character: For any directed system  $\langle M_i : i \in I \rangle$  of  $\leq_{\mathbf{K}^-}$  substructure of  $N \in \mathbf{K}$  and any b in N, there exists  $i_0 \in I$  such that<sup>6</sup>  $b \downarrow \bigcup_{i \in I} M_i$ .

$$b \bigsqcup_{M_{i_0}} \bigcup_{i \in I} M$$

Shelah has shown that in the first-order context a stable independence notion has finite local character if and only if the theory is *superstable* (another dividing line, see Definition 1.2.14; it suffices to say here that any first-order theory T categorical in a cardinal above |T| is superstable). Note that one of the usual definitions of first-order superstability is "every type does not fork over a finite set". We have defined finite local character slightly differently here, since it seems that it is much harder to consider independence relations over arbitrary sets in AECs.

The following questions are natural and have essentially been fully answered in the first-order case:

QUESTIONS 1.2.8.

- (1) Under what conditions does an AEC have a (super)stable independence relation?
- (2) Is such a relation unique?
- (3) What consequences does the existence of a (super)stable independence relation have?
- (4) How does this help to make progress toward Shelah's eventual categoricity conjecture?

Among other directions, discussed in Section 1.4, this thesis is motivated by Question 1.2.8. We will state and discuss our results toward it in Section 1.3. For now, observe that an arbitrary AEC can be very difficult to work with (for example, it may have models that do not have any proper extension). Thus we first give the definition of several structural properties that we will often assume.

**1.2.6.** Amalgamation, orbital types, and tameness. Let us say that an AEC **K** has *amalgamation* if whenever  $M_0 \leq_{\mathbf{K}} M_\ell$ ,  $\ell = 1, 2$ , there exists  $N \in \mathbf{K}$  and  $f_\ell : M_\ell \xrightarrow{M_0} N$  (the notion of embedding here is the natural one:  $f : M \to N$  is a **K**-embedding - we will just call it an embedding - if it is an injective homomorphism and  $f[M] \leq_{\mathbf{K}} N$ ). Another property that **K** may have is *joint embedding*: any two models in **K** embed into a common element. In the first-order case, it is well-known that (in nontrivial cases) amalgamation and no maximal models hold. Joint embedding will also hold if the theory is complete (i.e. it decides every formula). However amalgamation and joint embedding do *not* hold in every AEC: see the examples in [**KLH16**] or [**BKL**].

Given an AEC **K** with amalgamation, joint embedding, and no maximal models, one can build a monster model  $\mathfrak{C} \in \mathbf{K}$  that will intuitively be "very big and homogeneous". Formally, it will be universal (any "small" model of **K** embeds into it) and model-homogeneous (if  $M \leq_{\mathbf{K}} N$  and  $M \leq_{\mathbf{K}} \mathfrak{C}$ , then N embeds into  $\mathfrak{C}$  over

<sup>&</sup>lt;sup>6</sup>We define  $b \bigcup_{M_0}^{M_3} M_2$  to hold if and only if there exists  $M_1, M'_3 \in \mathbf{K}$  with  $M_3 \leq_{\mathbf{K}} M'_3, b \in |M_1|$ , and  $M_1 \bigcup_{M_0}^{M'_3} M_2$ .

M). Once such a model is fixed, we can assume all the objects we consider live inside it. We will say that an AEC has a monster model to mean that it has amalgamation, joint embedding, and no maximal models (and hence a monster model can be built).

It turns out it is very useful to study orbits of elements of  $\mathfrak{C}$  under automorphisms. This is the fundamental notion of a *Galois* (or orbital) type:

DEFINITION 1.2.9. For a fixed monster model  $\mathfrak{C}$ , a subset A of  $|\mathfrak{C}|$ , and an element b from  $\mathfrak{C}$ , we denote by gtp(b/A) (the *Galois type of b over A*) the orbit of b under the automorphisms of  $\mathfrak{C}$  fixing A pointwise. We write gS(A) for the set of Galois types over A.

It is possible (but more technical) to define Galois types in any AEC (not necessarily with a monster model), see Definition 2.2.17. This is done in the only possible way so that the Galois types are the roughest possible types satisfying gtp(a/M; N) = gtp(f(a)/f[M]; N') for any **K**-embedding  $f: M \to N'$ .

In the first-order case, orbital types have a very nice syntactic characterization: to check that two elements have the same orbital types over A, it is necessary and sufficient that they satisfy the same first-order formulas with parameters from A. In particular non-equality of two orbital types has a "small" (in that case finite) witness. Grossberg and VanDieren [**GV06b**] isolated this property from an earlier argument of Shelah [**She99**] and made it into a definition<sup>7</sup>:

DEFINITION 1.2.10. Let  $\mu$  be an infinite cardinal. An AEC **K** is  $\mu$ -tame (( $< \mu$ )-tame) if for any  $M \in \mathbf{K}$ , types over M are determined by their restrictions to subsets of M of size  $\mu$  (strictly less than  $\mu$ ). We say that an AEC is tame if it is  $\mu$ -tame for some cardinal  $\mu$ .

Any AEC induced from a first-order theory is  $(<\aleph_0)$ -tame, but there are other examples of tame classes (like the aforementioned classes of locally finite groups and non-Archimedean fields). In fact, it is quite hard to find counterexamples: Work of Makkai-Shelah [**MS90**] and Boney [**Bon14b**] have shown that assuming a large cardinal axiom *every* AEC is tame (but the  $\mu$  witnessing it will be very big: the least strongly compact cardinal above LS(**K**)). Recent work of Boney and Unger [**BU**] has established the converse: the statement "every AEC is tame" is *equivalent* to a large cardinal axiom.

Still, it seems that many examples have a monster model and are tame (see for example the recent survey [**BVd**]). Thus we believe that making these assumptions is reasonable, at least to obtain a first approximation to Questions 1.2.8. This is why a large part of this thesis focuses on the theory of tame AECs with a monster model. Nevertheless, we will also consider the following two deep questions of Grossberg:

QUESTIONS 1.2.11.

- (1) Does eventual amalgamation follow from categoricity in a high-enough cardinal?
- (2) Does tameness follow from categoricity in a high-enough cardinal?

<sup>&</sup>lt;sup>7</sup>Several variations exist: asking for the tameness to hold also for types of sequences, for types over arbitrary sets, or even for types of sequences to be determined by types of small subsequences (this is called shortness). We cite the variation that has been most studied in the literature.

#### 1.2. BACKGROUND

Grossberg and VanDieren [GV06c, GV06a] have proven an *upward* categoricity transfer in tame AECs with a monster model categorical in a *successor* cardinal. Earlier, Shelah [She99] had exhibited a downward transfer for classes with a monster model and again categorical *in a successor*. Combining the Shelah and Grossberg-VanDieren results, we see that Shelah's eventual categoricity conjecture holds for tame AECs with a monster model, categorical in a *successor cardinal*. As mentioned earlier, tameness follows from a large cardinal axiom and the work of Makkai and Shelah [MS90] and Boney [Bon14b] also showed how to derive the existence of a monster model from categoricity (thus the answer to Questions 1.2.11 is positive assuming large cardinals). Therefore [Bon14b, 7.5] assuming a large cardinal axiom, Shelah's eventual categoricity conjecture holds *provided* that the categoricity cardinal is a *successor*.

QUESTIONS 1.2.12.

- (1) What about Shelah's eventual categoricity conjecture when the categoricity cardinal is limit?
- (2) Is the dependency on large cardinal axioms needed in the above results?

**1.2.7. Stability theory.** A natural question is how big the set gS(M) from Definition 1.2.9 can be. Trivial bounds are  $||M|| \leq |gS(M)| \leq 2^{||M||}$ . When the lower bound is attained, the following name is given:

DEFINITION 1.2.13. An AEC **K** is stable in  $\lambda$  if for every  $M \in \mathbf{K}$  of size  $\lambda$ ,  $|gS(M)| \leq \lambda$ .

We want to study stability in AECs because it is closely connected to Question 1.2.8. Indeed, the reason for the name "stable" is that a first-order theory is stable in some cardinal exactly when it is stable in the sense given in Section 1.2.1 (i.e. it does not have the order property). Thus first-order stability is equivalent to the existence of a stable independence notion (see Section 1.2.5). This is a result of Shelah who more generally studied:

DEFINITION 1.2.14. The *stability spectrum* of a first-order theory (or of an AEC) is the class of cardinals in which it is stable. We say that a first-order theory is *superstable* if its stability spectrum is an end-segment of cardinals.

Shelah [She90] proved the following deep results; the proof shows that the stability spectrum is connected with properties of stable independence:

FACT 1.2.15 (The stability spectrum theorem). Let T be a stable first-order theory. Then there exists a cardinal  $\kappa(T) \leq |T|^+$  such that for any cardinal  $\lambda \geq 2^{|T|}$ , T is stable in  $\lambda$  if and only if  $\lambda = \lambda^{<\kappa(T)}$ . Moreover,  $\kappa(T)$  is connected to the local character of independence.

In their milestone paper on tameness, Grossberg and VanDieren [**GV06b**] showed that if a  $\mu$ -tame AEC with a monster model is stable in *some* cardinal above  $\mu$ , then it is stable in unboundedly many cardinals. Still the following questions were left unanswered:

QUESTIONS 1.2.16.

- (1) Is there an analog of the stability spectrum theorem for AECs?
- (2) Is an AEC unstable if and only if (in some sense) it can define an order?

**1.2.8. Saturated and limit models.** Other objects of study of stability theory are saturated models.

DEFINITION 1.2.17. For an AEC **K**, we say M is  $\lambda$ -saturated if for every  $A \subseteq |M|$ , if  $|A| < \lambda$  every element of gS(A) is realized in M. When  $\lambda = ||M||$ , we just say that M is saturated.

This can be seen as an analog of being algebraically closed (one can think of types as equations with solutions "somewhere", and a model is  $\lambda$ -saturated if every equation with parameters from a set of size less than  $\lambda$  which has a solution has a solution already inside the model). In fact (working in a fixed characteristic) algebraically closed fields are saturated in the AEC of fields. One can further show that (assuming a monster model) saturated models of size above  $LS(\mathbf{K})$  are unique, universal, and model-homogeneous.

A related concept is that of a *limit model*. We do not give the precise definition here but refer the reader to [**GVV16**] (or see Definition 4.2.6 in this thesis). Intuitively, a  $\kappa$ -limit model is what is obtained when trying to build a saturated models in  $\kappa$  steps. An important question is the *uniqueness of* ( $\geq \kappa$ )-*limit models*, which roughly means that we can build saturated models in  $\kappa$ -many steps. The default is  $\kappa = \aleph_0$ . Shelah has shown that in a stable first-order theory, one can build saturated models in  $\kappa(T)$ -many steps, and so in particular in a superstable theory uniqueness of limit models holds. In addition, the union of any chain of  $\lambda$ -saturated models of length at least  $\kappa(T)$  is also  $\lambda$ -saturated. The proof of these fundamental results heavily use independence.

One can ask whether there is a generalization to AECs. In fact, this is stated as [Bal09, Problem D.9.(3)], which we quote here:

PROBLEM 1.2.18 (Baldwin). Develop a theory of superstability [for AECs] that connects the stability spectrum function with properties in a fixed cardinal such as uniqueness of limit models and preservation of saturation under unions of chains.

Another central "property in a fixed cardinal" is having a so-called good frame.

**1.2.9. Good frames.** One of the main concepts in Shelah's AEC book [She09a] is that of a good  $\lambda$ -frame. Roughly (see [She09a, Definition II.2.1]) a good  $\lambda$ -frame is a superstable independence notion  $B \underset{M}{\cup} C$ , but we require that M and C have size  $\lambda$  and B has size one (i.e. is a single element<sup>8</sup>). The definition of a good  $\lambda$ -frame, also asks that its underlying AEC has amalgamation, no maximal models, and joint embedding for models of cardinality  $\lambda$ . One further asks that it be stable in  $\lambda$  (see Definition 1.2.13). Intuitively, **K** has a good  $\lambda$ -frame if it has "superstable-like" behavior at  $\lambda$ . In the first-order case, a theory has a good  $\lambda$ -frame if and only if it is stable in all  $\mu \geq \lambda$  (and hence in particular superstable). The three natural questions on good frames in AECs are:

QUESTIONS 1.2.19.

- (1) Existence: When does an AEC have a good  $\lambda$ -frame?
- (2) Extension: If an AEC has a good  $\lambda$ -frame, when does it have a good  $\mu$ -frame for  $\mu > \lambda$ ?

 $<sup>^{8}\</sup>mathrm{Formally},$  Shelah restricts further to a class of basic types, but we omit this technical point here.

#### 1.3. RESULTS

(3) Categoricity transfer: How do good frames help making progress toward Shelah's eventual categoricity conjecture?

The theory of good frames has been developed over hundreds of pages by Shelah [She09a, She09b] (see also [JS13]) who answers some of these questions using a mix of strong local model-theoretic and set-theoretic hypotheses. This thesis investigates how to trade these strong local assumptions for somewhat weaker but more natural global assumptions (like tameness and a monster model).

#### 1.3. Results

1.3.1. Canonicity of independence. Chapter 3 (a joint work with Will Boney, Rami Grossberg, and Alexei Kolesnikov) shows that in any AEC with a monster model, a stable independence relation (see Section 1.2.5) is unique if it exists. This answers Question 1.2.8(2) and generalizes a result of Harnik and Harrington [HH84] for first-order theories. The chapter also shows that the symmetry property follows from the others, generalizing the first-order argument of Shelah [She90, III.4.13]. This is used in other parts of the thesis to construct (relatives of) such an independence relation. In Theorem 6.9.7 of this thesis, we also show that categorical good frames are canonical. This gives evidence that these notions of independence are not ad-hoc.

**1.3.2.** Constructing and extending good frames and superstable independence relations. Toward an answer to Question 1.2.19(1), we give the first construction of a good frame from tameness and the existence of a monster model in Chapter 4. The result is extended in various ways throughout several (sometimes interconnected) chapters: 6, 7, 10, and 23. The argument can be extended to also build a superstable independence relation (see Section 1.2.5) assuming a stronger tameness hypothesis. We limit ourselves to quoting the strongest result:

COROLLARY 10.6.14. (joint with Monica VanDieren)

Let **K** be a  $\mu$ -tame AEC with a monster model. If **K** is  $\mu$ -superstable (a technical definition that follows from categoricity in some  $\lambda > \mu$ , see Definition 6.10.1) then **K** has a good  $\mu^+$ -frame on its saturated models of size  $\mu^+$ .

Interestingly, when **K** is stable in  $\aleph_0$ , then some amount of tameness follows and a good frame can be built directly:

THEOREM 1.3.1 (joint with Saharon Shelah, see Chapter 23). Assume that **K** is an AEC with  $LS(\mathbf{K}) = \aleph_0$  such that  $\mathbf{K}_{\aleph_0}$  has amalgamation, joint embedding, no maximal models, is categorical in  $\aleph_0$ , and is stable in  $\aleph_0$ . Then **K** is  $(<\aleph_0, \aleph_0)$ -tame (this means that Galois types over countable models are determined by their restrictions to finite sets) and there is a type-full good  $\aleph_0$ -frame on **K**.

One can also give an answer to Question 1.2.8(1):

COROLLARY 6.16.3. Let **K** be a fully  $(\langle \aleph_0 \rangle)$ -tame and short AEC<sup>9</sup> with a monster AEC. If **K** is  $\mu$ -superstable then **K** admits a superstable independence

<sup>&</sup>lt;sup>9</sup>This means, roughly, that types (of sequences, not elements) are determined by their restrictions to finite sets and finite sequences.

relation on its class of  $\lambda$ -saturated models (for some high-enough  $\lambda$ ).

If **K** is only fully (<  $\mu$ )-tame and short (for a possibly uncountable  $\mu$ ), the construction also gives a slightly weaker independence notion.

Finally, Will Boney has shown [**Bon14a**] that one can extend good frames (see Question 1.2.19(2)) if the class has a monster model and satisfies a slightly stronger condition than tameness (tameness for types of length two). In Chapter 5 (a joint work with Boney) we improve the condition to only tameness:

COROLLARY 5.6.9 (joint with Will Boney). Let  $\mathfrak{s}$  be a good  $\lambda$ -frame on the AEC **K**. If **K** has amalgamation and is  $\lambda$ -tame, then  $\mathfrak{s}$  extends to a good ( $\geq \lambda$ )-frame.

In Chapter 18 (also a joint work with Boney), we examine the well-known Hart-Shelah counterexample [**HS90**] (where good frames cannot always be extended) and isolate in a sense the precise reason why they cannot be extended. The work also provides new ways of building good frames.

These results show how to build various independence relations in tame AECs and are crucial in proving the results presented next.

1.3.3. Superstability theory for tame AECs. Starting with a joint work with Boney (Chapter 7), later with VanDieren (Chapter 10), and also in Chapters 17 and 19, we explore how one can prove questions around closure of saturated models under chains and uniqueness of limit models in the framework of tame AECs with a monster model. Most often we focus on the superstable case, starting by assuming a technical condition called  $\mu$ -superstability (mentioned earlier), building some kind of independence relation (see above), and using it to prove results on saturated and limit models. The  $\mu$ -superstability condition can be obtained from categoricity (this is a result of Shelah and Villaveces [SV99, 2.2.1], generalized in Chapter 20) but in Chapter 19 we establish that it also follows from tameness and stability on a tail:

COROLLARY 19.4.24. Let **K** be a  $\mu$ -tame AEC with a monster model. If **K** is stable on a tail of cardinals, then there exists  $\lambda$  such that **K** is  $\lambda$ -superstable.

From  $\mu$ -superstability and tameness, we completely solve the questions of uniqueness of limit models and behavior of saturated models:

THEOREM 1.3.2 (Some results also joint with VanDieren, see Chapter 10). Let **K** be a  $\mu$ -tame AEC with a monster model. If **K** is  $\mu$ -superstable, then:

- (1) For any  $\lambda \geq \mu$ , **K** is stable in  $\lambda$  and limit models of size  $\lambda$  are unique.
- (2) For any  $\lambda > \mu$ , the union of an increasing chain of  $\lambda$ -saturated models is  $\lambda$ -saturated. Moreover there is a saturated model of cardinality  $\lambda$ .

One can also obtain converses, deriving superstability from uniqueness of limit models or good behavior of saturation. In fact, in Chapter 9 (a joint work with Grossberg) we show that many definitions of superstability that are equivalent in the first-order case are also equivalent in the context of tame AECs with a monster model. We believe that these results show there is a satisfactory theory of superstability in tame AECs with a monster model, answering Problem 1.2.18 in that context.

**1.3.4.** Stability theory for tame AECs. Recall that in the first-order case, orbital types can be identified with sets of formulas, and a consequence of this is tameness. In Chapter 2, we show that in *any* tame AEC, orbital types can be seen as sets of formulas. This establishes a connection between the study of tame AECs and an earlier framework (introduced in Rami Grossberg's 1981's master thesis and well studied by Shelah, e.g. in [She09b, Chapter V.A]) called *stability theory inside a model*. One can then positively answer Question 1.2.16(2) for tame AECs with a monster model:

THEOREM 2.4.15. Let **K** be a tame AEC with a monster model. The following are equivalent<sup>10</sup>:

(1)  $\mathbf{K}$  is stable in some cardinal.

(2) K does not have the order property (defined in terms of orbital types).

The historically more recent Chapter 19 focuses on answering Question 1.2.16(1). We give a positive answer assuming the singular cardinal hypothesis (SCH), see Corollary 19.4.22.

The SCH is much weaker than the generalized continuum hypothesis (GCH). For example, there is a sense in which SCH holds assuming a large cardinal axiom [Sol74] so the eventual stability spectrum can be characterized unconditionally if one works above a large cardinal. Note also that the argument gives several interesting ZFC results. Chapter 19 also generalizes the superstable results on uniqueness of limit models and chains of saturated models to the stable context (some arguments use joint work with Will Boney from Chapter 7).

In the author's opinion, this shows that there is not only a *superstability* theory, but really a *stability* theory for tame AECs with a monster model. Hence Problem 1.2.18 can be solved more generally in this context.

**1.3.5.** Structure of categorical AECs with a monster model. This thesis is not limited to tame AECs. In fact often arguments developed in the tame context can be recycled and applied to a "tameless" framework. Further, Shelah [She99] has shown that there are ways to derive certain weak amount of tameness from categoricity (if, say, the AEC has a monster model). Shelah's argument depends critically on the categoricity cardinal being of "high-enough" cofinality in order to get a certain degree of saturation of the model. Chapter 17 answers a question of Baldwin and Shelah by showing it is not necessary:

COROLLARY 17.4.11. Let **K** be an AEC with a monster model. Let  $\lambda > LS(\mathbf{K})$ . If **K** is categorical in  $\lambda$ , then the model of cardinality  $\lambda$  is saturated.

As a consequence, we obtain the following nice pictures on the behavior of the class below the categoricity cardinal. This answers questions on uniqueness of limit models that had been open since **[SV99**]:

COROLLARY 17.5.7. Let **K** be an AEC with a monster model. Let  $\lambda > LS(\mathbf{K})$ . If **K** is categorical in  $\lambda$ , then:

<sup>&</sup>lt;sup>10</sup>The direction (1) implies (2) is known [She99, Claim 4.7(2)].

- (1) For any  $\mu \in [LS(\mathbf{K}), \lambda)$ , limit models of cardinality  $\mu$  are unique and (if  $\mu > LS(\mathbf{K})$ ) an increasing union of  $\mu$ -saturated models is  $\mu$ -saturated.
- (2) If  $\lambda \geq \beth_{(2^{LS(\mathbf{K})})^+}$ , then there exists  $\chi < \beth_{(2^{LS(\mathbf{K})})^+}$  such that:
  - (a) K is (χ, < λ)-weakly tame (i.e. only types over saturated models of size less than λ are determined by their restrictions to domains of size χ).</li>
  - (b) **K** has a good  $\mu$ -frame on the class of saturated models of cardinality  $\mu$  for any  $\mu \in (\chi, \lambda)$ .

The proof of these two theorems uses arguments from many of the earlier results on proving good behavior of saturated models and building good frames.

Chapter 17 also studies a property called *solvability*, introduced by Shelah as a possible definition of superstability in AECs, and proves that it behaves very well in AECs with a monster model. In particular, the consequences of categoricity above also follow from just solvability in  $\lambda$ .

**1.3.6.** Applications to the categoricity conjecture. From the theory of superstability discussed earlier, one can almost effortlessly obtain several partial categoricity transfers. For example:

THEOREM 15.3.8. Let **K** be a  $\mu$ -tame AEC with a monster model. If **K** is categorical in *some*  $\lambda > \mu$ , then **K** is categorical in *all*  $\lambda' > \mu$  of the form  $\lambda' = \beth_{\delta}$ , where  $\delta$  is divisible by  $\beth_{(2\mu)^+}$ . In particular, **K** is categorical in a proper class of cardinals.

Toward a full categoricity transfer, Shelah has isolated [She09a, Chapter III] a property he calls *having primes*. He gives several ideas at the end of [She09a, Chapter III] about how to use the property to transfer categoricity. Inspired by this, we prove in Chapters 8 and 11:

THEOREM 11.2.8. Let  $\mathfrak{s}$  be a good  $\lambda$ -frame on the AEC **K** that is categorical in  $\lambda$ . Assume that **K** is  $\lambda$ -tame and has primes. If **K** is categorical in *some*  $\mu > \lambda$ , then **K** is categorical in *all*  $\mu' > \lambda$ .

Roughly (say assuming the existence of a monster model and working inside it) a class has primes if it has a prime model N over sets of the form Ma. This means that whenever N' contains Ma, then N embeds inside it. The notion can also be defined without assuming a monster model. Vector spaces always have primes (take the span of Ma).

Notice that Theorem 11.2.8 did not assume the existence of a monster model. Thus one application is to deduce the amalgamation property (and thus by a standard argument the existence of a monster model) from categoricity in appropriate cardinals. This is a step toward answering Question 1.2.11(1). The argument uses deep results of Shelah:

COROLLARY 8.4.17. Let **K** be a tame AEC with primes. If **K** is categorical in cardinals of arbitrarily high cofinality, then there exists  $\lambda$  such that **K** has amalgamation for models of size at least  $\lambda$ .

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#### 1.3. RESULTS

Once one has a monster model, we can deduce the eventual categoricity conjecture for tame AECs with primes. This relies on many of the arguments given above, in particular to build good frames.

COROLLARY 15.4.9. Let **K** be a  $\mu$ -tame AEC with a monster model and primes. If **K** is categorical in *some*  $\lambda > \mu$ , then **K** is categorical in *all*  $\lambda' \ge \min(\lambda, \beth_{(2^{\mu})^+})$ .

We also improve existing results on categoricity in a successor, proving in the good frame context:

THEOREM 14.6.14. Let  $\mathfrak{s}$  be a good  $\lambda$ -frame on the AEC **K** that is categorical in  $\lambda$ . If **K** is  $\lambda$ -tame, has a monster model, and is categorical in *some successor*  $\mu > \lambda$ , then **K** is categorical in *all*  $\mu' > \lambda$ .

Compared to Theorem 11.2.8, Theorem 14.6.14 does not assume primes. Together, the two theorems give an interesting answer to Question 1.2.19(3). We can use Theorem 14.6.14 to improve a bound in a classical transfer of Shelah [She99] (but assuming that the AEC is tame), see the discussion in Chapter 14. Theorem 11.2.8 also proves new cases of the categoricity conjecture in homogeneous model theory (see Section 11.4).

We believe that the assumptions of having primes gives a partial answer to Question 1.2.12(1). It is open whether it follows from categoricity in a high-enough cardinal, but we can show that it is necessary: it follows from categoricity on a *tail* of cardinals. For example:

THEOREM 8.5.23 (see also Remark 8.5.24). Let **K** be an AEC and let  $\kappa >$  LS(**K**) be a strongly compact cardinal. Assume that **K** is categorical in *some*  $\lambda \geq \beth_{(2^{\kappa})^+}$ . The following are equivalent:

- (1) **K** is categorical in all  $\lambda' \geq \beth_{(2^{\kappa})^+}$ .
- (2) **K** has primes for models of size at least  $\beth_{(2^{\kappa})^+}$ .

In other words (assuming a large cardinal axiom), Shelah's eventual categoricity conjecture is equivalent to the statement that one can get existence of primes from categoricity.

It is natural to ask for a "concrete" framework where the categoricity transfers above apply. It turns out that *universal classes* are such a framework. A *universal class* is a class of structures in a fixed vocabulary closed under isomorphisms, substructure, and unions of chains (thus it is in particular an AEC, where the ordering is just "being a  $\tau$ -substructure"). There Shelah has developed a nice structure theory [**She87b**] and one can show (with a lot of additional work) that, roughly speaking, amalgamation follows from categoricity (and in fact Grossberg's Questions 1.2.11 have positive answers there). Further the class is known to be tame and have primes, so applying Theorem 15.4.9 we obtain Shelah's eventual categoricity conjecture in universal classes:

THEOREM 16.7.3. Let K be a universal class. If K is categorical in some  $\lambda \geq \exists_{\exists_{(2^{|\tau(\mathbf{K})|+\aleph_0})^+}}$ , then K is categorical in all  $\lambda' \geq \exists_{\exists_{(2^{|\tau(\mathbf{K})|+\aleph_0})^+}}$ .

This gives the first example of a non-trivial context where neither large cardinal nor model-theoretic properties such as the existence of a monster model are assumed to prove Shelah's eventual categoricity conjecture. One also does not need to assume that the categoricity cardinal is a successor. Note that the syntactic version of Theorem 16.7.3 (in countable vocabularies) also gives an approximation to Conjecture 1.2.7:

THEOREM 16.0.9. Let  $\psi$  be a universal  $\mathbb{L}_{\omega_1,\omega}$  sentence (in a countable vocabulary). If  $\psi$  is categorical in *some*  $\lambda \geq \beth_{\beth_{\omega_1}}$ , then  $\psi$  is categorical in *all*  $\lambda' \geq \beth_{\beth_{\omega_1}}$ .

## 1.4. Other results

**1.4.1. Building primes over saturated models.** We have already mentioned Theorem 8.5.23 which shows (under a large cardinal axiom) that Shelah's eventual categoricity conjecture is equivalent to the statement that every AEC categorical in a high-enough cardinal eventually has primes. The main result used in proving the left to right direction is a construction of prime models assuming the existence of a well-behaved independence notion. The technical statement is:

THEOREM 12.3.6. Let  $\mathbf{K}$  be an almost fully good AEC that is categorical in  $LS(\mathbf{K})$  and has the  $LS(\mathbf{K})$ -existence property for domination triples.

For any  $\lambda > \mathrm{LS}(\mathbf{K})$ ,  $\mathbf{K}_{\lambda}^{\lambda-\mathrm{sat}}$  has primes. That is, there is a prime model N over every set of the form  $M \cup \{a\}$  for  $M \in \mathbf{K}_{\lambda}$  saturated, but N is prime in the class of saturated models, i.e. it embeds over  $M \cup \{a\}$  into any saturated N' containing  $M \cup \{a\}$ .

When **K** is eventually categorical, then (using the large cardinal axiom), we can show that the hypotheses of Theorem 12.3.6 are satisfied on a tail of the AEC, and hence (since by eventual categoricity, all models are saturated) **K** will have primes. The proof of Theorem 12.3.6 generalizes an argument of Shelah who proved the result when  $\lambda$  was a successor cardinal.

**1.4.2.**  $\mu$ -AECs. Chapter 13 (a joint work with Will Boney, Rami Grossberg, Mike Lieberman, and Jiří Rosický) introduces  $\mu$ -AECs, a generalization of AECs where we require that the chain axiom holds only for chains of cofinality at least  $\mu$  (so  $\aleph_0$ -AECs are exactly AECs). We show that some classification-theoretic results (such as Shelah's argument for getting amalgamation in  $\lambda$  from few models in  $\lambda^+$ ) carry over to this framework. We also show that there is a correspondence between  $\mu$ -AECs and accessible categories whose morphisms are monomorphisms.

1.4.3. Quasiminimal AECs. Quasiminimal pregeometry classes were introduced by Zilber [Zil05a] in order to prove a categoricity theorem for the so-called pseudo-exponential fields. Quasiminimal pregeometry classes are a class of structures carrying a pregeometry satisfying several axioms. Roughly (see Definition 21.4.5) the axioms specify that the countable structures are quite homogeneous and that the generic type over them is unique (where types here are syntactic quantifier-free types). The original axioms included an "excellence" condition (but it has since been shown [BHH<sup>+</sup>14] that this follows from the rest). Zilber showed that a quasiminimal pregeometry class has at most one model in every uncountable cardinal, and in fact the structures are determined by their dimension. Note that quasiminimal pregeometry classes are typically non-elementary (see [Kir10, §5]): they are axiomatizable in  $\mathbb{L}_{\omega_1,\omega}(Q)$  (where Q is the quantifier "there exists uncountably many") but not even in  $\mathbb{L}_{\omega_1,\omega}$ .

In fact, quasiminimal pregeometry classes can be naturally seen as AECs (see Fact 21.4.8). In Chapter 21, we show that a converse is true: there is a natural class of AECs, which we call the *quasiminimal AECs*, that corresponds to quasiminimal pregeometry classes. Quasiminimal AECs are required to satisfy four purely semantic properties (see Definition 21.4.1), the most important of which are that the AEC must, in a technical sense, be closed under intersections (this is called "admitting intersections", see Definition 21.3.1) and over each countable model M there must be a *unique* Galois type that is not realized inside M.

This gives a simple purely semantic characterization of quasiminimal pregeometry classes (whose axioms are very syntactic: roughly they ask for some homogeneity of countable syntactic types). Along the way, we prove a new result in the theory of pregeometries: any homogeneous closure space with finite character is a pregeometry, i.e. it satisfies exchange (Corollary 21.2.12). This seems to be new (but see the remarks preceding the proof).

**1.4.4. On the uniqueness property of forking.** An important property of nonforking independence in a first-order stable theory is the uniqueness (or stationarity) of nonforking types over models: if  $M \leq N$  and  $p, q \in S(N)$  do not fork over M and are such that  $p \upharpoonright M = q \upharpoonright M$ , then p = q. This property is part of the definition of a good frame and we can prove various analogs in tame AECs. In Chapter 22, we prove an analog *without* assuming tameness.

We assume that we are working in a  $\mu$ -superstable AEC (where superstability is defined in terms of a technical condition, no long splitting chains; recall that in an AEC with amalgamation and no maximal models,  $\mu$ -superstability follows from categoricity above  $\mu$ ). For  $M \leq_{\mathbf{K}} N$  both limit models in  $\mathbf{K}_{\mu}$  and p a Galois type over N, we say p does not  $\mu$ -fork over M if there exists  $M_0 \in \mathbf{K}_{\mu}$  such that M is universal over  $M_0$  and p does not  $\mu$ -split over  $M_0$  (the exact definition of splitting is immaterial for this discussion). VanDieren has shown that non- $\mu$ -forking satisfies the definition of uniqueness if the "witnesses"  $M_0$  to the non- $\mu$ -forking are the same for both p and q. This weak form of uniqueness is fundamental to the development of the theory, but is not convenient to work with (to talk about nonforking, we have to mention the witnesses). In Chapter 22, we prove that the uniqueness property of  $\mu$ -nonforking holds regardless of the witnesses. This streamlines the theory and proves as a corollary the equivalence between several versions of the symmetry property defined in the literature.

**1.4.5.** Indiscernible extraction and Morley sequences. This chapter is not about AECs but about simple first-order theories. Simple theories are an extension of stable theories where it is still possible to develop some of the theories of nonforking independence. As a canonical example, the theory of the random graph is simple. In this setup an argument of Shelah shows how to build a fundamental object of study, Morley sequences. They are indiscernible sequences such that each element does not fork over the preceding ones. This argument uses the Erdős-Rado theorem on a sequence of length  $\beth_{(2^{|T|})^+}$ . It was asked by Grossberg, Iovino and Lessmann as well as by Baldwin whether it was possible to avoid using such big cardinals, i.e. whether it was possible to build Morley sequences and prove the basic

facts on simple theories using only "ordinary mathematics" (e.g. without using too much of the axiom of replacement).

Chapter 24 answers this question positively by giving an alternate proof of the existence of Morley sequences in simple theories that uses just Ramsey's theorem and compactness. The proof leads to a new characterization of simple theories: a theory is simple if and only if forking has the dual finite character (DFC) property (see Definition 24.3.2). The right to left direction of this result was contributed by Itay Kaplan after the initial circulation of [Vas17b] (which is Chapter 24 here).

#### 1.5. Categoricity in universal classes: a sketch

In this section, we sketch the proof of Shelah's eventual categoricity conjecture in universal classes. We try to be as self-contained as possible and still describe the main ideas somewhat accurately. The full proof is given in Theorem 16.7.3. This section is loosely based on [Vasb].

Recall that a universal class is a class K of structures in a fixed vocabulary  $\tau(K)$  closed under isomorphisms, substructures, and  $\subseteq$ -increasing chains. We will think of K as the AEC  $\mathbf{K} = (K, \subseteq)$ , and write  $\mathrm{LS}(\mathbf{K}) = |\tau(\mathbf{K})| + \aleph_0$ .

We will prove a weaker version than Theorem 16.7.3:

THEOREM 1.5.1. Let  $\mathbf{K}$  be a universal class. If  $\mathbf{K}$  is categorical in a proper class of cardinals, then  $\mathbf{K}$  is categorical on a tail of cardinals.

Note that by a Hanf number argument, this immediately implies the eventual categoricity conjecture for universal classes:

COROLLARY 1.5.2. There exists a map  $\mu \mapsto \lambda_{\mu}$  such that for any universal class **K**, with  $|\tau(\mathbf{K})| \leq \mu$ , if **K** is categorical in *some*  $\lambda \geq \lambda_{\mu}$ , then **K** is categorical on a tail of cardinals.

PROOF. Given  $\mu$ , there is only a set of (not equivalent up to renaming) universal classes with vocabulary of size at most  $\mu$ . Therefore we can let  $\lambda_{\mu}$  be the successor of the supremum of the set of cardinals  $\lambda$  such that there exists a universal class with vocabulary size at most  $\mu$  categorical in  $\lambda$  but not in a proper class of cardinals. Then any universal class  $\mathbf{K}$  with  $|\tau(\mathbf{K})| \leq \mu$  categorical in some  $\lambda \geq \lambda_{\mu}$  will be categorical in a proper class of cardinals, and hence on a tail by Theorem 1.5.1.  $\Box$ 

Theorem 16.7.3 shows that one can take  $\lambda_{\mu} = \beth_{\square_{(2^{\mu})^+}}$ . This is proven using more of the superstability theory. For example, given  $\mu$ , if we want to work at a "nice" class above  $\mu$  in the setup of Theorem 1.5.1 we can pick the next categoricity cardinal  $\lambda > \mu$  and work with  $\mathbf{K}_{\geq \lambda}$ . In the more general case, there may not be such a nice cardinal but we can work in the class of  $\mu^+$ -saturated models of  $\mathbf{K}$ , which by the superstability theory will (in cases of interest) be an AEC.

Theorem 1.5.1 will be proven in two steps. We will use the definitions from Section 1.2.6, in particular the notion of an orbital type and the tameness property.

**1.5.1. Two steps, and connecting them.** In general, a universal class may not have amalgamation. The first step derives amalgamation from categoricity. However the class will be changed:

THEOREM 1.5.3 (Structure of categorical universal classes). Let  $\mathbf{K}$  be a universal class categorical in a proper class of cardinals. Then there exists an AEC  $\mathbf{K}^*$  such that:

- (1) **K** and  $\mathbf{K}^*$  have the same vocabulary.
- (2)  $\operatorname{LS}(\mathbf{K}) \leq \operatorname{LS}(\mathbf{K}^*) < \beth_{(2^{\operatorname{LS}(\mathbf{K})})^+}.$
- (3) Whenever  $\mu > LS(\mathbf{K}^*)$ , **K** is categorical in  $\mu$  if and only if  $\mathbf{K}^*$  is categorical in  $\mu$ . In this case, the model of cardinality  $\mu$  is the same in both classes.
- (4) **K**<sup>\*</sup> has amalgamation, joint embedding, and arbitrarily large models (thus it has a monster model).
- (5)  $\mathbf{K}^*$  is  $\mathrm{LS}(\mathbf{K}^*)$ -tame (recall that this means that two different Galois types must differ over a domain of size at most  $\mathrm{LS}(\mathbf{K}^*)$ ).

Note that  $\mathbf{K}^*$  may not be a universal class, but we do know that it has some agreement with  $\mathbf{K}$ : it has the same vocabulary, its Löwenheim-Skolem-Tarski number is not too different, its categoricity spectrum is (eventually) the same and further the model in the categoricity cardinals agree. It turns out that this will be enough to show that  $\mathbf{K}$  and  $\mathbf{K}^*$  are eventually identical (so in particular their orderings eventually coincide). For an AEC  $\mathbf{K}$ , we write  $\mathbf{K}_{\geq\mu}$  for its restriction to models of size  $\mu$  (this is also an AEC with Löwenheim-Skolem-Tarski number  $\mu$ ).

LEMMA 1.5.4 (The connecting Lemma). Let  $\mathbf{K}^1$  and  $\mathbf{K}^2$  be AECs with the same vocabulary, the same Löwenheim-Skolem-Tarski number and such that for any  $\mu > \mathrm{LS}(\mathbf{K}^1) = \mathrm{LS}(\mathbf{K}^2)$ ,  $\mathbf{K}^1$  is categorical in  $\mu$  if and only if  $\mathbf{K}^2$  is categorical in  $\mu$  and further in this case the model of cardinality  $\mu$  is the same in  $\mathbf{K}^1$  and  $\mathbf{K}^2$ .

Assume that  $\mathbf{K}^1$  (and therefore  $\mathbf{K}^2$ ) is categorical in a proper class of cardinals. Assume also that  $\mathbf{K}^1$  has amalgamation, joint embedding, and arbitrarily large models. Then there exists  $\lambda > \mathrm{LS}(\mathbf{K}^1)$  such that  $\mathbf{K}^1_{\geq \lambda} = \mathbf{K}^2_{\geq \lambda}$  (in particular, also the orderings coincide above  $\lambda$ ).

The proof of Theorem 1.5.1 can be completed by appealing to a categoricity transfer in AECs satisfying enough structural properties:

THEOREM 1.5.5 (Categoricity for tame AECs with primes). Let **K** be an AEC that has amalgamation, joint embedding, and arbitrarily large models. Assume that **K** is  $LS(\mathbf{K})$ -tame and has primes over every set of the form  $M \cup \{a\}$  (i.e. working in the monster model  $\mathfrak{C}$ , there is N such that  $M \leq_{\mathbf{K}} N$ ,  $a \in N$ , and N embeds inside any other model containing  $M \cup \{a\}$ ).

If  $\mathbf{K}$  is categorical in a proper class of cardinals, then  $\mathbf{K}$  is categorical on a tail of cardinals.

Given the structure theorem, the connecting lemma, and the categoricity theorem, we can prove the eventual categoricity conjecture for universal classes:

PROOF OF THEOREM 1.5.1. Let **K** be a universal class categorical in a proper class of cardinals. Let  $\mathbf{K}^*$  be as given by the structure theorem for categorical universal classes (Theorem 1.5.3). We apply the connecting Lemma (Lemma 1.5.4 where  $\mathbf{K}^1$ ,  $\mathbf{K}^2$  there stand for  $\mathbf{K}^*$ ,  $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K}^*)}$  here). We get that there exists  $\lambda$  such that  $\mathbf{K}_{\geq \lambda} = \mathbf{K}^*_{\geq \lambda}$ .

Now,  $\mathbf{K}_{\geq\lambda}$  has primes over sets of the form  $M \cup \{a\}$ : one simply lets N to be the closure of  $M \cup \{a\}$  under the functions of  $\mathfrak{C}$ . Note that  $N \in \mathbf{K}$  as universal classes are closed under substructures and it is easy to see that N is prime over  $M \cup \{a\}$ .  $\mathbf{K}_{\geq\lambda}$  is also  $\lambda$ -tame and has amalgamation, joint embedding, and arbitrarily large models (because  $\mathbf{K}_{\geq\lambda} = \mathbf{K}^*_{\geq\lambda}$ , and  $\mathbf{K}^*$  has those properties). Therefore the categoricity theorem for tame AECs with primes (Theorem 1.5.5) applies and so  $\mathbf{K}_{\geq\lambda}$  (hence also  $\mathbf{K}$ ) is categorical on a tail of cardinals, as desired.

It remains to prove the structure theorem, the connecting lemma, and the categoricity theorem. We start with the connecting lemma, since it is conceptually the simplest of the three. The full proof appears as Theorem 16.3.8.

PROOF OF LEMMA 1.5.4. We proceed in several steps.

- (1) The categoricity spectrum of  $\mathbf{K}^1$  is closed: for any limit cardinal  $\lambda$ , if  $\mathbf{K}^1$  is categorical cofinally below  $\lambda$ , then  $\mathbf{K}^1$  is categorical in  $\lambda$ . Why? Again we proceed in several steps:
  - (a) For any categoricity cardinal  $\mu$ ,  $\mathbf{K}^1$  is stable below  $\mu$  (i.e. for any  $\mu_0 < \mu$ ,  $\mathbf{K}^1$  has at most  $\mu_0$ -many Galois type over any model of size  $\mu_0$ ) [Why? By what is now called Shelah's presentation theorem, one can do something like adding Skolem functions to any AECs. In particular, AECs with arbitrarily large models admit models generated by indiscernibles (Ehrenfeucht-Mostowski, or EM, models). Looking at an EM model generated by the linear order  $\lambda$ , we see that it must have few Galois types. This is essentially the argument that categoricity implies  $\aleph_0$ -stability in the proof of Morley's categoricity theorem.]
  - (b)  $\mathbf{K}^1$  is stable in all cardinals below  $\lambda$ . [Why? by the previous part and the assumption that  $\mathbf{K}^1$  is categorical cofinally below  $\lambda$ ]
  - (c) For any categoricity cardinal μ < λ, any model of size μ is saturated (i.e. it realizes every Galois type over a model of size less than μ). [Why? By stability in μ, one can build a μ<sub>0</sub><sup>+</sup>-saturated model in μ for every μ<sub>0</sub> < μ, then use categoricity.]</li>
  - (d) Any model M of size  $\lambda$  is saturated [Why? By the previous part and using categoricity in cofinally-many cardinals below  $\lambda$ , for any  $\leq_{\mathbf{K}}$ -submodel  $M_0$  of M of size less than  $\lambda$ , there is a saturated model of larger size (but still less than  $\lambda$ ) containing  $M_0$  and contained in M thus all types over  $M_0$  are realized in M.]
  - (e)  $\mathbf{K}^1$  is categorical in  $\lambda$ . [Why? By a lemma of Shelah, saturated models correspond to model-homogeneous models, and hence are unique.]
- (2) There is a categoricity cardinal  $\lambda$  such that  $\lambda = \lambda^{\text{LS}(\mathbf{K}^1)}$ . [Why? Using the previous part, recalling that  $\mathbf{K}^1$  is categorical in a proper class of cardinals by assumption].
- (3) Write  $\mathbf{K}^{\ell} = (K^{\ell}, \leq_{\mathbf{K}^{\ell}})$  for  $\ell = 1, 2$ . Then  $K_{\lambda}^{1} = K_{\lambda}^{2}$  [Why? Because they are categorical in  $\lambda$ , hence have the same model of cardinality  $\lambda$ .]
- (4) For  $\ell \in \{1,2\}$ ,  $M, N \in \mathbf{K}^{\ell}_{\lambda}$ , if  $M \leq_{\mathbf{K}^{\ell}} N$  then  $M \preceq_{\mathbb{L}_{\infty,\mathrm{LS}(\mathbf{K}^{\ell})^{+}}} N$  [Why? This is a result of Shelah [She09a, IV.1.12(2)]. The proof proceeds by induction on the formula, using Fodor's lemma (with  $\lambda = \lambda^{\mathrm{LS}(\mathbf{K}^{1})}$ ) and categoricity at appropriate steps. It does *not* rely on amalgamation.]
- (5) For  $\ell \in \{1,2\}$ ,  $M, N \in \mathbf{K}^{\ell}_{\lambda}$ , if  $M \preceq_{\mathbb{L}_{\infty, \mathrm{LS}(\mathbf{K}^{\ell})^{+}}} N$  then  $M \leq_{\mathbf{K}^{\ell}} N$  [Why? For any AEC **K**, the  $\leq_{\mathbf{K}}$  relation extends  $\preceq_{\mathbb{L}_{\infty, \mathrm{LS}(\mathbf{K})^{+}}}$ . This is a general result proven independently by Kueker [**Kue08**, 7.2(b)] and Shelah [**She09a**, IV.1.10(1)], using back and forth systems.]
- (6) For  $M, N \in K^1_{\lambda}$ ,  $M \leq_{\mathbf{K}^1} N$  if and only if  $M \leq_{\mathbf{K}^2} N$ . [Why? By the previous steps,  $\leq_{\mathbf{K}^{\ell}}$  coincides with  $\preceq_{\mathbb{L}_{\infty, \mathrm{LS}(\mathbf{K}^{\ell})}$ . Now use that  $\mathrm{LS}(\mathbf{K}^1) = \mathrm{LS}(\mathbf{K}^2)$ .]

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(7)  $\mathbf{K}_{\geq\lambda}^1 = \mathbf{K}_{\geq\lambda}^2$  [Why? Using resolutions into directed systems, one can show that  $\mathbf{K}^1$  and  $\mathbf{K}^2$  are determined by their models of size  $\lambda$  and the restriction of the orderings on models of size  $\lambda$ . The previous steps proved that  $\mathbf{K}_{\lambda}^1 = \mathbf{K}_{\lambda}^2$ , and hence it follows that  $\mathbf{K}_{>\lambda}^1 = \mathbf{K}_{>\lambda}^2$ .]

We discuss the proof of the structure theorem. Here we have to be a bit more vague as the details are very technical. A full proof is in Chapter 16, see Theorem 16.7.2.

PROOF OF THEOREM 1.5.3. The proof heavily uses Shelah's classification theory for universal classes, which first appeared in [She87b] and is revised in [She09b, Chapter V].

- (1) **K** does not have the order property: there is no quantifier-free formula  $\phi$ ,  $M \in \mathbf{K}$ , and sequence  $\langle \bar{a}_i : i < \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+} \rangle$  in M such that  $M \models \phi[\bar{a}_i, \bar{a}_j]$  if and only if i < j. [Why? Roughly, if there were such a sequence, then one could use Morley's method to get such a sequence ordered with a linear order I with more than |I|-many cuts. Thus **K** is unstable (in terms of quantifier-free types) somewhere below the categoricity cardinal, and we argued in the proof of the connecting lemma that this could not happen.]
- (2) By Shelah's structure theory for universal classes, there exists a relation "p does not fork over M" for  $M \subseteq N$  and p a quantifier-free type over N. We define  $M \leq^* N$  if and only if any quantifier-free type over M realized in N does not fork over M. We also define  $M_1 \stackrel{M_3}{\downarrow} M_2$  to mean that  $M_0 \leq^* M_\ell \leq^* M_3$  for  $\ell \in \{1, 2\}$  and for any  $\bar{a} \in {}^{<\omega}M_1$ , the quantifier-free type of  $\bar{a}$  over  $M_2$  (as computed in  $M_3$ ) does not fork over  $M_0$ .
- (3) The reader can think of  $M_1 \underset{M_0}{\downarrow} M_2$  as meaning that  $M_1$  and  $M_2$  are in nonforking amalgamation over  $M_0$  inside  $M_3$ . This has several nice properties: monotonicity (in the obvious directions), symmetry ( $M_1$  and  $M_2$ can be swapped) existence (any triple of models can be put in nonforking amalgamation), uniqueness (the non-forking amalgam is unique in the sense that if  $M_3^1$  and  $M_3^2$  are such that  $M_1 \underset{M_0}{\downarrow} M_2$ , then  $M_3^1$  and  $M_3^2$  embed into a common  $\leq^*$ - extension over  $M_1 \cup M_2$ ), and some continuity properties. Nonforking amalgamation also plays nicely with closure under functions. For example, if  $M_1 \underset{M_0}{\downarrow} M_2$  and  $M_3'$  is the closure of  $M_1 \cup M_2$ under the functions of  $M_1$  then  $M_3'$

under the functions of  $M_3$ , then  $M_1 \bigcup_{M_0}^{-3} M_2$ .

(4) Consider the class  $\mathbf{K}^{0,*} = (K, \leq^*)$ . By the existence of nonforking amalgamation,  $\mathbf{K}^{0,*}$  has amalgamation.  $\mathbf{K}^{0,*}$  also has arbitrarily large models (by the presentation theorem and classical calculations of Hanf number of infinitary languages). Moreover, any type does not fork over a "small"

set. Combining this with the uniqueness property of nonforking amalgamation, this gives tameness. One can also show that the Löwenheim-Skolem-Tarski number of  $\mathbf{K}^{0,*}$  satisfies the right conclusion. It is not clear that  $\mathbf{K}^{0,*}$  has joint embedding, but one can partition it into subclasses that each have joint embedding. One can pick a subclass  $\mathbf{K}^*$  that contains arbitrarily large models. Then  $\mathbf{K}^*$  satisfies all the right properties, except that...

- (5) The biggest problem is that  $\mathbf{K}^*$  may not be an AEC: it may not satisfy the smoothness axiom: if  $\langle M_i : i < \delta \rangle$  is  $\leq^*$ -increasing and  $M_i \leq^* N$  for all  $i < \delta$ , then it is not clear that  $\bigcup_{i < \delta} M_i \leq^* N$ .
- (6) Shelah has shown that failure of smoothness implies that K\* must have 2<sup>λ</sup>-many models for any sufficiently big *regular* cardinal λ. Thus if K is categorical in a regular cardinal we are done. However if not we have to do more work.
- (7) In Section 16.5, we show that any witness to the failure of smoothness  $\langle M_i : i < \delta \rangle$ , N can be copied into a tree  $\langle M_\eta : \eta \in {}^{\leq \delta}\lambda \rangle$ , where each of the branches are isomorphic to  $\langle M_i : i < \delta \rangle \frown \{N\}$  and the branches are "as

independent as possible". So for example for any  $\eta \in {}^{<\delta}\lambda, M_{\eta \frown 0} \bigcup_{M_{\eta}}^{M} M_{\eta \frown 1},$ 

where M is some model containing the entire tree (which can also be shown to exist). Constructing such a tree is quite technical, as we have to see that the induction can continue at limits without violating smoothness. This uses that  $\downarrow$  plays very well with the closure operator of the universal class, hence giving some weak version of smoothness for sequences independent of each other in a suitable sense.

(8) Once the tree is built, we can obtain many types from it, contradicting (essentially, but there are additional difficulties because  $\mathbf{K}^*$  is not an AEC) that  $\mathbf{K}^*$  should be stable below the categoricity cardinal.

We finish by sketching the proof of the categoricity theorem. For the full proof, see Theorem 11.3.8.

PROOF OF THEOREM 1.5.5. We proceed in several steps. We assume without loss of generality that all the models of **K** have size at least  $LS(\mathbf{K})$  (this can be achieved by replacing **K** with  $\mathbf{K}_{>LS(\mathbf{K})}$ ).

- (1) **K** is stable (in terms of counting the number of Galois types) in every cardinal. [Why? See the proof of the connecting lemma.]
- (2) For every  $M \in \mathbf{K}$ , there exists  $N \in \mathbf{K}$  such that N is universal over M and N has the same size as M. [Why? Using the argument that Shelah used to prove that model-homogeneous is equivalent to saturated.]
- (3) The model in any categoricity cardinal is saturated. [Why? See the proof of the connecting lemma.]
- (4) For  $\mu \geq \text{LS}(\mathbf{K})$  and  $M \leq_{\mathbf{K}} N$  both in  $\mathbf{K}_{\geq \mu}$ , let us say that a (Galois) type p over  $N \mu$ -splits over M if there exists  $N_1, N_2 \in \mathbf{K}_{\mu}$  with  $M \leq_{\mathbf{K}} N_1 \leq_{\mathbf{K}} N$ ,  $M \leq_{\mathbf{K}} N_2 \leq_{\mathbf{K}} N$ , and an automorphism f of  $\mathfrak{C}$  fixing M and sending  $N_1$  to  $N_2$  such that  $f(p \upharpoonright N_1) \neq p \upharpoonright N_2$ .

When p does not  $\mu$ -split over M, one can think that p is, in a weak sense, "determined" by  $p \upharpoonright M$ . As in the first-order superstable case, we

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want an approximation to "every type is determined/does not fork over a finite set". To this end, let us say that **K** has no long splitting chains in  $\mu$  if for every  $\delta < \mu^+$ , every increasing continuous  $\langle M_i : i \leq \delta \rangle$  and every p over  $M_{\delta}$ , if  $M_{i+1}$  is universal over  $M_i$  for all  $i < \delta$ , then there exists  $i < \delta$  such that p does not  $\mu$ -split over  $M_i$ .

- (5) Let  $\mu > LS(\mathbf{K})$  be a categoricity cardinal. Then **K** has no long splitting chains in  $\mu$ . [Why? One way is to use a theorem of Shelah and Villaveces [**SV99**, 2.2.1], an axiomatization of which is given in Chapter 20 here. Alternatively (this is Lemma 19.4.12), note that any increasing continuous chain  $\langle M_i : i \leq \delta \rangle$  in  $\mathbf{K}_{\mu}$  must consist of saturated models only (including at limits *i*). So let *p* be a type over  $M_{\delta}$ , assume for a contradiction that it  $\mu$ -splits over every  $M_i$ ,  $i < \delta$ . There are witnesses  $f_i, N_i^1, N_i^2$  to this splitting. Using tameness we can find smaller witnesses  $M_i^1, M_i^2$  (i.e. of size LS(**K**)). Let *N* be a substructure of  $M_{\delta}$  containing all the witnesses and of size  $\delta + LS(\mathbf{K})$ . Now without loss of generality  $\delta = \mathrm{cf} \, \delta < \mu$  (if  $\delta = \mu$  and  $\mu$  is regular, it is known [**She99**, 3.3(1)] that there must exist  $N_0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  such that *p* does not  $\mu$ -split over  $N_0$ , and then one can pick  $i < \delta$  such that *p* does not  $\mu$ -split over  $M_i$ ). Thus since  $M_{\delta}$  is saturated,  $p \upharpoonright N$  is realized in  $M_{\delta}$ , say by *b*. There is  $i < \delta$  such that  $b \in M_i$ . This ends up contradicting the splitting of *p* over  $M_i$ .]
- (6) Fix categoricity cardinals  $\mu$  and  $\lambda$  such that  $LS(\mathbf{K}) < \mu < \lambda$ . For  $M \leq_{\mathbf{K}} N$  both in  $\mathbf{K}_{\lambda}$  and p a type over N, say that p does not  $\mu$ -fork over M if there exists  $M_0 \in \mathbf{K}_{\mu}$  such that  $M_0 \leq_{\mathbf{K}} M$  and p does not  $\mu$ -split over  $M_0$ . Then  $\mu$ -nonforking induces what Shelah calls a good  $\mu$ -frame. This means that it has several of the basic properties of forking in a superstable first-order theory: monotonicity, invariance, every type has a nonforking extension, the nonforking extension is unique, nonforking has a certain symmetry property, and the following local character property: for any increasing continuous chain  $\langle M_i : i \leq \delta \rangle$ , p a type over  $M_{\delta}$ , there exists  $i < \delta$  such that p does not  $\mu$ -fork over  $M_i$  (note that as opposed to before, we do not require that  $M_{i+1}$  be universal over  $M_i$ ).

[Why does nonforking have all these properties? This is proven in Chapter 4. Extension and uniqueness can be established using tameness together with the weak extension and uniqueness properties of splitting proven by VanDieren in [Van06, I.4.10, I.4.12]. Symmetry is because otherwise one would get the order property, contradicting stability. The local character property is proven by contradiction: failure of local character would contradict no long splitting chains in  $\mu$ . We are using here that all the models of cardinality  $\lambda$  are saturated, since  $\lambda$  is a categoricity cardinal.]

(7) It is enough to show that **K** is categorical in  $\lambda^+$ . [Why? Once we have categoricity in  $\lambda^+$ , we can apply a known upward categoricity transfer from categoricity in a successor of Grossberg and VanDieren [**GV06c**, **GV06a**]. Alternatively, we can repeat the argument for categoricity in  $\lambda^+$  from categoricity in  $\lambda$  to get categoricity in  $\lambda^{++}$  from categoricity in  $\lambda$  to get categoricity in  $\lambda^{++}$  from categoricity in  $\lambda^+$ , and so on, obtaining categoricity in  $\lambda^{+n}$  for  $n < \omega$ . Recalling from the proof of the connecting lemma that the categoricity spectrum is closed, we can continue past the limit cardinals and get categoricity in every  $\lambda' \ge \lambda$ .]

- (8) Assume for a contradiction that **K** is not categorical in  $\lambda^+$ . Then there exists M of cardinality  $\lambda$  and a suitable type p over M that is omitted in an extension of M of cardinality  $\lambda^+$ . [Why? Because non-categoricity in  $\lambda^+$  implies the existence of a non-saturated model of cardinality  $\lambda^+$ ]
- (9) Fix such M and p. Let K<sub>¬p</sub> be the class of models omitting p. We add constant symbols for M to make the class closed under isomorphisms. Then K<sub>¬p</sub> is an AEC. Moreover it is λ-tame and has primes over sets of the form N∪{a} (at this point, we do not claim that K<sub>¬p</sub> has amalgamation; however there is a way to define prime models over M ∪ {a} without assuming amalgamation, basically using Galois types to code how N∪{a} is embedded).

[Why? That it is an AEC is easy to check (closure under unions is because any type realized in the union of a chain must be realized in one of the elements of the chain). That it has primes is because if  $N \cup \{a\}$ is contained in any element of  $\mathbf{K}_{\neg p}$  at all, then any model prime over it (which exists in  $\mathbf{K}$ ) can be embedded inside N, hence must also be in  $\mathbf{K}_{\neg p}$ . That it is tame is because Galois types are the same in  $\mathbf{K}$  and  $\mathbf{K}_{\neg p}$ : if  $gtp(a/N; N_1) = gtp(b/N; N_2)$  (in  $\mathbf{K}$ ), then there exists a map from the prime model over  $N \cup \{a\}$  inside  $N_1$  to the prime model over  $N \cup \{a\}$ inside  $N_2$  witnessing it. Thus  $gtp(a/N; N_1) = gtp(b/N; N_2)$  also in  $\mathbf{K}_{\neg p}$ . Conversely, if two types are equal in  $\mathbf{K}_{\neg p}$  then they are equal in  $\mathbf{K}$ .]

- (10)  $\mathbf{K}_{\neg p}$  has a good  $\lambda$ -frame (recall that this means there is a nice superstablelike forking notion; moreover the class has amalgamation in  $\lambda$ , no maximal models in  $\lambda$ , and joint embedding in  $\lambda$ ). [Why? This uses that pwas chosen "suitably" and some orthogonality calculus for good frames. See Theorem 11.2.7. Roughly, by noncategoricity in  $\lambda^+$  there must exist a type q that is "orthogonal" to p in the sense that one can realize it independently of p. This helps controlling the behavior of nonforking and amalgamation in  $\mathbf{K}_{\neg p}$ .]
- (11) If **K** is not categorical in  $\lambda^+$ ,  $\mathbf{K}_{\neg p}$  above has arbitrarily large models.

[Why? It is known (by a result of Will Boney [Bon14a] combined with Chapter 5) that in any AEC with a good  $\lambda$ -frame that is  $\lambda$ -tame and has amalgamation, the good  $\lambda$ -frame transfers up to a good ( $\geq \lambda$ )frame (i.e. the nonforking notion extends and remains nice, and moreover the class has no maximal models). By Theorem 8.4.16, it is also possible to prove this result if the AEC only has weak amalgamation instead of amalgamation. Weak amalgamation is a weakening of having primes, and we know that  $\mathbf{K}_{\neg p}$  has primes. Intuitively, what is happening is that we can prove the extension property of nonforking at a cardinal  $\theta$  using only amalgamation below  $\theta$ . Moreover having primes and knowing that all types can be extended implies amalgamation. Thus amalgamation can be proven by induction.

From our discussion,  $\mathbf{K}_{\neg p}$  must have a good  $(\geq \lambda)$ -frame. This means in particular that it has arbitrarily large models.

(12) For any  $\theta > \lambda$  there is a model of cardinality  $\theta$  omitting p. Thus there is a non-saturated model in every cardinal, contradicting that **K** is categorical in a proper class of cardinals above  $\lambda$ . [Why? Because we know that  $\mathbf{K}_{\neg p}$  has arbitrarily large models.]

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#### 1.6. Thesis overview

The rest of this thesis contains the author's papers. Some have already appeared in refereed journals, but others are still under review. We have put the material of Chapter 2 first, since it contains extensive preliminaries introducing our notation and the basic concepts. We have put the material of Chapter 24 last (even though it was written first) simply because it is not about AECs, so does not fit with the overall topic of the thesis (the author chose to include it simply because it is part of the work he did as a Ph.D. student). The other chapters are listed roughly in the order in which they were written.

We have edited the chapters to try to make the style of the thesis uniform and remove some redundancies but the differences compared with the original versions of the papers (from February 2017) are otherwise small.

Below, we list each chapter with a reference to the paper (or preprint) it is based on and a one sentence description of the material to be found there. Each chapter has an abstract which the reader can consult for a more in-depth overview. Note that several chapters are joint work. The author does NOT claim full credit for the results proven in those chapters; credit should be shared between the authors.

- Chapter 2 is [Vas16c]. It gives the background and notation on AECs used in this whole thesis and introduces *Galois Morleyization*, a way to think of Galois types as sets of infinitary formulas. Applications (using Shelah's stability theory inside a model) include the equivalence between stability in terms of Galois types and no order property in tame AECs, as well as some results on the coheir independence notion.
- Chapter 3 is [**BGKV16**] (a joint work with Will Boney, Rami Grossberg, and Alexei Kolesnikov). It presents a global framework for stable independence in any AECs with a monster model, and shows canonicity of forking in this setup. It also shows that symmetry of forking follows from no order property.
- Chapter 4 is [Vas16b]. It gives a construction of a good frame from tameness in a categorical AEC.
- Chapter 5 is [**BVe**] (a joint work with Will Boney). It studies independent sequences in good frames and as application proves that good frames can be extended using tameness, and that there is a natural notion of dimension of types in this case (given by the size of a maximal independent set of realizations).
- Chapter 6 is [Vas16a]. It axiomatizes the construction of independence relation from previous chapters and shows how to build a global notion as in Chapter 3.
- Chapter 7 is  $[\mathbf{BV17}]$  (a joint work with Will Boney). It gives results on when, in stable or superstable tame AECs, the union of a chain of  $\lambda$ -saturated model is  $\lambda$ -saturated. Of particular interest is the development of a theory of average types in this context.
- Chapter 8 is [Vasg]. It sets the fundamentals for the proof of Shelah's eventual categoricity conjecture in universal classes. However it does not completely solve the problem of deriving amalgamation from categoricity

in universal classes, and also the "high-enough" thresholds are improved in later chapters.

- Chapter 9 is [**GV**] (a joint work with Rami Grossberg). It shows that several definitions of superstability from the first-order context are also equivalent in tame AECs.
- Chapter 10 is [VV17] (a joint work with Monica VanDieren). It studies the symmetry property of splitting, both in tame and non-tame AECs, and derives it from no order property (this is a more technical result than in Chapter 3). Applications are given to construction of good frames and chains of saturated models.
- Chapter 11 is [Vasf]. It proves that any tame AEC with amalgamation and primes satisfies Shelah's eventual categoricity conjecture. In Chapter 8 this was proven assuming a stronger property than tameness (full tameness and type-shortness).
- Chapter 12 is [Vasa]. It shows how to build primes in classes of saturated models assuming the existence of a well-behaved independence notion. This shows assuming a large cardinal axioms that having primes follows from total categoricity.
- Chapter 13 is [BGL<sup>+</sup>16] (a joint work with Will Boney, Rami Grossberg, Mike Lieberman, and Jiří Rosický). It introduces μ-AECs, a generalization of AECs, and generalizes some results from the theory of AECs to this context. It also establishes that μ-AECs have an natural categorytheoretic analog: accessible categories whose morphisms are monomorphisms.
- Chapters 14 and 15 are based on the same paper, [Vas17a]. It proves a general categoricity transfer for global good frames categorical in a successor and gives improvements on the "high-enough" threshold of several known categoricity transfers. An exposition of Shelah's proof of the eventual categoricity conjecture in AECs with amalgamation (assuming some unpublished work and the weak GCH) is also given.
- Chapter 16 is [Vas17c]. It completes the proof of the eventual categoricity conjecture in universal classes by dealing with getting structure (most especially amalgamation) from categoricity. The main ingredient turns out to be a generalized symmetry lemma for independent, potentially non-smooth, trees.
- Chapter 17 is [Vase]. It shows that in an AEC K with a amalgamation and no maximal models categorical in  $\lambda > LS(\mathbf{K})$ , the model of cardinality  $\lambda$  is saturated. A downward solvability transfer is deduced and several other applications, outlining the structure of the AEC below the categoricity cardinal, are mentioned.
- Chapter 18 is  $[\mathbf{BVc}]$  (a joint work with Will Boney). It investigates good frames in the example of Hart and Shelah where categoricity holds at  $\aleph_0, \aleph_1, \ldots, \aleph_n$  but fails at  $\aleph_{n+1}$ . In particular the example has a non-weakly-successful good frame. The existence of such a frame was open. The chapter also investigates general ways of building good frames in "Hart-Shelah-like" setups.
## 1.6. THESIS OVERVIEW

- Chapter 19 is [Vash]. It gives the beginning of a stability theory for tame AECs, proving that superstability follows from stability on a tail and more generally (assuming SCH) characterizing the eventual stability spectrum.
- Chapter 20 is [**BGVV17**] (a joint work with Will Boney, Rami Grossberg, and Monica VanDieren). It gives an axiomatization of an argument of Shelah and Villaveces deriving a form of superstability from categoricity. The axiomatization gives more results and the proof fixes a gap in the Shelah-Villaveces theorem.
- Chapter 21 is [Vasd]. It proposes the definition of a quasiminimal AEC and shows that such classes correspond to Zilber's quasiminimal pregeometry classes.
- Chapter 22 is [Vasc]. It proves the uniqueness property of nonforking in a specific local context and derives equivalence between several definitions of nonforking symmetry.
- Chapter 23 is [SV] (a joint work with Saharon Shelah). It proves, roughly speaking, that ℵ<sub>0</sub>-stability implies ℵ<sub>0</sub>-superstability (just like in the first-order case) and (< ℵ<sub>0</sub>, ℵ<sub>0</sub>)-tameness.
- Chapter 24 is [Vas17b]. It gives an alternate proof of the existence of Morley sequences in simple theories that uses only principles from ordinary mathematics.

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# CHAPTER 2

# Infinitary stability theory

This chapter is based on [Vas16c]. I thank Will Boney for thoroughly reading this chapter and providing invaluable feedback. I also thank Alexei Kolesnikov for valuable discussions on the idea of thinking of Galois types as formulas. I thank John Baldwin, Jonathan Kirby, and a referee for valuable comments.

#### Abstract

We introduce a new device in the study of abstract elementary classes (AECs): Galois Morleyization, which consists in expanding the models of the class with a relation for every Galois (orbital) type of length less than a fixed cardinal  $\kappa$ . We show:

THEOREM 2.0.1 (The semantic-syntactic correspondence). An AEC K is fully  $(< \kappa)$ -tame and type short if and only if Galois types are syntactic in the Galois Morleyization.

This exhibits a correspondence between AECs and the syntactic framework of stability theory inside a model. We use the correspondence to make progress on the stability theory of tame and type short AECs. The main theorems are:

THEOREM 2.0.2. Let  $\mathbf{K}$  be a LS( $\mathbf{K}$ )-tame AEC with amalgamation. The following are equivalent:

- (1) **K** is Galois stable in some  $\lambda \geq LS(\mathbf{K})$ .
- (2) K does not have the order property (defined in terms of Galois types).
- (3) There exist cardinals  $\mu$  and  $\lambda_0$  with  $\mu \leq \lambda_0 < \beth_{(2^{\text{LS}(\mathbf{K})})^+}$  such that **K** is Galois stable in any  $\lambda \geq \lambda_0$  with  $\lambda = \lambda^{<\mu}$ .

THEOREM 2.0.3. Let **K** be a fully  $(< \kappa)$ -tame and type short AEC with amalgamation,  $\kappa = \beth_{\kappa} > \text{LS}(\mathbf{K})$ . If **K** is Galois stable, then the class of  $\kappa$ -Galois saturated models of **K** admits an independence notion (( $< \kappa$ )-coheir) which, except perhaps for extension, has the properties of forking in a first-order stable theory.

# 2.1. Introduction

Abstract elementary classes (AECs) are sometimes described as a purely semantic framework for model theory. It has been shown, however, that AECs are closely connected with more syntactic objects. See for example Shelah's presentation theorem [She87a, Lemma 1.8], or Kueker's [Kue08, Theorem 7.2] showing that an AEC with Löwenheim-Skolem number  $\lambda$  is closed under  $\mathbb{L}_{\infty,\lambda^+}$ -elementary equivalence. Another framework for non-elementary model theory is stability theory inside a model (introduced in Rami Grossberg's 1981 master thesis and studied for example<sup>1</sup> in [**Gro91a**, **Gro91b**] or [**She87b**, Chapter I], see [**She09b**, Chapter V.A] for a more recent version). There the methods are very syntactic but it is believed (see for example the remark on p. 116 of [**Gro91a**]) that they can help the resolution of more semantic questions, such as Shelah's categoricity conjecture for  $\mathbb{L}_{\omega_1,\omega}$ .

In this chapter, we establish a correspondence between these two frameworks. We show that results from stability theory inside a model directly translate to results about *tame* abstract elementary classes. Recall that an AEC is  $(< \kappa)$ -tame if its Galois (i.e. orbital) types are determined by their restrictions to domains of size less than  $\kappa$ . Tameness as a property of AEC was first isolated (from an argument in [She99]) by Grossberg and VanDieren [GV06b] and used to prove an upward categoricity transfer [GV06a, GV06c]. Boney [Bon14b] showed that tameness follows from the existence of large cardinals. Combined with the categoricity transfers of Grossberg-VanDieren and Shelah [She99], this showed assuming a large cardinal axiom that Shelah's eventual categoricity conjecture holds if the categoricity cardinal is a successor.

The basic idea of the translation is the observation (appearing for example in [Bon14b, p. 15] or [Lie11b, p. 206]) that in a ( $< \kappa$ )-tame abstract elementary class, Galois types over domains of size less than  $\kappa$  play a role analogous to first-order formulas. We make this observation precise by expanding the language of such an AEC with a relation symbol for each Galois type over the empty set of a sequence of length than  $\kappa$ , and looking at  $\mathbb{L}_{\kappa,\kappa}$ -formulas in the expanded language. We call this expansion the *Galois Morleyization*<sup>2</sup> of the AEC. Thinking of a type as the set of its small restrictions, we can then prove the *semantic-syntactic correspondence* (Theorem 2.3.15): Galois types in the AEC correspond to quantifier-free syntactic types in its Galois Morleyization.

The correspondence gives us a new method to prove results in tame abstract elementary classes:

- (1) Prove a syntactic result in the Galois Morleyization of the AEC (e.g. using tools from stability theory inside a model).
- (2) Translate to a semantic result in the AEC using the semantic-syntactic correspondence.
- (3) Push the semantic result further using known (semantic) facts about AECs, maybe combined with more hypotheses on the AEC (e.g. amalgamation).

As an application, we prove Theorem 2.0.2 in the abstract (see Theorem 2.4.15), which gives the equivalence between no order property and stability in tame AECs and generalizes one direction of the stability spectrum theorem of homogeneous model theory ([She70, Theorem 4.4], see also [GL02, Corollary 3.11]). The syntactic part of the proof is not new (it is a straightforward generalization of Shelah's first-order proof [She90, Theorem 2.10]) and we are told by Rami Grossberg that proving such results was one of the reason tameness was introduced (in fact theorems with the same spirit appear in [GV06b]). However we believe it is challenging

<sup>&</sup>lt;sup>1</sup>The definition of a model being stable appears already in [She78, Definition I.2.2] but (as Shelah notes in the introduction to [She87b, Chapter I]) this concept was not pursued further there.

<sup>&</sup>lt;sup>2</sup>We thank Rami Grossberg for suggesting the name.

#### 2.2. PRELIMINARIES

to give a transparent proof of the result using Galois types only. The reason is that the classical proof uses local types and it is not clear how to naturally define them semantically.

The method has other applications: Theorem 2.5.15 (formalizing Theorem 2.0.3 from the abstract) shows that in stable fully tame and short AECs, the coheir independence relation has some of the properties of a well-behaved independence notion. This is used in Chapter 6 to build a global independence notion from superstability. In Chapter 7, we also use syntactic methods to investigate chains of Galois-saturated models.

Precursors to this work include Makkai and Shelah's study of classes of models of an  $\mathbb{L}_{\kappa,\omega}$  theory for  $\kappa$  a strongly compact cardinal [**MS90**]: there they prove [**MS90**, Proposition 2.10] that Galois and syntactic  $\Sigma_1(\mathbb{L}_{\kappa,\kappa})$ -types are the same (so in particular those classes are ( $< \kappa$ )-tame). One can see the results of this chapter as a generalization to tame AECs. Also, the construction of the Galois Morleyization when  $\kappa = \aleph_0$  (so the language remains finitary) appears in [**HK16**, Section 2.4]. Moreover it has been pointed out to us<sup>3</sup> that a device similar to Galois Morleyization is used in [**Ros81**, Section 3] to present any concrete category as a class of models of an infinitary theory. However the use of Galois Morleyization to translate results of stability theory inside a model to AECs is new.

This chapter is organized as follows. In section 2.2, we review some preliminaries. In section 2.3, we introduce *functorial expansions*<sup>4</sup> of AECs and the main example: Galois Morleyization. We then prove the semantic-syntactic correspondence. In section 2.4, we investigate various order properties and prove Theorem 2.0.2. In section 2.5, we study the coheir independence relation. Several of these sections have *global hypotheses* which hold until the end of the section: see Hypotheses 2.3.9, 2.4.1, and 2.5.1.

We end with a note on how AECs compare to some other non first-order framework like homogeneous model theory (see [She70]). There is an example (due to Marcus, see [Mar72]) of an  $\mathbb{L}_{\omega_1,\omega}$ -axiomatizable class which is categorical in all uncountable cardinals but does not have an  $\aleph_1$ -sequentially-homogeneous model. For  $n < \omega$ , an example due to Hart and Shelah (see [HS90, BK09]) has amalgamation, no maximal models, and is categorical in all  $\aleph_k$  with  $k \leq n$ , but no higher. By [GV06c], the example cannot be  $\aleph_k$ -tame for k < n. However if  $\kappa$  is a strongly compact cardinal, the example will be fully ( $< \kappa$ )-tame and type short by the main result of [Bon14b]. The discussion on p. 74 of [Bal09] gives more non-homogeneous examples.

In general, classes from homogeneous model theory or quasiminimal pregeometry classes (see [**Kir10**]) are special cases of AECs that are always fully ( $< \aleph_0$ )-tame and type short. In this chapter we work with the much more general assumption of ( $< \kappa$ )-tameness and type shortness for a possibly uncountable  $\kappa$ .

#### 2.2. Preliminaries

We review some of the basics of abstract elementary classes and fix some notation. The reader is advised to skim through this section quickly and go back to it as needed.

<sup>&</sup>lt;sup>3</sup>By Jonathan Kirby.

<sup>&</sup>lt;sup>4</sup>These were called "abstract Morleyizations" in an early version of this chapter. We thank John Baldwin for suggesting the new name.

#### 2.2.1. Set theoretic terminology.

DEFINITION 2.2.1. Let  $\kappa$  be an infinite cardinal.

- (1) Let  $\kappa_r$  be the least regular cardinal greater than or equal to  $\kappa$ . That is,  $\kappa_r$  is  $\kappa^+$  if  $\kappa$  is singular and  $\kappa$  if  $\kappa$  is regular.
- (2) Let  $\kappa^-$  be  $\kappa$  if  $\kappa$  is limit or the unique  $\kappa_0$  such that  $\kappa_0^+ = \kappa$  if  $\kappa$  is a successor.
- (3) Let REG denote the class of regular cardinals.

We will often use the following function (see the notation in [Bal09, 4.24]):

DEFINITION 2.2.2 (Hanf function). For  $\lambda$  an infinite cardinal, define  $h(\lambda) := \beth_{(2^{\lambda})^+}$ . Also define  $h^*(\lambda) := h(\lambda^-)$ . When **K** is a fixed AEC (see Definition 2.2.14), write  $H_1$  for  $h(\text{LS}(\mathbf{K}))$  and  $H_2$  for  $h(H_1)$ .

Note that for  $\lambda$  infinite,  $\lambda = \beth_{\lambda}$  if and only if for all  $\mu < \lambda$ ,  $h(\mu) < \lambda$ .

**2.2.2. Syntax.** The notation of this chapter is standard, but since we will work with infinitary objects and need to be precise, we review the basics. We will often work with the logic  $\mathbb{L}_{\kappa,\kappa}$ , see [Dic75] for the definition and basic results.

DEFINITION 2.2.3. An *infinitary vocabulary* is a vocabulary where we also allow relation and function symbols of infinite arity. For simplicity, we require the arity to be an ordinal. An infinitary vocabulary is  $(< \kappa)$ -ary if all its symbols have arity strictly less than  $\kappa$ . A *finitary vocabulary* is a  $(< \aleph_0)$ -ary vocabulary.

For  $\tau$  an infinitary vocabulary,  $\phi$  an  $\mathbb{L}_{\kappa,\kappa}(\tau)$ -formula and  $\bar{x}$  a sequence of variables, we write  $\phi = \phi(\bar{x})$  to emphasize that the free variables of  $\phi$  appear among  $\bar{x}$  (recall that a  $\mathbb{L}_{\kappa,\kappa}$ -formula must have fewer than  $\kappa$ -many free variables, but not all elements of  $\bar{x}$  need to appear as free variables in  $\phi$ , so we allow  $\ell(\bar{x}) \geq \kappa$ ). We use a similar notation for sets of formulas. When  $\bar{a}$  is an element in some  $\tau$ -structure and  $\phi(\bar{x}, \bar{y})$  is a formula, we often abuse notation and say that  $\psi(\bar{x}) = \phi(\bar{x}, \bar{a})$  is a formula (again, we allow  $\ell(\bar{a}) \geq \kappa$ ). We say  $\phi(\bar{x}, \bar{a})$  is a formula over A if  $\bar{a} \in {}^{\infty}A$ .

DEFINITION 2.2.4. For  $\phi$  a formula over a set, let  $FV(\phi)$  denote an enumeration of the free variables of  $\phi$  (according to some canonical ordering on all variables). That is, fixing such an ordering,  $FV(\phi)$  is the smallest sequence  $\bar{x}$  such that  $\phi = \phi(\bar{x})$ . Let  $\ell(\phi) := \ell(FV(\phi))$  (it is an ordinal, but by permutting the variables we can usually assume without loss of generality that it is a cardinal), and dom  $\phi$  be the smallest set A such that  $\phi$  is over A. Define similarly the meaning of FV(p),  $\ell(p)$ , and dom p on a set p of formulas.

DEFINITION 2.2.5. For  $\tau$  an infinitary vocabulary,  $M \neq \tau$ -structure,  $A \subseteq |M|$ ,  $\bar{b} \in {}^{<\infty}|M|$ , and  $\Delta$  a set of  $\tau$ -formulas (in some logic), let<sup>5</sup>:

 $\operatorname{tp}_{\Delta}(\bar{b}/A;M) := \{ \phi(\bar{x};\bar{a}) \mid \phi(\bar{x},\bar{y}) \in \Delta, \, \bar{a} \in {}^{\ell(\bar{y})}A, \, \text{and} \, M \models \phi[\bar{b},\bar{a}] \}$ 

We will most often work with  $\Delta = qf \cdot \mathbb{L}_{\kappa,\kappa}$ , the set of *quantifier-free*  $\mathbb{L}_{\kappa,\kappa}$ -formulas.

<sup>&</sup>lt;sup>5</sup>Of course, we have in mind a canonical sequence of variables  $\bar{x}$  of order type  $\ell(\bar{b})$  that should really be part of the notation but (as is customary) we always omit this detail.

DEFINITION 2.2.6. For M a  $\tau$ -structure,  $\Delta$  a set of  $\tau$ -formulas,  $A \subseteq |M|$ ,  $\alpha$  an ordinal or  $\infty$ , let

$$S^{<\alpha}_{\Delta}(A;M) := \{ \operatorname{tp}_{\Delta}(\bar{b}/A;M) \mid \bar{b} \in {}^{<\alpha}|M| \}$$

Define similarly the variations for  $\leq \alpha$ ,  $\alpha$ , etc. We write  $S_{\Delta}(A; M)$  instead of  $S_{\Delta}^{1}(A; M)$ .

**2.2.3.** Abstract classes. We review the definition of an abstract elementary class. Abstract elementary classes (AECs) were introduced by Shelah in [She87a]. The reader unfamiliar with AECs can consult [Gro02] for an introduction.

We first review more general objects that we will sometimes use. Abstract classes are already defined in [**Gro**], while  $\mu$ -abstract elementary classes are introduced in Chapter 13. We will mostly use them to deal with functorial expansions and classes of saturated models of an AEC.

DEFINITION 2.2.7. An *abstract class* (AC for short) is a pair  $\mathbf{K} = (K, \leq_{\mathbf{K}})$ , where:

- (1) **K** is a class of  $\tau$ -structure, for some fixed infinitary vocabulary  $\tau$  (that we will denote by  $\tau(\mathbf{K})$ ). We say  $(K, \leq_{\mathbf{K}})$  is  $(<\mu)$ -ary if  $\tau$  is  $(<\mu)$ -ary.
- (2)  $\leq_{\mathbf{K}}$  is a partial order (that is, a reflexive and transitive relation) on **K**.
- (3) If  $M \leq_{\mathbf{K}} N$  are in  $\mathbf{K}$  and  $f : N \cong N'$ , then  $f[M] \leq_{\mathbf{K}} N'$  and both are in  $\mathbf{K}$ .
- (4) If  $M \leq_{\mathbf{K}} N$ , then  $M \subseteq N$ .

REMARK 2.2.8. We do not always strictly distinguish between K and  $\mathbf{K} = (K, \leq_{\mathbf{K}})$ . In particular we will often write  $M \in \mathbf{K}$  when we really mean  $M \in K$ .

NOTATION 2.2.9. For **K** an abstract class,  $M, N \in \mathbf{K}$ , we write  $M <_{\mathbf{K}} N$  when  $M \leq_{\mathbf{K}} N$  and  $M \neq N$ .

DEFINITION 2.2.10. For **K** an abstract class and  $\lambda$  a cardinal, write  $I(\mathbf{K}, \lambda)$  for the number of non-isomorphic models of cardinality  $\lambda$  in **K**. If  $I(\mathbf{K}, \lambda) = 1$ , we say that **K** is *categorical in*  $\lambda$ .

DEFINITION 2.2.11. Let **K** be an abstract class. A sequence  $\langle M_i : i < \delta \rangle$  of elements of **K** is *R*-increasing if for all  $i < j < \delta$ ,  $M_i R M_j$ . Strictly increasing means that in addition  $M_i \neq M_j$  for i < j.  $\langle M_i : i < \delta \rangle$  is continuous if for all limit  $i < \delta$ ,  $M_i = \bigcup_{j < i} M_j$ . When *R* is omitted, we mean  $R = \leq_{\mathbf{K}}$ .

NOTATION 2.2.12. For **K** an abstract class, we use notations such as  $\mathbf{K}_{\lambda}, \mathbf{K}_{\geq \lambda}, \mathbf{K}_{<\lambda}, \mathbf{K}_{\mathcal{F}}$  for the restriction of the class to models in **K** of size  $\lambda, \geq \lambda, < \lambda$ , contained in the set  $\mathcal{F}$ , respectively.

DEFINITION 2.2.13. Let  $(I, \leq)$  be a partially-ordered set.

- (1) We say that I is  $\mu$ -directed provided for every  $J \subseteq I$  if  $|J| < \mu$  then there exists  $r \in I$  such that  $r \geq s$  for all  $s \in J$  (thus  $\aleph_0$ -directed is the usual notion of directed set)
- (2) Let **K** be an abstract class. An indexed system  $\langle M_i : i \in I \rangle$  of models in **K** is  $\mu$ -directed if I is a  $\mu$ -directed set and i < j implies  $M_i \leq_{\mathbf{K}} M_j$ .

DEFINITION 2.2.14. Let  $\mu$  be a regular cardinal and let **K** be a (<  $\mu$ )-ary abstract class. We say that **K** is a  $\mu$ -abstract elementary class ( $\mu$ -AEC for short) if:

- (1) Coherence: If  $M_0, M_1, M_2 \in \mathbf{K}$  satisfy  $M_0 \leq_{\mathbf{K}} M_2, M_1 \leq_{\mathbf{K}} M_2$ , and  $M_0 \subseteq M_1$ , then  $M_0 \leq_{\mathbf{K}} M_1$ ;
- (2) Tarski-Vaught axioms: Suppose  $\langle M_i \in \mathbf{K} : i \in I \rangle$  is a  $\mu$ -directed system. Then:
  - (a)  $\bigcup_{i \in I} M_i \in \mathbf{K}$  and, for all  $j \in I$ , we have  $M_j \leq_{\mathbf{K}} \bigcup_{i \in I} M_i$ .
  - (b) If there is some  $N \in \mathbf{K}$  so that for all  $i \in I$  we have  $M_i \leq_{\mathbf{K}} N$ , then we also have  $\bigcup_{i \in I} M_i \leq_{\mathbf{K}} N$ .
- (3) Löwenheim-Skolem-Tarski axiom: There exists a cardinal  $\lambda = \lambda^{<\mu} \geq |\tau(\mathbf{K})| + \mu$  such that for any  $M \in \mathbf{K}$  and  $A \subseteq |M|$ , there is some  $M_0 \leq_{\mathbf{K}} M$  such that  $A \subseteq |M_0|$  and  $||M_0|| \leq |A|^{<\mu} + \lambda$ . We write  $\mathrm{LS}(\mathbf{K})$  for the minimal such cardinal<sup>6</sup>.

When  $\mu = \aleph_0$ , we omit it and simply call **K** an *abstract elementary class* (AEC for short).

In any abstract class, we can define a notion of embedding:

DEFINITION 2.2.15. Let **K** be an abstract class. We say a function  $f: M \to N$  is a **K**-embedding if  $M, N \in \mathbf{K}$  and  $f: M \cong f[M] \leq_{\mathbf{K}} N$ . For  $A \subseteq |M|$ , we write  $f: M \to N$  to mean that f fixes A pointwise. Unless otherwise stated, when we write  $f: M \to N$  we mean that f is an embedding.

Here are three key structural properties an abstract class can have:

DEFINITION 2.2.16. Let  $\mathbf{K}$  be an abstract class.

- (1) **K** has amalgamation if for any  $M_0 \leq_{\mathbf{K}} M_\ell$  in **K**,  $\ell = 1, 2$ , there exists  $N \in \mathbf{K}$  and  $f_\ell : M_\ell \xrightarrow[M_0]{} N$ .
- (2) **K** has *joint embedding* if for any  $M_{\ell}$  in **K**,  $\ell = 1, 2$ , there exists  $N \in \mathbf{K}$  and  $f_{\ell} : M_{\ell} \to N$ .
- (3) **K** has no maximal models if for any  $M \in \mathbf{K}$  there exists  $N \in \mathbf{K}$  with  $M <_{\mathbf{K}} N$ .

**2.2.4.** Galois types. Let **K** be an abstract class. There is a well-known a semantic notion of types for **K**, Galois types, that was first introduced by Shelah in [She87b, Definition II.1.9]. While Galois types are usually only defined over models, here we allow them to be over any set. This is not harder and is often notationally convenient<sup>7</sup>. Note however that Galois types over sets are in general not too well-behaved. For example, they can sometimes fail to have an extension (in the sense that if we have  $N, N' \in \mathbf{K}, A \subseteq |N| \cap |N'|$  and p a Galois type over A realized in N, then we may not be able to extend p to a type over N') if their domain is not an amalgamation base.

Definition 2.2.17.

- (1) Let  $\mathbf{K}^3$  be the set of triples of the form  $(\bar{b}, A, N)$ , where  $N \in \mathbf{K}$ ,  $A \subseteq |N|$ , and  $\bar{b}$  is a sequence of elements from N.
- (2) For  $(\bar{b}_1, A_1, N_1), (\bar{b}_2, A_2, N_2) \in \mathbf{K}^3$ , we say  $(\bar{b}_1, A_1, N_1) E_{\mathrm{at}}(\bar{b}_2, A_2, N_2)$  if  $A := A_1 = A_2$ , and there exists  $f_{\ell} : N_{\ell} \xrightarrow{} N$  such that  $f_1(\bar{b}_1) = f_2(\bar{b}_2)$ .

<sup>&</sup>lt;sup>6</sup>Pedantically, LS(**K**) really depends on  $\mu$  but  $\mu$  will always be clear from context.

<sup>&</sup>lt;sup>7</sup>For example, types over the empty sets are used here in the definition of the Galois Morleyization. They appear implicitly in the definition of the order property in [**She99**, Definition 4.3] and explicitly in [**GV06b**, Notation 1.9]. They are also used in [**HK06**].

#### 2.2. PRELIMINARIES

- (3) Note that  $E_{\text{at}}$  is a symmetric and reflexive relation on  $\mathbf{K}^3$ . We let E be the transitive closure of  $E_{\text{at}}$ .
- (4) For  $(\bar{b}, A, N) \in \mathbf{K}^3$ , let  $\operatorname{gtp}(\bar{b}/A; N) := [(\bar{b}, A, N)]_E$ . We call such an equivalence class a *Galois type*. We write  $\operatorname{gtp}_{\mathbf{K}}(\bar{b}/A; N)$  when **K** is not clear from context.
- (5) For  $p = \operatorname{gtp}(\overline{b}/A; N)$  a Galois type, define<sup>8</sup>  $\ell(p) := \ell(\overline{b})$  and dom p := A.

We can go on to define the restriction of a type (if  $A_0 \subseteq \text{dom } p$ ,  $I \subseteq \ell(p)$ , we will write  $p^I \upharpoonright A_0$  when the realizing sequence is restricted to I and the domain is restricted to  $A_0$ ), the image of a type under an isomorphism, or what it means for a type to be realized. Just as in [She09a, Observation II.1.11.4], we have:

FACT 2.2.18. If **K** has amalgamation, then  $E = E_{at}$ .

Note that the proof goes through, even though we only have amalgamation over models, not over all sets.

REMARK 2.2.19. To gain further insight into the difference between E and  $E_{\text{at}}$ , consider the following situation. Let **K** be an AEC that does *not* have amalgamation and assume we are given  $M \leq_{\mathbf{K}} N$ ,  $a_1, a_2 \in |M|$ , and  $A \subseteq |M|$ . Suppose we know that  $(a_1, A, M)E_{\text{at}}(a_2, A, M)$ . Then because  $(a_\ell, A, N)E_{\text{at}}(a_\ell, A, M)$  for  $\ell =$ 1, 2, we have that  $(a_1, A, N)E(a_2, A, N)$ , but we may *not* have that  $(a_1, A, N)E_{\text{at}}(a_2, A, N)$ .

We also have the basic monotonicity and invariance properties [She09a, Observation II.1.11], which follow directly from the definition:

PROPOSITION 2.2.20. Let **K** be an abstract class. Let  $N \in \mathbf{K}$ ,  $A \subseteq |N|$ , and  $\bar{b} \in {}^{<\infty}|N|$ .

- (1) Invariance: If  $f: N \cong_A N'$ , then  $gtp(\bar{b}/A; N) = gtp(\underline{f}(\bar{b})/A; N')$ .
- (2) Monotonicity: If  $N \leq_{\mathbf{K}} N'$ , then  $\operatorname{gtp}(\overline{b}/A; N) = \operatorname{gtp}(\overline{b}/A; N')$ .

Monotonicity says that when  $N \leq_{\mathbf{K}} N'$ , the set of Galois types (over a fixed set A) realized in N' is at least as big as the set of Galois types over A realized in N (using the notation below,  $gS(A; N) \subseteq gS(A; N')$ ). When A = M for  $M \leq_{\mathbf{K}} N$ (or  $A = \emptyset$ ), we can further define the class gS(A) of all Galois types over A in the natural way. Assuming the existence of a monster model  $\mathfrak{C}$  containing A, this is the same as the usual definition: all types over A realized in  $\mathfrak{C}$ .

Definition 2.2.21.

(1) Let  $N \in \mathbf{K}$ ,  $A \subseteq |N|$ , and  $\alpha$  be an ordinal. Define:

$$gS^{\alpha}(A;N) := \{gtp(\bar{b}/A;N) \mid \bar{b} \in {}^{\alpha}|N|\}$$

(2) For  $M \in \mathbf{K}$  and  $\alpha$  an ordinal, let:

 $gS^{\alpha}(M) := \{ p \mid \exists N \in \mathbf{K} : M \leq_{\mathbf{K}} N \text{ and } p \in gS^{\alpha}(M; N) \}$ 

(3) For  $\alpha$  an ordinal, let:

$$\mathrm{gS}^\alpha(\emptyset) := \bigcup_{N \in \mathbf{K}} \mathrm{gS}^\alpha(\emptyset; N)$$

When  $\alpha = 1$ , we omit it. Similarly define  $gS^{<\alpha}$ , where  $\alpha$  is allowed to be  $\infty$ .

 $<sup>^{8}</sup>$ It is easy to check that this does not depend on the choice of representatives.

REMARK 2.2.22. When  $\alpha$  is an ordinal,  $gS^{\alpha}(M)$  and  $gS^{\alpha}(\emptyset)$  could a priori be proper classes. However in reasonable cases (e.g. when **K** is a  $\mu$ -AEC) they are sets. For example when **K** is a  $\mu$ -AEC, an upper bound for  $|gS^{\alpha}(M)|$  is  $2^{(||M||+\alpha+LS(\mathbf{K}))^{<\mu}}$ .

Next, we recall the definition of tameness), a locality property of types. Tameness was introduced by Grossberg and VanDieren in [**GV06b**] and used to get an upward stability transfer (and an upward categoricity transfer in [**GV06c**]). Later on, Boney showed in [**Bon14b**] that it followed from large cardinals and also introduced a dual property he called *type shortness*.

DEFINITION 2.2.23 (Definitions 3.1 and 3.3 in [Bon14b]). Let **K** be an abstract class and let  $\Gamma$  be a class (possibly proper) of Galois types in **K**. Let  $\kappa$  be an infinite cardinal.

- (1) **K** is  $(<\kappa)$ -tame for  $\Gamma$  if for any  $p \neq q$  in  $\Gamma$ , if  $A := \operatorname{dom} p = \operatorname{dom} q$ , then there exists  $A_0 \subseteq A$  such that  $|A_0| < \kappa$  and  $p \upharpoonright A_0 \neq q \upharpoonright A_0$ .
- (2) **K** is  $(<\kappa)$ -type short for  $\Gamma$  if for any  $p \neq q$  in  $\Gamma$ , if  $\alpha := \ell(p) = \ell(q)$ , then there exists  $I \subseteq \alpha$  such that  $|I| < \kappa$  and  $p^I \neq q^I$ .
- (3)  $\kappa$ -tame means ( $< \kappa^+$ )-tame, similarly for type short.
- (4) We usually just say "short" instead of "type short".
- (5) Usually,  $\Gamma$  will be a class of types over models only, and we often specify it in words. For example,  $(<\kappa)$ -short for types of length  $\alpha$  means  $(<\kappa)$ short for  $\bigcup_{M \in \mathbf{K}} \mathrm{gS}^{\alpha}(M)$ .
- (6) We say **K** is  $(< \kappa)$ -tame if it is  $(< \kappa)$ -tame for types of length one.
- (7) We say **K** is fully  $(< \kappa)$ -tame if it is  $(< \kappa)$ -tame for  $\bigcup_{M \in \mathbf{K}} \mathrm{gS}^{<\infty}(M)$ , similarly for short.

We review the natural notion of stability in this context. The definition here is slightly unusual compared to the rest of the litterature: we define what it means for a *model* to be stable in a given cardinal, and get a local notion of stability that is equivalent (in AECs) to the usual notion if amalgamation holds, but behaves better if amalgamation fails. Note that we count the number of types over an arbitrary set, not (as is common in AECs) only over models. In case the abstract class has a Löwenheim-Skolem number and we work above it this is equivalent, as any type in  $gS^{<\alpha}(A;N)$  can be extended<sup>9</sup> to  $gS^{<\alpha}(B;N)$  when  $A \subseteq B$ , so  $|gS^{<\alpha}(A;N)| \leq |gS^{<\alpha}(B;N)|$ .

DEFINITION 2.2.24 (Stability). Let **K** be an abstract class. Let  $\alpha$  be a cardinal,  $\mu$  be a cardinal. A model  $N \in \mathbf{K}$  is  $(< \alpha)$ -stable in  $\mu$  if for all  $A \subseteq |N|$  of size  $\leq \mu$ ,  $|gS^{<\alpha}(A;N)| \leq \mu$ . Here and below,  $\alpha$ -stable means  $(< (\alpha^+))$ -stable. We say "stable" instead of "1-stable".

**K** is  $(< \alpha)$ -stable in  $\mu$  if every  $N \in \mathbf{K}$  is  $(< \alpha)$ -stable in  $\mu$ . **K** is  $(< \alpha)$ -stable if it is  $(< \alpha)$ -stable in unboundedly many cardinals.

Define similarly syntactically stable for syntactic types (in this chapter the quantifier-free  $\mathbb{L}_{\kappa,\kappa}$ -types, where  $\kappa$  is clear from context).

The next fact spells out the connection between stability for types of different lengths and tameness.

FACT 2.2.25. Let **K** be an AEC and let  $\mu \geq LS(\mathbf{K})$ .

 $<sup>^{9}</sup>$ Note that this does not use any amalgamation because we work inside the same model N.

- (1) [Bon17, Theorem 3.1]: If **K** is stable in  $\mu$ ,  $\mathbf{K}_{\mu}$  has amalgamation, and  $\mu^{\alpha} = \mu$ , then **K** is  $\alpha$ -stable in  $\mu$ .
- (2) **[GV06b**, Corollary 6.4]<sup>10</sup>: If **K** has amalgamation, is  $\mu$ -tame, and stable in  $\mu$ , then **K** is stable in all  $\lambda$  such that  $\lambda^{\mu} = \lambda$ .
- (3) If **K** has amalgamation, is  $\mu$ -tame, and is stable in  $\mu$ , then **K** is  $\alpha$ -stable (in unboundedly many cardinals), for all cardinals  $\alpha$ .

PROOF OF (3). Given cardinals  $\lambda_0 \geq \text{LS}(\mathbf{K})$  and  $\alpha$ , let  $\lambda := (\lambda_0)^{\alpha+\mu}$ . Combining the first two statements gives us that  $\mathbf{K}$  is  $\alpha$ -stable in  $\lambda$ .

Finally, we review the natural definition of saturation using Galois types. Note that we again give the local definitions (but they are equivalent to the usual ones assuming amalgamation).

DEFINITION 2.2.26. Let **K** be an abstract class,  $M \in \mathbf{K}$  and  $\mu$  be an infinite cardinal.

- (1) For  $N \geq_{\mathbf{K}} M$ , M is  $\mu$ -saturated<sup>11</sup> in N if for any  $A \subseteq |M|$  of size less than  $\mu$ , any  $p \in \mathrm{gS}^{<\mu}(A; N)$  is realized in M.
- (2) M is  $\mu$ -saturated if it is  $\mu$ -saturated in N for all  $N \ge_{\mathbf{K}} M$ . When  $\mu = ||M||$ , we omit it.
- (3) We write  $\mathbf{K}^{\mu\text{-sat}}$  for the class of  $\mu$ -saturated models of  $\mathbf{K}_{\geq\mu}$  (ordered by the ordering of  $\mathbf{K}$ ).

Remark 2.2.27.

- (1) We defined saturation also when  $\mu \leq \text{LS}(\mathbf{K})$ . This is why we look at types over sets and not only over models. In an AEC, when  $\mu > \text{LS}(\mathbf{K})$ , this is equivalent to the usual definition (see also the remark before Definition 2.2.24).
- (2) We could similarly define what it means for a *set* to be saturated in a model (this is useful in Chapter 7).
- (3) It is easy to check that if **K** is an AEC with amalgamation and  $\mu > LS(\mathbf{K})$ , then  $\mathbf{K}^{\mu\text{-sat}}$  is a  $\mu_r$ -AEC (recall Definitions 2.2.1 and 2.2.14) with  $LS(\mathbf{K}^{\mu\text{-sat}}) \leq LS(\mathbf{K})^{<\mu_r}$ .

## 2.3. The semantic-syntactic correspondence

## 2.3.1. Functorial expansions and the Galois Morleyization.

DEFINITION 2.3.1.  $\widehat{K}$ —see functorial expansion Let **K** be an abstract class. A functorial expansion of **K** is a class  $\widehat{K}$  satisfying the following properties:

- (1)  $\hat{K}$  is a class of  $\hat{\tau}$ -structures, where  $\hat{\tau}$  is a fixed (possibly infinitary) vocabulary extending  $\tau(\mathbf{K})$ .
- (2) The map  $\widehat{M} \mapsto \widehat{M} \upharpoonright \tau(\mathbf{K})$  is a bijection from  $\widehat{K}$  onto  $\mathbf{K}$ . For  $M \in \mathbf{K}$ , we will write  $\widehat{M}$  for the unique element of  $\widehat{K}$  whose reduct is M. When we write " $\widehat{M} \in \widehat{K}$ ", it is understood that  $M = \widehat{M} \upharpoonright \tau(\mathbf{K})$ .
- (3) Invariance: For  $M, N \in \mathbf{K}$ , if  $f: M \cong N$ , then  $f: \widehat{M} \cong \widehat{N}$ .

 $<sup>^{10}</sup>$ The result we want can easily be seen to follow from the proof there: see [**Bal09**, Theorem 12.10].

 $<sup>^{11}</sup>$ Pedantically, we should really say "Galois-saturated" to differentiate this from being syntactically saturated. In this chapter, we will only discuss Galois saturation.

(4) Monotonicity: If  $M \leq_{\mathbf{K}} N$  are in  $\mathbf{K}$ , then  $\widehat{M} \subseteq \widehat{N}$ .

We say a functorial expansion  $\widehat{K}$  is  $(<\kappa)$ -ary if  $\tau(\widehat{K})$  is  $(<\kappa)$ -ary.

Example 2.3.2.

- (1) For **K** an abstract class, **K** is a functorial expansion of **K** itself. This is because  $\leq_{\mathbf{K}}$  must extend  $\subseteq$ .
- (2) Let **K** be an abstract class with  $\tau := \tau(\mathbf{K})$  and let  $\kappa$  be an infinite cardinal. Add a  $(<\kappa)$ -ary predicate P to  $\tau$ , forming a language  $\hat{\tau}$ . Expand each  $M \in \mathbf{K}$  to a  $\hat{L}$ -structure by defining  $P^{\widehat{M}}(\bar{a})$  (where  $P^{\widehat{M}}$  is the interpretation of P inside  $\widehat{M}$ ) to hold if and only if  $\bar{a}$  is the universe of a  $\leq_{\mathbf{K}}$ -submodel of M (this is more or less what Shelah does in [She09a, Definition IV.1.9.1]). Then the resulting class  $\hat{K}$  is a functorial expansion of  $\mathbf{K}$ .
- (3) Let T be a complete first-order theory in a vocabulary  $\tau$ . Let  $\mathbf{K} := (\operatorname{Mod}(T), \preceq)$ . It is common to expand  $\tau$  to  $\hat{\tau}$  by adding a relation symbol for every first-order  $\tau$ -formula. We then expand T (to  $\hat{T}$ ) and every model M of T in the expected way (to some  $\widehat{M}$ ) and obtain a new theory in which every formula is equivalent to an atomic one (this is commonly called the *Morleyization* of the theory). Then  $\widehat{K} := \operatorname{Mod}(\widehat{T})$  is a functorial expansion of  $\mathbf{K}$ .
- (4) Let T be a first-order complete theory. Expanding each model M of T to its canonical model  $M^{\text{eq}}$  of  $T^{\text{eq}}$  (see [She90, III.6]) also describes a functorial expansion.
- (5) The canonical structures of [CHL85] also induce a functorial expansion.

The main example of functorial expansion used in this chapter is the *Galois Morleyization*:

DEFINITION 2.3.3. Let **K** be an abstract class and let  $\kappa$  be an infinite cardinal. Define an expansion  $\hat{\tau}$  of  $\tau(\mathbf{K})$  by adding a relation symbol  $R_p$  of arity  $\ell(p)$  for each  $p \in \mathrm{gS}^{<\kappa}(\emptyset)$ . Expand each  $N \in \mathbf{K}$  to a  $\hat{\tau}$ -structure  $\hat{N}$  by specifying that for each  $\bar{a} \in {}^{<\kappa}|\hat{N}|, R_p^{\hat{N}}(\bar{a})$  (where  $R_p^{\hat{N}}$  is the interpretation of  $R_p$  inside  $\hat{N}$ ) holds exactly when  $\mathrm{gtp}(\bar{a}/\emptyset; N) = p$ . We call  $\hat{K}$  the  $(<\kappa)$ -Galois Morleyization of  $\mathbf{K}$ .

REMARK 2.3.4. Let **K** be an AEC and  $\kappa$  be an infinite cardinal. Let  $\widehat{K}$  be the (<  $\kappa$ )-Galois Morleyization of **K**. Then  $|\tau(\widehat{K})| \leq |gS^{<\kappa}(\emptyset)| + |\tau(\mathbf{K})| \leq 2^{<(\kappa + LS(\mathbf{K})^+)}$ .

It is straightforward to check that the Galois Morleyization is a functorial expansion. We include a proof here for completeness.

PROPOSITION 2.3.5. Let **K** be an abstract class and let  $\kappa$  be an infinite cardinal. Let  $\hat{K}$  be the  $(< \kappa)$ -Galois Morleyization of **K**. Then  $\hat{K}$  is a functorial expansion of **K**.

PROOF. Let  $\tau := \tau(\mathbf{K})$  be the vocabulary of  $\mathbf{K}$ . Looking at Definition 2.3.1, there are four properties to check:

- (1) By definition of the Galois Morleyization,  $\hat{K}$  is a class of  $\hat{\tau}$ -structure, for a fixed vocabulary  $\hat{\tau}$ .
- (2) The map  $M \mapsto M \upharpoonright \tau$  is a bijection: It is a surjection by definition of the Galois Morleyization. It is an injection: Assume that  $M' := \widehat{M} \upharpoonright \tau = \widehat{N} \upharpoonright$

 $\tau$  but  $\widehat{M} \neq \widehat{N}$ . Then there must exist a  $p \in \mathrm{gS}(\emptyset)$  and an  $\overline{a} \in {}^{<\kappa}|M'|$  such that  $\mathrm{gtp}(\overline{a}/\emptyset;M') = p$  but  $\mathrm{gtp}(\overline{a}/\emptyset;M') \neq p$ . Thus  $p \neq p$ , a contradiction.

- (3) Let  $M, N \in \mathbf{K}$  and  $f : M \cong N$ . We have to see that  $f : \widehat{M} \cong \widehat{N}$ . Let  $p \in \mathrm{gS}(\emptyset)$  and let  $\overline{a} \in {}^{<\kappa}|M|$ . Assume that  $\widehat{M} \models R_p(\overline{a})$ . Then by definition  $p = \mathrm{gtp}(\overline{a}/\emptyset; M)$ . Therefore by Proposition 2.2.20.(1),  $p = \mathrm{gtp}(f(\overline{a})/\emptyset; N)$ . Hence  $\widehat{N} \models R_p(f(\overline{a}))$ . The steps can be reversed to obtain the converse.
- (4) Let  $M \leq_{\mathbf{K}} N$  be in  $\mathbf{K}$ . We want to see that  $\widehat{M} \subseteq \widehat{N}$ . So let  $p \in \mathrm{gS}(\emptyset), \bar{a} \in {}^{<\kappa}|M|$ . Assume first that  $\widehat{M} \models R_p(\bar{a})$ . Then  $p = \mathrm{gtp}(\bar{a}/\emptyset; M)$ . Therefore by Proposition 2.2.20.(2),  $p = \mathrm{gtp}(\bar{a}/\emptyset; N)$ . Therefore  $\widehat{N} \models R_p(\bar{a})$ . The steps can be reversed to obtain the converse.

Note that a functorial expansion can naturally be seen as an abstract class:

DEFINITION 2.3.6. Let  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  be an abstract class and let  $\widehat{K}$  be a functorial expansion of  $\mathbf{K}$ . Define an ordering  $\leq_{\widehat{\mathbf{K}}}$  on  $\widehat{K}$  by  $\widehat{M} \leq_{\widehat{\mathbf{K}}} \widehat{N}$  if and only if  $M \leq_{\mathbf{K}} N$ . Let  $\widehat{\mathbf{K}}$  be the abstract class  $(\widehat{K}, \leq_{\widehat{\mathbf{K}}})$ .

The next propositions are easy but conceptually interesting.

PROPOSITION 2.3.7. Let  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  be an abstract class with  $\tau := \tau(\mathbf{K})$ . Let  $\widehat{\mathbf{K}}$  be a functorial expansion of  $\mathbf{K}$  and let  $\widehat{\tau} := \tau(\widehat{\mathbf{K}})$ .

- (1)  $\widehat{\mathbf{K}}$  is an abstract class.
- (2) If every chain in  $\mathbf{K}$  has an upper bound, then every chain in  $\mathbf{K}$  has an upper bound.
- (3) Galois types are the same in **K** and  $\widehat{\mathbf{K}}$ : gtp<sub>**K**</sub> $(\bar{a}_1/A; N_1) = \text{gtp}_{\mathbf{K}}(\bar{a}_2/A; N_2)$ if and only if gtp<sub> $\widehat{\mathbf{K}}$ </sub> $(\bar{a}_1/A; \widehat{N}_1) = \text{gtp}_{\widehat{\mathbf{K}}}(\bar{a}_2/A; \widehat{N}_2)$ .
- (4) Assume **K** is a  $\mu$ -AEC and  $\widehat{\mathbf{K}}$  is a  $(<\mu)$ -ary Morleyization of **K**. Then  $\widehat{\mathbf{K}}$  is a  $\mu$ -AEC with  $\mathrm{LS}(\widehat{\mathbf{K}}) = \mathrm{LS}(\mathbf{K}) + |\widehat{\tau}|^{<\mu}$ .
- (5) Let  $\tau \subseteq \hat{\tau}' \subseteq \hat{\tau}$ . Then  $\widehat{\mathbf{K}} \upharpoonright \hat{\tau}' := \{\widehat{M} \upharpoonright \hat{\tau}' \mid \widehat{M} \in \widehat{K}\}$  is a functorial expansion of  $\mathbf{K}$ .
- (6) If  $\widehat{\mathbf{K}}$  is a functorial expansion of  $\widehat{\mathbf{K}}$ , then  $\widehat{\mathbf{K}}$  is a functorial expansion of  $\mathbf{K}$ .

PROOF. All are straightforward. As an example, we show that if **K** is a  $\mu$ -AEC,  $\widehat{\mathbf{K}}$  is a  $(<\mu)$ -ary functorial expansion of **K**, and  $\langle \widehat{M}_i : i \in I \rangle$  is a  $\mu$ -directed system in  $\widehat{\mathbf{K}}$ , then letting  $M := \bigcup_{i \in I} M_i$ , we have that  $\bigcup_{i \in I} \widehat{M}_i = \widehat{M}$  (so in particular  $\bigcup_{i \in I} \widehat{M}_i \in \widehat{\mathbf{K}}$ ). Let R be a relation symbol in  $\widehat{\tau}$  of arity  $\alpha$ . Let  $\overline{a} \in {}^{\alpha}|\widehat{M}|$ . Assume  $\widehat{M} \models R[\overline{a}]$ . We show  $\bigcup_{i \in I} \widehat{M}_i \models R[\overline{a}]$ . The converse is done by replacing R by  $\neg R$ , and the proof with function symbols is similar. Since  $\widehat{\tau}$  is  $(<\mu)$ -ary,  $\alpha < \mu$ . Since Iis  $\mu$ -directed,  $\overline{a} \in {}^{\alpha}|M_j|$  for some  $j \in I$ . Since  $M_j \leq_{\mathbf{K}} M$ , the monotonicity axiom implies  $\widehat{M}_j \subseteq \widehat{M}$ . Thus  $\widehat{M}_j \models R[\overline{a}]$ , and this holds for all  $j' \geq j$ . Thus by definition of the union,  $\bigcup_{i \in I} \widehat{M}_i \models R[\overline{a}]$ .

REMARK 2.3.8. A word of warning: if **K** is an AEC and  $\hat{\mathbf{K}}$  is a functorial expansion of **K**, then **K** and  $\hat{\mathbf{K}}$  are isomorphic as categories. In particular, any directed system in  $\hat{\mathbf{K}}$  has a colimit. However, if  $\tau(\hat{\mathbf{K}})$  is not finitary the colimit of

a directed system in  ${\bf K}$  may not be the union: relations may need to contain more elements.

**2.3.2. Formulas and syntactic types.** From now on until the end of the section, we assume:

HYPOTHESIS 2.3.9. **K** is an abstract class with  $\tau := \tau(\mathbf{K})$ ,  $\kappa$  is an infinite cardinal,  $\hat{\mathbf{K}}$  is an arbitrary (<  $\kappa$ )-ary functorial expansion of **K** with vocabulary  $\hat{\tau} := \tau(\hat{\mathbf{K}})$ .

At the end of this section, we will specialize to the case when  $\widehat{\mathbf{K}}$  is the  $(\langle \kappa \rangle)$ -Galois Morleyization of  $\mathbf{K}$ . Recall from Section 2.2.2 that the set qf- $\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})$  denotes the quantifier-free  $L_{\kappa,\kappa}(\widehat{\tau})$  formulas.

PROPOSITION 2.3.10. Let  $\phi(\bar{x})$  be a quantifier-free  $\mathbb{L}_{\kappa,\kappa}(\hat{\tau})$  formula,  $M \in \mathbf{K}$ , and  $\bar{a} \in M$ . If  $f: M \to N$ , then  $\widehat{M} \models \phi[\bar{a}]$  if and only if  $\widehat{N} \models \phi[f(\bar{a})]$ .

Proof. Directly from the invariance and monotonicity properties of functorial expansions.  $\hfill \square$ 

In general, Galois types (computed in  $\mathbf{K}$ ) and syntactic types (computed in  $\hat{\mathbf{K}}$ ) are different. However, Galois types are always at least as fine as quantifier-free syntactic types (this is a direct consequence of Proposition 2.3.10 but we include a proof for completeness).

LEMMA 2.3.11. Let  $N_1, N_2 \in \mathbf{K}$ ,  $A \subseteq |N_\ell|$  for  $\ell = 1, 2$ . Let  $\bar{b}_\ell \in N_\ell$ . If  $\operatorname{gtp}(\bar{b}_1/A; N_1) = \operatorname{gtp}(\bar{b}_2/A; N_2)$ , then  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\bar{b}_1/A; \widehat{N}_1) = \operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\bar{b}_2/A; \widehat{N}_2)$ .

PROOF. By transitivity of equality, it is enough to show that if  $(\bar{b}_1, A, N_1) E_{\mathrm{at}}(\bar{b}_2, A, N_2)$ , then  $\mathrm{tp}_{\mathrm{qf}-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\bar{b}_1/A; \widehat{N}_1) = \mathrm{tp}_{\mathrm{qf}-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\bar{b}_2/A; \widehat{N}_2)$ . So assume  $(\bar{b}_1, A, N_1)E_{\mathrm{at}}(\bar{b}_2, A, N_2)$ . Then there exists  $N \in \mathbf{K}$  and  $f_{\ell} : N_{\ell} \xrightarrow{A} N$  such that  $f_1(\bar{b}_1) = f_2(\bar{b}_2)$ . Let  $\phi(\overline{x})$ be a quantifier-free  $\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})$  formula over A. Assume  $\widehat{N}_1 \models \phi[\bar{b}_1]$ . By Proposition 2.3.10,  $\widehat{N} \models \phi[f_1(\bar{b}_1)]$ , so  $\widehat{N} \models \phi[f_2(\bar{b}_2)]$ , so by Proposition 2.3.10 again,  $\widehat{N}_2 \models \phi[\bar{b}_2]$ . Replacing  $\phi$  by  $\neg \phi$ , we get the converse, so  $\mathrm{tp}_{\mathrm{qf}-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\bar{b}_1/A; \widehat{N}_1) =$  $\mathrm{tp}_{\mathrm{qf}-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\bar{b}_2/A; \widehat{N}_2)$ .

Note that this used that the types were quantifier-free. We have justified the following definition:

DEFINITION 2.3.12. For a Galois type p, let  $p^s$  be the corresponding quantifierfree syntactic type in the functorial expansion. That is, if  $p = \text{gtp}(\bar{b}/A; N)$ , then  $p^s := \text{tp}_{\text{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\bar{b}/A; \widehat{N}).$ 

PROPOSITION 2.3.13. Let  $N \in \mathbf{K}$ ,  $A \subseteq |N|$ . Let  $\alpha$  be an ordinal. The map  $p \mapsto p^s$  from  $gS^{\alpha}(A; N)$  to  $S^{\alpha}_{qf-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(A; \widehat{N})$  (recall Definition 2.2.6) is a surjection.

PROOF. If  $\operatorname{tp}_{qf-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{b}/A;\widehat{N}) = q \in \operatorname{S}_{qf-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}^{\alpha}(A;\widehat{N})$ , then by definition  $\left(\operatorname{gtp}(\overline{b}/A;N)\right)^s = q$ .

REMARK 2.3.14. To investigate formulas with quantifiers, we could define a different version of Galois types using isomorphisms rather than embeddings, and remove the monotonicity axiom from the definition of a functorial expansion. As we have no use for it here, we do not discuss this approach further.

**2.3.3.** On when Galois types are syntactic. We have seen in Proposition 2.3.13 that  $p \mapsto p^s$  is a surjection, so Galois types are always at least as fine as quantifier-free syntactic type in the expansion. It is natural to ask when they are the same, i.e. when  $p \mapsto p^s$  is a *bijection*. When  $\hat{\mathbf{K}}$  is the  $(< \kappa)$ -Galois Morleyization of  $\mathbf{K}$  (see Definition 2.3.3), we answer this using shortness and tameness (Definition 2.2.23). Note that we make no hypothesis on  $\mathbf{K}$ . In particular, amalgamation is not needed.

THEOREM 2.3.15 (The semantic-syntactic correspondence). Assume  $\hat{\mathbf{K}}$  is the  $(<\kappa)$ -Galois Morleyization of  $\mathbf{K}$ .

Let  $\Gamma$  be a family of Galois types. The following are equivalent:

- (1) **K** is  $(< \kappa)$ -tame and short for  $\Gamma$ .
- (2) The map  $p \mapsto p^s$  is a bijection from  $\Gamma$  onto  $\Gamma^s := \{p^s \mid p \in \Gamma\}.$

Proof.

- (1) implies (2): By Lemma 2.3.11, the map  $p \mapsto p^s$  with domain  $\Gamma$  is well-defined and it is clearly a surjection onto  $\Gamma^s$ . It remains to see it is injective. Let  $p, q \in \Gamma$  be distinct. If they do not have the same domain or the same length, then  $p^s \neq q^s$ , so assume that  $A := \operatorname{dom} p = \operatorname{dom} q$ and  $\alpha := \ell(p) = \ell(q)$ . Say  $p = \operatorname{gtp}(\bar{b}/A; N), q = \operatorname{gtp}(\bar{b}'/A; N')$ . By the tameness and shortness hypotheses, there exists  $A_0 \subseteq A$  and  $I \subseteq$  $\alpha$  of size less than  $\kappa$  such that  $p_0 := p^I \upharpoonright A_0 \neq q^I \upharpoonright A_0 =: q_0$ . Let  $\bar{a}_0$  be an enumeration of  $A_0$ , and let  $\bar{b}_0 := \bar{b} \upharpoonright I$ ,  $\bar{b}'_0 := \bar{b}' \upharpoonright I$ . Let  $p'_0 := \operatorname{gtp}(\bar{b}_0\bar{a}_0/\emptyset; N)$ , and let  $\phi := R_{p'_0}(\bar{x}_0, \bar{a}_0)$ , where  $\bar{x}_0$  is a sequence of variables of type I. Since  $\bar{b}_0$  realizes  $p_0$  in N,  $\hat{N} \models \phi[\bar{b}_0]$ , and since  $\bar{b}'_0$ realizes  $q_0$  in N' and  $q_0 \neq p_0$ ,  $\hat{N'} \models \neg \phi[\bar{b}'_0]$ . Thus  $\phi(\bar{x}_0) \in p^s$ ,  $\neg \phi(\bar{x}_0) \in q^s$ . By definition,  $\phi(\bar{x}_0) \notin q$  so  $p^s \neq q^s$ .
- (2) implies (1): Let  $p, q \in \Gamma$  be distinct with domain A and length  $\alpha$ . Say  $p = \operatorname{gtp}(\overline{b}/A; N), q = \operatorname{gtp}(\overline{b}'/A; N')$ . By hypothesis,  $p^s \neq q^s$  so there exists  $\phi(\overline{x})$  over A such that (without loss of generality)  $\phi(\overline{x}) \in p$  but  $\neg \phi(\overline{x}) \in q$ . Let  $A_0 := \operatorname{dom} \phi, \overline{x}_0 := \operatorname{FV}(\phi)$  (note that  $A_0$  and  $\overline{x}_0$  have size strictly less than  $\kappa$ ). Let  $\overline{b}_0, \overline{b}'_0$  be the corresponding subsequences of  $\overline{b}$  and  $\overline{b}'$  respectively. Let  $p_0 := \operatorname{gtp}(\overline{b}_0/A_0; N), q_0 := \operatorname{gtp}(\overline{b}'_0/A_0; N')$ . Then it is straightforward to check that  $\phi \in p_0^s, \neg \phi \in q_0^s$ , so  $p_0^s \neq q_0^s$  and hence (by Lemma 2.3.11)  $p_0 \neq q_0$ . Thus  $A_0$  and I witness tameness and shortness respectively.

REMARK 2.3.16. The proof shows that (2) implies (1) is valid when  $\mathbf{K}$  is any functorial expansion of  $\mathbf{K}$ .

COROLLARY 2.3.17. Assume  $\widehat{\mathbf{K}}$  is the  $(<\kappa)$ -Galois Morleyization of  $\mathbf{K}$ .

- (1) **K** is fully  $(< \kappa)$ -tame and short if and only if for any  $M \in \mathbf{K}$  the map  $p \mapsto p^s$  from  $gS^{<\infty}(M)$  to  $S_{\mathrm{qf}-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}^{<\infty}(M)$  is a bijection<sup>12</sup>.
- (2) **K** is  $(< \kappa)$ -tame if and only if for any  $M \in \mathbf{K}$  the map  $p \mapsto p^s$  from gS(M) to  $S_{qf-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(M)$  is a bijection.

PROOF. By Theorem 2.3.15 applied to  $\Gamma := \bigcup_{M \in \mathbf{K}} \mathrm{gS}^{<\infty}(M)$  and  $\Gamma := \bigcup_{M \in \mathbf{K}} \mathrm{gS}(M)$  respectively.

REMARK 2.3.18. For  $M \in \mathbf{K}$ ,  $p, q \in \mathrm{gS}(M)$ , say  $pE_{<\kappa}q$  if and only if  $p \upharpoonright A_0 = q \upharpoonright A_0$  for all  $A_0 \subseteq |M|$  with  $|A_0| < \kappa$ . Of course, if  $\mathbf{K}$  is  $(<\kappa)$ -tame, then  $E_{<\kappa}$  is just equality. More generally, the proof of Theorem 2.3.15 shows that if  $\widehat{\mathbf{K}}$  is the  $(<\kappa)$ -Galois Morleyization of  $\mathbf{K}$ , then  $pE_{<\kappa}q$  if and only if  $p^s = q^s$ . Thus in that case quantifier-free syntactic types in the Morleyization can be seen as  $E_{<\kappa}$ -equivalence classes of Galois types. Note that  $E_{<\kappa}$  appears in the work of Shelah, see for example [She99, Definition 1.8].

# 2.4. Order properties and stability spectrum

In this section, we start applying the semantic-syntactic correspondence (Theorem 2.3.15) to prove new structural results about AECs. In the introduction, we described a three-step general method to prove a result about AECs using syntactic methods. In the proof of Theorem 2.4.15, Corollary 2.4.13 gives the first step, Theorem 2.3.15 gives the second, while Facts 2.4.7 (AECs have a Hanf number for the order property) and 2.2.25 (In tame AECs with amalgamation, stability behaves reasonably well) are keys for the third step.

Throughout this section, we work with the (<  $\kappa$ )-Galois Morleyization of a fixed AEC **K**:

Hypothesis 2.4.1.

- (1)  $\mathbf{K}$  is an abstract elementary class.
- (2)  $\kappa$  is an infinite cardinal.
- (3)  $\mathbf{K}$  is the  $(<\kappa)$ -Galois Morleyization of  $\mathbf{K}$  (recall Definition 2.3.3). Set  $\widehat{\tau} := \tau(\widehat{\mathbf{K}})$ .

**2.4.1. Several order properties.** The next definition is a natural syntactic extension of the first-order order property. A related definition appears already in **[She72]** and has been well studied (see for example **[GS86b, GS**]).

DEFINITION 2.4.2 (Syntactic order property). Let  $\alpha$  and  $\mu$  be cardinals with  $\alpha < \kappa$ . A model  $\widehat{M} \in \widehat{\mathbf{K}}$  has the syntactic  $\alpha$ -order property of length  $\mu$  if there exists  $\langle \bar{a}_i : i < \mu \rangle$  inside  $\widehat{M}$  with  $\ell(\bar{a}_i) = \alpha$  for all  $i < \mu$  and a quantifier-free  $\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})$ -formula  $\phi(\bar{x},\bar{y})$  such that for all  $i, j < \mu$ ,  $\widehat{M} \models \phi[\bar{a}_i, \bar{a}_j]$  if and only if i < j.

Let  $\beta \leq \kappa$  be a cardinal.  $\widehat{M}$  has the syntactic  $(\langle \beta \rangle)$ -order property of length  $\mu$  if it has the syntactic  $\alpha$ -order property of length  $\mu$  for some  $\alpha < \beta$ .  $\widehat{M}$  has the syntactic order property of length  $\mu$  if it has the syntactic  $(\langle \kappa \rangle)$ -order property of length  $\mu$ .

 $\widehat{\mathbf{K}}$  has the syntactic  $\alpha$ -order of length  $\mu$  if some  $\widehat{M} \in \widehat{\mathbf{K}}$  has it.  $\widehat{\mathbf{K}}$  has the syntactic order property if it has the syntactic order property for every length.

We emphasize that the syntactic order property is always considered inside the Galois Morleyization  $\hat{\mathbf{K}}$  and must be witnessed by a *quantifier-free* formula. Also, since any such formula has fewer than  $\kappa$  free variables, nothing would be gained by defining the  $(\alpha)$ -syntactic order property for  $\alpha \geq \kappa$ . Thus we talk of the syntactic order property instead of the  $(< \kappa)$ -syntactic order property.

Arguably the most natural semantic definition of the order property in AECs appears in [She99, Definition 4.3].

DEFINITION 2.4.3. Let  $\alpha$ ,  $\kappa$ , and  $\mu$  be cardinals. A model  $M \in \mathbf{K}$  has the Galois  $(\alpha, \kappa)$ -order property of length  $\mu$  if there exists  $\langle \bar{a}_i : i < \mu \rangle$  inside M with  $\ell(\bar{a}_i) = \alpha$  for all  $i < \mu$  and a set  $A \subseteq |N|$  with  $|A| \leq \kappa$  such that for any  $i_0 < j_0 < \mu$  and  $i_1 < j_1 < \mu$ ,  $\operatorname{gtp}(\bar{a}_{i_0}\bar{a}_{j_0}/A; N) \neq \operatorname{gtp}(\bar{a}_{j_1}\bar{a}_{j_1}/A; N)$ .

When  $\kappa = 0$  (as will be the case in this chapter), we drop it. We usually drop the "Galois" and define variations such as "**K** has the  $\alpha$ -order property" as in Definition 2.4.2.

REMARK 2.4.4. If M has the  $(\alpha, \kappa)$ -order property of length  $\mu$ , then it has the  $(\alpha + \kappa)$ -order property of length  $\mu$ .

REMARK 2.4.5. For T a first-order theory and **K** its corresponding AEC of models, the following are equivalent:

- (1) T is unstable.
- (2) **K** has the  $(\alpha, 0)$ -order property, for some  $\alpha < \aleph_0$ .
- (3) **K** has the  $(\alpha, \kappa)$ -order property, for some cardinals  $\alpha$  and  $\kappa$ .

Notice that the definition of the Galois  $\alpha$ -order property is more general than that of the syntactic  $\alpha$ -order property, since  $\alpha$  is not required to be less than  $\kappa$ . However the next result shows that the two properties are equivalent when  $\alpha < \kappa$ . Notice that this does not use any tameness.

PROPOSITION 2.4.6. Let  $\alpha$ ,  $\mu$ , and  $\lambda$  be cardinals with  $\alpha < \kappa$ . Let  $N \in \mathbf{K}$ .

- (1) If  $\widehat{N}$  has the syntactic  $\alpha$ -order property of length  $\mu$ , then N has the  $\alpha$ -order property of length  $\mu$ .
- (2) Conversely, let  $\chi := |gS^{\alpha+\alpha}(\emptyset)|$ , and assume that  $\mu \ge (2^{\lambda+\chi})^+$ . If N has the  $\alpha$ -order property of length  $\mu$ , then  $\widehat{N}$  has the syntactic  $\alpha$ -order property of length  $\lambda$ .

In particular, **K** has the  $\alpha$ -order property if and only if  $\widehat{\mathbf{K}}$  has the syntactic  $\alpha$ -order property.

Proof.

- (1) This is a straightforward consequence of Proposition  $2.3.10^{13}$ .
- (2) Let  $\langle \bar{a}_i : i < \mu \rangle$  witness that N has the Galois  $\alpha$ -order property of length  $\mu$ . By the Erdős-Rado theorem used on the coloring  $(i < j) \mapsto$  $\operatorname{gtp}(\bar{a}_i \bar{a}_j / \emptyset; N)$ , we get that (without loss of generality),  $\langle \bar{a}_i : i < \lambda \rangle$  is such that whenever i < j,  $\operatorname{gtp}(\bar{a}_i \bar{a}_j / \emptyset; N) = p \in \operatorname{gS}^{\alpha + \alpha}(\emptyset)$ . But then (since by assumption  $\operatorname{gtp}(\bar{a}_i \bar{a}_j / \emptyset; N) \neq \operatorname{gtp}(\bar{a}_j \bar{a}_i / \emptyset; N)$ ),  $\phi(\bar{x}, \bar{y}) := R_p(\bar{x}, \bar{y})$  witnesses that  $\hat{N}$  has the syntactic  $\alpha$ -order property of length  $\lambda$ .

We will see later (Theorem 2.4.15) that assuming some tameness, even when  $\alpha \geq \kappa$ , the  $\alpha$ -order property implies the syntactic order property.

In the next section, we heavily use the assumption of no syntactic order property of length  $\kappa$ . We now look at how that assumption compares to the order property (of arbitrary long length). Note that Proposition 2.4.6 already tells us that the  $(< \kappa)$ -order property implies the syntactic order property of length  $\kappa$ . To get an equivalence, we will assume  $\kappa$  is a fixed point of the Beth function. The key is:

<sup>&</sup>lt;sup>13</sup>We are using that everything in sight is quantifier-free. Note that this part works for any functorial expansion  $\hat{\mathbf{K}}$  of  $\mathbf{K}$ .

FACT 2.4.7. Let  $\alpha$  be a cardinal. If **K** has the  $\alpha$ -order property of length  $\mu$  for all  $\mu < h(\alpha + LS(\mathbf{K}))$ , then **K** has the  $\alpha$ -order property.

PROOF. By the same proof as [She99, Claim 4.5.3].

COROLLARY 2.4.8. Assume  $\beth_{\kappa} = \kappa > LS(\mathbf{K})$ . Then  $\widehat{\mathbf{K}}$  has the syntactic order property of length  $\kappa$  if and only if  $\mathbf{K}$  has the ( $\langle \kappa \rangle$ )-order property.

PROOF. If  $\mathbf{\hat{K}}$  has the syntactic order property of length  $\kappa$ , then for some  $\alpha < \kappa$ ,  $\mathbf{\hat{K}}$  has the syntactic  $\alpha$ -order property of length  $\kappa$ , and thus by Proposition 2.4.6 the  $\alpha$ -order property of length  $\kappa$ . Since  $\kappa = \beth_{\kappa}$ ,  $h(|\alpha| + \text{LS}(\mathbf{K})) < \kappa$ , so by Fact 2.4.7,  $\mathbf{K}$  has the  $\alpha$ -order property.

Conversely, if **K** has the  $(< \kappa)$ -order property, Proposition 2.4.6 implies that  $\hat{\mathbf{K}}$  has the syntactic order property, so in particular the syntactic order property of length  $\kappa$ .

For completeness, we also discuss the following semantic variation of the syntactic order property of length  $\kappa$  that appears in [**BG**, Definition 4.2] (but is adapted from a previous definition of Shelah, see there for more background):

DEFINITION 2.4.9. For  $\kappa > \text{LS}(\mathbf{K})$ , **K** has the weak  $\kappa$ -order property if there are  $M \in \mathbf{K}_{<\kappa}$ ,  $N \geq_{\mathbf{K}} M$ , types  $p \neq q \in \text{gS}^{<\kappa}(M)$ , and sequences  $\langle \bar{a}_i : i < \kappa \rangle$ ,  $\langle \bar{b}_i : i < \kappa \rangle$  from N so that for all  $i, j < \kappa$ :

- (1)  $i \leq j$  implies  $gtp(\bar{a}_i \bar{b}_j / M; N) = p$ .
- (2) i > j implies  $gtp(\bar{a}_i \bar{b}_j / M; N) = q$ .

LEMMA 2.4.10. Let  $\kappa > LS(\mathbf{K})$ .

- (1) If **K** has the  $(< \kappa)$ -order property, then **K** has the weak  $\kappa$ -order property.
- (2) If **K** has the weak  $\kappa$ -order property, then **K** has the syntactic order property of length  $\kappa$ .

In particular, if  $\kappa = \beth_{\kappa}$ , then the weak  $\kappa$ -order property, the  $(< \kappa)$ -order property of length  $\kappa$ , and the  $(< \kappa)$ -order property are equivalent.

### Proof.

(1) Assume **K** has the  $(<\kappa)$ -order property. To see the weak order property, let  $\alpha < \kappa$  be such that **K** has the  $\alpha$ -order property. Fix an  $N \in \mathbf{K}$  such that N has a long-enough  $\alpha$ -order property. Pick any  $M \in \mathbf{K}_{<\kappa}$  with  $M \leq_{\mathbf{K}} N$ . By using the Erdős-Rado theorem twice, we can assume we are given  $\langle \bar{c}_i : i < \kappa \rangle$  such that whenever  $i < j < \kappa$ ,  $\operatorname{gtp}(\bar{c}_i \bar{c}_j / M; N) = p$ , and  $\operatorname{gtp}(\bar{c}_j \bar{c}_i / M; N) = q$ , for some  $p \neq q \in \operatorname{gS}(M)$ .

For  $l < \kappa$ , let  $j_l := 2l$ , and  $k_l := 2l + 1$ . Then  $j_l, k_l < \kappa$ , and  $l \le l'$ implies  $j_l < k_{l'}$ , whereas l > l' implies  $j_l > k_{l'}$ . Thus the sequences defined by  $\bar{a}_l := \bar{c}_{j_l}, \bar{b}_l := \bar{c}_{k_l}$  are as required.

(2) Assume **K** has the weak  $\kappa$ -order property and let  $M, N, p, q, \langle \bar{a}_i : i < \kappa \rangle$ ,  $\langle \bar{b}_i : i < \kappa \rangle$  witness it. For  $i < \kappa$ , Let  $\bar{c}_i := \bar{a}_i \bar{b}_i$  and  $\phi(\bar{x}_1 \bar{x}_2; \bar{y}_1 \bar{y}_2) := R_p(\bar{y}_1, \bar{x}_2)$ . This witnesses the syntactic order property of length  $\kappa$  in  $\hat{\mathbf{K}}$ .

The last sentence follows from Proposition 2.4.6 and Corollary 2.4.8.

**2.4.2.** Order property and stability. We now want to relate stability in terms of the number of types (see Definition 2.2.24) to the order property and use this to find many stability cardinals.

Note that stability in  $\mathbf{K}$  (in terms of Galois types, see Definition 2.2.24) coincides with syntactic stability in  $\hat{\mathbf{K}}$  given enough tameness and shortness (see Theorem 2.3.15). In general, they could be different, but by Proposition 2.3.10, stability always implies syntactic stability (and so syntactic unstability implies unstability). This contrasts with the situation with the order properties, where the syntactic and regular order property are equivalent without tameness (see Proposition 2.4.6).

The basic relationship between the order property and stability is given by:

FACT 2.4.11. If **K** has the  $\alpha$ -order property and  $\mu \geq |\alpha| + \text{LS}(\mathbf{K})$ , then **K** is not  $\alpha$ -stable in  $\mu$ . If in addition  $\alpha < \kappa$ , then  $\widehat{\mathbf{K}}$  is not even syntactically  $\alpha$ -stable in  $\mu$ .

PROOF. [She99, Claim 4.8.2] is the first sentence. The proof (see Fact 3.5.12) generalizes (using the syntactic order property) to get the second sentence.  $\Box$ 

This shows that the order property implies unstability and we now work towards a syntactic converse. The key is [She09b, Theorem V.A.1.19], which shows that if a model does not have the (syntactic) order property of a certain length, then it is (syntactically) stable in certain cardinals. Here, syntactic refers to Shelah's very general context, where any subset  $\Delta$  of formulas from any abstract logic is allowed. Shelah assumes that the vocabulary is finitary but the proof goes through just as well with an infinitary vocabulary (the proof only deals with formulas, which *are* allowed to be infinitary). Thus specializing the result to the context of this chapter (working with the logic  $\mathbb{L}_{\kappa,\kappa}(\hat{\tau})$  and  $\Delta = qf-\mathbb{L}_{\kappa,\kappa}(\hat{\tau})$ ), we obtain:

FACT 2.4.12. Let  $\widehat{N} \in \widehat{\mathbf{K}}$ . Let  $\alpha < \kappa$ . Let  $\chi \ge (|\widehat{\tau}| + 2)^{<\kappa}$  be a cardinal. If  $\widehat{N}$  does not have the syntactic order property of length  $\chi^+$ , then whenever  $\lambda = \lambda^{\chi} + \beth_2(\chi)$ ,  $\widehat{N}$  is (syntactically) (<  $\kappa$ )-stable in  $\lambda$ .

The next corollary does not need any amalgamation or tameness. Intuitively, this is because every property involved ends up being checked inside a single model (for example,  $\hat{\mathbf{K}}$  syntactically stable in some cardinal means that all of its *models* are syntactically stable in the cardinal).

COROLLARY 2.4.13. The following are equivalent:

- (1) For every  $\kappa_0 < \kappa$  and every  $\alpha < \kappa$ ,  $\widehat{\mathbf{K}}$  is syntactically  $\alpha$ -stable in *some* cardinal greater than or equal to  $\mathrm{LS}(\mathbf{K}) + \kappa_0$ .
- (2) **K** does not have the  $(< \kappa)$ -order property.
- (3) There exist<sup>14</sup> cardinals  $\mu$  and  $\lambda_0$  with  $\mu \leq \lambda_0 < h^*(\kappa + \mathrm{LS}(\mathbf{K})^+)$  (recall Definition 2.2.2) such that  $\widehat{\mathbf{K}}$  is syntactically ( $< \kappa$ )-stable in any  $\lambda \geq \lambda_0$  with  $\lambda^{<\mu} = \lambda$ . In particular,  $\widehat{\mathbf{K}}$  is syntactically ( $< \kappa$ )-stable.

PROOF. (3) says in particular that  $\widehat{\mathbf{K}}$  is syntactically  $(\langle \kappa \rangle)$ -stable in a proper class of cardinals, so it clearly implies (1). (1) implies (2): We prove the contrapositive. Assume that  $\mathbf{K}$  has the  $(\langle \kappa \rangle)$ -order property. In particular,  $\mathbf{K}$  has the  $(\langle \kappa \rangle)$ -order property of length  $h(\kappa + \mathrm{LS}(\mathbf{K}))$ . By definition, this means that for

<sup>&</sup>lt;sup>14</sup>The cardinal  $\mu$  is closely related to the local character cardinal  $\bar{\kappa}$  for nonsplitting. See for example [**GV06b**, Theorem 4.13].

some  $\alpha < \kappa$ , **K** has the  $\alpha$ -order property of length  $h(\kappa + \text{LS}(\mathbf{K}))$ . By Fact 2.4.7, **K** has the  $\alpha$ -order property. By Fact 2.4.11,  $\widehat{\mathbf{K}}$  is not syntactically  $\alpha$ -stable in any cardinal above  $\text{LS}(\mathbf{K}) + |\alpha|$  (that is, for each  $\lambda \geq \text{LS}(\mathbf{K}) + |\alpha|$ , there is  $\widehat{N} \in \widehat{\mathbf{K}}$  such that  $\widehat{N}$  is not syntactically  $\alpha$ -stable in  $\lambda$ ). Thus taking  $\kappa_0 := |\alpha|$ , we get that (1) fails.

Finally (2) implies (3). Assume **K** does not have the  $(<\kappa)$ -order property. By the contrapositive of Fact 2.4.7, for each  $\alpha < \kappa$ , there exists  $\mu_{\alpha} < h(|\alpha| + \mathrm{LS}(\mathbf{K})) \le$  $h^*(\kappa + \mathrm{LS}(\mathbf{K})^+)$  such that **K** does not have the  $\alpha$ -order property of length  $\mu_{\alpha}$ . Since  $2^{<(\kappa+\mathrm{LS}(\mathbf{K})^+)} < h^*(\kappa + \mathrm{LS}(\mathbf{K})^+)$ , we can without loss of generality assume that  $2^{<(\kappa+\mathrm{LS}(\mathbf{K})^+)} \le \mu_{\alpha}$  for all  $\alpha < \kappa$ . Let  $\chi := \sup_{\alpha < \kappa} \mu_{\alpha}$ . Then **K** does not have the  $(<\kappa)$ -order property of length  $\chi$ . Now if  $\kappa$  is a successor (say  $\kappa = \kappa_0^+$ ), then  $\chi = \mu_{\kappa_0} < h(\kappa_0) \le h^*(\kappa + \mathrm{LS}(\mathbf{K})^+)$ . Otherwise  $h^*(\kappa + \mathrm{LS}(\mathbf{K})^+) = h(\kappa + \mathrm{LS}(\mathbf{K}))$ and cf  $h(\kappa + \mathrm{LS}(\mathbf{K})) = (2^{\kappa+\mathrm{LS}(\mathbf{K})})^+ > \kappa$ , so  $\chi < h^*(\kappa + \mathrm{LS}(\mathbf{K})^+)$ . Let  $\mu := \chi^+$ and  $\lambda_0 := \beth_2(\chi)$ . It is easy to check that  $\mu \le \lambda_0 < h^*(\kappa + \mathrm{LS}(\mathbf{K})^+)$ . Finally, note that by Remark 2.3.4,  $|\hat{\tau}| \le 2^{<(\kappa+\mathrm{LS}(\mathbf{K})^+)}$ , so  $\chi \ge (|\hat{\tau}| + 2)^{<\kappa}$ . Now apply Fact 2.4.12 to each  $\hat{N} \in \hat{\mathbf{K}}$  (note that by definition of  $\lambda_0$ , if  $\lambda = \lambda^{\chi} \ge \lambda_0$ , then  $\lambda = \lambda^{\chi} + \beth_2(\chi)$ ).

REMARK 2.4.14. Shelah [Shec, Theorem 3.3] claims (without proof) a version of (1) implies (3).

Assuming  $(< \kappa)$ -tameness for types of length less than  $\kappa$ , we can of course convert the above result to a statement about Galois types. To replace " $(< \kappa)$ -stable" by just "stable" (recall that this means stable for types of length one) and also get away with only tameness for types of length one, we will use amalgamation together with Fact 2.2.25.

THEOREM 2.4.15. Assume **K** has amalgamation and is  $(< \kappa)$ -tame. The following are equivalent:

- (1) **K** is stable in *some* cardinal greater than or equal to  $LS(\mathbf{K}) + \kappa^{-}$  (recall Definition 2.2.1).
- (2)  $\mathbf{K}$  does not have the order property.
- (3) **K** does not have the  $(< \kappa)$ -order property.
- (4) There exist cardinals  $\mu$  and  $\lambda_0$  with  $\mu \leq \lambda_0 < h^*(\kappa + \mathrm{LS}(\mathbf{K})^+)$  (recall Definition 2.2.2) such that  $\mathbf{K}$  is stable in any  $\lambda \geq \lambda_0$  with  $\lambda^{<\mu} = \lambda$ .

In particular, **K** is stable if and only if **K** does not have the order property.

PROOF. Clearly, (4) implies (1) and (2) implies (3). (1) implies (2): If **K** has the  $\alpha$ -order property, then by Fact 2.4.11 it cannot be  $\alpha$ -stable in any cardinal above  $LS(\mathbf{K}) + |\alpha|$ . By Fact 2.2.25.(3), **K** is not stable in any cardinal greater than or equal to  $\kappa^- + LS(\mathbf{K})$ , so (1) fails. Finally, (3) implies (4) by combining Corollary 2.4.13 and Corollary 2.3.17.

PROOF OF THEOREM 2.0.2. Set  $\kappa := \mathrm{LS}(\mathbf{K})^+$  in Theorem 2.4.15. Note that in that case  $\kappa^- = \mathrm{LS}(\mathbf{K})$  (Definition 2.2.1) and  $h^*(\kappa + \mathrm{LS}(\mathbf{K})^+) = h^*(\mathrm{LS}(\mathbf{K})^+) =$  $h(\mathrm{LS}(\mathbf{K}))$  by Definition 2.2.2.

## 2.5. Coheir

We look at the natural generalization of coheir (introduced in [LP79] for firstorder logic) to the context of this chapter. A definition of coheir for classes of models of an  $\mathbb{L}_{\kappa,\omega}$  theory was first introduced in [**MS90**] and later adapted to general AECs in [**BG**]. We give a slightly more conceptual definition here and show that coheir has several of the basic properties of forking in a stable first-order theory. This improves on [**BG**] which assumed that coheir had the extension property.

Hypothesis 2.5.1.

- (1)  $\mathbf{K}^0$  is an AEC with amalgamation.
- (2)  $\kappa > \text{LS}(\mathbf{K}^0)$  is a fixed cardinal.
- (3)  $\mathbf{K} := (\mathbf{K}^0)^{\kappa\text{-sat}}$  is the class of  $\kappa\text{-saturated}$  models of  $\mathbf{K}^0$ .
- (4)  $\widehat{\mathbf{K}}$  is the  $(\langle \kappa \rangle)$ -Galois Morleyization of  $\mathbf{K}$ . Set  $\widehat{\tau} := \tau(\widehat{\mathbf{K}})$ .

The reader can see  $\widehat{\mathbf{K}}$  as the class in which coheir is computed syntactically, while  $\mathbf{K}$  is the class in which it is used semantically.

For the sake of generality, we do not assume stability or tameness yet. We will do so in parts (2) and (3) of Theorem 2.5.15, the main theorem of this section. After the proof of Theorem 2.5.15, we give a proof of Theorem 2.0.3 in the abstract.

Note that by Remark 2.2.27, **K** is a  $\kappa_r$ -AEC (see Definition 2.2.14). Moreover by saturation the ordering has some elementarity. More precisely, let  $\Sigma_1(\mathbb{L}_{\kappa,\kappa}(\hat{\tau}))$ denote the set of  $\mathbb{L}_{\kappa,\kappa}(\hat{\tau})$ -formulas of the form  $\exists \bar{x}\psi(\bar{x};\bar{y})$ , where  $\psi$  is quantifier-free. We then have:

PROPOSITION 2.5.2. If  $M, N \in \mathbf{K}$  and  $M \leq_{\mathbf{K}} N$ , then  $\widehat{M} \preceq_{\Sigma_1(\mathbb{L}_{\kappa,\kappa}(\widehat{\tau}))} \widehat{N}$ .

PROOF. Assume that  $\widehat{N} \models \exists \bar{x}\psi(\bar{x};\bar{a})$ , where  $\bar{a} \in {}^{<\kappa}|M|$  and  $\psi$  is a quantifierfree  $\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})$ -formula. Let A be the range of  $\bar{a}$ . Let  $\bar{b} \in {}^{<\kappa}|N|$  be such that  $\widehat{N} \models \psi[\bar{b},\bar{a}]$ . Since M is  $\kappa$ -saturated, there exists  $\bar{b}' \in {}^{<\kappa}|M|$  such that  $\operatorname{gtp}(\bar{b}'/A;M) = \operatorname{gtp}(\bar{b}/A;N)$ . Now it is easy to check using Proposition 2.3.11 that  $\widehat{M} \models \psi[\bar{b}';\bar{a}]$ .  $\Box$ 

Also note that if  $\kappa$  is suitably chosen and  $\mathbf{K}^0$  is stable, then we have a strong failure of the order property in  $\widehat{\mathbf{K}}$ :

PROPOSITION 2.5.3. If  $\kappa = \beth_{\kappa}$  and  $\mathbf{K}^0$  is stable (in unboundedly many cardinals, see Definition 2.2.24), then  $\hat{\mathbf{K}}$  does not have the syntactic order property of length  $\kappa$ .

PROOF. By Fact 2.2.25,  $\mathbf{K}^0$  is  $(< \kappa)$ -stable in unboundedly many cardinals. By Fact 2.4.11,  $\mathbf{K}^0$  does not have the  $(< \kappa)$ -order property.

Let  $\widehat{\mathbf{K}^0}$  be the  $(\langle \kappa \rangle)$ -Galois Morleyization of  $\mathbf{K}^0$ . By Corollary 2.4.8,  $\widehat{\mathbf{K}^0}$  does not have the syntactic order property of length  $\kappa$ .

Now note that Galois types are the same in **K** and  $\mathbf{K}^0$ : for  $N \in \mathbf{K}$ ,  $A \subseteq |N|$ , and  $\bar{b}, \bar{b}' \in {}^{<\infty}|N|$ ,  $\operatorname{gtp}_{\mathbf{K}^0}(\bar{b}/A; N) = \operatorname{gtp}_{\mathbf{K}^0}(\bar{b}'/A; N)$  if and only if  $\operatorname{gtp}_{\mathbf{K}}(\bar{b}/A; N) =$  $\operatorname{gtp}_{\mathbf{K}}(\bar{b}'/A; N)^{15}$ . To see this, use amalgamation together with the fact that every model in  $\mathbf{K}^0$  can be  $\leq_{\mathbf{K}}$ -extended to a model in  $\mathbf{K}$ .

It follows that  $\widehat{\mathbf{K}} \subseteq \mathbf{K}^0$ . By definition of the syntactic order property, this means that also  $\widehat{\mathbf{K}}$  does not have the syntactic order property of length  $\kappa$ , as desired.

DEFINITION 2.5.4. Let  $\widehat{N} \in \widehat{\mathbf{K}}$ ,  $A \subseteq |\widehat{N}|$ , and p be a set of formulas (in some logic) over  $\widehat{N}$ .

<sup>&</sup>lt;sup>15</sup>Recall that  $gtp_{\mathbf{K}}$  denotes Galois types as computed in  $\mathbf{K}$  and  $gtp_{\mathbf{K}^0}$  Galois types computed in  $\mathbf{K}^0$  (see Definition 2.2.17).

- (1) p is a  $(<\kappa)$ -heir over A if for any formula  $\phi(\bar{x}; \bar{b}) \in p$  over A, there exists  $\bar{a} \in {}^{<\kappa}A$  such that  $\phi(\bar{x}; \bar{a}) \in p \upharpoonright A$ .
- (2) p is a  $(<\kappa)$ -coheir over A in  $\widehat{N}$  if for any  $\phi(\overline{x}) \in p$  there exists  $\overline{a} \in {}^{<\kappa}A$  such that  $\widehat{N} \models \phi[\overline{a}]$ . When  $\widehat{N}$  is clear from context, we drop it.

REMARK 2.5.5. Here,  $\kappa$  is fixed (Hypothesis 2.5.1), so we will just remove it from the notation and simply say that p is a (co)heir over A.

REMARK 2.5.6. In this section, p will be  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{c}/B;\widehat{N})$  for a fixed B such that  $A \subseteq B \subseteq |\widehat{N}|$ .

REMARK 2.5.7. Working in  $\widehat{N} \in \widehat{\mathbf{K}}$ , let  $\overline{c}$  be a permutation of  $\overline{c}'$ , and A, B be sets. Then  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{c}/B;\widehat{N})$  is a coheir over A if and only if  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{c}'/B;\widehat{N})$ is a coheir over A. Similarly for heir. Thus we can talk about  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(C/B;\widehat{N})$ being a heir/coheir over A without worrying about the enumeration of C.

We will mostly look at coheir, but the next proposition tells us how to express one in terms of the other.

PROPOSITION 2.5.8.  $\operatorname{tp}_{qf-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}/A\overline{b};\widehat{N})$  is a heir over A if and only if  $\operatorname{tp}_{qf-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{b}/A\overline{a};\widehat{N})$  is a coheir over A.

PROOF. Straightforward.

It is convenient to see coheir as an independence relation:

NOTATION 2.5.9. Write  $A \stackrel{N}{\downarrow} B$  if  $M, N \in \mathbf{K}$ ,  $M \leq_{\mathbf{K}} N$ , and  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(A/|M| \cup B; \widehat{N})$  is a coheir over |M| in  $\widehat{N}$ . We also say<sup>16</sup> that  $\operatorname{gtp}(A/B; N)$  is a coheir over M.

REMARK 2.5.10. The definition of  $\downarrow$  depends on  $\kappa$  but we hide this detail.

Interestingly, Definition 2.5.4 is equivalent to the semantic definition of Boney and Grossberg [**BG**, Definition 3.2]:

PROPOSITION 2.5.11. Let  $N \in \mathbf{K}$ . Then  $p \in \mathrm{gS}^{<\infty}(B; N)$  is a coheir over  $M \leq_{\mathbf{K}} N$  if and only if for any  $I \subseteq \ell(p)$  and any  $B_0 \subseteq B$ , if  $|I_0| + |B_0| < \kappa$ ,  $p^I \upharpoonright B_0$  is realized in M.

**PROOF.** Straightforward

For completeness, we show that the definition of heir also agrees with the semantic definition of Boney and Grossberg [**BG**, Definition 6.1].

PROPOSITION 2.5.12. Let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  be in  $\mathbf{K}, \bar{a} \in {}^{<\infty}|\hat{N}|$ .

Then  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}/M;\widehat{N})$  is a heir over  $M_0$  if and only if for all  $(<\kappa)$ -sized  $I \subseteq \ell(\overline{a})$  and  $(<\kappa)$ -sized  $M_0^- \leq_{\mathbf{K}} M_0, \ M_0^- \leq_{\mathbf{K}} M^- \leq_{\mathbf{K}} M$  (where we also allow  $M_0^-$  to be empty), there is  $f: M^- \xrightarrow{M_0^-} M_0$  such that  $\operatorname{gtp}(\overline{a}/M; N)$  extends

 $f(\operatorname{gtp}((\bar{a}\restriction I)/M^-;N)).$ 

 $<sup>^{16}</sup>$ It is easy to check this does not depend on the choice of representatives.

#### 2.5. COHEIR

PROOF. Assume first  $\operatorname{tp}_{\operatorname{qf}-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}/M;\widehat{N})$  is a heir over  $M_0$  and let  $I \subseteq \ell(\overline{a})$ ,  $M_0^- \leq_{\mathbf{K}} M^- \leq_{\mathbf{K}} M$  be  $(<\kappa)$ -sized, with  $M_0^-$  possibly empty. Let  $p := \operatorname{gtp}((\overline{a} \upharpoonright I)/M^-; N)$ . Let  $\overline{b}_0$  be an enumeration of  $M_0^-$  and let  $\overline{b}$  be an enumeration of  $|M^-|\backslash|M_0^-|$ . Let  $q := \operatorname{gtp}((\overline{a} \upharpoonright I)\overline{b}_0\overline{b}/\emptyset; N)$ . Consider the formula  $\phi(\overline{x};\overline{b};\overline{b}_0) := R_q(\overline{x};\overline{b};\overline{b}_0)$ , where  $\overline{x}$  are the free variables in  $\operatorname{tp}_{\operatorname{qf}-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}/M;\widehat{N})$  and we assume for notational simplicity that the *I*-indiced variables are picked out by  $R_q(\overline{x},\overline{b},\overline{b}_0)$ . Then  $\phi$  is in  $\operatorname{tp}_{\operatorname{qf}-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}/M;\widehat{N})$ . By the syntactic definition of heir, there is  $\overline{c} \in \langle \kappa | M_0 |$  such that  $\phi(\overline{x};\overline{c};\overline{b}_0)$  is in  $\operatorname{tp}_{\operatorname{qf}-\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}/M_0;\widehat{N})$ . By definition of the  $(<\kappa)$ -Galois Morleyization this means that  $\operatorname{gtp}((\overline{a} \upharpoonright I)\overline{b}\overline{b}_0/\emptyset; N) = \operatorname{gtp}((\overline{a} \upharpoonright I)\overline{c}\overline{b}_0/\emptyset; N)$ .

By definition of Galois types and amalgamation (see Fact 2.2.18), there exists  $N' \geq_{\mathbf{K}} N$  and  $g: N \to N'$  such that  $g((\bar{a} \upharpoonright I)\bar{b}\bar{b}_0) = (\bar{a} \upharpoonright I)\bar{c}\bar{b}_0$ . Let  $f := g \upharpoonright M^-$ . Then from the definitions of  $\bar{b}_0$ ,  $\bar{b}$ , and  $\bar{c}$ , we have that  $f: M^- \longrightarrow M_0^-$ .

 $M_0$ . Moreover,  $f(\operatorname{gtp}((\bar{a} \upharpoonright I)/M^-; N)) = \operatorname{gtp}(\bar{a} \upharpoonright I/f[M^-]; N)$ , which is clearly extended by  $\operatorname{gtp}(\bar{a}/M; N)$ .

The converse is similar.

REMARK 2.5.13. The notational difficulties encountered in the above proof and the complexity of the semantic definition of heir show the convenience of using a syntactic notation rather than working purely semantically.

We now investigate the properties of coheir. For the convenience of the reader, we explicitly prove the uniqueness property (we have to slightly adapt the proof of (U) from [**MS90**, Proposition 4.8]). For the others, they are either straightforward or we can just quote.

LEMMA 2.5.14. Let  $M, N, N' \in \mathbf{K}$  with  $M \leq_{\mathbf{K}} N, M \leq_{\mathbf{K}} N'$ . Assume  $\widehat{M}$  does not have the syntactic order property of length  $\kappa$ . Let  $\overline{a} \in {}^{<\infty}|N|, \overline{a}' \in {}^{<\infty}|N'|, \overline{b} \in {}^{<\infty}|M|$  be given such that:

- (1)  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}/M;\widehat{N}) = \operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}'/M;\widehat{N'})$
- (2)  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}/M\overline{b};\widehat{N})$  is a coheir over M.
- (3)  $\operatorname{tp}_{\operatorname{af-}\mathbb{L}_{\kappa}}(\widehat{\tau})(\overline{b}/M\overline{a}';\widehat{N'})$  is a coheir over M.

Then  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}/M\overline{b};\widehat{N}) = \operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}'/M\overline{b};\widehat{N'}).$ 

PROOF. We suppose not and prove that  $\widehat{M}$  has the syntactic order property of length  $\kappa$ . Assume that  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}/M\overline{b};\widehat{N}) \neq \operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}'/M\overline{b};\widehat{N}')$  and pick  $\phi(\overline{x},\overline{y})$  a formula over M witnessing it:

(1) 
$$\widehat{N} \models \phi[\bar{a};\bar{b}] \text{ but } \widehat{N'} \models \neg \phi[\bar{a}';\bar{b}]$$

(note that we can assume without loss of generality that  $\ell(\bar{a}) + \ell(\bar{b}) < \kappa$ ). Define by induction on  $i < \kappa \ \bar{a}_i, \ \bar{b}_i$  in M such that for all  $i, j < \kappa$ :

- (1)  $\widehat{M} \models \phi[\overline{a}_i, \overline{b}].$
- (2)  $\widehat{M} \models \phi[\overline{a}_i, \overline{b}_j]$  if and only if  $i \leq j$ .
- (3)  $\widehat{N} \models \neg \phi[\overline{a}, \overline{b}_i].$
- Note that since  $\bar{b}_i \in {}^{<\kappa}|M|$ , (3) is equivalent to  $\widehat{N'} \models \neg \phi[\bar{a}', \bar{b}_i]$ .

This is enough: Then  $\chi(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2) := \phi(\bar{x}_1, \bar{y}_2) \wedge \bar{x}_1 \bar{y}_1 \neq \bar{x}_2 \bar{y}_2$  together with the sequence  $\langle \bar{a}_i \bar{b}_i : i < \kappa \rangle$  witness the syntactic order property of length  $\kappa$ .

This is possible: Suppose that  $\bar{a}_j$ ,  $\bar{b}_j$  have been defined for all j < i. Note that by the induction hypothesis and (1) we have:

$$\widehat{N} \models \bigwedge_{j < i} \phi[\bar{a}_j, \bar{b}] \land \bigwedge_{j < i} \neg \phi[\bar{a}, \bar{b}_j] \land \phi[\bar{a}, \bar{b}]$$

Since  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{a}/A\overline{b};\widehat{N})$  is a coheir over M, there is  $\overline{a}'' \in {}^{<\kappa}|M|$  such that:

$$\widehat{N} \models \bigwedge_{j < i} \phi[\bar{a}_j, \bar{b}] \land \bigwedge_{j < i} \neg \phi[\bar{a}'', \bar{b}_j] \land \phi[\bar{a}'', \bar{b}]$$

Note that all the data in the equation above is in M, so as  $M \leq_{\mathbf{K}} N$ , the monotonicity axiom of functorial expansions implies  $\widehat{M} \subseteq \widehat{N}$ , so  $\widehat{M}$  also models the above. By monotonicity again,  $\widehat{N'}$  models the above. We also know that  $\widehat{N'} \models \neg \phi[\overline{a'}, \overline{b}]$ . Thus we have:

$$\widehat{N'} \models \bigwedge_{j < i} \phi[\bar{a}_j, \bar{b}] \land \bigwedge_{j < i} \neg \phi[\bar{a}'', \bar{b}_j] \land \phi[\bar{a}'', \bar{b}] \land \neg \phi[\bar{a}', \bar{b}]$$

Since  $\operatorname{tp}_{\operatorname{qf-}\mathbb{L}_{\kappa,\kappa}(\widehat{\tau})}(\overline{b}/M\overline{a}';\widehat{N})$  is a coheir over M, there is  $\overline{b}'' \in {}^{<\kappa}|M|$  such that:

$$\widehat{N'} \models \bigwedge_{j < i} \phi[\bar{a}_j, \bar{b}''] \land \bigwedge_{j < i} \neg \phi[\bar{a}'', \bar{b}_j] \land \phi[\bar{a}'', \bar{b}''] \land \neg \phi[\bar{a}', \bar{b}'']$$

Let  $\bar{a}_i := \bar{a}'', \bar{b}_i := \bar{b}''$ . It is easy to check that this works.

THEOREM 2.5.15 (Properties of coheir).

- (1) (a) Invariance: If  $f: N \cong N'$  and  $A \underset{M}{\stackrel{N}{\downarrow}} B$ , then  $f[A] \underset{f[M]}{\stackrel{N'}{\downarrow}} f[B]$ .
  - (b) Monotonicity: If  $A \stackrel{N}{\underset{M}{\downarrow}} B$  and  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N, A_0 \subseteq A$ ,  $B_0 \subseteq B, |M'| \subseteq B, A_0 \cup B_0 \subseteq |N_0|$ , then  $A_0 \stackrel{N_0}{\downarrow} B_0$ .
  - $B_0 \subseteq B, |M'| \subseteq B, A_0 \cup B_0 \subseteq |N_0|, \text{ then } A_0 \underset{M'}{\stackrel{N_0}{\downarrow}} B_0.$ (c) Normality: If  $A \underset{M}{\stackrel{N}{\downarrow}} B, \text{ then } A \cup |M| \underset{M}{\stackrel{N}{\downarrow}} B \cup |M|.$
  - (d) Disjointness: If  $A \underset{M}{\stackrel{N}{\downarrow}} B$ , then  $A \cap B \subseteq |M|$ .
  - (e) Left and right existence:  $A \stackrel{N}{\underset{M}{\downarrow}} M$  and  $M \stackrel{N}{\underset{M}{\downarrow}} B$ .
  - (f) Left and right  $(<\kappa)$ -set-witness:  $A \underset{M}{\stackrel{N}{\downarrow}} B$  if and only if for all  $A_0 \subseteq A$ and  $B_0 \subseteq B$  of size less than  $\kappa$ ,  $A_0 \underset{M}{\stackrel{N}{\downarrow}} B_0$ .

(g) Strong left transitivity: If 
$$M_1 \underset{M_0}{\overset{N}{\downarrow}} B$$
 and  $A \underset{M_1}{\overset{N}{\downarrow}} B$ , then  $A \underset{M_0}{\overset{N}{\downarrow}} B$ 

(2) If 
$$\widehat{\mathbf{K}}$$
 does not have the syntactic order property of length  $\kappa$ , then<sup>17</sup>:  
(a) Symmetry:  $A \bigcup_{M}^{N} B$  if and only if  $B \bigcup_{M}^{N} A$ .

<sup>&</sup>lt;sup>17</sup>Note that (by Proposition 2.5.3) this holds in particular if  $\kappa = \beth_{\kappa}$  and  $\mathbf{K}^0$  is stable.

#### 2.5. COHEIR

- (b) Strong right transitivity: If  $A \underset{M_0}{\downarrow} M_1$  and  $A \underset{M_1}{\downarrow} B$ , then  $A \underset{M_0}{\downarrow} B$ . (c) Set local character: For all cardinals  $\alpha$ , all  $p \in gS^{\alpha}(M)$ , there exists  $M_0 \leq_{\mathbf{K}} M$  with  $\|M_0\| \leq \mu_{\alpha} := (\alpha + 2)^{<\kappa_r}$  such that p is a coheir over  $M_0$ .
- (d) Syntactic uniqueness: If  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N_\ell$  for  $\ell = 1, 2, |M_0| \subseteq B \subseteq$ |M|.  $q_{\ell} \in \mathcal{S}_{\mathrm{qf-L}_{\kappa,\kappa}(\widehat{\tau})}^{<\infty}(B;\widehat{N_{\ell}}), q_1 \upharpoonright M_0 = q_2 \upharpoonright M_0 \text{ and } q_{\ell} \text{ is a coheir}$ over  $M_0$  in  $\widehat{N}_{\ell}$  for  $\ell = 1, 2$ , then  $q_1 = q_2$ .
- (e) Syntactic stability: For  $\alpha$  a cardinal,  $\widehat{\mathbf{K}}$  is syntactically  $\alpha$ -stable in all  $\lambda \geq \mathrm{LS}(\mathbf{K}^0)$  such that  $\lambda^{\mu_{\alpha}} = \lambda$ .
- (3) If  $\widehat{\mathbf{K}}$  does not have the syntactic order property of length  $\kappa$  and  $\mathbf{K}^0$  is  $(<\kappa)$ -tame and short for types of length less than  $\alpha$ , then:
  - (a) Uniqueness: If  $p, q \in gS^{<\alpha}(M)$  are coheir over  $M_0 \leq_{\mathbf{K}} M$  and  $p \upharpoonright$  $M_0 = q \upharpoonright M_0$ , then p = q.
  - (b) Stability: For all  $\beta < \alpha$ ,  $\mathbf{K}^0$  is  $\beta$ -stable in all  $\lambda \ge \mathrm{LS}(\mathbf{K}^0)$  such that  $\lambda^{\mu_{\beta}} = \lambda.$

**PROOF.** Observe that (except for part (3)), one can work in  $\widehat{\mathbf{K}}$  and prove the properties there using purely syntactic methods (so amalgamation is never needed for example). More specifically, (1) is straightforward. As for (2), symmetry is exactly as in<sup>18</sup> [Pil82, Proposition 3.1] (Lemma 2.5.14 is not needed here), strong right transitivity follows from strong left transitivity and symmetry, syntactic uniqueness is by symmetry and Lemma 2.5.14, and set local character is as in the proof of  $(B)_{\mu}$ in [MS90, Proposition 4.8]. Note that the proofs in [MS90] and [Pil82] use that the ordering has some elementarity. In our case, this is given by Proposition 2.5.2.

The proof of stability is as in the first-order case. To get part (3), use the translation between Galois and syntactic types (Theorem 2.3.15).  $\square$ 

**PROOF OF THEOREM 2.0.3.** If the hypotheses of Theorem 2.0.3 in the abstract hold for the AEC  $\mathbf{K}^0$ , then the hypothesis of each parts of Theorem 2.5.15 hold (see Proposition 2.5.3). 

REMARK 2.5.16. We can give more localized version of some of the above results. For example in the statement of the symmetry property it is enough to assume that M does not have the syntactic order property of length  $\kappa$ . We could also have been more precise and state the uniqueness property in terms of being  $(<\kappa)$ -tame and short for  $\{q_1,q_2\}$ , where  $q_1,q_2$  are the two Galois types we are comparing.

REMARK 2.5.17. We can use Theorem 2.5.15.(2e) to get another proof of the equivalence between (syntactic) stability and no order property in AECs.

REMARK 2.5.18. The extension property (given  $p \in gS^{<\infty}(M), N \geq_{\mathbf{K}} M, p$ has an extension to N which is a coheir over M) seems more problematic. In **[BG]**, Boney and Grossberg simply assumed it (they also showed that it followed from  $\kappa$  being strongly compact [**BG**, Theorem 8.2(1)]). Here we do not need to

 $<sup>^{18}</sup>$ Note that a proof of symmetry of nonforking from no order property already appears in [She78], but Pillay's proof for coheir is the one we use here.

assume it but are still unable to prove it. In Chapter 6, we prove it assuming a superstability-like hypothesis and more locality  $^{19}$ .

 $<sup>^{19}\</sup>mathrm{A}$  word of caution: In [**HL02**, Section 4], the authors give an example of an  $\omega$ -stable class that does not have extension. However, the extension property they consider is *over all sets*, not only over models.

# CHAPTER 3

# Canonical forking in AECs

This chapter is based on [**BGKV16**] and is joint work with Will Boney, Rami Grossberg, and Alexei Kolesnikov.

### Abstract

Boney and Grossberg [**BG**] proved that every nice AEC has an independence relation. We prove that this relation is unique: in any given AEC, there can exist at most one independence relation that satisfies existence, extension, uniqueness and local character. While doing this, we study more generally the properties of independence relations for AECs and also prove a canonicity result for Shelah's good frames. The usual tools of first-order logic (like the finite equivalence relation theorem or the type amalgamation theorem in simple theories) are not available in this context. In addition to the loss of the compactness theorem, we have the added difficulty of not being able to assume that types are sets of formulas. We work axiomatically and develop new tools to understand this general framework.

# 3.1. Introduction

Let **K** be an abstract elementary class (AEC) which satisfies amalgamation, joint embedding, and which does not have maximal models. These assumptions allow us to work inside its monster model  $\mathfrak{C}$ . The main results of this chapter are:

- (1) There is at most one independence relation satisfying existence, extension, uniqueness and local character (Corollary 3.5.18).
- (2) Under some reasonable conditions, the coheir relation of [**BG**] has local character and is canonical (Theorems 3.6.3 and 3.6.6).
- (3) Shelah's weakly successful good  $\lambda$ -frames are canonical: an AEC can have at most one such frame (Theorem 3.6.12).

To understand the relevance of the results, some history is necessary.

In 1970, Shelah discovered the notion "tp( $\bar{a}/B$ ) forks over A" (for  $A \subseteq B$ ), a generalization of Morley's rank in  $\omega$ -stable theories. Its basic properties were published in [She78].

In 1974, Lascar [Las76, Theorem 4.9] established that for superstable theories, any relation between  $\bar{a}$ , B, A satisfying the basic properties of forking is Shelah's forking relation. In 1984, Harnik and Harrington [HH84, Theorem 5.8] extended Lascar's abstract characterization to stable theories. Their main device was the finite equivalence relation theorem. In 1997, Kim and Pillay [KP97, Theorem 4.2] published an extension to simple theories, using the independence theorem (also known as the type-amalgamation theorem).

This chapter deals with the characterization of independence relations in various non-elementary classes. An early attempt on this problem can be found in Kolesnikov's [Kol05], which focuses on some important particular cases (e.g. homogeneous model theory and classes of atomic models). We work in a more general context, and only rely on the abstract properties of independence. We cannot assume that types are sets of formulas, so work only with Galois (i.e. orbital) types.

In [She87b, Chapter II] (which later appeared as [She09b, Chapter V.B]), Shelah gave the first axiomatic definition of independence in AECs, and showed that it generalized first-order forking. In [She09a, Chapter II], Shelah gave a similar definition, localized to models of a particular size  $\lambda$  (the so-called "good  $\lambda$ frame"). Shelah proved that a good frame existed, under very strong assumptions (typically, the class is required to be categorical in two consecutive cardinals).

Recently, working with a different set of assumptions (the existence of a monster model and tameness), Boney and Grossberg [**BG**] gave conditions (namely a form of Galois stability and the extension property for coheir) under which an AEC has a global independence relation. This showed that one could study independence in a broad family of AECs. The chapter is strongly motivated by both [**She09a**, Chapter II] and [**BG**].

The chapter is structured as follows. In Section 3.2, we fix our notation, and review some of the basic concepts in the theory of AECs. In Section 3.3, we introduce independence relations, the main object of study of this chapter, as well as some important properties they could satisfy, such as extension and uniqueness. We consider two examples: coheir and nonsplitting.

In Section 3.4, we prove a weaker version of (1) (Corollary 3.4.14) that has some extra assumptions. This is the core of the chapter.

In Section 3.5, we go back to the properties listed in Section 3.3 and investigate relations between them. We show that some of the hypotheses in Corollary 3.4.14 are redundant. For example, we show that the symmetry and transitivity properties follow from existence, extension, uniqueness, and local character. We conclude by proving (1). Finally, in Section 3.6, we apply our methods to the coheir relation considered in [**BG**] and to Shelah's good frames, proving (2) and (3).

While we work in a more general framework, the basic results of Sections 3.2-3.3 often have proofs that are very similar to their first-order analogs. Readers feeling confident in their knowledge of first-order nonforking can start reading directly from Section 3.4 and refer back to Sections 3.2-3.3 as needed.

An early version of this chapter was circulated already in early 2014. Since that time, Theorem 3.5.13 has been used to build a good frame from amalgamation, tameness, and categoricity in a suitable cardinal (Chapter 4). We can also use it to deduce a certain symmetry property for nonsplitting in classes with amalgamation categorical in a high-enough cardinal (Chapter 10), with consequences on the uniqueness of limit models. The question of canonicity of forking in more local setups (e.g. when the independence relation is only defined for certain types over models of a certain size) is pursued further in Chapter 6. The latter preprint addresses Questions 3.5.5, 3.6.13, 3.7.1, and 3.7.2 posed in this chapter.

# 3.2. Notation and prerequisites

We assume the reader is familiar with abstract elementary classes and the basic related concepts. We briefly review what we need in this chapter, and set up some notation. HYPOTHESIS 3.2.1. We work in a fixed abstract elementary class  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  which satisfies amalgamation and joint embedding, and has no maximal models.

## 3.2.1. The monster model.

DEFINITION 3.2.2. Let  $\mu > LS(\mathbf{K})$  be a cardinal. For models  $M \leq_{\mathbf{K}} N$ , we say N is a  $\mu$ -universal extension of M if for any  $M' \geq_{\mathbf{K}} M$ , with  $||M'|| < \mu$ , M' can be embedded inside N over M, i.e. there exists a  $\mathbf{K}$ -embedding  $f: M' \to N$  fixing M pointwise. We say N is a universal extension of M if it is a  $||M||^+$ -universal extension of M.

DEFINITION 3.2.3. Let  $\mu > LS(\mathbf{K})$  be a cardinal. We say a model N is  $\mu$ -model homogeneous if for any  $M \leq_{\mathbf{K}} N$ , N is a  $\mu$ -universal extension of M. We say M is  $\mu$ -saturated if it is  $\mu$ -model homogeneous (this is equivalent to the classical definition by [She01a, Lemma 0.26]).

DEFINITION 3.2.4 (Monster model). Since **K** has amalgamation and joint embedding properties and has no maximal models, we can build a strictly increasing continuous chain  $(\mathfrak{C}_i)_{i\in OR}$ , where for all i,  $\mathfrak{C}_{i+1}$  is universal over  $\mathfrak{C}_i$ . We call the union  $\mathfrak{C} := \bigcup_{i\in OR} \mathfrak{C}_i$  the monster model<sup>1</sup> of **K**.

Any model of **K** can be embedded inside the monster model, so we will adopt the convention that any set or model we consider is a subset or a substructure of  $\mathfrak{C}$ .

We write  $\operatorname{Aut}_A(\mathfrak{C})$  for the set of automorphisms of  $\mathfrak{C}$  fixing A pointwise. When  $A = \emptyset$ , we omit it.

We will use the following without comments.

REMARK 3.2.5. Let M, N be models. By our convention,  $M \leq_{\mathbf{K}} \mathfrak{C}$  and  $N \leq_{\mathbf{K}} \mathfrak{C}$ , thus by the coherence axiom,  $M \subseteq N$  implies  $M \leq_{\mathbf{K}} N$ .

DEFINITION 3.2.6. Let I be an index set. Let  $\overline{A} := (A_i)_{i \in I}, \overline{B} := (B_i)_{i \in I}$ be sequences of sets, and let C be a set. We write  $f : \overline{A} \equiv_C \overline{B}$  to mean that  $f \in \operatorname{Aut}_C(\mathfrak{C})$ , and for all  $i \in I, f[A_i] = B_i$ . We write  $\overline{A} \equiv_C \overline{B}$  to mean that  $f : \overline{A} \equiv_C \overline{B}$  for some f. When C is empty, we omit it.

We will most often use this notation when I has a single element, or when all the sets are singletons. In the later case, we identify a set with the corresponding singleton, i.e. if  $\bar{a} = (a_i)_{i \in I}$  and  $\bar{b} := (b_i)_{i \in I}$  are sequences, we write  $f : \bar{a} \equiv_C \bar{b}$ instead of  $f : \bar{A} \equiv_C \bar{B}$ , with  $A_i := \{a_i\}, B_i := \{b_i\}$ . We write  $gtp(\bar{a}/C)$  for the  $\equiv_C$ equivalence class of  $\bar{a}$ . This corresponds to the usual notion of Galois types from Definition 2.2.17.

Note that for sets A, B, we have  $f : A \equiv_C B$  precisely when there are enumerations  $\bar{a}, \bar{b}$  of A and B respectively such that  $f : \bar{a} \equiv_C \bar{b}$ .

## 3.3. Independence relations

In this section, we define independence relations, the main object of study of this chapter. We then consider two examples: coheir and nonsplitting.

<sup>&</sup>lt;sup>1</sup>Since  $\mathfrak{C}$  is a proper class, it is strictly speaking not an element of **K**. We ignore this detail, since we could always replace OR in the definition of  $\mathfrak{C}$  by a cardinal much bigger than the size of the models under discussion.

#### 3.3.1. Basic definitions.

DEFINITION 3.3.1 (Independence relation). An independence relation  $\downarrow$  is a set of triples of the form (A, M, N) where A is a set, M, N are models (i.e.  $M, N \in \mathbf{K}$ ),  $M \leq_{\mathbf{K}} N$ . Write  $A \downarrow_{M} N$  for  $(A, M, N) \in \downarrow$ . When  $A = \{a\}$ , we may write  $a \downarrow_{M} N$  for  $A \downarrow_{M} N$ . We require that  $\downarrow$  satisfies the following properties:

- (I) Invariance: Assume  $(A, M, N) \equiv (A', M', N')$ . Then  $A \downarrow N$  if and only if  $A' \underset{M'}{\bigcup} N'$ .
- (M) Left and right monotonicity: If  $A \underset{M}{\cup} N$ ,  $A' \subseteq A$ ,  $M \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ ,

then  $A' \underset{M}{\downarrow} N'$ . • (B) Base monotonicity: If  $A \underset{M}{\downarrow} N$ , and  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N$ , then  $A \underset{M'}{\downarrow} N$ . We write  $\underset{M}{\downarrow}$  for  $\downarrow$  restricted to the base set M, and similarly for e.g.  $A \underset{M}{\downarrow}$ .

In what follows,  $\downarrow$  always denotes an independence relation.

REMARK 3.3.2. To avoid relying on a monster model, we could introduce an ambient model  $\widehat{N}$  as a fourth parameter in the above definition (i.e. we would write  $A \downarrow N$ ). This would match the approach in [She09b, Chapter V.B] and [She09a, Chapter II] where the existence of a monster model is not assumed. We would require that N contains the other parameters A, M and N. To avoid cluttering the notation, we will not adopt this approach, but generalizing most of our results to this context should cause no major difficulty. Some simple cases will be treated in the discussion of good frames in Section 3.6. In Chapter 6, many of the results of this chapter are stated in a "monsterless" framework.

We will consider the following properties of independence<sup>2</sup>:

- $(C)_{\kappa}$  Continuity: If  $A \not \downarrow N$ , then there exists  $A^{-} \subseteq A$ ,  $B^{-} \subseteq N$  of size strictly less than  $\kappa$  such that for all  $N_0 \geq_{\mathbf{K}} M$  containing  $B^{-}$ ,  $A^{-} \not \downarrow N_0$ .
- (T) Left transitivity: If  $M_1 \underset{M_0}{\downarrow} N$ , and  $M_2 \underset{M_1}{\downarrow} N$ , with  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ , then  $M_2 \underset{M_0}{\downarrow} N$ .
- $(T_*)$  Right transitivity: If  $A \underset{M_0}{\cup} M_1$ , and  $A \underset{M_1}{\cup} M_2$ , with  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}}$  $M_2$ , then  $A \underset{M_0}{\downarrow} M_2$ .
- (S) Symmetry: If  $A \underset{M}{\downarrow} N$ , then there is  $M' \ge_{\mathbf{K}} M$  with  $A \subseteq M'$  such that  $N \underset{M}{\downarrow} M'$ . If A is a model extending M, one can take  $M' = A^{-3}$ .
- (U)<sup>M</sup> Uniqueness: If  $A \downarrow N$ ,  $A' \downarrow N$ , and  $f : A \equiv_M A'$ , then  $g : A \equiv_N A'$  for some g so that  $g \upharpoonright^M A = f \upharpoonright^M A$ .

<sup>&</sup>lt;sup>2</sup>Continuity, transitivity, uniqueness, existence and extension are adapted from [MS90]. Symmetry comes from [She09a, Chapter II].

<sup>&</sup>lt;sup>3</sup>This second part actually follows from monotonicity and the first part.

- (E) The following properties hold:
  - $(E_0)$  Existence: for all sets A and models  $M, A \downarrow M$ .
  - $(E_1)$  Extension: Given a set A, and  $M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} N'$ , if  $A \downarrow_M N$ , then there is  $A' \equiv_N A$  such that  $A' \bigcup_M N'$ .
- (L) Local character:  $\kappa_{\alpha}(\downarrow) < \infty$  for all  $\alpha$ , where  $\kappa_{\alpha}(\downarrow) := \min\{\lambda \in$  $\operatorname{REG} \cup \{\infty\}$ : for all  $\mu = \operatorname{cf} \mu \geq \lambda$ , all increasing, continuous chains  $\langle M_i :$  $i \leq \mu$  and all sets A of size  $\alpha$ , there is some  $i_0 < \mu$  so  $A \downarrow M_{\mu}$ .
- $(E_{+})$  Strong extension: A technical property used in the proof of canonicity. See Definition 3.4.4.

For (P) a property that is not local character, and M a model, when we say  $\downarrow$ has  $(P)_M$ , we mean  $\downarrow_M$  has (P) (i.e.  $\downarrow$  has (P) when the base is restricted to be M). If P is either (T) or  $(T_*)$ ,  $(P)_M$  means we assume  $M_0 = M$  in the definition.

Whenever we are considering two independence relations  $\stackrel{(1)}{\downarrow}$  and  $\stackrel{(2)}{\downarrow}$ , we write

 $(P^{(1)})$  as a shorthand for " $\downarrow$  has (P)", and similarly for  $(P^{(2)})$ . Notice the following important consequence of (E):

REMARK 3.3.3. Assume  $\downarrow$  has  $(E)_M$ . Then for any A, and  $N \geq_{\mathbf{K}} M$ , there is  $A' \equiv_M A$  such that  $A' \downarrow_M N$  (use  $(E_0)_M$  to see  $A \downarrow_M M$ , and then use  $(E_1)_M$ ). Assuming  $(T^*)_M$ , this last statement is actually equivalent to  $(E)_M$ .

The property  $(E_{+})$  will be introduced and motivated later in the chapter. For now, we note that there is an asymmetry in our definition of an independence relation: the parameter on the left is allowed to be an arbitrary set, while the parameter on the right must be a model extending the base. This is because we have in mind the analogy " $a \underset{M}{\cup} N$  if and only if tp(a/N) does not fork over M",

and in AECs, types over models are much better behaved than types over sets.

The price to pay is that the statement of symmetry is not easy to work with. Assume for example we know an independence relation satisfies (T) and (S). Should it satisfy  $(T_*)$ ? Surprisingly, this is not easy to show. We prove it in Lemma 3.5.9, assuming (E). For now, we prepare the ground by showing how to extend an independence relation to take arbitrary sets on the right hand side.

DEFINITION 3.3.4 (Closure of an independence relation). We call  $\downarrow$  a *closure* of  $\downarrow$  if  $\overline{\downarrow}$  is a relation defined on all triples of the form (A, M, B), where M is a model (but maybe  $M \not\subseteq B$ ). We require it satisfies the following properties:

- For all A, and all M ≤<sub>K</sub> N, A ↓ N if and only if A ↓ M.
  (I) Invariance: If (A, M, B) ≡ (A', M', B'), then A ↓ B if and only if  $A' {igstarrow}_{M'} B'.$
- (M) Left and right monotonicity: If  $A \overline{\bigcup}_M B$  and  $A' \subseteq A, B' \subseteq B$ , then  $A' {\downarrow}_M B'$ .
- (B) Base monotonicity: If  $A \overline{\downarrow}_M B$ , and  $M \leq_{\mathbf{K}} M' \subseteq M \cup B$ , then  $A {\perp}_{M'} B.$

The minimal closure of  $\downarrow$  is the relation  $\overline{\downarrow}$  defined by  $A \overline{\downarrow} C$  if and only if there exists  $N \geq_{\mathbf{K}} M$ , with  $C \subseteq N$ , so that  $A \downarrow N$ .

It is straightforward to check that the minimal closure of  $\downarrow$  is the smallest closure of  $\downarrow$  but there might be others (and they also sometimes turn out to be useful), see the coheir and explicit nonsplitting examples below.

We can adapt the list of properties to a closure  $\bot$ .

Definition 3.3.5.

- We say  $\overline{\downarrow}$  has (S) if for all sets A, B,  $A\overline{\downarrow}_M B$  if and only if  $B\overline{\downarrow}_M A$ .
- We say that  $\downarrow$  has  $(C)_{\kappa}$  if whenever  $A \not \perp o_M B$ , there exists  $A^- \subseteq A$ ,  $B^- \subseteq B$  of size strictly less than  $\kappa$  such that  $A^- \not \perp o_M B^-$ .
- We say that  $\overline{\downarrow}$  has  $(E_1)$  if whenever  $A\overline{\downarrow}_M C$ , and  $C \subseteq C'$ , there exists  $A' \equiv_{MC} A$  such that  $A'\overline{\downarrow}_M C'$ .
- We say that  $\overline{\bigcup}$  has (U) if whenever  $A\overline{\bigcup}_M C$ ,  $A'\overline{\bigcup}_M C$ , and  $f: A \equiv_M A'$ , there is  $g: A \equiv_{MC} A'$  with  $g \upharpoonright A = f \upharpoonright A$ .
- We say that  $\overline{\downarrow}$  has (T) if whenever  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2, M_2 \overline{\downarrow}_{M_1} C$ , and  $M_1 \overline{\downarrow}_{M_0} C$ , we have  $M_2 \overline{\downarrow}_{M_0} C$ .
- The statements of  $(T_*)$ ,  $(E_0)$ , (L) are unchanged. We will not need to use  $(E_+)$  on a closure.

For an arbitrary closure, we cannot say much about the relationship between the properties satisfied by  $\downarrow$  and those satisfied by  $\overline{\downarrow}$ . The situation is different for the minimal closure, but we defer our analysis to section 3.5.

REMARK 3.3.6. Shelah's notion of a good  $\lambda$ -frame introduced in [She09a, Chapter II] is another axiomatic approach to independence in AECs. There are several key differences with our framework. In particular, good  $\lambda$ -frames only operate on  $\lambda$ -sized models and singleton sets. On the other hand, the theory of good  $\lambda$ -frames is very developed; see e.g. [She09a, JS12, JS13].

An earlier framework which is closer to our own is the "Existential framework" AxFr<sub>3</sub> (see [**She09b**, Definition V.B.1.9]). The key differences are that AxFr<sub>3</sub> only defines  $M_1 \stackrel{\widehat{N}}{\underset{M}{\downarrow}} M_2$  when  $M \leq_{\mathbf{K}} M_\ell$ ,  $\ell = 1, 2$ , AxFr<sub>3</sub> (essentially) assumes  $(C)_{\aleph_0}$ , while we seldom need continuity, and local character (a property crucial to our canonicity proof) is absent from the axioms of AxFr<sub>3</sub>.

**3.3.2. Examples.** Though so far developed abstractly, this framework includes many previously studied independence relations.

DEFINITION 3.3.7 (Coheir, [**BG**]). Fix a cardinal  $\kappa > LS(\mathbf{K})$ . We call a set small if it is of size less than  $\kappa$ . For  $M \leq_{\mathbf{K}} N$ , define

 $\begin{array}{cc} A \stackrel{(\mathrm{ch})}{\underset{M}{\downarrow}} N \iff \text{ for every small } A^{-} \subseteq A \text{ and } N^{-} \leq_{\mathbf{K}} N, \\ & \text{ there is } B^{-} \subseteq M \text{ such that } B^{-} \equiv_{N^{-}} A^{-}. \end{array}$ 

(ch)

One can readily check that  $\downarrow$  satisfies the properties of an independence re-

lation.  $\downarrow$  was first studied in [**BG**], based on results of [**MS90**] and [**Bon14b**], and generalizes the first-order notion of coheir. An alternative name for this notion is ( $< \kappa$ ) satisfiability. Sufficient conditions for this relation to be well-behaved (i.e. to have most of the properties listed above) are given in [**BG**, Theorem 5.1], reproduced here as Fact 3.3.16.

DEFINITION 3.3.8. We define a natural closure for  $\downarrow$ :

$$\begin{array}{cc} A \stackrel{(\bar{ch})}{\underset{M}{\downarrow}} C \iff & \text{for every small } A^- \subseteq A \text{ and } C^- \subseteq C, \\ & \text{there is } B^- \subseteq M \text{ such that } B^- \equiv_{C^-} A^- \end{array}$$

It is straightforward to check that  $\downarrow$  is indeed a closure of  $\downarrow$ , but it is not clear at all that this is the *minimal* one. This closure will be useful in the proof of local character (Theorem 3.6.3) Note that  $\downarrow$  differs from the notion of coheir given in [**MS90**]; there, types are consistent sets of formulas from a fragment of  $L_{\kappa,\kappa}$  for  $\kappa$  strongly compact and the notion there (see [**MS90**, Definition 4.5]) allows parameters from C and |M|.

DEFINITION 3.3.9 ( $\mu$ -nonsplitting, [She99]). Let  $\mu \geq LS(\mathbf{K})$ . For  $M \leq_{\mathbf{K}} N$ , we say  $A \underset{M}{\stackrel{(\mu-ns)}{\longrightarrow}} N$  if and only if for for all  $N_1, N_2 \in \mathbf{K}_{\leq \mu}$  with  $M \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} N$ ,  $\ell = 1, 2$ , if  $f: N_1 \equiv_M N_2$ , then there is  $g: N_1 \equiv_{AM} N_2$  such that  $f \upharpoonright N_1 = g \upharpoonright N_1$ .

There is also a definition of nonsplitting that does not depend on a cardinal  $\mu$ .

DEFINITION 3.3.10 (Nonsplitting). For  $M \leq_{\mathbf{K}} N$ ,

$$A \stackrel{(\mathrm{ns})}{\underset{M}{\downarrow}} N \iff A \stackrel{(\mu\text{-ns})}{\underset{M}{\downarrow}} N \text{ for all } \mu.$$

An equivalent definition of nonsplitting is given by the following.

PROPOSITION 3.3.11.  $A \stackrel{(ns)}{\downarrow} N$  if and only if for all  $N_1, N_2 \in \mathbf{K}$  with  $M \leq_{\mathbf{K}} N_{\ell} \leq_{\mathbf{K}} N$ ,  $\ell = 1, 2$ , if  $h : N_1 \equiv_M N_2$ , then  $f : A \equiv_{N_2} h[A]$  for some f with  $f \upharpoonright A = h \upharpoonright A$  (equivalently,  $\bar{a} \equiv_{N_2} h(\bar{a})$  for all enumerations  $\bar{a}$  of A). The analog statement also holds for  $\mu$ -nonsplitting.

The analog statement also holds for  $\mu$ -holdsplitting.

PROOF. Assume  $h: N_1 \equiv_M N_2$ , and  $f: A \equiv_{N_2} h[A]$  is such that  $f \upharpoonright A = h \upharpoonright A$ . Let  $g := f^{-1} \circ h$ . Then  $g \upharpoonright N_1 = h \upharpoonright N_1$ , and g fixes AM. In other words,  $g: N_1 \equiv_{AM} N_2$  is as needed. Conversely, assume  $h: N_1 \equiv_M N_2$ . Find  $g: N_1 \equiv_{AM} N_2$  such that  $h \upharpoonright N_1 = g \upharpoonright N_1$ . Then  $f := h \circ g^{-1}$  is the desired witness that  $A \equiv_{N_2} h[A]$ .

Using Proposition 3.3.11 to check base monotonicity, it is easy to see that both  $(ns) \qquad (\mu-ns)$ 

 $<sup>\</sup>downarrow$  and  $\downarrow$  are independence relations. These notions of splitting in AECs were

first explored in [She99], but have seen a wide array of uses; see [SV99, Van06, **Van13, GVV16**] or Chapter 4 for examples.  $\mu$ -nonsplitting is more common in the literature, but we focus on nonsplitting here. Using tameness, there is a correspondence between the two:

PROPOSITION 3.3.12. Let  $M \leq_{\mathbf{K}} N$  and  $\mu \geq LS(\mathbf{K})$ . If **K** is  $\mu$ -tame for |A|-length types and  $\mu' \in [\mu, \|N\|]$ , then

$$A \stackrel{(\mu\text{-ns})}{\underset{M}{\downarrow}} N \implies A \stackrel{(\mu'\text{-ns})}{\underset{M}{\downarrow}} N$$

**PROOF.** We use the equivalence given by Proposition 3.3.11. Let  $\mu' \in [\mu, ||N||]$ , and suppose  $A \overset{(\mu'-\mathrm{ns})}{\underset{M}{\not \downarrow}} N$ . Then there are  $N_{\ell} \in \mathbf{K}_{\mu'}$  so  $M \leq_{\mathbf{K}} N_{\ell} \leq_{\mathbf{K}} N$  for  $\ell = 1, 2$ and  $h: N_1 \equiv_M \overset{\sim}{N_2}$ , but  $\bar{a} \not\equiv_{N_2} h(\bar{a})$  for some enumeration  $\bar{a}$  of A. By tameness, there is  $N_2^- \in \mathbf{K}_{\leq \mu}$  so that  $\bar{a} \not\equiv_{N_2^-} h(\bar{a})$ . Without loss of generality,  $M \leq_{\mathbf{K}} N_2^-$ . Let  $N_1^- := h^{-1}[N_2^-]$ . Then  $N_1^-$  and  $N_2^-$  witness that  $A \overset{(\mu-\mathrm{ns})}{\underset{M}{\not M}} N$ . 

A variant is explicit nonsplitting, which allows the  $N_i$ 's to be sets instead of requiring models; this is based on explicit non-strong splitting from [She99, Definition 4.11.2].

DEFINITION 3.3.13 (Explicit Nonsplitting). For  $M \leq_{\mathbf{K}} N$ , we say  $A \bigcup_{M}^{(\text{nes})} N$  if and only if for for all  $C_1, C_2 \subseteq N$ , if  $f: C_1 \equiv_M C_2$ , then there is  $g: C_1 \equiv_{AM} C_2$ such that  $f \upharpoonright C_1 = g \upharpoonright C_1$ .

From the definition, we see immediately that  $\bigcup_{(nes)}^{(nes)} \subseteq \bigcup_{(nes)}^{(ns)}$ . Of course, the corresponding version of Proposition 3.3.11 also holds for  $\bigcup$ , so it is again straightform (nes) (nes) ward to check that  $\bigcup$  is an independence relation. One advantage of using  $\bigcup$ (ns) over  $\bigcup$  is that it has a natural closure:

DEFINITION 3.3.14. We say  $A \stackrel{(\overline{\text{nes}})}{\underset{M}{\bigcup}} C$  if and only if for for all  $C_1, C_2 \subseteq C$ , if  $f: C_1 \equiv_M C_2$ , then there is  $g: C_1 \equiv_{AM} C_2$  such that  $f \upharpoonright C_1 = g \upharpoonright C_1$ .

Again, it is not clear this is the minimal closure. We will have no use for this closure, so for most of the chapter we will stick with regular nonsplitting.

Nonsplitting will be used mostly as a technical tool to state and prove intermediate lemmas, while coheir will be relevant only in Section 3.6.

3.3.3. Properties of coheir and nonsplitting. We now investigate the properties satisfied by coheir and nonsplitting. Here is what holds in general:

PROPOSITION 3.3.15. Let  $\kappa > LS(\mathbf{K})$ .

- (1)  $\stackrel{(ch)}{\downarrow}$  and  $\stackrel{(c\bar{h})}{\downarrow}$  have  $(C)_{\kappa}$ , and (T). (2) If M is  $\kappa$ -saturated,  $\stackrel{(ch)}{\downarrow}$  and  $\stackrel{(c\bar{h})}{\downarrow}$  have  $(E_0)_M$ .

(3) 
$$\stackrel{(ns)}{\downarrow}$$
,  $\stackrel{(nes)}{\downarrow}$ , and  $\stackrel{(\overline{nes})}{\downarrow}$  have  $(E_0)$ .

PROOF. Just check the definitions.

While extension and uniqueness are usually considered very strong assumptions, it is worth noting that nonsplitting satisfies a weak version of them, see [Van06, Theorems I.4.10, I.4.12]. It is also well known that nonsplitting has local character assuming tameness and stability (see e.g. [GV06b, Fact 4.6]). This will not be used.

Regarding coheir, the following<sup>4</sup> appears in [BG]:

FACT 3.3.16. Let  $\kappa > LS(\mathbf{K})$  be regular. Assume  $\mathbf{K}$  is fully  $(< \kappa)$ -tame, fully

 $(<\kappa)$ -type short, has no weak  $\kappa$ -order property<sup>5</sup> and  $\stackrel{(cn)}{\downarrow}$  has  $(E)^6$ .

Then  $\stackrel{(ch)}{\downarrow}$  has (U) and (S).

Moreover, if  $\kappa$  is strongly compact, then the tameness and type-shortness hy-  $_{\rm (ch)}$ 

potheses hold for free,  $\bigcup$  has  $(E_1)$ , and "no weak  $\kappa$  order property" is implied by " $\exists \lambda > \kappa$  so  $I(\lambda, \mathbf{K}) < 2^{\lambda}$ ."

As we will see, right transitivity  $(T_*)$  can be deduced either from symmetry and (T) (Lemma 3.5.9) or from uniqueness (Lemma 3.5.11). Local character will be shown to follow from symmetry (Theorem 3.6.3).

## 3.4. Comparing two independence relations

In this section, we prove the main result of this chapter (Canonicity of forking), modulo some extra hypotheses that will be eliminated in Section 3.5. After discussing some preliminary lemmas, we introduce a strengthening of the extension property,  $(E_+)$ , which plays a crucial role in the proof. We then prove canonicity using  $(E_+)$  (Corollary 3.4.8). Finally, we show  $(E_+)$  follows from some of the more classical properties that we had previously introduced (Corollary 3.4.13), obtaining the main result of this section (Corollary 3.4.14). We conclude by giving some examples showing our hypotheses are close to optimal.

For the rest of this section, we fix two independence relations  $\downarrow$  and  $\downarrow$ . Recall from Definition 3.3.1 that this means they satisfy (I), (M) and (B). We aim to

show that if  $\downarrow$  and  $\downarrow$  satisfy enough of the properties introduced in Section 3.3, (1) (2) then  $\downarrow = \downarrow$ .

The first easy observation is that given some uniqueness, only one direction is necessary<sup>7</sup>:

LEMMA 3.4.1. Let M be a model. Assume:

	(1)	(2)
(1)	$\downarrow$	
	M	M

<sup>&</sup>lt;sup>4</sup>Since this chapter was first circulated, a stronger result has been proven (for example one need not assume (E)). See Theorem 2.5.15.

<sup>5</sup>See [**BG**, Definition 4.2].

<sup>6</sup>All the properties mentioned in this Lemma are valid for models of size  $\geq \kappa$  only.

<sup>&</sup>lt;sup>7</sup>Shelah states as an exercise a variation of this lemma in [She09a, Exercise II.6.6.(1)].

(2) 
$$(E^{(1)})_M, (U^{(2)})_M$$
  
 $\Gamma hen \downarrow_M^{(1)} = \downarrow_M^{(2)}.$ 

PROOF. Assume  $A \stackrel{(2)}{\underset{M}{\downarrow}} N$ . By  $(E^{(1)})_M$ , find  $A' \equiv_M A$  so that  $A' \stackrel{(1)}{\underset{M}{\downarrow}} N$ . By hypothesis (1),  $A' \stackrel{(2)}{\underset{M}{\downarrow}} N$ . By  $(U^{(2)})_M$ ,  $A' \equiv_N A$ . By  $(I^{(1)})_M$ ,  $A \stackrel{(1)}{\underset{M}{\downarrow}} N$ .

With a similar idea, one can relate an arbitrary independence relation to non-splitting<sup>8</sup>:

LEMMA 3.4.2. Assume  $(U)_M$ . Then  $\bigcup_M \subseteq \bigcup_M^{(ns)}$ .

PROOF. Assume  $A \underset{M}{\downarrow} N$ . Let  $M \leq_{\mathbf{K}} N_1, N_2 \leq_{\mathbf{K}} N$  and  $h : N_1 \equiv_M N_2$ . By monotonicity,  $A \underset{M}{\downarrow} N_\ell$  for  $\ell = 1, 2$ . By invariance,  $h[A] \underset{M}{\downarrow} N_2$ . By  $(U)_M$ , there is  $f : A \equiv_{N_2} h[A]$  with  $f \upharpoonright A = h \upharpoonright A$ . By Proposition 3.3.11,  $A \underset{M}{\overset{(ns)}{\downarrow}} N$ .

A similar result holds for  $\bigcup$ , see Lemma 3.5.6.

The following consequence of invariance will be used repeatedly:

LEMMA 3.4.3. Assume  $\downarrow$  satisfies  $(E_1)_M$ . Assume  $A \underset{M}{\downarrow} N$ , and  $N' \ge_{\mathbf{K}} N$ . Then there is  $N'' \equiv_N N'$  such that  $A \underset{M}{\downarrow} N''$ .

PROOF. By  $(E_1)_M$ , there is  $f : A' \equiv_N A$ ,  $A' \underset{M}{\downarrow} N'$ . Thus  $f : (A', N') \equiv_N (A, f[N'])$ , so letting N'' := f[N'] and applying invariance, we obtain  $A \underset{M}{\downarrow} N''$ .  $\Box$ 

Even though we will not use it, we note that an analogous result holds for left extension, see Lemma 3.5.8.

We now would like to strengthen Lemma 3.4.3 as follows: suppose we are given  $A, M \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$ , and assume N is "very big" (e.g. it is  $(2^{|A|+||N_0||})^+$ -saturated), but does not contain A. Can we find  $N'_0 \equiv_M N_0$  with  $A \downarrow N'_0$ , and  $N'_0 \leq_{\mathbf{K}} N$ ?

We give this property a name:

DEFINITION 3.4.4 (Strong extension). An independence relation  $\downarrow$  has  $(E_+)$  (strong extension) if for any  $M \leq_{\mathbf{K}} N_0$  and any set A, there is  $N \geq_{\mathbf{K}} N_0$  such that for all  $N' \equiv_{N_0} N$ , there is  $N'_0 \equiv_M N_0$  with  $A \downarrow N'_0$  and  $N'_0 \leq_{\mathbf{K}} N'$ .

Intuitively,  $(E_+)$  says that no matter which isomorphic copy N' of N we pick, even if N' does not contain A, N' is so big that we can still find  $N'_0$  inside N' with the right property. This is stronger than (E) in the following sense:

PROPOSITION 3.4.5. If  $\downarrow$  has  $(E_+)$ ,  $\downarrow$  has  $(E_0)$ . If in addition  $\downarrow$  has  $(T_*)$ , then  $\downarrow$  has  $(E_1)$ . Thus if  $\downarrow$  has  $(E_+)$  and  $(T_*)$ , it has (E).

<sup>&</sup>lt;sup>8</sup>Shelah gives a variation of this lemma in [She09a, Claim III.2.20.(1)].
PROOF. Use monotonicity and Remark 3.3.3.

REMARK 3.4.6. Example 3.4.15 shows  $(E_+)$  does *not* follow from (E). Strong extension allows us to prove canonicity:

LEMMA 3.4.7. Assume  $(E_1^{(1)})_M, (E_+^{(2)})_M$ . Assume also that  $\bigcup_M^{(1)} \subseteq \bigcup_M^{(ns)}$ . Then  $\bigcup_M^{(1)} \subseteq \bigcup_M^{(2)}$ .

PROOF. Assume  $A \downarrow_{M}^{(1)} N_{0}$ . We show  $A \downarrow_{M}^{(2)} N_{0}$ . Fix  $N \geq_{\mathbf{K}} N_{0}$  as described by  $(E_{+}^{(2)})_{M}$ . By Lemma 3.4.3, we can find  $N' \equiv_{N_{0}} N$  such that  $A \downarrow_{M}^{(1)} N'$ . By definition

of N, one can pick  $N'_0 \equiv_M N_0$  with  $N'_0 \leq_{\mathbf{K}} N'$  and  $A \bigcup_{M}^{(2)} N'_0$ .

We have  $A \stackrel{(ns)}{\underset{M}{\bigcup}} N', M \leq_{\mathbf{K}} N'_0, N_0 \leq_{\mathbf{K}} N'$ , and  $N'_0 \equiv_M N_0$ , so by definition of (2)

nonsplitting,  $N'_0 \equiv_{AM} N_0$ . By invariance,  $A \bigcup_{M}^{(2)} N_0$ , as needed.  $\Box$ 

COROLLARY 3.4.8 (Canonicity of forking from strong extension). Assume:

•  $(U^{(1)})_M, (E^{(1)})_M.$ •  $(U^{(2)})_M, (E^{(2)}_+)_M.$ Then  $\bigcup_{M=1}^{(1)} = \bigcup_{M=1}^{(2)}.$ 

PROOF. By Lemma 3.4.1, it is enough to see  $\bigcup_{M}^{(1)} \subseteq \bigcup_{M}^{(2)}$ . By Lemma 3.4.2,  $\bigcup_{M}^{(1)} \subseteq \bigcup_{M}^{(ns)}$ . The result now follows from Lemma 3.4.7.

We now proceed to show that  $(E_+)$  follows from (E),  $(T_*)$ , (S) and (L). We will use the following important concept:

DEFINITION 3.4.9 (Independent sequence). Let I be a linearly ordered set. A sequence of sets  $(A_i)_{i \in I}$  is *independent* over a model M if there is a strictly increasing continuous chain of models  $(N_i)_{i \in I}$  such that for all  $i \in I$ :

(1)  $M \cup \bigcup_{j < i} A_j \subseteq N_i$  and  $N_0 = M$ . (2)  $A_i \underset{M}{\downarrow} N_i$ .

This generalizes the notion of independent sequence from the first-order case. The most natural definition would only require  $A_i \overline{\bigcup}_M \bigcup_{j < i} A_j$  (for some closure  $\overline{\bigcup}$  of  $\overline{\bigcup}$ ) but it turns out it is convenient to have a sequence of models  $(N_i)_{i \in I}$  witnessing the independence in a uniform way.

We note that very similar definitions appear already in the litterature. See [JS12, Definition 3.2], [She09a, Section III.5], or [She09b, Definition V.D.3.15].

Just like in the first-order case, the extension property allows us to build independent sequences:

LEMMA 3.4.10 (Existence of independent sequences). Assume  $(E)_M$ . Let A be a set, and let  $\delta$  be an ordinal. Then there is a sequence  $(A_i)_{i<\delta}$  independent over M so that  $A_i \equiv_M A$  for all  $i < \delta$ , and  $A_0 = A$ .

PROOF. Define the  $(A_i)_{i<\delta}$  and the  $(N_i)_{i<\delta}$  witnessing the independence of the sequence by induction on  $i < \delta$ . Take  $N_0 = M$  and  $A_0 = A$ . Assume inductively  $(A_j)_{j<i}, (N_j)_{j<i}$  have been defined. If i is a limit, let  $N_i := \bigcup_{j<i} N_j$ . If i is a successor, let  $N_i$  be any model containing  $M \cup \bigcup_{j<i} (A_j \cup N_j)$  and strictly extending the previous  $N_j$ 's. By  $(E)_M$ , there is  $A_i \equiv_M A$  such that  $A_i \underset{M}{\downarrow} N_i$ . Thus  $(A_i)_{i<\delta}$  is as desired.

The next result is key to the proof of  $(E_+)$ . It is adapted from [**Bal88**, Theorem II.2.18].

LEMMA 3.4.11. Assume  $\downarrow$  has  $(S), (T_*)_M, (L)$ . Let A be a set, and let  $\mu := \kappa_{|A|}(\downarrow)$ . Then whenever  $(M_i)_{i < \mu}$  is an independent sequence over M with  $M \leq_{\mathbf{K}} M_i$  for all i, there is  $i < \mu$  with  $A \downarrow M_i$ .

PROOF. Let  $(N_i)_{i < \mu}$  witness independence of the  $M_i$ 's. Let  $N_{\mu} := \bigcup_{i < \mu} N_i$ . By definition of  $\mu$ , there is  $i < \mu$  so that  $A \underset{N_i}{\downarrow} N_{\mu}$ . By (S), there is a model  $N_A$ with  $N_i \leq_{\mathbf{K}} N_A$ ,  $A \subseteq N_A$ , and  $N_{\mu} \underset{N_i}{\downarrow} N_A$ . By (M),  $M_i \underset{N_i}{\downarrow} N_A$ . Since the  $M_i$ 's are independent, we also have  $M_i \underset{M}{\downarrow} N_i$ . By  $(T_*)_M$ ,  $M_i \underset{M}{\downarrow} N_A$ . By (S) (recall that  $M \leq_{\mathbf{K}} M_i$ ),  $N_A \underset{M}{\downarrow} M_i$ . By (M),  $A \underset{M}{\downarrow} M_i$ , as desired.  $\Box$ 

REMARK 3.4.12. The same proof works if we replace  $\downarrow$  by its minimal closure  $\overline{\downarrow}$ , and  $(M_i)_{i < \mu}$  by an arbitrary sequence  $(B_i)_{i < \mu}$  independent over M.

COROLLARY 3.4.13. Assume  $(E)_M$ , (S),  $(T_*)_M$ , and (L). Then  $(E_+)_M$ .

PROOF. Fix A and  $N_0 \geq_{\mathbf{K}} M$ . Let  $\mu := \kappa_{|A|}(\downarrow)$ . By Lemma 3.4.10, there is a sequence  $(M_i)_{i < \mu}$  independent over M such that  $M_i \equiv_M N_0$  for all  $i < \mu$ , and  $M_0 = N_0$ . Let  $(N'_i)_{i < \mu}$  witness independence of the  $M_i$ 's. We claim  $N := \bigcup_{i < \mu} N'_i$ is as required. By construction,  $N_0 = M_0 \leq_{\mathbf{K}} N$ .

Now let  $f : N \equiv_{N_0} N'$ . Let  $M'_i := f[M_i]$ . Invariance implies  $(M'_i)_{i < \mu}$  is an independent sequence over M inside N', with  $M'_i \equiv_M N_0$  for all  $i < \mu$ . By Lemma 3.4.11, there is  $i < \mu$  so that  $A \downarrow M'_i$ , so  $N'_0 := M'_i$  is exactly as needed.

COROLLARY 3.4.14. Assume:

•  $(E^{(1)})_M, (U^{(1)})_M.$ •  $(E^{(2)})_M, (U^{(2)})_M, (L^{(2)}), (S^{(2)}), (T^{(2)}_*)_M.$ Then  $\bigcup_M^{(1)} = \bigcup_M^{(2)}.$ 

PROOF. Combine Corollaries 3.4.8 and 3.4.13.

We will see (Corollary 3.5.17) that (S) and  $(T_*)$  follow from (E), (U), and (L). We now argue that the other hypotheses are necessary. The following example (versions of which appears at various places in the literature, e.g. [She09a, Example II.6.4], [Adl09a, Example 6.6]) shows we cannot remove the local character assumption from Corollary 3.4.14. In particular,  $(E_{+})$  does not follow from (E) and (U) alone. The example also shows the AxFr<sub>3</sub> framework (see [She09b, Definition V.B.1.9) is not canonical.

EXAMPLE 3.4.15. Let  $T_{ind}$  be the first-order theory of the random graph, and let **K** be the class of models of  $T_{ind}$ , ordered by first-order elementary substructure. Define

- A ↓ N iff A ∩ N ⊆ M, and there are no edges between A\M and N\M.
  A ↓ N iff A ∩ N ⊆ M, and all the possible cross edges between A\M and N\M are present.

(2)(1)

It is routine to check that both  $\bigcup_{i=1}^{n-1}$  and  $\bigcup_{i=1}^{n-1}$  are independence relations with (E),

 $(U), (S), (T), (T_*), (C)_{\aleph_0}$ . Yet  $\stackrel{(1)}{\downarrow} \neq \stackrel{(2)}{\downarrow}$ , so one knows from Corollary 3.4.14 (or from first-order stability theory) that K can have no independence relation which in addition has (L) or  $(E_+)$ .

Of course,  $T_{ind}$  is simple, so first-order nonforking will actually have  $(E_+)$ , local character, transitivity and symmetry (but not uniqueness).

A concrete reason  $(E_+)$  does not hold e.g. for  $\bigcup^{(1)}$  is that given  $M \leq_{\mathbf{K}} N_0$  one can pick  $a \notin N_0$  such that there is an edge from a to any element of  $N_0$ . Then for any  $N \geq_{\mathbf{K}} N_0$ , one can again pick  $N' \equiv_{N_0} N$ , disjoint from  $\{a\} \cup (N \setminus N_0)$  such that there is an edge from a to any element of N'. Then  $a \bigcup_{M}^{(1)} N'_0$  for  $N'_0 \leq_{\mathbf{K}} N$  implies  $N'_0 = M$ . Local character fails for a similar reason.

EXAMPLE 3.4.16. It is also easy to see that  $(E^{(2)})$  and  $(U^{(2)})$  are necessary in Corollary 3.4.14. Assume  $\downarrow$  has (E), (U), (S),  $(T_*)$ , and (L). Then the independence relation  $\downarrow^{(2)}$  defined by  $A \downarrow^{(2)}_M N$  for all A and  $M \leq_{\mathbf{K}} N$  satisfies

 $(E), (S), (T_*), (L), \text{ but not } (U), \text{ so is distinct from } \overset{(1)}{\downarrow}.$ Similarly define  $A \underset{M}{\overset{(2)}{\downarrow}} N$  if and only if  $M \leq_{\mathbf{K}} N$  and either both  $A \underset{M}{\overset{(1)}{\downarrow}} N$  and

 $||M|| \geq LS(\mathbf{K})^+$ , or M = N. Then  $\bigcup$  has  $(E_0)$ , (U), (S),  $(T_*)$  and (L), but does not have  $(E_1)_M$  if M is a model of size  $LS(\mathbf{K})$ . This last example was adapted from [Adl09a, Example 6.4].

REMARK 3.4.17. After the initial submission of the paper this chapter is based on, it was shown in Lemma 6.9.1 that (E) can be removed from the hypotheses of Corollary 3.4.14 (but one has to replace it by  $(C)_{\kappa}$ ) if one only wants the independence relations to agree over sufficiently saturated models.

#### 3.5. Relationship between various properties

In this section, we investigate some of relations between the properties introduced earlier. We first discuss the interaction between properties of an independence relation and properties of its closures, and show how to obtain transitivity from various other properties. We then show how to obtain symmetry from existence, extension, uniqueness, and local character (Corollary 3.5.17). This second part has a stability-theoretic flavor and most of it does not depend on the first part.

Most of the material in the first part of this section is not used in the rest of the chapter, but the concept of closure (Definition 3.3.4) felt unmotivated without it. Our investigation remains far from exhaustive, and leaves a lot of room for further work.

**3.5.1.** Properties of the minimal closure. Recall the notion of closure of an independence relation (Definition 3.3.4). We would like to know when we can transfer properties from an independence relation to its closures and vice-versa.

For an arbitrary closure, we can say little:

LEMMA 3.5.1. Let  $\overline{\downarrow}$  be a closure of  $\downarrow$ . Then:

- (1) A property in the following list holds for  $\downarrow$  if and only if it holds for  $\overline{\downarrow}$ :  $(T_*)_M, (E_0)_M, (L).$
- (2) If a property in the following list holds for  $\overline{\downarrow}$ , then it holds for  $\downarrow$ :  $(C)_{\kappa}$ ,  $(T)_M, (E_1)_M, (U)_M$ .

Proof.

- (1) Because those properties have the same definition for  $\downarrow$  and  $\overline{\downarrow}$ .
- (2) Straightforward from the definitions.

The minimal closure is more interesting. We start by generalizing Lemma 3.4.3:

LEMMA 3.5.2. Assume  $\downarrow$  satisfies  $(E_1)_M$ . Let  $\overline{\downarrow}$  be the minimal closure of  $\downarrow$ . Assume  $A \overline{\downarrow}_M C$ , and let B be an arbitrary set. Then there is  $B' \equiv_{MC} B$  such that  $A \overline{\downarrow}_M B'$ .

PROOF. Let N be a model containing C and M such that  $A \underset{M}{\downarrow} N$ . Let N' be a model containing NB. By Lemma 3.4.3, there is  $N'' \equiv_N N'$  such that  $A \underset{M}{\downarrow} N''$ . Now use monotonicity to get the result.

The next lemma tells us that the minimal closure is the only one that will keep the extension property:

LEMMA 3.5.3. Let  $\overline{\downarrow}$  be a closure of  $\downarrow$  and let  $\overline{\downarrow}$  be the minimal closure of  $\downarrow$ . Assume  $\downarrow$  has  $(E_1)_M$ . Then  $\overline{\downarrow}_M = \overline{\downarrow}_M$  if and only if  $\overline{\downarrow}$  has  $(E_1)_M$ .

PROOF. Assume first  $\overline{\bigcup}_M = \overline{\bigcup}_M$ . Let  $C \subseteq C'$ , and assume  $A \overline{\bigcup}_M C$ . Then by definition of the minimal closure, there exists  $N \ge_{\mathbf{K}} M$  containing C such that  $A \bigcup_M N$ . Let N' be a model containing N and C'. By  $(E_1)_M$  for  $\bigcup$ , there is  $A' \equiv_N A$ so that  $A' \bigcup_M N'$ . By monotonicity,  $A' \overline{\bigcup}_M C'$ , and since N contains  $C, A' \overline{\bigcup}_M C$ . Conversely, assume  $\overline{\downarrow}_M$  has  $(E_1)_M$ . We know already that  $\overline{\downarrow} \subseteq \overline{\downarrow}$ , so assume  $A \overline{\downarrow}_M C$ . Let N be a model containing M and C. By Lemma 3.5.2, there is  $N' \equiv_{MC} N$  so that  $A \overline{\downarrow}_M N'$ , so  $A \overline{\downarrow}_M C$ , as needed.

LEMMA 3.5.4. Let  $\overline{\bigcup}$  be the *minimal* closure of  $\bigcup$ . Then

- (1)  $(E)_M$  holds for  $\downarrow$  if and only if it holds for  $\overline{\downarrow}$ .
- (2)  $(S)_M$  holds for  $\downarrow$  if and only if it holds for  $\downarrow$ .
- (3) If  $\downarrow$  has  $(E)_M$ , then it has  $(U)_M$  if and only if  $\overline{\downarrow}$  does.
- (4) If  $\downarrow$  has (E), then it has (T) if and only if  $\downarrow$  does<sup>9</sup>.

Proof.

- (1) By Lemmas 3.5.1 and 3.5.3.
- (2) Straightforward from the definition of symmetry and monotonicity.
- (3) One direction holds by Lemma 3.5.1. For the other direction, assume  $\overline{\downarrow}$  has  $(E_1)_M$  and  $\downarrow$  has  $(U)_M$ . Assume  $A \overline{\downarrow}_M C$  and  $A' \overline{\downarrow}_M C$ , with  $f: A \equiv_M A'$ . Let N be a model containing MC such that  $A \downarrow N$ . By extension again, find  $h: A' \equiv_{MC} A''$  such that  $A'' \downarrow N$ . We know  $h' := h \circ f: A \equiv_M A''$ , so by uniqueness, there is  $h'': A \equiv_{N'} A''$ , and  $h'' \upharpoonright A = h' \upharpoonright A = (h \circ f) \upharpoonright A$ , so  $f \upharpoonright A = (h^{-1} \circ h'') \upharpoonright A$ . Therefore  $g:=h^{-1} \circ h''$  is the desired witness that  $A \equiv_{MC} A'$ .
- (4) Let  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ , and assume  $M_1 \overline{\bigcup}_{M_0} C$ ,  $M_2 \overline{\bigcup}_{M_1} C$ . Let N be an extension of  $M_1$  containing C such that  $M_2 \overline{\bigcup}_{M_1} N$ . Let  $\chi$  be a big cardinal, so that  $(V_{\chi}, \in)$  reflects enough set theory and contains  $NM_2$ . Let N' be what  $V_{\chi}$  believes is the monster model.

By Lemma 3.4.3, there is  $f: N'' \equiv_N N'$  such that  $M_2 \underset{M_1}{\bigcup} N''$ . Notice that  $CM_1 \subseteq N \subseteq N'$ , so since we took  $\chi$  big enough, we can apply the definition of the minimal closure *inside*  $V_{\chi}$  to get  $N'_0 \leq_{\mathbf{K}} N'$  containing  $M_0$ and C so that  $M_1 \underset{M_0}{\bigcup} N'_0$ . Let  $N_0 := f[N'_0]$ . By invariance,  $M_1 \underset{M_0}{\bigcup} N_0$ , and  $N_0 \leq_{\mathbf{K}} N''$ , so by monotonicity,  $M_2 \underset{M_1}{\bigcup} N_0$ , so by  $(T)_{M_0}$  for  $\bigcup$ ,  $M_2 \underset{M_0}{\bigcup} N_0$ . By monotonicity again,  $M_2 \underset{M_0}{\bigcup} C$ .

The following remains to be investigated:

QUESTION 3.5.5. Let  $\overline{\downarrow}$  be the minimal closure of  $\downarrow$ . Under what conditions does  $(C)_{\kappa}$  for  $\downarrow$  imply  $(C)_{\kappa}$  for  $\overline{\downarrow}$ ?

We can use Lemma 3.5.4 to prove a variation on Lemma 3.4.2.

LEMMA 3.5.6. Assume  $\downarrow$  has  $(E)_M$  and  $(U)_M$ . Then  $\downarrow_M \subseteq \bigcup_M^{\text{(nes)}}$ .

<sup>9</sup>More precisely, if  $\downarrow$  has  $(E)_{M_1}$ , and for  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ , we have that  $M_2 \downarrow N$ ,  $M_1 \downarrow N_0$  implies  $M_2 \downarrow N$ , then  $M_2 \overline{\downarrow}_{M_1} C$ ,  $M_1 \overline{\downarrow}_{M_0} C$  implies  $M_2 \overline{\downarrow}_{M_0} C$ .

PROOF. Let  $\overline{\downarrow}$  be the minimal closure of  $\downarrow$ . By Lemma 3.5.4,  $\overline{\downarrow}$  has  $(E)_M$  and  $(U)_M$ .

Assume  $A \underset{M}{\downarrow} N$ . Let  $C_1, C_2 \subseteq N$ , and  $h : C_1 \equiv_M C_2$ . By monotonicity,  $A \underset{M}{\downarrow} C_\ell$  for  $\ell = 1, 2$ . By invariance,  $h[A] \underset{M}{\downarrow} C_2$ . By  $(U)_M$ , there is  $f : A \equiv_{MC_2} h[A]$  with  $f \upharpoonright A = h \upharpoonright A$ . By (the proof of) Proposition 3.3.11,  $A \underset{M}{\overset{(\text{nes})}{\downarrow}} N$ .

QUESTION 3.5.7. Is the  $(E)_M$  hypothesis necessary?

We can also obtain a left version of Lemma 3.4.3:

LEMMA 3.5.8. Let  $\overline{\downarrow}$  be a closure of  $\downarrow$ . Assume  $\downarrow$  has  $(E)_N$ , and  $\overline{\downarrow}$  has  $(T)_{M_1}$ . Suppose that  $N \underset{M_1}{\downarrow} M_2$ , with  $N \ge_{\mathbf{K}} M_1$ . Then for all  $N' \ge_{\mathbf{K}} N$ , there exists  $N'' \equiv_N N'$  such that  $N'' \underset{M_1}{\downarrow} M_2$ .

In particular, this holds if  $\downarrow$  has (E) and (T).

PROOF. The last line follows from part (4) of Lemma 3.5.4 by taking  $\overline{\downarrow}$  to be the minimal closure of  $\downarrow$ .

To see the rest, let  $N_3$  be a model containing  $M_2N$ . By  $(E)_N$ , there is  $N'' \equiv_N N'$  such that  $N'' \underset{N}{\downarrow} N_3$ . Since  $M_2 \subseteq N_3$ ,  $N'' \underset{N}{\downarrow} M_2$ . By hypothesis,  $N \underset{M_1}{\downarrow} M_2$ . So since  $\overline{\downarrow}$  has  $(T)_{M_1}$ ,  $N'' \underset{M_1}{\downarrow} M_2$ . Since  $M_2 \geq_{\mathbf{K}} M_1$ ,  $N'' \underset{M_1}{\downarrow} M_2$ .  $\Box$ 

Finally, we can also use symmetry to translate between the transitivity properties:

LEMMA 3.5.9. Assume  $\downarrow$  has (S). Then:

(1) If  $\downarrow$  has  $(T_*)_{M_0}$ , then  $\downarrow$  has  $(T)_{M_0}$ .

(2) If  $\downarrow$  has  $(T)_{M_0}$  and (E), then it has  $(T_*)_{M_0}$ .

PROOF. Let  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ . Let  $\overline{\downarrow}$  be the minimal closure of  $\downarrow$ . By Lemma 3.5.4,  $\overline{\downarrow}$  has (S).

(1) By Lemma 3.5.1,  $\overline{\bigcup}$  has  $(T_*)_{M_0}$ . Now use symmetry.

(2) By part (4) of Lemma 3.5.4,  $\downarrow$  has  $(T)_{M_0}$ . Now use symmetry.

This gives us one way to obtain right transitivity for coheir:

COROLLARY 3.5.10. Assume 
$$\stackrel{(ch)}{\downarrow}$$
 has  $(S)$  and  $(E)$ . Then  $\stackrel{(ch)}{\downarrow}$  has  $(T_*)$ .  
PROOF. By Proposition 3.3.15,  $\stackrel{(ch)}{\downarrow}$  has  $(T)$ . Apply Lemma 3.5.9.

Another way to obtain right transitivity from other properties appears in [She09a, Claim II.2.18]:

LEMMA 3.5.11. Assume  $\downarrow$  has  $(E_1)_M$  and (U). Then  $\downarrow$  has  $(T_*)_M$ .

PROOF. Let  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ , and assume  $A \underset{M_0}{\downarrow} M_1$  and  $A \underset{M_1}{\downarrow} M_2$ . By  $(E_1)_M$ , there exists  $A' \equiv_{M_1} A$  such that  $A' \underset{M_0}{\downarrow} M_2$ . By base monotonicity,  $A' \underset{M_1}{\downarrow} M_2$ . By uniqueness,  $A \equiv_{M_2} A'$ . By invariance,  $A \underset{M_0}{\downarrow} M_2$ .

**3.5.2.** Getting symmetry. We prove that symmetry follows from (E), uniqueness and local character and deduce the main theorem of this chapter (Corollary 3.5.18). We start by assuming some stability. Recall the definition of the order property (Definition 2.4.3). It is stronger than unstability:

FACT 3.5.12. Let  $\alpha$  be a cardinal. If **K** has the  $\alpha$ -order property, then **K** is  $\alpha$ -unstable.

PROOF SKETCH. This is [She99, Claim 4.7.2]. Shelah's proof is "Straight.", so we elaborate a little.

Let  $\lambda > LS(\mathbf{K})$ . We show **K** is  $\alpha$ -unstable in  $\lambda$ . Let  $I \subseteq I$  be linear orderings such that  $||I|| \leq \lambda$ ,  $||\hat{I}|| > \lambda$ , and I is dense in  $\hat{I}$ . Combining Shelah's presentation theorem with Morley's method, we can get a sequence  $\hat{\mathbf{I}} := \left\langle \bar{a}_i \mid i \in \hat{I} \right\rangle$  with  $\ell(\bar{a}_i) =$  $\alpha$  and  $i_0 < j_0, i_1 < j_1$  implies  $\bar{a}_{i_0} \bar{a}_{j_0} \neq \bar{a}_{j_1} \bar{a}_{i_1}$ . Let  $\mathbf{I} := \langle \bar{a}_i \mid i \in I \rangle$ .

Now for any i < j in  $\hat{I}$ ,  $\bar{a}_i \not\equiv_{\mathbf{I}} \bar{a}_j$ . Indeed, pick i < k < j with  $k \in I$ . Then  $\bar{a}_i \bar{a}_k \neq \bar{a}_j \bar{a}_k$  by construction, so  $\bar{a}_i \neq_{\bar{a}_k} \bar{a}_j$ . This completes the proof that **K** is  $\alpha$ -unstable in  $\lambda$ . 

We are now ready to prove symmetry. The argument is similar to [She90, Theorem III.4.13] or [She75a, Theorem 5.1].

THEOREM 3.5.13 (Symmetry). Assume  $\downarrow$  has  $(E)_M$  and  $\downarrow \subseteq \bigcup_M^{(nes)}$ . Assume in addition that **K** does not have the order property. Then  $\downarrow$  has  $(S)_M$ .

**PROOF.** Let  $\overline{\downarrow}$  be the minimal closure of  $\downarrow$ . Recall that by Lemma 3.5.4,  $\downarrow$ has  $(S)_M$  if and only if  $\overline{\bigcup}$  has  $(S)_M$ .

Assume for a contradiction  $\bigcup$  does not have  $(S)_M$ . Pick A and  $M \leq_{\mathbf{K}} N$  such that  $A \downarrow N$ , but  $N \not\downarrow o_M A$ . Let  $\lambda$  be an arbitrary uncountable cardinal. We will show that **K** has the (||N|| + |A|)-order property of length  $\lambda$ . This will contradict the assumption that **K** does not have the order property.

We will build increasing continuous  $\langle M_{\alpha} \in \mathbf{K} : \alpha < \lambda \rangle$ , and  $\langle A_{\alpha}, M'_{\alpha}, N_{\alpha} : \alpha < \lambda \rangle$  $\lambda$  by induction so

- (1)  $M_0 \geq_{\mathbf{K}} N$  and  $A \subseteq |M_0|$ .
- (2)  $N_{\alpha} \equiv_M N$  and  $N_{\alpha} \leq_{\mathbf{K}} M'_{\alpha}$
- (3)  $A_{\alpha} \equiv_N A$  and  $A_{\alpha} \subseteq M_{\alpha+1}$ .
- (4)  $M_{\alpha} \leq_{\mathbf{K}} M'_{\alpha} \leq_{\mathbf{K}} M_{\alpha+1}$ . (5)  $N_{\alpha} \downarrow M_{\alpha}$  and  $A_{\alpha} \downarrow M'_{\alpha}$ .

This is possible. Let  $M_0$  be any model containing AN. At  $\alpha$  limits, let  $M_{\alpha} :=$  $\bigcup_{\beta < \alpha} M_{\beta}$ . Now assume inductively that  $M_{\beta}$  has been defined for  $\beta \leq \alpha$ , and  $A_{\beta}$ ,  $N_{\beta}, M_{\beta}'$  have been defined for  $\beta < \alpha$ . Use  $(E)_M$  to find  $N_{\alpha} \equiv_M N$  with  $N_{\alpha} \downarrow M_{\alpha}$ . Now pick  $M'_{\alpha} \geq M_{\alpha}$  containing  $N_{\alpha}$ . Now, by  $(E)_M$  again, find  $A_{\alpha} \equiv_N \stackrel{M}{A}$  with  $A_{\alpha} \downarrow \stackrel{M}{\downarrow} \stackrel{M'_{\alpha}}{}$ . Pick  $M_{\alpha+1} \geq_{\mathbf{K}} M_{\alpha}$  containing  $A_{\alpha}$  and  $M'_{\alpha}$ .

This is enough. We show that for  $\alpha, \beta < \lambda$ :

- (1) If  $\beta < \alpha$ ,  $(A, N) \not\equiv_M (A_\beta, N_\alpha)$ .
- (2) If  $\beta \ge \alpha$ ,  $(A, N) \equiv_M (A_\beta, N_\alpha)$ .

For (1), suppose  $\beta < \alpha$ . Since  $A \subseteq M \leq_{\mathbf{K}} M_{\alpha}$ , we have  $N_{\alpha} \downarrow_{M} A$ . Then we can use the invariance of  $\downarrow$  and the assumption of no symmetry to conclude  $(A, N_{\alpha}) \not\equiv_{M}$ (A, N). On the other hand, we know that  $N_{\alpha} \downarrow_{M}^{(nes)} M_{\alpha}$ . Since  $A, A_{\beta} \subseteq M_{\alpha}$  and  $A \equiv_{M} A_{\beta}$ , we must have  $(A, N_{\alpha}) \equiv_{M} (A_{\beta}, N_{\alpha})$ . Thus  $(A, N) \not\equiv_{M} (A_{\beta}, N_{\alpha})$ . To see (2), suppose  $\beta \geq \alpha$  and recall that  $(A, N) \equiv_{M} (A_{\beta}, N)$ . We also have

that  $A_{\beta} \stackrel{(\text{nes})}{\underset{M}{\longrightarrow}} M'_{\beta}$ .  $N \equiv_M N_{\alpha}$  and  $N, N_{\alpha} \subseteq M'_{\beta}$ , the definition of non explicit splitting implies that  $(A_{\beta}, N) \equiv_M (A_{\beta}, N_{\alpha})$ . This gives us that  $(A, N) \equiv_M (A_{\beta}, N_{\alpha})$  as desired.

REMARK 3.5.14. The same proof can be used to obtain symmetry in the good frame framework. This is used in the construction of a good frame of Chapter 4.

COROLLARY 3.5.15. Assume **K** does not have the order property. Assume  $\downarrow$  has  $(E)_M$  and  $(U)_M$ . Then  $\downarrow$  has  $(S)_M$ .

PROOF. By Lemma 3.5.6, 
$$\bigcup_{M} \subseteq \bigcup_{M}^{\text{(nes)}}$$
. Now apply Theorem 3.5.13.

If in addition we assume local character, we obtain the "no order property" hypothesis:

LEMMA 3.5.16. Assume  $\downarrow$  has (U) and (L) (or just  $\kappa_1(\downarrow) < \infty$ ). Then **K** is  $\alpha$ -stable for all  $\alpha$ . In particular, it does not have the order property.

PROOF. That  $\alpha$ -stability implies no  $\alpha$ -order property is the contrapositive of Fact 3.5.12. Now, assume (U) and let  $\mu := \kappa_1(\downarrow) < \infty$ . Fix a cardinal  $\alpha \ge 1$ . We want to see **K** is  $\alpha$ -stable. Since stability for types of length  $\alpha$  implies stability for types of length  $\beta$  when  $\beta < \alpha$ , we can assume without loss of generality  $\alpha \ge \mu + \text{LS}(\mathbf{K})$ .

Let  $\lambda := \beth_{\alpha^+}$ . Then:

(1)  $\lambda$  is strong limit.

(2)  $\operatorname{cf}(\lambda) = \alpha^+ > \mu + \operatorname{LS}(\mathbf{K}).$ 

(3)  $\lambda^{\alpha} = \sup_{\gamma < \lambda} \gamma^{\alpha} = \lambda.$ 

We claim that **K** is  $\alpha$ -stable in  $\lambda$ . By Fact 2.2.25(1), it is enough to see it is 1stable in  $\lambda$ . Suppose not. Then there exists  $M \in \mathbf{K}_{\lambda}$ , and  $\{a_i\}_{i < \lambda^+}$  such that i < jimplies  $a_i \not\equiv_M a_j$ . Let  $(M_i)_{i < \lambda}$  be increasing continuous such that  $M = \bigcup_{i < \lambda} M_i$ and  $||M_i|| < \lambda$ . By definition of  $\mu$ , for each  $i < \lambda^+$ , there exists  $k_i < \lambda$  such that  $a_i \downarrow M$ . By the pigeonhole principle, we can shrink  $\{a_i\}_{i < \lambda^+}$  to assume without  $M_{k_i}$ 

loss of generality that  $k_i = k_0$  for all  $i < \lambda^+$ . Since there are at most  $2^{\|M_{k_0}\|} < \lambda$ many types over  $M_{k_0}$ , there exists  $i < j < \lambda^+$  such that  $a_i \equiv_{M_{k_0}} a_j$ . By uniqueness,  $a_i \equiv_M a_j$ , a contradiction.

COROLLARY 3.5.17. Assume  $\downarrow$  has  $(E)_M$ , (U) and (L) (or just  $\kappa_1(\downarrow) < \infty$ ). Then  $\downarrow$  has  $(S)_M$  and  $(T_*)_M$ .

PROOF. Lemma 3.5.11 gives  $(T_*)_M$ . Combine Lemma 3.5.16 and Corollary 3.5.15 to obtain  $(S)_M$ .

Thus we obtain another version of the canonicity theorem:

COROLLARY 3.5.18 (Canonicity of forking). Let  $\bigcup_{i=1}^{n-1}$  and  $\bigcup_{i=1}^{n-1}$  be independence relations. Assume:

•  $(E^{(1)})_M, (U^{(1)})_M.$ •  $(E^{(2)})_M, (U^{(2)}), (L^{(2)}).$ Then  $\bigcup_{M=1}^{(1)} = \bigcup_{M=1}^{(2)}.$ 

In particular, there can be at most one independence relation satisfying existence, extension, uniqueness, and local character.

PROOF. Combine Corollaries 3.4.14 and 3.5.17.

# 3.6. Applications

**3.6.1. Canonicity of coheir.** Fix a regular  $\kappa > LS(\mathbf{K})$ . Below, when we say coheir has a given property, we mean that it has that property for base models in  $\mathbf{K}_{\geq\kappa}$ .

We are almost ready to show that coheir is canonical, but we first need to show it has local character. We will use the following strengthening that deals with subsets instead of chains of models:

DEFINITION 3.6.1. Let  $\downarrow$  be an independence relation. For  $\alpha$  a cardinal, let  $\bar{\kappa}_{\alpha} = \bar{\kappa}_{\alpha}(\downarrow)$  be the smallest cardinal such that for all N, and all A with  $|A| = \alpha$ , there exists  $M \leq N$  with  $||M|| < \bar{\kappa}_{\alpha}$  and  $A \downarrow N$ .  $\bar{\kappa}_{\alpha} = \infty$  if there is no such cardinal.

REMARK 3.6.2. For all  $\alpha$ ,  $\kappa_{\alpha}(\downarrow) \leq \bar{\kappa}_{\alpha}(\downarrow)^{+}$ . Thus  $\bar{\kappa}_{\alpha}(\downarrow) < \infty$  implies  $\kappa_{\alpha}(\downarrow) < \infty$ .

THEOREM 3.6.3 (Local character for coheir). Assume  $\stackrel{(ch)}{\downarrow}$  has (S). Then  $\bar{\kappa}_{\alpha}(\downarrow) \leq ((\alpha+2)^{<\kappa})^+$ . In particular,  $\downarrow$  has (L).

The proof is similar to that of  $[Adl09b, Theorem 1.6]^{10}$ . The key is that  $\stackrel{(c\bar{h})}{\downarrow}$  always satisfies a dual to local character:

LEMMA 3.6.4. Let N, C be given. Then there is  $M \leq N$ ,  $||M|| \leq (|C|+2)^{<\kappa} + LS(\mathbf{K})$  such that  $N \underset{M}{\overset{(c\bar{\mathbf{h}})}{\underset{M}{\downarrow}}} C$ .

PROOF SKETCH. For each of the  $|C|^{<\kappa}$  small subsets of C, look at the  $\leq 2^{<\kappa}$  small types over that set (realized in N), and collect a realization of each in a set  $A \subseteq |N|$ . Then pick  $M \leq_{\mathbf{K}} N$  to contain A and be of the appropriate size.

We will also use the following application of the fact  $\stackrel{(ch)}{\downarrow}$  has  $(C)_{\kappa}$  and a strong form of base monotonicity.

<sup>&</sup>lt;sup>10</sup>After proving the result, we noticed that a similar argument also appears in the proof of  $(B)_{\mu}$  in [MS90, Proposition 4.8].

LEMMA 3.6.5. Let  $\lambda$  be such that  $\operatorname{cf} \lambda \geq \kappa$ . Let  $(A_i)_{i < \lambda}, (M_i)_{i < \lambda}, (C_i)_{i < \lambda}$  be (not necessarily strictly) increasing chains. Assume  $A_i \underset{M_i}{\stackrel{(ch)}{\longrightarrow}} C_i$  for all  $i < \lambda$ . Let  $A_{\lambda} := \bigcup_{i < \lambda} A_i$ , and define  $M_{\lambda}, C_{\lambda}$  similarly. Then  $A_{\lambda} \underset{M_{\lambda}}{\stackrel{(ch)}{\longrightarrow}} C_{\lambda}$ .

PROOF. From the definition of  $\stackrel{(\bar{ch})}{\downarrow}$ , we see that for all  $i < \lambda$ ,  $A_i \stackrel{(\bar{ch})}{\downarrow}_{M_{\lambda}} C_i$ . Now

use the fact that  $\stackrel{\text{(CI)}}{\downarrow}$  has  $(C)_{\kappa}$  (Proposition 3.3.15).

PROOF OF THEOREM 3.6.3. Fix  $\alpha$ , and let A and N be given with  $|A| = \alpha$ . Let  $\mu := (|A|+2)^{<\kappa}$ . Inductively build  $(M_i)_{i \leq \mu}$ ,  $(N_i)_{i \leq \mu}$  increasing continuous such that for all  $i < \mu$ :

(1) 
$$A \subseteq N_i$$
.  
(2)  $M_i \leq_{\mathbf{K}} N$ ,  $||M_i|| \leq \mu$ .  
(3)  $M_i \leq_{\mathbf{K}} N_{i+1}$ .  
(4)  $N \underset{M_{i+1}}{\bigcup} N_{i+1}$ .

This is enough: By König's lemma, cf  $\mu \geq \kappa$  so by Lemma 3.6.5,  $N \underset{M_{\mu}}{\bigcup} N_{\mu}$ . Moreover, by (2), (3) and the chain axioms,  $M_{\mu} \leq_{\mathbf{K}} N_{\mu}, N$ , and by (2),  $||M_{\kappa}|| \leq \mu$ . Thus  $N \underset{M_{\mu}}{\bigcup} N_{\mu}$ , and one can apply (S) to get  $N_{\mu} \underset{M_{\mu}}{\bigcup} N$ . By monotonicity,  $A \underset{M_{\mu}}{\bigcup} N$ , exactly as needed.

This is possible: Pick any  $A \subseteq N_0$  with  $||N_0|| \leq \mu$  (this is possible since  $\mu \geq |A| + \kappa > \mathrm{LS}(\mathbf{K})$ ). Now, given *i* non-limit,  $(N_j)_{j \leq i}$  and  $(M_j)_{j < i}$ , use Lemma 3.6.4 to find  $M_i \leq_{\mathbf{K}} N$ ,  $||M_i|| \leq (||N_i||)^{<\kappa} \leq \mu$ , such that  $N \bigcup_{M_i}^{(ch)} N_i$ . Then pick any  $N_{i+1}$  extending both  $M_i$  and  $N_i$ , with  $||N_{i+1}|| \leq \mu$ .

We finally have all the machinery to prove:

THEOREM 3.6.6 (Canonicity of coheir). Assume **K** is fully  $(< \kappa)$ -tame, fully  $(< \kappa)$ -type short, and has no weak  $\kappa$ -order property<sup>11</sup>.

Assume  $\stackrel{(ch)}{\downarrow}$  has (E). Then:

(ch)

- (1)  $\bigcup$  has  $(C)_{\kappa}$ , (T),  $(T_*)$ , (S), (U), and (L).
- (2) Any independence relation satisfying (*E*) and (*U*) must be  $\bigcup$  (for base models in  $\mathbf{K}_{\geq \kappa}$ ).

PROOF. By Proposition 3.3.15,  $\stackrel{\text{(ch)}}{\downarrow}$  has  $(C)_{\kappa}$  and (T). By Fact 3.3.16,  $\stackrel{\text{(ch)}}{\downarrow}$ 

has (U) and (S). By Corollary 3.5.10 (or Lemma 3.5.11),  $\stackrel{(m)}{\downarrow}$  also has  $(T_*)$ . By

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<sup>&</sup>lt;sup>11</sup>See [**BG**, Definition 4.2].

Theorem 3.6.3,  $\stackrel{\text{(ch)}}{\downarrow}$  has (L). This takes care of (1). (2) follows from (1) and Corollary 3.4.14.

COROLLARY 3.6.7 (Canonicity of coheir, assuming a strongly compact). Assume  $\kappa$  is strongly compact, all models in  $\mathbf{K}_{\geq\kappa}$  are  $\kappa$ -saturated, and there exists  $\lambda > \kappa$  such that  $I(\lambda, \mathbf{K}) < 2^{\lambda}$ . Then:

(1)  $\bigcup$  has  $(E), (C)_{\kappa}, (T), (T_*), (S), (U), \text{ and } (L).$ 

(ch)

(2) Any independence relation satisfying (E) and (U) must be  $\bigcup$  (for base models in  $\mathbf{K}_{>\kappa}$ ).

PROOF. By Proposition 3.3.15,  $\stackrel{(ch)}{\downarrow}$  has  $(E_0)$ . Thus by the moreover part of  $\stackrel{(ch)}{}_{(ch)}$  Fact 3.3.16,  $\stackrel{(ch)}{\downarrow}$  has (E). Now apply Theorem 3.6.6.

**3.6.2.** Canonicity of good frames. As has already been noted, the framework AxFri<sub>3</sub> defined in [She09b, Chapter V.B] is a precursor to our own, but Example 3.4.15 shows it is not canonical. Shelah also investigated an extension of AxFri<sub>3</sub> axiomatizing primeness (the "primal framework") but it is outside the scope of this chapter.

We will however briefly discuss the canonicity of good frames. Good frames were first defined in [She09a, Chapter II]. We will assume the reader is familiar with their definition and basic properties. As already noted, the main difference with our framework is that a good frame is local: For a fixed  $\lambda \geq LS(\mathbf{K})$ , a good

 $\lambda$ -frame assumes the existence of a nice independence relation  $\downarrow$  where only  $a \downarrow_M^{'} N$ 

is defined, for a an element of  $\widehat{N}$  and  $M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} \widehat{N}$  models of size  $\lambda$ .

In [She09a, Section II.6], Shelah shows that, assuming a technical condition (that the frame is weakly successful), one can extend it uniquely to a *non-forking* 

*frame*: basically an independence relation  $\downarrow$  where  $M_1 \downarrow_M^{\widehat{N}} M_2$  is defined for  $M \leq_{\mathbf{K}}$ 

 $M_{\ell} \leq_{\mathbf{K}} \widehat{N}$  in  $\mathbf{K}_{\lambda}$ ,  $\ell = 1, 2$ . For the rest of this section, we fix  $\lambda \geq \mathrm{LS}(\mathbf{K})$  and we do *not* assume the existence of a monster model (Hypothesis 3.2.1). Recall however that the definition of a good frame implies  $\mathbf{K}_{\lambda}$  has some nice properties, i.e. it has amalgamation, joint embedding, no maximal model, is stable<sup>12</sup> in  $\lambda$ , and has a superlimit model.

FACT 3.6.8. If  $\mathfrak{s}$  is a weakly successful good  $\lambda$ -frame, then it extends uniquely to a non-forking frame (i.e. using Shelah's terminology, there is a unique non-forking frame NF that respects  $\mathfrak{s}$ ).

PROOF. Uniqueness is [She09a, Claim II.6.3] and existence is [She09a, Conclusion II.6.34].  $\Box$ 

As Shelah observed, Example 3.4.15 shows that a non-forking frame by itself need not be unique: we need to know it comes from a good frame, or at least that there is a good frame around. Shelah showed:

(ch)

<sup>&</sup>lt;sup>12</sup>Really only stable for basic types, but full stability follows (see [She09a, Claim II.4.2.1]).

FACT 3.6.9. Assume that  $\mathfrak{s}$  is a good<sup>+</sup>  $\lambda$ -frame and NF is a non-forking frame, both with underlying AEC K. Then NF respects  $\mathfrak{s}$ .

PROOF. See [She09a, Claim II.6.7].

Here, good<sup>+</sup> is a technical condition asking for slightly more than just the original axioms of a good frame.

We also have that a non-forking frame induces a good frame:

FACT 3.6.10. Assume  $\mathbf{K}_{\lambda}$  has a superlimit, is stable in  $\lambda$ , and carries a nonforking frame NF (so in particular it has amalgamation) with independence relation

(defined for models in  $\mathbf{K}_{\lambda}$ )  $\downarrow$ . Then the relation  $a \downarrow N$  holds iff there is  $\widehat{N}' \geq_{\mathbf{K}} \widehat{N}$ and  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} \widehat{N}'$  with  $a \in M'$  so that  $M' \downarrow N$  defines a type-full (i.e. the basic types are all the nonalgebraic types) good  $\lambda$ -frame t. If in addition NF comes from a type-full weakly successful good  $\lambda$ -frame  $\mathfrak{s}$ , then  $\mathfrak{s} = \mathfrak{t}$ .

PROOF. See [She09a, Claim II.6.36].

Thus we obtain the following canonicity result:

COROLLARY 3.6.11. Assume that  $\mathfrak{s}_1$  is a weakly successful good<sup>+</sup>  $\lambda$ -frame and  $\mathfrak{s}_2$  is a weakly successful good  $\lambda$ -frame in the same underlying AEC K. Assume further  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  are type-full (i.e. their basic types are all the nonalgebraic types). Then  $\mathfrak{s}_1 = \mathfrak{s}_2$ .

**PROOF.** Using Fact 3.6.8, let  $NF_{\ell}$  be the non-forking frame extending  $\mathfrak{s}_{\ell}$  for  $\ell = 1, 2$ . By Fact 3.6.9, NF<sub>2</sub> respects  $\mathfrak{s}_1$ , so NF<sub>1</sub> = NF<sub>2</sub>. By Fact 3.6.10 (the existence of a good frame implies the stability and superlimit hypotheses), we must also have  $\mathfrak{s}_1 = \mathfrak{s}_2$ . 

The methods of this chapter can show slightly more: we can get rid of the  $good^+$ .

THEOREM 3.6.12 (Canonicity of good frames). Let  $\mathfrak{s}_1, \mathfrak{s}_2$  be weakly successful good  $\lambda$ -frames with underlying AEC **K** and the same basic types. Then  $\mathfrak{s}_1 = \mathfrak{s}_2$ .

**PROOF SKETCH.** Using Fact 3.6.8, let  $NF_{\ell}$  be the non-forking frame extending  $\mathfrak{s}_{\ell}$  for  $\ell = 1, 2$ . Let  $\stackrel{(1)}{\downarrow}, \stackrel{(2)}{\downarrow}$  be the independence relations (for models in  $\mathbf{K}_{\lambda}$ ) associated to  $NF_1$ ,  $NF_2$  respectively. By Fact 3.6.10, one can extend their domain to allow a single element on the left hand side. Thus without loss of generality we may assume  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  are type-full. Let  $M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} \widehat{N}$  and let  $a \in \widehat{N}$ . Assume  $a \stackrel{(1),\hat{N}}{\underset{M}{\downarrow}} N_0$ . We show  $a \stackrel{(2),\hat{N}}{\underset{M}{\downarrow}} N_0$ . The symmetric proof will show the converse is true,

and hence that  $\mathfrak{s}_1 = \mathfrak{s}_2$ .

First observe that stability, amalgamation, joint embedding and no maximal model in  $\lambda$  implies we can build a saturated (hence model-homogeneous) model  $\mathcal{M}$ of size  $\lambda^+$ . Since (as we will show) the argument below only uses objects of size  $\lambda$ , we can take  $\mathcal{M}$  to be our monster model for this argument (i.e. we assume any

set we consider comes from  $\mathcal{M}$ ). Then we have  $a \stackrel{(1),\widehat{N}}{\underset{M}{\downarrow}} N_0$  if and only if  $a \stackrel{(1),\mathcal{M}}{\underset{M}{\downarrow}} N_0$ 

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so below we drop  $\mathcal{M}$  and only talk about  $a \stackrel{(1)}{\underset{M}{\downarrow}} N_0$ , and similarly for  $\stackrel{(2)}{\downarrow}$ . Note that

working inside  $\mathcal{M}$  is not essential (we could always make the ambient model  $\tilde{N}$  grow bigger as our proof proceeds) but simplifies the notation and lets us quote our previous proofs verbatim.

Now, we observe that our proof of Corollary 3.4.14 is local-enough (i.e. it can be carried out inside  $\mathcal{M}$ ). We sketch the details: First build a sequence  $(M_i)_{i < \omega}$ independent (in the sense of  $\downarrow$ ) over  $\mathcal{M}$  so that  $M_0 = N_0$ ,  $M_i \equiv_{\mathcal{M}} N_0$ . Let  $N := \bigcup_{i < \omega} N'_i$ , where  $N'_i$  witness the independence of the sequence. Notice that we can take  $N \in \mathbf{K}_{\lambda}$ , by cardinality considerations. By extension, find  $f : N \equiv_{\mathcal{M}} \mathcal{N}'$ so that  $a \downarrow_{\mathcal{M}}^{(1)} \mathcal{N}'$ . Let  $M'_i := f[M_i]$ . By the proof of Lemma 3.4.11 (and recalling that  $\kappa_1(\downarrow) \le \omega$  in good frames), there is  $i < \omega$  such that  $a \downarrow_{\mathcal{M}}^{(2)} M'_i$  (notice that Fact

3.6.10 is what makes the argument go through). Finally, use the proof of Lemma 3.4.7 to conclude that  $a \downarrow N_0$ .

We do not know whether one can say more, namely:

QUESTION 3.6.13. Let  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  be good  $\lambda$ -frames with the same underlying AEC and the same basic types. Is  $\mathfrak{s}_1 = \mathfrak{s}_2$ ?

## 3.7. Conclusion

We have shown that an AEC with a monster model can have at most one "forking-like" notion. On the other hand, we believe the question of when such a forking-like notion exists is still poorly understood. For example, is there a natural condition implying that coheir has extension in Fact 3.3.16? Even the following is open:

QUESTION 3.7.1. Assume **K** is fully  $(< \kappa)$ -tame, fully  $(< \kappa)$ -type short and categorical in some high-enough  $\lambda > \kappa$ . Does **K** have an independence relation with (E), (U) and (L)?

Using the good frames machinery, an approximation is proven in [Bon14a] using some GCH-like hypotheses. However, the global assumptions of tameness and a monster model gives us a lot more power than just the local assumptions used to obtain a good frame.

It is also open whether such an independence relation has to be coheir (i.e. even if coheir does not satisfy (E)):

QUESTION 3.7.2. Assume  $\downarrow$  is an independence relation with (E), (U) and (L). Let M be sufficiently saturated. Under what conditions does  $\downarrow = \bigcup_{M}^{(ch)} M$ ?

Finally, we note that while some of our results are local and can be adapted to the good frames context (see e.g. Theorem 3.6.12), some are not (e.g. Theorem 3.5.13, Lemma 3.5.4.4). It would be interesting to know how much non-locality is really necessary for such results. This would help us understand how much power the globalness of our definition of independence relations really gives us.

# CHAPTER 4

# Forking and superstability in tame AECs

This chapter is based on [Vas16b]. I thank John T. Baldwin, Will Boney, Adi Jarden, Alexei Kolesnikov, and the anonymous referee for valuable comments that helped improve the presentation of this chapter.

## Abstract

We prove that any tame abstract elementary class categorical in a suitable cardinal has an eventually global good frame: a forking-like notion defined on all types of single elements. This gives the first known general construction of a good frame in ZFC. We show that we already obtain a well-behaved independence relation assuming only a superstability-like hypothesis instead of categoricity. These methods are applied to obtain an upward stability transfer theorem from categoricity and tameness, as well as new conditions for uniqueness of limit models.

# 4.1. Introduction

In 2009, Shelah published a two volume book [She09a, She09b] on classification theory for abstract elementary classes. The central new structural notion is that of a good  $\lambda$ -frame (for a given abstract elementary class (AEC) **K**): a generalization of first-order forking to types over models of size  $\lambda$  in **K** (see Section 4.2.2 below for the precise definition). The existence of a good frame shows that **K** is very well-behaved at  $\lambda$  and the aim was to use this frame to deduce more on the structure of **K** above  $\lambda$ . Part of this program has already been accomplished through several hundreds of pages of hard work (see for example [She01a], [She09a, Chapter 2 and 3], [JS12, JS13, JS, Jar]). Among many other results, Shelah shows that good frames exist under strong categoricity assumptions and additional set-theoretic hypotheses:

FACT 4.1.1 (Theorem II.3.7 in [She09a]). Assume  $2^{\lambda} < 2^{\lambda^+} < 2^{\lambda^{++}}$  and the weak diamond ideal in  $\lambda^+$  is not  $\lambda^{++}$ -saturated.

Let **K** be an abstract elementary class with  $LS(\mathbf{K}) \leq \lambda$ . Assume:

- (1) **K** is categorical in  $\lambda$  and  $\lambda^+$ .
- (2)  $0 < I(\lambda^{++}, \mathbf{K}) < \mu_{\text{unif}}(\lambda^{++}, 2^{\lambda^{+}})$

Then **K** has a good  $\lambda^+$ -frame.

It is a major open problem whether the set-theoretic hypotheses in Fact 4.1.1 are necessary. In this chapter, we show that if the class already has some *global* structure, then good frames are much easier to build. For example we prove, *in* ZFC (see Theorem 4.7.4):

THEOREM 4.1.2. Let **K** be an abstract elementary class with amalgamation and no maximal models. Assume **K** is categorical in a high-enough<sup>1</sup> successor  $\lambda^+$ . Then **K** has a type-full good  $\lambda$ -frame.

By the main theorem of [She99], the hypotheses of Theorem 4.1.2 imply **K** is categorical in  $\lambda$ . On the other hand, we do not need any set-theoretic hypothesis and we do not need to know anything about the number of models in  $\lambda^{++}$ . Moreover, the frame Shelah constructs typically defines a notion of forking only for a restricted class of basic types (the minimal types). With a lot of effort, he then manages to show [She09a, Section III.9] that under some set-theoretic hypotheses one can always extend a frame to be type-full. In our frame, forking is directly defined for every type. This is technically very convenient and closer to the first-order intuition. Of course, we pay for this luxury by assuming amalgamation and no maximal models<sup>2</sup>.

Our proof relies on two key properties of AECs. The first one is tameness (a locality property of Galois types, see Definition 4.2.3), and assuming it lets us relax the "high-enough successor" assumption in Theorem 4.1.2, see Theorem 4.7.3:

THEOREM 4.1.3. Let **K** be an abstract elementary class with amalgamation and no maximal models. Assume **K** is  $\mu$ -tame and categorical in some cardinal  $\lambda$ such that  $cf(\lambda) > \mu$ . Then **K** has a type-full good  $\geq \lambda$ -frame.

That is, not only do we obtain a good  $\lambda$ -frame, but we can also extend this frame to any model of size  $\geq \lambda$  (this last step essentially follows from earlier work of Boney [**Bon14a**]). Hence we obtain a global forking notion above  $\lambda$ , although only defined for 1-types. A forking notion for types of all lengths is obtained in [**BG**] (using stronger tameness hypotheses than ours) but the authors assume the extension property for coheir, and it is unclear when this holds, even assuming categoricity everywhere. Thus our result partially answers Question 3.7.1 (which asked when categoricity together with tameness implies the existence of a forking-like notion for types of all lengths satisfying uniqueness, local character, and extension). We also obtain new theorems whose statements do not mention frames:

COROLLARY 4.1.4. Let **K** be an abstract elementary class with amalgamation and no maximal models. Assume **K** is  $\mu$ -tame and categorical in some cardinal  $\lambda$ such that  $cf(\lambda) > \mu$ . Then **K** is stable everywhere.

REMARK 4.1.5. Shelah already established in [She99] that categoricity in  $\lambda >$  LS(K) implies stability below  $\lambda$  (assuming amalgamation and no maximal models). The first upward stability transfer for tame AECs appeared in [GV06b]. Later, [BKV06] gave some variations, showing for example  $\aleph_0$ -stability and a strong form of tameness implies stability everywhere. Our upward stability transfer improves on [BKV06, Corollary 4.7] which showed that categoricity in a successor  $\lambda$  implies stability in  $\lambda$ .

<sup>&</sup>lt;sup>1</sup>In fact,  $\lambda$  can be taken to be above  $h(h(h(\mathrm{LS}(\mathbf{K})))^+)$ , where  $h(\mu) = \beth_{(2^{\mu})^+}$ .

 $<sup>^{2}</sup>$ After submitting this chapter, we discovered that Shelah claims to build a good frame in ZFC from categoricity in a high-enough cardinal in Chapter IV of [She09a]. We were unable to fully check Shelah's proof. At the very least, our construction using tameness is simpler and gives much lower Hanf numbers.

#### 4.2. PRELIMINARIES

COROLLARY 4.1.6. Let **K** be an abstract elementary class with amalgamation and no maximal models. Assume **K** is  $\mu$ -tame and categorical in some cardinal  $\lambda$ such that  $cf(\lambda) > \mu$ . Then **K** has a unique limit model<sup>3</sup> in every  $\lambda' \ge \lambda$ .

REMARK 4.1.7. This is also new and complements the conditions for uniqueness of limit models given in [She99], [Van06], and [GVV16].

The second key property in our proof is a technical condition we call local character of  $\mu$ -splitting for  $\triangleleft$ -chains (see Definition 4.3.10). This follows from categoricity in a cardinal of cofinality larger than  $\mu$  and we believe it is a good candidate for a definition of superstability, at least in the tame context. Under this hypothesis, we already obtain a forking notion that is well-behaved for  $\mu^+$ -saturated base models and can prove the upward stability transfer given by Corollary 4.1.4. Local character of splitting already played a key role in other papers such as [SV99], [Van06], and [GVV16].

Even if this notion of superstability fails to hold, we can still look at the length of the chains for which  $\mu$ -splitting has local character (analogous to the cardinal  $\kappa(T)$  in the first-order context). Using GCH, we can generalize one direction of the first-order characterization of the stability spectrum (Theorem 4.7.6).

The chapter is structured as follows: In Section 4.2, we review background in the theory of AECs and give the definition of good frames. In Section 4.3, we fix a cardinal  $\mu$  and build a  $\mu$ -frame-like object named a *skeletal frame*. This is done using the weak extension and uniqueness properties of splitting isolated by VanDieren [**Van02**], together with the assumption of local character of splitting. In Section 4.4, we show that some of the properties of our skeletal frame in  $\mu$  lift to cardinals above  $\mu$  (and in fact become better than they were in  $\mu$ ). This is done using the same methods as in [**She09a**, Section II.2].

In Section 4.5, we show assuming tameness that the other properties of the skeletal frame lift as well and similarly become better, so that we obtain (if we restrict ourselves to  $\mu^+$ -saturated models and so, assuming categoricity in the right cardinal, to all models) all the properties of a good frame except perhaps symmetry. This uses the ideas from [**Bon14a**]. Next in Section 4.6 we show how to get symmetry by using more tameness together with the order property (this is where we really use that we have structure properties holding globally and not only at a few cardinals). Finally, we put everything together in Section 4.7. In Section 4.8, we conclude.

At the beginning of Sections 4.3, 4.4, 4.5, and 4.6, we give hypotheses that are assumed to hold everywhere in those sections. We made an effort to show clearly how much of the structural properties (amalgamation, tameness, superstability, etc.) are used at each step, but our construction is new even for the case of a totally categorical AEC  $\mathbf{K}$  with amalgamation, no maximal models, and LS( $\mathbf{K}$ )-tameness. It might help the reader to keep this case in mind throughout.

### 4.2. Preliminaries

We will use the following facts about transfering basic properties of AECs across cardinals:

FACT 4.2.1. Let  $\mathcal{F}$  be an interval of cardinals as above.

 $<sup>^{3}</sup>$ This holds even in the stronger sense of [**SV99**, Theorem 3.3.7], i.e. two limit models over the same base are isomorphic over the base.

- (1) If  $\mathbf{K}_{\mu}$  has no maximal models for all  $\mu \in \mathcal{F}$ , then  $\mathbf{K}_{\mathcal{F}}$  has no maximal models.
- (2) If  $\mathbf{K}_{\mu}$  has amalgamation for all  $\mu \in \mathcal{F}$ , then  $\mathbf{K}_{\mathcal{F}}$  has amalgamation.

PROOF. No maximal models is straightforward and amalgamation is [She09a, Conclusion I.2.12].  $\hfill \Box$ 

We will also use:

LEMMA 4.2.2. Let  $\mathcal{F} = [\lambda, \theta)$  be an interval of cardinals as above. If  $\mathbf{K}_{\mathcal{F}}$  has amalgamation and  $\mathbf{K}_{\lambda}$  has joint embedding, then  $\mathbf{K}_{\mathcal{F}}$  has joint embedding.

PROOF SKETCH. Let  $M_{\ell} \in \mathbf{K}_{\mathcal{F}}$ ,  $\ell = 1, 2$ . Pick  $M'_{\ell} \leq_{\mathbf{K}} M_{\ell}$  of size  $\lambda$ . Use joint embedding on  $M'_1, M'_2$ , then use amalgamation.

We write  $gS^{na}(M)$  for the set of *nonalgebraic* 1-types over M, that is:

$$gS^{na}(M) := \{gtp(a/M; N) \mid a \in N \setminus M, M \leq_{\mathbf{K}} N \in \mathbf{K}\}\$$

We will use the following notation for tameness:

DEFINITION 4.2.3 (Tameness). Let  $\lambda > \kappa \geq \mathrm{LS}(\mathbf{K})$ . Let  $\alpha$  be a cardinal. We say that  $\mathbf{K}$  is  $(\kappa, \lambda)$ -tame for  $\alpha$ -length types if for any  $M \in \mathbf{K}_{\leq \lambda}$  and any  $p, q \in S^{\alpha}(M)$ , if  $p \neq q$ , then there exists  $M_0 \in \mathbf{K}_{\leq \kappa}$  with  $M_0 \leq_{\mathbf{K}} M$  such that  $p \upharpoonright M_0 \neq q \upharpoonright M_0$ . We define similarly  $(\kappa, < \lambda)$ -tame,  $(< \kappa, \lambda)$ -tame, etc. When  $\lambda = \infty$ , we omit it. When  $\alpha = 1$ , we omit it. We say that  $\mathbf{K}$  is fully  $\kappa$ -tame if it is  $\kappa$ -tame for all lengths.

REMARK 4.2.4. If  $\alpha < \beta$ , and **K** is  $\beta$ -stable in  $\lambda$ , then **K** is  $\alpha$ -stable in  $\lambda$ .

The following follows from [Bon17, Theorem 3.1].

FACT 4.2.5. Let  $\lambda \geq \text{LS}(\mathbf{K})$ . Let  $\alpha$  be a cardinal. Assume  $\mathbf{K}$  is stable in  $\lambda$  and  $\lambda^{\alpha} = \lambda$ . Then  $\mathbf{K}$  is  $\alpha$ -stable in  $\lambda$ .

# 4.2.1. Universal and limit extensions.

DEFINITION 4.2.6 (Universal and limit extensions). For  $M, N \in \mathbf{K}$ , we say that N is universal over M (written  $M <_{\mathbf{K}}^{\mathrm{univ}} N$ ) if and only if M < N and for any  $M' \in \mathbf{K}_{\|M\|}$  with  $M' \geq_{\mathbf{K}} M$ , M' can be embedded inside N over M. We also write  $N >_{\mathbf{K}}^{\mathrm{univ}} M$  for  $M <_{\mathbf{K}}^{\mathrm{univ}} N$ .

For  $\mu \geq \mathrm{LS}(\mathbf{K})$  and  $0 < \delta < \mu^+$  an ordinal, we say that N is  $(\mu, \delta)$ -limit over M (written  $M <_{\mathbf{K}}^{\mu,\delta} N$ ) if and only if  $M, N \in \mathbf{K}_{\mu}, M \leq_{\mathbf{K}} N$ , and there is a  $<_{\mathbf{K}}^{\mathrm{univ}}$ increasing chain  $(M_i)_{i \leq \delta}$  with  $M_0 = M, M_{\delta} = N$  and  $M_{\delta} = \bigcup_{i < \delta} M_i$  if  $\delta$  is limit. We also write  $N >_{\mathbf{K}}^{\mu,\delta} M$  for  $M <_{\mathbf{K}}^{\mu,\delta} N$ .

We say that a model N is *limit* if it is  $(||N||, \gamma)$ -limit over M for some  $M \leq_{\mathbf{K}} N$ and some limit ordinal  $\gamma < \mu^+$ .

DEFINITION 4.2.7. A model  $N \in \mathbf{K}$  is  $\mu$ -model-homogeneous if for any  $M \leq_{\mathbf{K}} N$  with  $||M|| < \mu$ , we have  $M <_{\mathbf{K}}^{\text{univ}} N$ . N is model-homogeneous if it is ||N||-model-homogeneous.

FACT 4.2.8. Let  $\mu \geq \text{LS}(\mathbf{K})$ . Assume  $\mathbf{K}_{\mu}$  has amalgamation, no maximal models, and is stable. For any  $M \in \mathbf{K}_{\mu}$ , there exists  $N \in \mathbf{K}_{\mu}$  such that  $M <_{\mathbf{K}}^{\text{univ}} N$ . Therefore there is a model-homogeneous  $N \in \mathbf{K}_{\mu^+}$  with  $M <_{\mathbf{K}} N$ .

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**PROOF.** The first part is by [She09a, Claim II.1.16.1(a)]. The second part follows from iterating the first part  $\mu^+$  many times.

REMARK 4.2.9. By [She01a, Lemma 0.26], for  $\mu > LS(\mathbf{K})$ , N is  $\mu$ -modelhomogeneous if and only if it is  $\mu$ -saturated.

The next proposition is folklore and the results appear in several places in the literature (see for example [She99, Lemma 2.2]). For the convenience of the reader, we have included the proofs.

PROPOSITION 4.2.10. Let  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}, \mu \geq \mathrm{LS}(\mathbf{K})$  and  $0 < \delta < \mu^+$ . Then:

- (1)  $M_0 <_{\mathbf{K}}^{\mu,\delta} M_1$  implies  $M_0 <_{\mathbf{K}}^{\text{univ}} M_1$ . (2)  $M_0 <_{\mathbf{K}}^{\text{univ}} M_1 \leq_{\mathbf{K}} M_2$  implies  $M_0 <_{\mathbf{K}}^{\text{univ}} M_2$ .
- (3) Assume  $\mathbf{K}_{\mu}$  has amalgamation. Then  $M_0 \leq_{\mathbf{K}} M_1 <_{\mathbf{K}}^{\mu,\delta} M_2$  implies  $M_0 <^{\mu,\delta}_{\mathbf{K}} M_2.$
- (4) Assume  $\mathbf{K}_{\mu}$  has amalgamation, no maximal models, and is stable. Then there exists  $M'_0$  such that  $M_0 <^{\mu,\delta}_{\mathbf{K}} M'_0$ .
- (5) Conversely, if for every  $M_0 \in \mathbf{K}_{\mu}$  there exists  $M'_0 \in \mathbf{K}_{\mu}$  such that  $M_0 <_{\mathbf{K}}^{\mathrm{univ}}$  $M'_0$ , then  $\mathbf{K}_{\mu}$  has amalgamation, no maximal models, and is stable.

Proof.

- (1) Fix  $(N_i)_{i \leq \delta}$  witnessing that  $M_0 <_{\mathbf{K}}^{\mu,\delta} M_1$ . Let  $M'_0 \geq_{\mathbf{K}} M_0$  have size  $\mu$ . Since  $\delta > 0$ ,  $N_1$  is well defined, and is universal over  $N_0 = M_0$ , hence  $M'_0$ can be embedded inside  $N_1$  over  $M_0$ , and hence since  $N_1 \leq_{\mathbf{K}} M_1$  can be embedded inside  $M_1$  over  $M_0$ .
- (2) Let  $M'_0 \geq_{\mathbf{K}} M_0$  have size  $\mu$ . Since  $M'_0$  embed inside  $M_1$  over  $M_0$ , it also embeds inside  $M_2$  over  $M_0$ . (3) Let  $(N_i)_{i \leq \delta}$  witness  $M_0 <_{\mathbf{K}}^{\mu,\delta} M_1$ . We show that  $M_0 <_{\mathbf{K}}^{\text{univ}} N_1$ . This is
- enough since then  $M_0 \frown (N_i)_{0 \le i \le \delta}$  will witness that  $M_0^{-1} <_{\mathbf{K}}^{\mu,\delta} M_2$ . Let  $M'_0 \geq_{\mathbf{K}} M_0$  have size  $\mu$ . By amalgamation, find  $M'_1 \geq_{\mathbf{K}} M_1$  and  $h : M'_0 \xrightarrow{M_0} M'_1$ . Now use universality of  $M_2$  over  $M_1$  to find  $g : M'_1 \xrightarrow{M_1} M_2$ . Let  $f := g \circ h$ . Then  $f : M'_0 \xrightarrow{M_0} M_2$ , as desired.
- (4) Iterate Fact 4.2.8  $\delta$  many times.
- (5) Let  $M_0 \in \mathbf{K}_{\mu}$  and let  $M'_0 >^{\text{univ}}_{\mathbf{K}} M_0$  be in  $\mathbf{K}_{\mu}$ .  $M'_0$  witnesses that  $M_0$  is not maximal in  $\mathbf{K}_{\mu}$ . Moreover,  $M_0$  is an amalgamation base, since any two models of size  $\mu$  extending  $M_0$  can amalgamated over  $M_0$  inside  $M'_0$ . Finally, all types over  $M_0$  are realized in  $M'_0$  which has size  $\mu$ , there can be at most  $\mu$  many of them, so stability follows.

We give orderings satisfying the conclusion of Proposition 4.2.10 a name:

DEFINITION 4.2.11 (Abstract universal ordering). An abstract universal ordering on  $\mathbf{K}_{\mu}$  is a binary relation  $\triangleleft$  on  $\mathbf{K}_{\mu}$  satisfying the following properties. For any  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$ :

- (1)  $M_0 \triangleleft M_1$  implies  $M_0 <_{\mathbf{K}}^{\text{univ}} M_1$ .
- (2) There exists  $N_0 \in \mathbf{K}_{\mu}$  such that  $M_0 \triangleleft N_0$ .
- (3)  $M_0 \leq_{\mathbf{K}} M_1 \triangleleft M_2$  implies  $M_0 \triangleleft M_2$ .

(4) Closure under isomorphism: if  $M_0 \triangleleft M_1$  and  $f: M_1 \cong M'_1$ , then  $f[M_0] \triangleleft M'_1$ .

Note that this implies that  $\triangleleft$  is a strict partial ordering on  $\mathbf{K}_{\mu}$  extending  $<_{\mathbf{K}}$ . For  $0 < \delta < \mu^+$ , a model  $M \in \mathbf{K}_{\mu}$  is  $(\delta, \triangleleft)$ -limit if there exists a  $\triangleleft$ -increasing chain  $(M_i)_{i < \delta}$  in  $\mathbf{K}_{\mu}$  such that  $M = \bigcup_{i < \delta} M_i$ . M is  $\triangleleft$ -limit if there exists a limit  $\delta$  such that M is  $(\delta, \triangleleft)$ -limit.

REMARK 4.2.12. Assume  $\mathbf{K}_{\mu}$  has amalgamation, no maximal models, and is stable. Then by Proposition 4.2.10, for any  $0 < \delta < \mu^+$ ,  $<_{\mathbf{K}}^{\mu,\delta}$  is an abstract universal ordering on  $\mathbf{K}_{\mu}$ . Moreover, the existence of *any* abstract universal ordering on  $\mathbf{K}_{\mu}$  implies that  $<_{\mathbf{K}}^{\text{univ}}$  is an abstract universal ordering, and hence that  $\mathbf{K}_{\mu}$  has amalgamation, no maximal models, and is stable.

Let  $LS(\mathbf{K}) \leq \mu < \lambda$ . Even assuming stability everywhere, is is unclear whether there should be any model-homogeneous model in  $\lambda$  (think for example of the case  $cf(\lambda) = \omega$ ). The following tells us that we can at least get an approximation to one: we can do the usual construction of special models in a cardinal  $\lambda$  if **K** is stable below  $\lambda$ . This will be used in the proof of the superstability theorem (Theorem 4.5.6).

LEMMA 4.2.13. Let  $LS(\mathbf{K}) \leq \mu^+ < \lambda$ . Assume  $\mathbf{K}_{[\mu,\lambda)}$  has amalgamation, no maximal models, and is stable in  $\mu'$  for unboundedly many  $\mu < \mu' < \lambda$  (that is, for any  $\mu < \mu' < \lambda$ , there exists  $\mu' \leq \mu'' < \lambda$  such that  $\mathbf{K}_{\mu''}$  is stable).

For any  $N_0 \in \mathbf{K}_{[\mu,\lambda)}$ , there exists  $(N_i)_{i<\lambda} <_{\mathbf{K}}^{\text{univ-increasing continuous in } \mathbf{K}_{[\mu,\lambda)}$ with each  $N_{i+1} \mu^+$ -model-homogeneous. Moreover any  $M \in \mathbf{K}_{[\mu,\lambda]}$  such that  $N_0 \leq_{\mathbf{K}} M$  can be embedded inside  $N := \bigcup_{i<\lambda} N_i$  over  $N_0$ .

PROOF. We build  $(N_i)_{i < \lambda}$  by induction.  $N_0$  is already given and without loss of generality  $||N_0|| \ge \mu^+$ . Take unions at limits and for a given  $N_i$ , first take  $N'_i \ge N_i$  such that  $\mathbf{K}_{||N'_i||}$  is stable, and iterate Fact 4.2.8  $\mu^+$ -many times to pick  $N_{i+1} \in \mathbf{K}_{||N'_i||}$  which is also  $\mu^+$ -model-homogeneous such that  $N'_i <_{\mathbf{K}}^{\mathrm{univ}} N_{i+1}$  (and so by Proposition 4.2.10 also  $N_i <_{\mathbf{K}}^{\mathrm{univ}} N_{i+1}$ ). Now given  $M \in \mathbf{K}_{[\mu,\lambda]}$  with  $N_0 \leq_{\mathbf{K}} M$ , let  $(M_i)_{i \le \lambda}$  be an increasing continuous

Now given  $M \in \mathbf{K}_{[\mu,\lambda]}$  with  $N_0 \leq_{\mathbf{K}} M$ , let  $(M_i)_{i \leq \lambda}$  be an increasing continuous resolution of M such that  $||M_i|| < \lambda$  for all  $i < \lambda$  and  $M_0 = N_0$ . Inductively build  $(f_i)_{i \leq \lambda}$  an increasing continuous chain of  $\mathbf{K}$ -embeddings such that for each  $i \leq \lambda$ ,  $f_i : M_i \xrightarrow[M_0]{} N_i$ . This is easy since  $N_{i+1} >_{\mathbf{K}}^{\mathrm{univ}} N_i$  for all  $i < \lambda$ . Then  $f_{\lambda}$  embeds Minto N.

**4.2.2. Good frames.** Good frames were first defined in [She09a, Chapter II]. The idea is to provide a localized (i.e. only for base models of a given size  $\lambda$ ) axiomatization of a forking-like notion for (a "nice enough" set of) 1-types. Jarden and Shelah (in [JS13]) later gave a slightly more general definition, not assuming the existence of a superlimit model and dropping some of the redundant clauses. We will use a slight variation here: we assume the models come from  $\mathbf{K}_{\mathcal{F}}$ , for  $\mathcal{F}$  an interval, instead of just  $\mathbf{K}_{\lambda}$ . We first adapt the definition of a pre- $\lambda$ -frame from [She09a, Definition III.0.2.1] to such an interval:

DEFINITION 4.2.14 (Pre-frame). Let  $\mathcal{F}$  be an interval of the form  $[\lambda, \theta)$ , where  $\lambda$  is a cardinal, and  $\theta > \lambda$  is either a cardinal or  $\infty$ .

A pre- $\mathcal{F}$ -frame is a triple  $\mathfrak{s} = (\mathbf{K}, \downarrow, \mathbf{gS}^{\mathrm{bs}})$ , where:

#### 4.2. PRELIMINARIES

- (1) **K** is an abstract elementary class<sup>4</sup> with  $\lambda \geq \text{LS}(\mathbf{K}), \mathbf{K}_{\lambda} \neq \emptyset$ .
- (1) **K** is an abstract elementary class with  $\lambda \geq \text{LS}(\mathbf{R}), \mathbf{K}_{\lambda} \neq \emptyset$ . (2)  $\text{gS}^{\text{bs}} \subseteq \bigcup_{M \in \mathbf{K}_{\mathcal{F}}} \text{gS}^{\text{na}}(M)$ . For  $M \in \mathbf{K}_{\mathcal{F}}$ , we write  $\text{gS}^{\text{bs}}(M)$  for  $\text{gS}^{\text{bs}} \cap \text{gS}^{\text{na}}(M)$ . (3)  $\downarrow$  is a relation on quadruples of the form  $(M_0, M_1, a, N)$ , where  $M_0 \leq_{\mathbf{K}}$   $M_1 \leq_{\mathbf{K}} N, a \in N$ , and  $M_0, M_1, N$  are all in  $\mathbf{K}_{\mathcal{F}}$ . We write  $\downarrow (M_0, M_1, a, N)$ or  $a \bigcup_{M_0}^N M_1$  instead of  $(M_0, M_1, a, N) \in \bigcup$ .
- (4) The following properties hold:
  - (a) Invariance: If  $f: N \cong N'$  and  $a \downarrow_{M_0}^N M_1$ , then  $f(a) \downarrow_{f[M_0]}^{N'} f[M_1]$ . If  $gtp(a/M_1; N) \in gS^{bs}(M_1)$ , then  $gtp(f(a)/f[M_1]; N') \in gS^{bs}(f[M_1])$ .
  - (b) Monotonicity: If  $a \bigcup_{M_0}^N M_1$ ,  $M_0 \leq_{\mathbf{K}} M'_0 \leq_{\mathbf{K}} M'_1 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N' \leq_{\mathbf{K}}$  $N \leq_{\mathbf{K}} N'' \text{ with } a \in N' \text{ and } N'' \in \mathbf{K}_{\mathcal{F}}, \text{ then } a \bigcup_{M'_0}^{N'} M'_1 \text{ and } a \bigcup_{M'_0}^{N''} M'_1.$ (c) Nonforking types are basic: If  $a \bigcup_{M}^{N} M$ , then  $gtp(a/M; N) \in gS^{bs}(M)$ .

We write  $\lambda$ -frame instead of  $\{\lambda\}$ -frame,  $(\geq \lambda)$ -frame instead of  $[\lambda, \infty)$ -frame. We sometimes drop the  $\mathcal{F}$  when it is clear from context.

A pre-frame is *type-full* if  $gS^{bs}(M) = gS^{na}(M)$  for all  $M \in \mathbf{K}_{\mathcal{F}}$ .

For  $\mathcal{F}' \subseteq \mathcal{F}$  an interval, we let  $\mathfrak{s} \upharpoonright \mathcal{F}'$  denote the pre- $\mathcal{F}'$ -frame defined in the obvious way by restricting the basic types and  $\downarrow$  to models in  $\mathbf{K}_{\mathcal{F}'}$ . For  $\lambda' \in \mathcal{F}$ , we write  $\mathfrak{s} \upharpoonright \lambda'$  instead of  $\mathfrak{s} \upharpoonright \{\lambda'\}$ .

By the invariance and monotonicity properties,  $\downarrow$  is really a relation on types. This justifies the next definition.

DEFINITION 4.2.15. If  $\mathfrak{s} = (\mathbf{K}, \bigcup, \mathrm{gS}^{\mathrm{bs}})$  is a pre- $\mathcal{F}$ -frame,  $p \in \mathrm{gS}(M_1)$  is a type, we say p does not  $\mathfrak{s}$ -fork over  $M_0$  if  $a \bigcup_{M_0} M_1$  for some (equivalently any) a and Nsuch that  $p = \operatorname{gtp}(a/M_1; N)$ .

REMARK 4.2.16. A pre-frame defines an *abstract* notion of forking. That is, we only know that the relation  $\downarrow$  satisfies some axioms but it could a-priori be defined arbitrarily. Later in the chapter, we will study a specific definition of forking (based on splitting). While the specific definition we will give will coincide (over sufficiently saturated models) with first-order forking when the AEC is a class of models of a first-order theory, the reader should remember that we are working in much more generality than the first-order framework, hence most of the properties of first-order forking need not hold here.

REMARK 4.2.17. We could have started from  $(\mathbf{K}, \downarrow)$  and defined the basic types as those that do not fork over their own domain. The existence property of good frames (see below) would then hold for free. Since we are sometimes interested in studying frames that only satisfy existence over a certain class of models (like the saturated models), we will not adopt this approach.

<sup>&</sup>lt;sup>4</sup>In [She09a, Definition III.0.2.1], Shelah only asks that **K** contains the models of size  $\mathcal{F}$  of an AEC. For easy of exposition, we do not adopt this approach.

REMARK 4.2.18 (Monotonicity of  $\mathfrak{s}$ -forking). If  $\mathfrak{s} = (\mathbf{K}, \bigcup, \mathrm{gS}^{\mathrm{bs}})$  is a pre- $\mathcal{F}$ -frame,  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N_1 \leq_{\mathbf{K}} N_0$  are in  $\mathbf{K}_{\mathcal{F}}$ , and  $p \in \mathrm{gS}^{\mathrm{bs}}(N_0)$  does not  $\mathfrak{s}$ -fork over  $M_0$ , then by the monotonicity axiom,  $p \upharpoonright N_1$  does not  $\mathfrak{s}$ -fork over  $M_1$ . We will use this fact freely.

DEFINITION 4.2.19 (Good frame). Let  $\mathcal{F}$  be as above.

A good  $\mathcal{F}$ -frame is a pre- $\mathcal{F}$ -frame  $(\mathbf{K}, \, \downarrow, gS^{bs})$  satisfying in addition:

- (1)  $\mathbf{K}_{\mathcal{F}}$  has amalgamation, joint embedding, and no maximal model.
- (2) bs-Stability:  $|\mathbf{gS}^{\mathrm{bs}}(M)| \leq ||M||$  for all  $M \in \mathbf{K}_{\mathcal{F}}$ .
- (3) Density of basic types: If  $M <_{\mathbf{K}} N$  and  $M, N \in \mathbf{K}_{\mathcal{F}}$ , then there is  $a \in N$  such that  $gtp(a/M; N) \in gS^{bs}(M)$ .
- (4) Existence: If  $M \in \mathbf{K}_{\mathcal{F}}$  and  $p \in \mathrm{gS}^{\mathrm{bs}}(M)$ , then p does not  $\mathfrak{s}$ -fork over M.
- (5) Extension: If  $p \in gS(N)$  does not  $\mathfrak{s}$ -fork over M, and  $N' \in \mathbf{K}_{\mathcal{F}}$  is such that  $N' \geq_{\mathbf{K}} N$ , then there is  $q \in gS(N')$  extending p that does not  $\mathfrak{s}$ -fork over M.
- (6) Uniqueness: If  $p, q \in gS(N)$  do not  $\mathfrak{s}$ -fork over M and  $p \upharpoonright M = q \upharpoonright M$ , then p = q.
- (7) Symmetry: If  $a_1 \downarrow_{M_0}^N M_2$ ,  $a_2 \in M_2$ , and  $gtp(a_2/M_0; N) \in gS^{bs}(M_0)$ , then

there is  $M_1$  containing  $a_1$  and there is  $N' \geq_{\mathbf{K}} N$  such that  $a_2 \bigcup_{M_0}^{N'} M_1$ .

- (8) Local character: If  $\delta$  is a limit ordinal,  $(M_i)_{i \leq \delta}$  is an increasing chain in  $\mathbf{K}_{\mathcal{F}}$  with  $M_{\delta} = \bigcup_{i < \delta} M_i$ , and  $p \in \mathrm{gS}^{\mathrm{bs}}(M_{\delta})$ , then there exists  $i < \delta$  such that p does not  $\mathfrak{s}$ -fork over  $M_i$ .
- (9) Continuity: If  $\delta$  is a limit ordinal,  $(M_i)_{i \leq \delta}$  is an increasing chain in  $\mathbf{K}_{\mathcal{F}}$  with  $M_{\delta} = \bigcup_{i < \delta} M_i$ ,  $p \in \mathrm{gS}(M_{\delta})$  is so that  $p \upharpoonright M_i$  does not  $\mathfrak{s}$ -fork over  $M_0$  for all  $i < \delta$ , then p does not  $\mathfrak{s}$ -fork over  $M_0$ .
- (10) Transitivity<sup>5</sup>: If  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ ,  $p \in gS(M_2)$  does not  $\mathfrak{s}$ -fork over  $M_1$ and  $p \upharpoonright M_1$  does not  $\mathfrak{s}$ -fork over  $M_0$ , then p does not  $\mathfrak{s}$ -fork over  $M_0$ .

For  $\mathbb{L}$  a list of properties, a good<sup> $-\mathbb{L}$ </sup>  $\mathcal{F}$ -frame is a pre- $\mathcal{F}$ -frame that satisfies all the properties of good frames except possibly the ones in  $\mathbb{L}$ . In this chapter,  $\mathbb{L}$  will only contain symmetry and/or bs-stability. We abbreviate symmetry by S, bs-stability by St, and write good<sup>-(S,St)</sup>.

We say that **K** has a good  $\mathcal{F}$ -frame if there is a good  $\mathcal{F}$ -frame where **K** is the underlying AEC (and similarly for good<sup>-</sup>).

REMARK 4.2.20. Using  $\mathcal{F}$  instead of a single cardinal  $\lambda$  is only a convenience: just like an abstract elementary class **K** is determined by  $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}$ , a good<sup>-</sup>  $\mathcal{F}$ -frame  $\mathfrak{s}$  is determined by  $\mathfrak{s} \upharpoonright \lambda$ , where  $\lambda := \min(\mathcal{F})$ . More precisely, if  $\mathfrak{t}$  is a good<sup>-</sup>  $\mathcal{F}$ frame such that  $\mathfrak{t} \upharpoonright \lambda = \mathfrak{s} \upharpoonright \lambda$ , then the arguments from [**She09a**, Section II.2] show that  $\mathfrak{t} = \mathfrak{s}$ .

Note that local character implies nonforking is always witnessed by a model of small size:

PROPOSITION 4.2.21. Assume  $\mathcal{F}$  is an interval of cardinals with minimum  $\lambda$ . Assume  $\mathfrak{s} = (\mathbf{K}, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  is a pre- $\mathcal{F}$ -frame satisfying local character and transitivity.

<sup>&</sup>lt;sup>5</sup>This actually follows from uniqueness and extension, see [She09a, Claim II.2.18].

If  $M \in \mathbf{K}_{\mathcal{F}}$  and  $p \in \mathrm{gS}^{\mathrm{bs}}(M)$ , then there exists  $M' \in \mathbf{K}_{\lambda}$  such that p does not  $\mathfrak{s}$ -fork over M'.

PROOF. By induction on  $\lambda' := ||M||$ . If  $\lambda' = \lambda$ , then since local character implies existence, we can take M' := M. Otherwise,  $\lambda' > \lambda$  so we can take a resolution  $(M_i)_{i < \lambda'}$  of M such that  $\lambda \leq ||M_i|| < \lambda'$  for all  $i < \lambda'$ . By local character, there exists  $i < \lambda'$  such that p does not  $\mathfrak{s}$ -fork over  $M_i$ . By monotonicity,  $p \upharpoonright M_i$  does not  $\mathfrak{s}$ -fork over  $M_i$ , so must be basic. By the induction hypothesis, there exists  $M' \in \mathbf{K}_{\lambda}$  such that  $p \upharpoonright M_i$  does not  $\mathfrak{s}$ -fork over M'. By transitivity, pdoes not  $\mathfrak{s}$ -fork over M'.

# 4.3. A skeletal frame from splitting

Hypothesis 4.3.1.

- (1) **K** is an abstract elementary class.  $\mu \ge LS(\mathbf{K})$  is a cardinal.  $\mathbf{K}_{\mu} \neq \emptyset$ .
- (2)  $\mathbf{K}_{\mu}$  has amalgamation.

In this section, we start our quest for a good frame. Note that we do *not* assume that any abstract notion of forking is available to us at the start. Recall the following variations on first-order splitting from [She99, Definition 3.2]:

DEFINITION 4.3.2. For  $p \in gS(N)$ , we say that  $p \mu$ -splits over M if  $M \leq_{\mathbf{K}} N$ and there exists  $N_1, N_2 \in \mathbf{K}_{\mu}$  so that  $M \leq_{\mathbf{K}} N_{\ell} \leq N$  for  $\ell = 1, 2$ , and  $h : N_1 \cong_M N_2$ such that  $h(p \upharpoonright N_1) \neq p \upharpoonright N_2$ .

When  $\mu$  is clear from context, we drop it.

REMARK 4.3.3 (Monotonicity of splitting). If  $p \in gS(N)$  does not  $\mu$ -split over M and  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$  are all in  $\mathbf{K}_{\mu}$ , then  $p \upharpoonright N'$  does not  $\mu$ -split over M'.

REMARK 4.3.4. If  $\mathfrak{s}$  is a good<sup>-</sup>  $\mu$ -frame, and p does not  $\mathfrak{s}$ -fork over M, then p does not  $\mu$ -split over M (this will not be used but follows from the uniqueness property, see e.g. Lemma 3.4.2). Thus splitting can be seen as a first approximation to a forking notion.

Our starting point will be the following extension and uniqueness properties of splitting, first isolated by VanDieren [Van02, Theorem II.7.9, Theorem II.7.11]. Intuitively, they tell us that the usual uniqueness and extension property of a forking notion hold of splitting provided we have enough room (concretely, the base model has to be "shifted" by a universal extension).

FACT 4.3.5. Let  $M_0 <_{\mathbf{K}}^{\text{univ}} M \leq_{\mathbf{K}} N$  with  $M_0, M, N \in \mathbf{K}_{\mu}$ . Then:

- (1) Weak uniqueness: If  $p_{\ell} \in gS(N)$  does not split over  $M_0$ ,  $\ell = 1, 2, p_1 \upharpoonright M = p_2 \upharpoonright M$ , then  $p_1 = p_2$ .
- (2) Weak extension: If  $p \in gS(M)$  does not split over  $M_0$ , then there exists  $q \in gS(N)$  extending p that does not split over  $M_0$ . Moreover, q can be taken to be nonalgebraic if p is nonalgebraic.

PROOF. See [Van06, Theorem I.4.12] for weak uniqueness. For weak extension, use universality to get  $h: N \xrightarrow[M_0]{} M$ . Further extend h to an isomorphism  $\hat{h}: \widehat{N} \cong_{M_0} \widehat{M}$ . So that  $\widehat{M}$  contains a realization a of p. Let  $a' := \hat{h}^{-1}(a)$ , and let  $q := \operatorname{gtp}(a/N; \widehat{N})$ . The proof of [Van06, Theorem I.4.10] shows q is indeed an extension of p that does not split over  $M_0$ . In addition if q is algebraic,  $a' \in N$  so  $a = h(a') \in M$ , so p is algebraic. 

We will mostly use those two properties instead of the exact definition of splitting. However, they characterize splitting in the following sense:

PROPOSITION 4.3.6. Assume  $\mathbf{K}_{\mu}$  has amalgamation, no maximal models, and is stable. Let  $\mathfrak{s}$  be a type-full pre- $\mu$ -frame with underlying AEC K. The following are equivalent.

- (1) For all  $M, N \in \mathbf{K}_{\mu}$  with  $M \leq_{\mathbf{K}} N$  and all types  $p \in gS(N)$ , if p does not  $\mathfrak{s}$ -fork over M, then for any  $M <_{\mathbf{K}}^{\mathrm{univ}} M' \leq_{\mathbf{K}} N$ , p does not split over M'.
- (2) s-forking satisfies weak uniqueness and weak extension (i.e. the conclusion of Fact 4.3.5 holds with "split" replaced by "fork").

**PROOF.** Chase the definitions (not used).

We also obtain a weak transitivity property:

PROPOSITION 4.3.7 (Weak transitivity of splitting). Let  $M_0 \leq_{\mathbf{K}} M_1 <_{\mathbf{K}}^{\mathrm{univ}}$  $M'_1 \leq_{\mathbf{K}} M_2$  all be in  $\mathbf{K}_{\mu}$ . Let  $p \in \mathrm{gS}(M_2)$ . If  $p \upharpoonright M'_1$  does not split over  $M_0$  and pdoes not split over  $M_1$ , then p does not split over  $M_0$ .

**PROOF.** By weak extension, find  $q \in gS(M_2)$  extending  $p \upharpoonright M'_1$  and not splitting over  $M_0$ . By monotonicity, q does not split over  $M_1$ . By weak uniqueness, p = q, as needed. 

We now turn to building a forking notion that will satisfy a version of uniqueness and extension (see Definition 4.2.19) in  $\mathbf{K}_{\mu}$ . The idea is simple enough: we want to say that a type does not fork over M if there is a "small" substructure  $M_0$  of M over which the type does not *split*. Fact 4.3.5 suggests that "small" should mean "such that M is a universal extension of  $M_0$ ", and this is exactly how we define it:

DEFINITION 4.3.8 ( $\mu$ -forking). Let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  be models in  $\mathbf{K}_{\mu}$ . We say  $p \in gS(N)$  explicitly does not  $\mu$ -fork over  $(M_0, M)$  if:

(1)  $M_0 <_{\mathbf{K}}^{\text{univ}} M \leq_{\mathbf{K}} N.$ (2) p does not  $\mu$ -split over  $M_0$ .

We say p does not  $\mu$ -fork over M if there exists  $M_0$  so that p explicitly does not  $\mu$ -fork over  $(M_0, M)$ .

The reader should note that the word "forking" is used in two different senses in this chapter:

- In the sense of an "abstract notion": this depends on a pre- $\mathcal{F}$ -frame  $\mathfrak{s}$  and is called  $\mathfrak{s}$ -forking in Definition 4.2.15. This is defined for models of sizes in  $\mathcal{F}$ .
- In the concrete sense of Definition 4.3.8. This is called  $\mu$ -forking and is only defined for types over models of size  $\mu$ . Later this will be extended to models of sizes at least  $\mu$  and we will get a (concrete) notion called  $(\geq \mu)$ -forking (Definition 4.4.2). Of course, the two notions will coincide over models of size  $\mu$ .

When we say that a type p explicitly does not  $\mu$ -fork over  $(M_0, M)$ , we think of M as the base, and  $M_0$  as the *explicit* witness to the  $\mu$ -nonforking. It would be nice if we could get rid of the witness entirely and get that  $\mu$ -nonforking satisfies extension and uniqueness, but uniqueness seems to depend on the particular witness.

Transitivity is also problematic: although we manage to get a weak version depending on the particular witnesses, we still do not know how to prove the witnessfree version. This was stated as [**Bal09**, Exercise 12.9] but Baldwin later realized [**Bal**] there was a mistake in his proof.

If instead we define "p does not  $\mu$ -fork" over M" to mean "for all  $M_0 <_{\mathbf{K}}^{\mathrm{univ}} M$ both in  $\mathbf{K}_{\mu}$  there exists  $M'_0$  in  $\mathbf{K}_{\mu}$  with  $M_0 \leq_{\mathbf{K}} M'_0 <_{\mathbf{K}}^{\mathrm{univ}} M$  and p explicitly does not  $\mu$ -fork over  $(M'_0, M)$ " then extension and uniqueness (and thus transitivity) hold, but local character (assuming local character of splitting) is problematic. Thus it seems we have to carry along the witness in our definition of forking, and this makes the resulting independence notion quite weak (hence the name "skeletal"). However, we will see in the next sections that (assuming some tameness and homogeneity) our skeletal  $\mu$ -frame transfers to a much better-behaved frame *above*  $\mu$ . In particular, full uniqueness and transitivity will hold there.

LEMMA 4.3.9 (Basic properties of  $\mu$ -forking). Below, all models are in  $\mathbf{K}_{\mu}$ .

- (1) Monotonicity: If  $p \in gS(N)$  explicitly does not  $\mu$ -fork over  $(M_0, M)$ ,  $M_0 \leq_{\mathbf{K}} M'_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$  and  $M'_0 <_{\mathbf{K}}^{\mathrm{univ}} M'$ , then  $p \upharpoonright N'$ explicitly does not  $\mu$ -fork over  $(M'_0, M')$ . In particular, if  $p \in gS(N)$  does not  $\mu$ -fork over M and  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ , then  $p \upharpoonright N'$  does not  $\mu$ -fork over M'.
- (2) Extension: If  $p \in gS(N)$  explicitly does not  $\mu$ -fork over  $(M_0, M)$  and  $N' \geq_{\mathbf{K}} N$ , then there is  $q \in gS(N')$  extending p that explicitly does not  $\mu$ -fork over  $(M_0, M)$ . If p is nonalgebraic, then q is nonalgebraic.
- (3) Uniqueness: If  $p_{\ell} \in gS(N)$  explicitly does not  $\mu$ -fork over  $(M_0, M)$ ,  $\ell = 1, 2$ , and  $p_1 \upharpoonright M = p_2 \upharpoonright M$ , then  $p_1 = p_2$ .
- (4) Transitivity: Let  $M_1 \leq_{\mathbf{K}} M_2 \leq_{\mathbf{K}} M_3$  and let  $p \in \mathrm{gS}(M_3)$ . If  $p \upharpoonright M_2$  explicitly does not  $\mu$ -fork over  $(M_0, M_1)$  and p explicitly does not  $\mu$ -fork over  $(M'_0, M_2)$  for  $M_0 \leq_{\mathbf{K}} M'_0$ , then p explicitly does not  $\mu$ -fork over  $(M_0, M_1)$ .
- (5) Nonalgebraicity: If  $p \in gS(N)$  does not  $\mu$ -fork over M and  $p \upharpoonright M$  is not algebraic, then p is not algebraic.

PROOF. Monotonicity follows directly from the definition (and Proposition 4.2.10.(2)), extension and uniqueness are just restatements of Fact 4.3.5, and transitivity is a restatement of Proposition 4.3.7. For nonalgebraicity, assume  $p \upharpoonright M$  is nonalgebraic. Then it has a nonalgebraic nonforking extension to N by extension, and this extension must be p by uniqueness, so the result follows.

Assuming some local character for splitting, we obtain weak versions of the local character and continuity properties:

DEFINITION 4.3.10. Let R be a binary relation on  $\mathbf{K}_{\mu}$ , and let  $\kappa$  be a regular cardinal. We say that  $\mu$ -splitting has  $\kappa$ -local character for R-increasing chains if for any R-increasing  $(M_i)_{i \leq \delta}$  with  $\mathrm{cf}(\delta) \geq \kappa$ ,  $M_{\delta} = \bigcup_{i < \delta} M_i$ , and any  $p \in \mathrm{gS}(M_{\delta})$ , there is  $i < \delta$  so that p does not split over  $M_i$ .

REMARK 4.3.11. If  $\mathbf{K}_{\mu}$  is stable, then by [**GV06b**, Fact 4.6]  $\mu$ -splitting has  $\mu^+$ -local character for  $\leq$ -increasing chains.

LEMMA 4.3.12. Let  $\triangleleft$  be an abstract universal ordering on  $\mathbf{K}_{\mu}$ , and let  $\kappa$  be a regular cardinal. Assume splitting has  $\kappa$ -local character for  $\triangleleft$ -increasing chains. Then:

- (1)  $\kappa$ -local character for  $\triangleleft$ -increasing chains: If  $(M_i)_{i \leq \delta}$  is a  $\triangleleft$ -increasing chain in  $\mathbf{K}_{\mu}$  with  $\mathrm{cf}(\delta) \geq \kappa$ ,  $M_{\delta} = \bigcup_{i < \delta} M_i$  and  $p \in \mathrm{gS}(M_{\delta})$ , then there exists  $i < \delta$  so that p explicitly does not  $\mu$ -fork over  $(M_i, M_{i+1})$ .
- (2)  $\kappa$ -continuity for  $\triangleleft$ -increasing chains: If  $(M_i)_{i \leq \delta}$  is a  $\triangleleft$ -increasing chain in  $\mathbf{K}_{\mu}$  with  $\mathrm{cf}(\delta) \geq \kappa$ ,  $M_{\delta} = \bigcup_{i < \delta} M_i$  and  $p \in \mathrm{gS}(M_{\delta})$  such that  $p \upharpoonright M_i$  does not  $\mu$ -fork over  $M_0$  for all  $i < \delta$ , then p does not  $\mu$ -fork over  $M_0$ . Moreover, if in addition  $p \upharpoonright M_i$  explicitly does not  $\mu$ -fork over  $(M'_0, M_0)$  for all  $i < \delta$  (i.e. the witness is always the same), then p explicitly does not  $\mu$ -fork over  $(M'_0, M_0)$ .
- (3) Existence over  $(\geq \kappa, \triangleleft)$ -limits: If  $M \in \mathbf{K}_{\mu}$  is  $(\delta, \triangleleft)$ -limit for some  $\delta$  with  $\mathrm{cf}(\delta) \geq \kappa$ , then any  $p \in \mathrm{gS}(M)$  does not  $\mu$ -fork over M. In fact, if  $p_0, \ldots, p_{n-1} \in \mathrm{gS}(M), n < \omega$ , then there exists  $M_0 <_{\mathbf{K}}^{\mathrm{univ}} M$  such that  $p_i$  explicitly does not  $\mu$ -fork over  $(M_0, M)$  for all i < n.

## Proof.

- (1) Follows from  $\kappa$ -local character of splitting for  $\triangleleft$ -increasing chains.
- (2) By  $\kappa$ -local character, there exists  $i < \delta$  so that p explicitly does not  $\mu$ -fork over  $(M_i, M_{i+1})$ . By assumption, there exists  $M'_0 <_{\mathbf{K}}^{\mathrm{univ}} M_0$  so that  $p \upharpoonright M_{i+1}$  explicitly does not  $\mu$ -fork over  $(M'_0, M_0)$ . Since  $M'_0 \leq_{\mathbf{K}} M_i$ , we can apply transitivity to obtain that p explicitly does not  $\mu$ -fork over  $(M'_0, M_0)$ . The proof of the moreover part is similar.
- (3) By local character and monotonicity.

Thus if splitting has  $\aleph_0$ -local character for  $\triangleleft$ -increasing chains for some abstract universal ordering  $\triangleleft$  and if all models in  $\mathbf{K}_{\mu}$  are  $\triangleleft$ -limit (e.g. if  $\mathbf{K}_{\mu}$  is categorical), then it seems we are very close to having a good<sup>-S</sup>  $\mu$ -frame, but the witnesses must be carried along, which as observed above is rather annoying. Also, local character and continuity only hold for  $\triangleleft$ -chains.

In the next sections, we show that these problems disappear when we transfer our skeletal frame above  $\mu$ . Note that Shelah's construction of a good frame in [**She09a**, Theorem II.3.7] already takes advantage of that phenomenon. A similar idea is also exploited in the definition of a rooted minimal type in Grossberg and VanDieren's categoricity transfer from tameness [**GV06a**, Definition 2.6].

## 4.4. Going up without assuming tameness

Hypothesis 4.4.1.

- (1) **K** is an abstract elementary class.  $\mu \ge LS(\mathbf{K})$  is a cardinal.  $\mathbf{K}_{\mu} \neq \emptyset$ .
- (2)  $\triangleleft$  is an abstract universal ordering on  $\mathbf{K}_{\mu}$ . In particular (by Remark 4.2.12),  $\mathbf{K}_{\mu}$  has amalgamation, no maximal models, and is stable.

In [She09a, Section II.2], Shelah showed how to extend a good  $\mu$ -frame to all models in  $\mathbf{K}_{\geq\mu}$ . The resulting object will in general not be a good ( $\geq \mu$ )-frame, but several of the properties are nevertheless preserved. In this section, we apply the same procedure on our skeletal  $\mu$ -frame (induced by  $\mu$ -forking defined in the previous section) and show Shelah's arguments still go through, assuming the base

models are  $\mu^+$ -homogeneous. In the next section, we will assume tameness to prove more properties of  $(\geq \mu)$ -forking.

We define  $(\geq \mu)$ -forking from  $\mu$ -forking in exactly the same way Shelah extends a good  $\mu$ -frame to a  $(\geq \mu)$ -frame:

DEFINITION 4.4.2. Assume  $M, N \in \mathbf{K}_{\geq \mu}$  and  $p \in \mathrm{gS}^{\mathrm{na}}(N)$ . We say that p does not  $(\geq \mu)$ -fork over M if  $M \leq_{\mathbf{K}} N$  and there exists M' in  $\mathbf{K}_{\mu}$  with  $M' \leq_{\mathbf{K}} M$  such that for all  $N' \in \mathbf{K}_{\mu}$  with  $M' \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ ,  $p \upharpoonright N'$  does not  $\mu$ -fork over M'.

For technical reasons, we also need to define explicit ( $\geq \mu$ )-forking over a model of size  $\mu$ :

DEFINITION 4.4.3 (Explicit  $(\geq \mu)$ -forking in  $\mathbf{K}_{\geq \mu}$ ). Assume  $N \in \mathbf{K}_{\geq \mu}$ ,  $M_0 \leq_{\mathbf{K}} M$  are in  $\mathbf{K}_{\mu}$ , and  $p \in \mathrm{gS}^{\mathrm{na}}(N)$ . We say that p explicitly does not  $(\geq \mu)$ -fork over  $(M_0, M)$  if p does not  $\mu$ -split over  $M_0$  and  $M_0 <_{\mathbf{K}}^{\mathrm{univ}} M \leq_{\mathbf{K}} N$ . Equivalently, for all  $N' \in \mathbf{K}_{\mu}$  with  $M \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ , we have  $p \upharpoonright N'$  explicitly does not  $\mu$ -fork over  $(M_0, M)$  (see Definition 4.3.8).

REMARK 4.4.4. The following easy propositions follow from the definitions. We will use them without further comments in the rest of this chapter.

- (1) The definitions of  $(\geq \mu)$ -forking and  $\mu$ -forking coincide over models of size  $\mu$ . That is, if  $M_0, M, N \in \mathbf{K}_{\mu}$  and  $p \in \mathrm{gS^{na}}(N)$ , then p does not  $\mu$ -fork over M if and only if p does not  $(\geq \mu)$ -fork over M and p explicitly does not  $(\geq \mu)$ -fork over  $(M_0, M)$  if and only if p explicitly does not  $\mu$ -fork over  $(M_0, M)$ .
- (2) For  $M \leq_{\mathbf{K}} N$  both in  $\mathbf{K}_{\geq\mu}$ ,  $p \in \mathrm{gS}^{\mathrm{na}}(N)$  does not  $(\geq \mu)$ -fork over M if and only if there exists  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{\mu}$  such that p does not  $(\geq \mu)$ -fork over  $M_0$ .
- (3) For  $M \leq_{\mathbf{K}} N$  with  $M \in \mathbf{K}_{\mu}$ ,  $N \in \mathbf{K}_{\geq \mu}$ ,  $p \in \mathrm{gS}^{\mathrm{na}}(N)$  does not  $(\geq \mu)$ -fork over M if and only if for all  $N' \leq_{\mathbf{K}} N$  with  $M \leq_{\mathbf{K}} N'$ ,  $p \upharpoonright N'$  does not  $\mu$ -fork over M.

DEFINITION 4.4.5. We define a nonforking relation  $\downarrow$  on  $\mathbf{K}_{\geq \mu}$  by  $a \downarrow_{M}^{\widehat{N}} N$  if and only if  $M, N, \widehat{N} \in \mathbf{K}_{\geq \mu}, a \in \widehat{N}$ , and  $gtp(a/N; \widehat{N})$  does not  $(\geq \mu)$ -fork over M.

PROPOSITION 4.4.6.  $\mathfrak{s}_0 := (\mathbf{K}, \bigcup, \mathrm{gS^{na}})$  is a type-full pre- $[\mu, \infty)$ -frame.

PROOF. The properties to check follow directly from the definition of  $(\geq \mu)$ nonforking.  $\mathfrak{s}_0$  is type-full since we defined the basic types to be all the nonalgebraic
types.

In  $\mathbf{K}_{\mu}$  we had by definition that a type which does not  $\mu$ -fork over M also explicitly does not  $\mu$ -fork over  $(M_0, M)$  for some witness  $M_0$ . This is not necessarily the case for  $(\geq \mu)$ -nonforking: take for example  $N \in \mathbf{K}_{>\mu}$  and  $M \in \mathbf{K}_{\mu}$  and assume  $p \in \mathrm{gS}(N)$  does not  $(\geq \mu)$ -fork over M. Then for all  $N' \in \mathbf{K}_{\mu}$  with  $M \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N, p \upharpoonright N'$  does not  $\mu$ -fork over M, i.e. there is a witness  $M'_0$  such that  $p \upharpoonright N'$  explicitly does not  $\mu$ -fork over  $(M'_0, M)$ , but there could be different witnesses  $M'_0$  for different N'.

The next lemma shows that this can be avoided if we have enough homogeneity. This is crucial to our proofs of transitivity, uniqueness, and extension. LEMMA 4.4.7. Assume  $M \leq_{\mathbf{K}} N$  are both in  $\mathbf{K}_{\geq \mu^+}$  and M is  $\mu^+$ -modelhomogeneous. Assume  $p \in gS(N)$  does not  $(\geq \mu)$ -fork over M. Then there exists  $M'_0, M' \in \mathbf{K}_{\mu}$  with  $M'_0 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M$  such that p explicitly does not  $(\geq \mu)$ fork over  $(M'_0, M')$  (i.e.  $(\geq \mu)$ -nonforking over M' is witnessed by the same  $M'_0$ uniformly, see the discussion above).

PROOF. By definition, there is  $M'_0$  in  $\mathbf{K}_{\mu}$  with  $M'_0 \leq_{\mathbf{K}} M$  such that p does not  $(\geq \mu)$ -fork over  $M'_0$ . Since M is  $\mu^+$ -model-homogeneous, one can pick  $M' >_{\mathbf{K}}^{\mathrm{univ}} M'_0$  in  $\mathbf{K}_{\mu}$  with  $M' \leq_{\mathbf{K}} M$ . By monotonicity (Lemma 4.3.9.(1)), p explicitly does not  $(\geq \mu)$ -fork over  $(M'_0, M')$ .

Using Lemma 4.4.7, we can give a simpler definition of  $(\geq \mu)$ -forking. This will not be used but shows that our forking is the same as that defined in [She09a, Definition III.9.5.2].

PROPOSITION 4.4.8. Assume  $M \leq_{\mathbf{K}} N$  are both in  $\mathbf{K}_{\geq \mu^+}$  and M is  $\mu^+$ -modelhomogeneous. Let  $p \in \mathrm{gS}^{\mathrm{na}}(N)$ . Then p does not  $(\geq \mu)$ -fork over M if and only if there exists  $M_0 \in \mathbf{K}_{\mu}$  such that  $M_0 \leq_{\mathbf{K}} M$  and p does not  $\mu$ -split over  $M_0$ .

PROOF. If p does not  $(\geq \mu)$ -fork over M, use Lemma 4.4.7 to get  $M'_0, M' \in \mathbf{K}_{\mu}$ with  $M_0 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M$  such that p explicitly does not  $(\geq \mu)$ -fork over  $(M_0, M')$ . By definition, this means that p does not  $\mu$ -split over  $M_0$ . Conversely, assume  $M_0 \in \mathbf{K}_{\mu}$  is such that  $M_0 \leq_{\mathbf{K}} M$  and p does not  $\mu$ -split over  $M_0$ . Since M is  $\mu^+$ -model-homogeneous, there exists  $M' \in \mathbf{K}_{\mu}$  such that  $M_0 <_{\mathbf{K}}^{\mathrm{univ}} M' \leq_{\mathbf{K}} M$ . Thus p explicitly does not  $(\geq \mu)$ -fork over  $(M_0, M')$ , so it does not  $(\geq \mu)$ -fork over M.

LEMMA 4.4.9 (Existence). Let  $M \in \mathbf{K}_{\geq \mu^+}$  be  $\mu^+$ -model-homogeneous. Then  $p \in \mathrm{gS}^{\mathrm{na}}(M)$  if and only if p does not  $(\geq \mu)$ -fork over M.

PROOF. If p does not fork over M, then p is nonalgebraic by definition. Now assume p is nonalgebraic. By [**GV06b**, Fact 4.6], there is  $M'_0 \in \mathbf{K}_{\mu}$  with  $M'_0 \leq_{\mathbf{K}} M$ such that p does not  $\mu$ -split over  $M_0$ . Pick  $M' \in \mathbf{K}_{\mu}$  with  $M' >_{\mathbf{K}}^{\mathrm{univ}} M'_0$  so that  $M' \leq_{\mathbf{K}} M$ . This is possible by  $\mu^+$ -model-homogeneity. We have that p explicitly does not  $(\geq \mu)$ -fork over  $(M'_0, M')$ , so does not  $(\geq \mu)$ -fork over M', as needed.  $\Box$ 

LEMMA 4.4.10 (Transitivity). If  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$  are all in  $\mathbf{K}_{\geq \mu}$ ,  $M_1$  is  $\mu^+$ -model-homogeneous,  $p \in \mathrm{gS}^{\mathrm{na}}(M_2)$  is such that  $p \upharpoonright M_1$  does not  $(\geq \mu)$ -fork over  $M_0$  and p does not  $(\geq \mu)$ -fork over  $M_1$ , then p does not  $(\geq \mu)$ -fork over  $M_0$ .

PROOF. Find  $M'_0 \in \mathbf{K}_{\mu}$  with  $M'_0 \leq_{\mathbf{K}} M_0$  such that  $p \upharpoonright M_1$  does not  $(\geq \mu)$ -fork over  $M'_0$ . Using monotonicity and Lemma 4.4.7, we can also find  $M'_1, M''_1 \in \mathbf{K}_{\mu}$ with  $M'_0 \leq_{\mathbf{K}} M'_1 <_{\mathbf{K}}^{\text{univ}} M''_1 \leq_{\mathbf{K}} M_1$  such that p explicitly does not  $(\geq \mu)$ -fork over  $(M'_1, M''_1)$ . By transitivity in  $\mathbf{K}_{\mu}$  (Lemma 4.3.9.(4)), p does not  $(\geq \mu)$ -fork over  $M'_0$ , and hence over  $M_0$ .

LEMMA 4.4.11 (Local character). Assume splitting has  $\kappa$ -local character for  $\triangleleft$ -increasing chains. If  $cf(\delta) \geq \kappa$ ,  $(M_i)_{i \leq \delta}$  is an increasing chain in  $\mathbf{K}_{\geq \mu^+}$  with  $M_{\delta} = \bigcup_{i < \delta} M_i$ ,  $M_i$  is  $\mu^+$ -model-homogeneous for  $i < \delta$ , and  $p \in gS^{\mathrm{na}}(M_{\delta})$ , then there is  $i < \delta$  such that p does not  $(\geq \mu)$ -fork over  $M_i$ .

PROOF. Without loss of generality,  $\delta$  is regular. If  $\delta \ge \mu^+$ , then  $M_{\delta}$  is also  $\mu^+$ model-homogeneous so one can pick  $N^* \in \mathbf{K}_{\mu}$  with  $N^* \le_{\mathbf{K}} M_{\delta}$  witnessing existence (use Lemma 4.4.9) and find  $i < \delta$  with  $N^* \le_{\mathbf{K}} M_i$ , so p does not ( $\ge \mu$ )-fork over  $M_i$  as needed. Now assume  $\delta < \mu^+$ . We imitate the proof of [She09a, Claim II.2.11.5]. Assume the conclusion fails. Build  $(N_i)_{i \leq \delta} \triangleleft$ -increasing continuous in  $\mathbf{K}_{\mu}, (N'_i)_{i \leq \delta}$  $\leq_{\mathbf{K}}$ -increasing continuous in  $\mathbf{K}_{\mu}$  such that for all  $i < \delta$ :

- (1)  $N_i \leq M_i$ .
- (2)  $N_i \leq N'_i \leq M_{\delta}$ .
- (3)  $p \upharpoonright N'_{i+1}$  explicitly  $\mu$ -forks over  $(N_i, N_{i+1})$ . (4)  $\bigcup_{j \le i} (N'_j \cap M_{i+1}) \le |N_{i+1}|$ .

This is possible. For i = 0, let  $N_0 \in \mathbf{K}_{\mu}$  be any model with  $N_0 \leq M_0$ , and let  $N'_0 := N_0$ . For *i* limit, take unions. For the successor case, assume i = j + 1. Choose  $N_i \leq M_i$  satisfying (4) with  $N_i \geq N_j$  (possible since  $M_i$  is  $\mu^+$ -modelhomogeneous). By assumption,  $p (\geq \mu)$ -forks over  $M_i$ , hence explicitly  $(\geq \mu)$ -forks over  $(N_i, N_i)$ , and so by definition of forking and monotonicity there exists  $N'_i \in \mathbf{K}_{\mu}$ with  $M_{\delta} \geq_{\mathbf{K}} N'_i \geq_{\mathbf{K}} N_i, N'_i \geq_{\mathbf{K}} N'_j$ , and  $p \upharpoonright N'_i$  explicitly  $\mu$ -forking over  $(N_j, N_i)$ . It is as required.

This is enough. By local character in  $\mathbf{K}_{\mu}$ , there is  $i < \delta$  such that  $p \upharpoonright N_{\delta}$ explicitly does not  $\mu$ -fork over  $(N_i, N_{i+1})$ . By (2) and (4),  $N'_{\delta} \leq_{\mathbf{K}} N_{\delta}$ . Thus  $p \upharpoonright N'_{i+1}$  explicitly does not  $\mu$ -fork over  $(N_i, N_{i+1})$ , contradicting (3).

LEMMA 4.4.12 (Continuity). Assume splitting has  $\kappa$ -local character for  $\triangleleft$ -increasing chains. If  $cf(\delta) \ge \kappa$ ,  $(M_i)_{i \le \delta}$  is an increasing chain in  $\mathbf{K}_{\ge \mu^+}$  with  $M_{\delta} = \bigcup_{i < \delta} M_i$ ,  $M_i \ \mu^+$ -model-homogeneous for  $i < \delta$ , and  $p \in gS(M_\delta)$  is so that  $p \upharpoonright M_i$  does not  $(\geq \mu)$ -fork over  $M_0$  for all  $i < \delta$ , then  $p_{\delta}$  does not  $(\geq \mu)$ -fork over  $M_0$ .

PROOF. In a type-full frame such as ours, this follows directly from  $\kappa$ -local character and transitivity, see [She09a, Claim II.2.17.3]. 

REMARK 4.4.13. In the statements of local character and continuity, we assumed that  $M_i$  was  $\mu^+$ -model-homogeneous for all  $i < \delta$ , but not that their union  $M_{\delta}$  was  $\mu^+$ -model-homogeneous.

# 4.5. A tame good frame, perhaps without symmetry

Boney showed in [Bon14a] that given a good  $\mu$ -frame, tameness implies that Shelah's extension of the frame to  $\geq \mu$  is actually a good ( $\geq \mu$ )-frame. In this section, we apply the ideas of his proof (assuming the base models are  $\mu^+$ -modelhomogeneous) to our skeletal  $\mu$ -frame.

More precisely, we fix a cardinal  $\lambda > \mu$ , assume enough tameness, and build a  $good^{-S} \lambda$ -frame (i.e. we have all the properties of a good  $\lambda$ -frame except perhaps symmetry). We will prove symmetry in the next section.

Hypothesis 4.5.1.

- (1) **K** is an abstract elementary class.  $\mu \geq \text{LS}(\mathbf{K})$  is a cardinal.  $\mathbf{K}_{\mu} \neq \emptyset$ .
- (2)  $\triangleleft$  is an abstract universal ordering on  $\mathbf{K}_{\mu}$ . In particular (by Remark 4.2.12),  $\mathbf{K}_{\mu}$  has amalgamation, no maximal models, and is stable.
- (3)  $\kappa$  is the least regular cardinal such that splitting has  $\kappa$ -local character for  $\triangleleft$ -increasing chains in  $\mathbf{K}_{\mu}$ .
- (4)  $\lambda > \mu$  is such that:
  - (a) **K** is  $(\mu, \lambda)$ -tame<sup>6</sup>.

 $<sup>^{6}</sup>$ Recall (Definition 4.2.3) that this means that the Galois types over models of size at most  $\lambda$  are determined by their restrictions to submodels of size  $\mu$ .

- (b)  $\mathbf{K}_{[\mu,\lambda]}$  has amalgamation.
- (c)  $\mathbf{K}_{[\mu,\lambda)}$  has no maximal models.

REMARK 4.5.2.  $\kappa$  plays a similar role as the cardinal  $\kappa(T)$  in the first-order context. By Remark 4.3.11 and Hypothesis 4.5.1.(2),  $\kappa \leq \mu^+$ . In the end, we will be able to obtain a good frame only when  $\kappa = \aleph_0$ , but studying the general case leads to results on the stability spectrum.

Note that uniqueness is actually equivalent to  $(\mu, \lambda)$ -tameness by [Bon14a, Theorem 3.2]. The easiest case is when  $\lambda = \mu^+$ . Then we know a model-homogeneous model exists in  $\mathbf{K}_{\lambda}$ , and this simplifies some of the proofs.

LEMMA 4.5.3 (Uniqueness). Let  $M \leq_{\mathbf{K}} N$  be models in  $\mathbf{K}_{[\mu,\lambda]}$ . Let  $p, q \in gS(N)$ . Assume  $p \upharpoonright M = q \upharpoonright M$ .

- (1) If  $M \in \mathbf{K}_{\mu}$  and p, q explicitly do not  $(\geq \mu)$ -fork over  $(M_0, M)$  for some  $M_0 <_{\mathbf{K}}^{\mathrm{univ}} M$ , then p = q.
- (2) If  $M \in \mathbf{K}_{[\mu^+,\lambda]}$  is  $\mu^+$ -model-homogeneous and p, q do not  $(\geq \mu)$ -fork over M, then p = q.

PROOF. (1) follows from uniqueness in  $\mathbf{K}_{\mu}$  (Lemma 4.3.9.(3)) and tameness. To see (2), use monotonicity and Lemma 4.4.7, to find  $M'_0, M' \in \mathbf{K}_{\mu}$  with  $M'_0 <_{\mathbf{K}}^{\mathrm{univ}}$  $M' \leq_{\mathbf{K}} M$  such that both p and q explicitly do not  $(\geq \mu)$ -fork over  $(M'_0, M')$ . Now apply (1).

Interestingly, we already have enough machinery to obtain a stability transfer theorem. First recall:

FACT 4.5.4.  $\mathbf{K}_{\mu^+}$  is stable.

PROOF. This could be done using the method of proof of Theorem 4.5.6, but this is also  $[\mathbf{BKV06}, \text{Theorem 1}]$ .

Recall that  $\kappa$  is the local character cardinal, see Hypothesis 4.5.1.(3.

LEMMA 4.5.5. Assume that  $\lambda > \mu^+$ ,  $cf(\lambda) \ge \kappa$ , and there are unboundedly (in the same sense as in the statement of Lemma 4.2.13) many  $\mu \le \lambda' < \lambda$  such that  $\mathbf{K}_{\lambda'}$  is stable. Then  $\mathbf{K}_{\lambda}$  is stable.

PROOF. Let  $M \in \mathbf{K}_{\lambda}$ . By Lemma 4.2.13, M can be embedded inside some  $\widehat{M} \in \mathbf{K}_{\lambda}$  which can be written as  $\bigcup_{i < \lambda} M_i$ , with  $(M_i)_{i < \lambda}$  an increasing chain<sup>7</sup> of  $\mu^+$ -model-homogeneous models in  $\mathbf{K}_{[\mu^+,\lambda)}$ . From amalgamation, we know that Galois types can be extended, so  $|gS(M)| \leq |gS(\widehat{M})|$ , and so we can assume without loss of generality that  $M = \widehat{M}$ . Let  $(p_j)_{j < \lambda^+}$  be types in gS(M). By  $\kappa$ -local character, for each  $j < \lambda^+$  there is  $i_j < \lambda$  such that  $p_j$  does not  $(\geq \mu)$ -fork over  $M_{i_j}$ . By the pigeonhole principle, we may assume  $i_j = i_0$  for all  $j < \lambda^+$ . Taking  $i_0$  bigger if necessary, we may assume that  $\mathbf{K}_{\|M_{i_0}\|}$  is stable. Thus  $|gS(M_{i_0})| \leq \|M_{i_0}\| \leq \lambda$ , so by the pigeonhole principle again, we can assume that there is  $q \in gS(M_{i_0})$  such that  $p_j \upharpoonright M_{i_0} = q$  for all  $j < \lambda^+$ . By uniqueness,  $p_j = p_{j'}$  for each  $j, j' < \lambda^+$ , so the result follows.

<sup>&</sup>lt;sup>7</sup>Explicitly, we take  $(N_i)_{i < \lambda}$  as given by Lemma 4.2.13 for some  $N_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{\mu^+}$ , and let  $M_i := N_{i+1}$ . Note that the chain  $(M_i)_{i < \lambda}$  will not be continuous.

We can now prove that stability transfers up if the locality cardinal  $\kappa$  of Hypothesis 4.5.1.(3) is  $\aleph_0$ . Recall that  $\lambda$  is the cardinal above  $\mu$  fixed in Hypothesis 4.5.1.(4). Recall also that we already have stability in  $\mu$  by Hypothesis 4.5.1.(2).

THEOREM 4.5.6 (The superstability theorem). If  $\kappa = \aleph_0$ , then  $\mathbf{K}_{\lambda}$  is stable.

**PROOF.** We work by induction on  $\lambda$ . If  $\lambda = \mu^+$ , this is Fact 4.5.4 and if  $\lambda > \mu^+$ this is given by Lemma 4.5.5 and the induction hypothesis.  $\square$ 

Assuming the generalized continuum hypothesis (GCH), we can also say something for arbitrary  $\kappa$  (this will not be used):

THEOREM 4.5.7. Assume GCH. If  $\lambda^{<\kappa} = \lambda$ , then  $\mathbf{K}_{\lambda}$  is stable.

**PROOF.** By induction on  $\lambda$ . If  $\lambda = \mu^+$ , this is Fact 4.5.4, so assume  $\lambda > \mu^+$ . By König's theorem,  $cf(\lambda) \geq \kappa$ . If  $\lambda$  is successor, then  $\lambda^{\mu} = \lambda$  by GCH, so by **[GV06b**, Corollary 6.4], **K** is stable in  $\lambda$ . If  $\lambda$  is limit there exists a sequence of successor cardinals  $(\lambda_i)_{i < cf(\lambda)}$  increasing cofinal in  $\lambda$  with  $\lambda_0 \ge \mu^+$ . Since without loss of generality  $\kappa \leq \mu^+$  (Remark 4.3.11), GCH implies that  $\lambda_i^{<\kappa} = \lambda_i$ , so by the induction hypothesis, **K** is stable in  $\lambda_i$  for all  $i < cf(\lambda)$ . Apply Lemma 4.5.5 to conclude. 

We now prove extension. This follows from compactness in the first-order case, but we make crucial use of the superstability hypothesis  $\kappa = \aleph_0$  in the general case (recall from the hypotheses of this section that  $\kappa$  is the local character cardinal for  $\mu$ -splitting).

LEMMA 4.5.8. Assume  $\kappa = \aleph_0$ . Let  $\delta < \lambda^+$  be a limit ordinal. Assume  $(M_i)_{i < \delta}$ is an increasing continuous sequence in  $\mathbf{K}_{[\mu,\lambda)}$  with  $M_0 \in \mathbf{K}_{\mu}$ . Let  $(p_i)_{i < \delta}$  be an increasing continuous sequence of types with  $p_i \in gS(M_i)$  for all  $i < \delta$ , and  $p_i$ explicitly does not  $(\geq \mu)$ -fork over  $(M'_0, M_0)$ . Assume that one of the following holds:

- (1)  $(M_i)_{i < \delta}$  is  $\triangleleft$ -increasing in  $\mathbf{K}_{\mu}$ .
- (2) For all  $i < \delta$ ,  $M_{i+1}$  is  $\mu^+$ -model-homogeneous.

Then there exists a unique  $p_{\delta} \in gS(M_{\delta})$  extending each  $p_i$  and explicitly not  $(\geq \mu)$ -forking over  $(M'_0, M_0)$ .

PROOF. This is similar to the argument in [GV06a, Corollary 2.22], but we give some details. We focus on (1) (the proof of the other case is completely similar). Build by induction  $(f_{i,j})_{i < j < \delta}$ ,  $(a_i)_{i < \delta}$ , and increasing continuous  $(N_i)_{i < \delta}$  such that for all  $i < j < \delta$ :

- (1)  $M_i \leq_{\mathbf{K}} N_i, a_i \in N_i$ .
- (2)  $f_{i,j}: N_i \to N_j.$ (3) For  $j < k < \delta$ ,  $f_{j,k} \circ f_{i,j} = f_{i,k}.$
- (4)  $f_{i,j}$  fixes  $M_i$ .
- (5)  $f_{i,j}(a_i) = a_j$ .
- (6)  $p_i = \text{gtp}(a_i/M_i; N_i).$

This is enough. Let  $(N_{\delta}, (f_{i,\delta}))_{i < \delta}$  be the direct limit of the system  $(N_i, f_{i,j})_{i < j < \delta}$ , and let  $a_{\delta} := f_{0,\delta}(a_0), p_{\delta} := \operatorname{gtp}(a_{\delta}/M_{\delta}; N_{\delta})$ . One easily checks that  $p_{\delta}$  extends each  $p_i$ ,  $i < \delta$ , and so using continuity for  $\triangleleft$ -increasing chains (Lemma 4.3.12.(2)), explicitly does not  $(\geq \mu)$ -fork over  $(M'_0, M_0)$ . Finally,  $p_{\delta}$  is unique by Lemma 4.5.3.

This is possible. For i = 0, we take  $a_0$  and  $N_0$  so that  $gtp(a_0/M_0; N_0) = p_0$ . For i limit, we let  $(N_i, f_{i_0,i})_{i_0 < i}$  be the direct limit of the system  $(N_{i_0}, f_{i_0,j_0})_{i_0 < j_0 < i}$ , and let  $a_i := f_{0,i}(a_0)$ . By continuity for  $\triangleleft$ -increasing chains,  $gtp(a_i/M_i; N_i)$  explicitly does not  $(\geq \mu)$ -fork over  $(M'_0, M_0)$ , and so by uniqueness, it must equal  $p_i$ . For  $i = i_0 + 1$  successor, find  $a_i$  and  $N'_i \geq M_i$  such that  $p_i = gtp(a_i/M_i; N'_i)$ . Since  $p_i \upharpoonright M_{i_0} = p_{i_0}$ , we can use the definition of types to amalgamate  $N_{i_0}$  and  $N'_i$  over  $M_{i_0}$ : there exists  $N_i \geq_{\mathbf{K}} N'_i$  and  $f_{i_0,i} : N_{i_0} \xrightarrow{M_{i_0}} N_i$  so that  $f_{i_0,i}(a_{i_0}) = a_i$ . Define  $f_{i'_0,i} := f_{i_0,i} \circ f_{i'_0,i_0}$  for all  $i'_0 < i_0$ .

LEMMA 4.5.9 (Extension). Assume  $\kappa = \aleph_0$ . Let  $M \leq_{\mathbf{K}} N$  both be in  $\mathbf{K}_{[\mu^+,\lambda]}$  with M and  $N \mu^+$ -model-homogeneous, and let  $p \in \mathrm{gS}^{\mathrm{na}}(M)$ . Then there is  $q \in \mathrm{gS}(N)$  extending p that does not fork over M.

PROOF. We imitate the proof of [**Bon14a**, Theorem 5.3]. By existence and Lemma 4.4.7, there exists  $M'_0, M_0 \in \mathbf{K}_{\mu}$  with  $M'_0 <_{\mathbf{K}}^{\text{univ}} M_0 \leq_{\mathbf{K}} M$  and p explicitly  $(\geq \mu)$ -nonforking over  $(M'_0, M_0)$ . Work by induction on  $\lambda$ . If  $N \in \mathbf{K}_{<\lambda}$ , use the induction hypothesis, so assume  $N \in \mathbf{K}_{\lambda}$ . There are two cases: either  $\lambda = \mu^+$  or  $\lambda > \mu^+$ .

Assume first  $\lambda > \mu^+$ . By transitivity and Lemma 4.2.13, we can assume without loss of generality that  $N = \bigcup_{i < \lambda} N_i$ , where  $(N_i)_{i \leq \lambda}$  is a  $<_{\mathbf{K}}^{\text{univ}}$ -increasing continuous chain in  $\mathbf{K}_{[\mu^+,\lambda)}$ , each  $N_{i+1}$  is  $\mu^+$ -model-homogeneous, and  $N_0$  extends  $M_0$ . Now inductively build a  $\leq$ -increasing continuous  $(M_i)_{i \leq \lambda}$  with  $M_{\lambda} = M$  so that  $M_0 \leq_{\mathbf{K}}$  $M_i \leq_{\mathbf{K}} N_i$  for all  $i < \lambda$  (we allow repetitions). Set  $p_i := p \upharpoonright M_i$  and note that by monotonicity,  $p_i$  explicitly does not  $(\geq \mu)$ -fork over  $(M'_0, M_0)$ .

We inductively build an increasing  $(q_i)_{i \leq \lambda}$  with  $q_i \in \mathrm{gS}(N_i)$ ,  $p_i \leq_{\mathbf{K}} q_i$ , and  $q_i$ explicitly does not  $(\geq \mu)$ -fork over  $(M'_0, M_0)$ . For i = 0, use extension in  $\mathbf{K}_{<\lambda}$  to find  $q_0$  as needed. For i = j + 1, use extension to find a  $(\geq \mu)$ -nonforking extension  $q_i \in \mathrm{gS}(N_i)$  of  $q_j$  that explicitly does not  $(\geq \mu)$ -fork over  $(M'_0, M_0)$ . By uniqueness,  $q_i \geq_{\mathbf{K}} p_i$ . At limits, use Lemma 4.5.8 and uniqueness.  $q := q_\lambda$  is as desired.

If  $\lambda = \mu^+$ , the construction is exactly the same except we use extension in  $\mathbf{K}_{\lambda}$  at successor steps and the first case of Lemma 4.5.8 at limit steps. Note that since N is  $\mu^+$ -model-homogeneous,  $N = \bigcup_{i < \mu^+} N_i$ , where  $(N_i)_{i < \mu^+}$  is a  $\triangleleft$ -increasing continuous chain in  $\mathbf{K}_{\mu}$ .

DEFINITION 4.5.10. Let  $\mathfrak{s} := \mathfrak{s}_0 \upharpoonright \lambda$ , where  $\mathfrak{s}_0$  is the pre-frame from Proposition 4.4.6.

Corollary 4.5.11. Assume:

- (1)  $\kappa = \aleph_0$ .
- (2)  $\mathbf{K}_{\mu}$  has joint embedding.
- (3)  $\mathbf{K}_{\lambda}$  has no maximal models.
- (4) All the models in  $\mathbf{K}_{\lambda}$  are  $\mu^+$ -model-homogeneous.

Then  $\mathfrak{s}$  is a type-full good<sup>-S</sup>  $\lambda$ -frame.

PROOF. It is easy to see  $\mathfrak{s}$  is a type-full pre- $\lambda$ -frame.  $\mathbf{K}_{\lambda}$  has amalgamation and no maximal models by hypothesis. It has joint embedding since  $\mathbf{K}_{\mu}$  has joint embedding and  $\mathbf{K}_{[\mu,\lambda]}$  has amalgamation (see Lemma 4.2.2). Stability holds by Theorem 4.5.6. Density of basic types is always true in a type-full frame. For the other properties, see Lemmas 4.4.9, 4.4.10, 4.4.11, 4.4.12, 4.5.3, and 4.5.9 (note that the original statement of extension in Definition 4.2.19 follows from Lemma 4.5.9 and transitivity).  $\Box$ 

LEMMA 4.5.12. Assume **K** is categorical in  $\lambda$  and  $\kappa = \aleph_0$ . Then:

- (1)  $\mathbf{K}_{[\mu,\lambda]}$  has joint embedding and  $\mathbf{K}_{\lambda}$  (and hence  $\mathbf{K}_{[\mu,\lambda]}$ ) has no maximal models.
- (2) All the models in  $\mathbf{K}_{\lambda}$  are  $\mu^+$ -model-homogeneous.

PROOF. To see (2), assume first that  $\mathbf{K}_{\lambda}$  has no maximal models. Use stability to build  $(M_i)_{i \leq \mu^+} <_{\mathbf{K}}^{\text{univ}}$ -increasing continuous with  $M_i \in \mathbf{K}_{\lambda}$  for all  $i < \mu^+$ . Then  $M_{\mu^+}$  is  $\mu^+$ -model-homogeneous. If  $\mathbf{K}_{\lambda}$  has a maximal model, then it is easy to check directly that the maximal model is  $\mu^+$ -model-homogeneous.

For (1),  $\mathbf{K}_{\lambda}$  has joint embedding by categoricity. Now since  $\mathbf{K}_{[\mu,\lambda)}$  has no maximal models, any  $M \in \mathbf{K}_{[\mu,\lambda)}$  embeds into an element of  $\mathbf{K}_{\lambda}$ , so joint embedding for  $\mathbf{K}_{[\mu,\lambda]}$  follows. To see  $\mathbf{K}_{\lambda}$  has no maximal model, let  $N \in \mathbf{K}_{\lambda}$  be given. First assume  $\lambda = \mu^+$ . Build a  $\triangleleft$ -increasing continuous chain  $(M_i)_{i \leq \mu^+}$ , and  $a \in N$  such that for all  $i < \mu^+$ :

(1) 
$$M_i \in \mathbf{K}_{\mu}, M_i \leq_{\mathbf{K}} N.$$

- (2)  $a \notin M_0$ .
- (3)  $gtp(a/M_i; N)$  does not  $\mu$ -fork over  $M_0$ .

This is enough  $M_{\mu^+} \in \mathbf{K}_{\lambda^+}$ . Moreover by Lemma 4.3.9.(5),  $a \notin M_i$  for all  $i < \mu^+$ , so  $a \notin M_{\mu^+}$ . Thus  $M_{\mu^+} <_{\mathbf{K}} N$ . By categoricity, the result follows.

This is possible. Pick a  $\triangleleft$ -limit  $M_0 \in \mathbf{K}_{\mu}$  with  $M_0 \leq_{\mathbf{K}} N$  (this is possible by model-homogeneity of N), and pick any  $a \in N \setminus M_0$ . At limits, take unions and use continuity (Lemma 4.3.12.(2)) to see the requirements are maintained. For a successor i = j + 1, use extension and some renaming. In details, pick an arbitrary  $M'_i \geq M_j$  with  $M'_i \leq_{\mathbf{K}} N$  (possible by model-homogeneity). By extension (Lemma 4.3.9.(2)), there is  $q \in \mathrm{gS}(M'_i)$  that does not  $\mu$ -fork over  $M_0$  and extends  $p_j := \mathrm{gtp}(a/M_j; N)$ . Since N is saturated, there is  $a' \in N$  realizing q. Pick  $N \geq_{\mathbf{K}} N_i \geq_{\mathbf{K}} M'_i$  containing a' and a. By assumption,  $\mathrm{gtp}(a'/M_j; N_i) = p_j =$  $\mathrm{gtp}(a/M_j; N_i)$ . Thus there is  $N'_i \geq_{\mathbf{K}} N_i$  and  $f : N_i \xrightarrow{M_j} N'_i$  such that f(a') = aand without loss of generality  $N'_i \leq_{\mathbf{K}} N$ . Let  $M_i := f[M'_i]$  and use invariance to see it is as desired.

If  $\lambda > \mu^+$ , the proof is completely similar: if there is  $N_1 >_{\mathbf{K}} N$ , we are done, so assume not. Then amalgamation implies N must be model-homogeneous. Build a  $<^{\text{univ}}_{\mathbf{K}}$ -increasing continuous  $(M_i)_{i \leq \lambda}$  and  $a \in N$  such that for all  $i < \lambda$ :

- (1)  $M_i \in \mathbf{K}_{[\mu^+,\lambda)}, M_i \leq_{\mathbf{K}} N.$
- (2)  $M_{i+1}$  is  $\mu^+$ -model-homogeneous.
- (3)  $gtp(a/M_i; N)$  does not  $(\geq \mu)$ -fork over  $M_0$ .

As before, this is possible and the result follows.

COROLLARY 4.5.13. If **K** is categorical in  $\lambda$  and  $\kappa = \aleph_0$ , then  $\mathfrak{s}$  is a type-full good<sup>-S</sup>  $\lambda$ -frame.

PROOF. Lemma 4.5.12 tells us all the hypotheses of Corollary 4.5.11 are satisfied.  $\hfill \Box$ 

Note that categoricity in  $\lambda$  is not the only hypothesis giving that all models in  $\mathbf{K}_{\lambda}$  are  $\mu^+$ -model-homogeneous. For example:

FACT 4.5.14 (Theorem 5.4 in [**BG**]). Assume **K** has amalgamation, is categorical in a cardinal  $\theta$  so that  $\mathbf{K}_{\theta}$  has a  $\mu^+$ -model-homogeneous model (this holds if e.g.  $\theta^{\mu} = \theta$ ). Then every member of  $\mathbf{K}_{\geq \chi}$  is  $\mu^+$ -model-homogeneous, where  $\chi := \min(\theta, \sup_{\gamma < \mu} \beth_{(2^{\gamma})^+})$ .

# 4.6. Getting symmetry

From Corollary 4.5.11, we obtain from reasonable assumptions a forking notion that satisfies all the properties of a good  $\lambda$ -frame except perhaps symmetry. Note that assuming more tameness, the frame can also be extended (see Fact 4.6.9) to models of size above  $\lambda$ :

FACT 4.6.1. Let  $\mathfrak{s} = (\mathbf{K}, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  be a good<sup>-S</sup>  $\lambda$ -frame. Let  $\theta > \lambda$  and let  $\mathcal{F} := [\lambda, \theta)$ . Assume  $\mathbf{K}_{\mathcal{F}}$  has amalgamation and no maximal models, and  $\mathbf{K}$  is  $(\lambda, < \theta)$ -tame. Then  $\mathfrak{s}$  can be extended to a good<sup>-S</sup>  $\mathcal{F}$ -frame. If  $\mathfrak{s}$  is type-full, then the extended frame will also be type-full.

PROOF. Apply [**Bon14a**, Theorem 1.1]: its proof only uses the tameness for 2-types hypothesis to obtain symmetry. Note that if (as there) we start with a good  $\lambda$ -frame, then no maximal models follows. Here we do not have symmetry, so we assume it as an additional hypothesis. The proof of Lemma 4.4.11 gives us that the extended frame is type-full if  $\mathfrak{s}$  is.

We have justified:

HYPOTHESIS 4.6.2.  $\mathfrak{s} = (\mathbf{K}, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  is a good<sup>-</sup>  $\mathcal{F}$ -frame, where  $\mathcal{F}$  is an interval of cardinals of the form  $[\lambda, \theta)$  for  $\lambda$  a cardinal and  $\theta > \lambda$  either a cardinal or  $\infty$ .

In this section, we will prove that  $\mathfrak{s}$  also satisfies symmetry if  $\theta$  is big-enough. Note that we do not need to assume tameness since enough tameness for what we want follows from the uniqueness and local character properties of  $\mathfrak{s}$ -forking, see [Bon14a, Theorem 3.2].

Note that (see the definition of good<sup>-</sup> in 4.2.19) we do *not* assume  $\mathfrak{s}$  satisfies bs-stability. It will hold in the setup of the previous sections, but the arguments of this section work just as well without it. Note in passing that bs-stability and stability are equivalent:

FACT 4.6.3 ([She09a], Claim II.4.2.1). For any  $\lambda' \in \mathcal{F}$ ,  $\mathfrak{s} \upharpoonright \lambda'$  satisfies bsstability if and only if **K** is stable in  $\lambda'$ .

Moreover, eventual stability will follow from the structural properties of forking:

**PROPOSITION 4.6.4.** 

- (1) If  $2^{\lambda} \in \mathcal{F}$ , then **K** is stable in  $2^{\lambda}$ .
- (2) Assume  $\chi_0 \in \mathcal{F}$  and **K** is stable in  $\chi_0$ . Then **K** is stable in every  $\chi \geq \chi_0$  with  $\chi \in \mathcal{F}$ .

In particular, if  $\chi$  is a cardinal with  $2^{\lambda} \leq \chi < \theta$ , then **K** is stable in  $\chi$ .

Proof.

(1) Let  $\chi := 2^{\lambda}$ . By Fact 4.6.3, it is enough to show that  $\mathfrak{s} \upharpoonright \chi$  satisfies bs-stability. Let  $M \in \mathbf{K}_{\chi}$ , and let  $(p_i)_{i < \chi^+}$  be elements of  $\mathrm{gS}^{\mathrm{bs}}(M)$ . Let  $(M_i)_{i < \chi}$  be a resolution of M. For each  $i < \chi^+$ , local character implies there exists  $j_i < \chi$  such that  $p_i$  does not  $\mathfrak{s}$ -fork over  $M_{j_i}$ . By the pigeonhole principle, we can assume without loss of generality that  $j_i = j_0$  for all  $i < \chi^+$ . By Proposition 4.2.21 and transitivity, there exists  $M' \in \mathbf{K}_{\lambda}$  such that  $M' \leq_{\mathbf{K}} M_{j_0}$  and  $p_i$  does not  $\mathfrak{s}$ -fork over M' for all  $i < \chi^+$ . We know that  $|\mathrm{gS}(M')| \leq 2^{\lambda} = \chi$ , so by the pigeonhole principle again, we can assume that there is  $q \in \mathrm{gS}(M')$  such that  $p_i \upharpoonright M' = q$  for all  $i < \chi^+$ . By uniqueness,  $p_i = p_{i'}$  for all  $i, i' < \chi^+$ , and the result follows.

(2) By the proof of stability in Fact 4.6.1.

We would like to give conditions under which  $\mathfrak{s}$  has symmetry. A useful fact<sup>8</sup> is that it is enough to look at  $\mathfrak{s} \upharpoonright \lambda$ :

FACT 4.6.5 (Theorem 5.6.8).  $\mathfrak{s}$  has symmetry if and only if  $\mathfrak{s} \upharpoonright \lambda$  has symmetry.

Since we are not assuming anything about how  $\mathfrak{s}$  is defined, we will work by contradiction: We will show that if  $\theta$  is big enough and symmetry fails, then we get the order property, a nonstructure property which implies unstability. This is how the symmetry property of forking was originally proven in the first-order context, see [She90, Theorem III.4.13]. The same approach was later used in a non-elementary setup in [She75a, Theorem 5.1], and generalized in Theorem 3.5.13. We will rely on the proof of the latter.

Fact 4.6.6.

- (1) If **K** has the  $(\alpha, \chi)$ -order property of length  $h(\alpha + \chi + LS(\mathbf{K}))$ , then **K** has the  $(\alpha, \chi)$ -order property.
- (2) If **K** has the  $(\alpha, \chi)$ -order property, then it is  $\alpha$ -unstable in  $\chi'$  for all  $\chi' \ge \chi$ .

PROOF. This combines Fact 2.4.7 and 2.4.11.

FACT 4.6.7. If  $\mathfrak{s}$  does not have symmetry, then **K** has the  $(2, \lambda)$ -order property of length  $\theta$ .

PROOF. By Fact 4.6.5,  $\mathfrak{s} \upharpoonright \lambda$  does not have symmetry. The result now follows by exactly the same proof as Theorem 3.5.13.

COROLLARY 4.6.8. If  $\theta \ge h(\lambda)$ , then  $\mathfrak{s}$  has symmetry.

PROOF. If  $\mathfrak{s}$  does not have symmetry, then by Fact 4.6.7 and Fact 4.6.6.(1), **K** has the  $(2, \lambda)$ -order property and hence by Fact 4.6.6.(2) is 2-unstable in  $2^{\lambda}$ . By Theorem 4.2.5, **K** is unstable in  $2^{\lambda}$ , contradicting Proposition 4.6.4 (note that  $2^{\lambda} < h(\lambda) \leq \theta$ ).

Thus it seems quite a big gap between  $\lambda$  and  $\theta$  is needed. On the other hand the proof of Fact 4.6.1 tells us that with enough tameness we can make  $\mathcal{F}$  bigger:

FACT 4.6.9. Let  $\theta' \geq \theta$  and let  $\mathcal{F}' := [\lambda, \theta')$ . Assume  $\mathbf{K}_{\mathcal{F}'}$  has amalgamation and no maximal models, and  $\mathbf{K}$  is  $(\lambda, \theta')$ -tame. Then  $\mathfrak{s}$  can be extended to a good<sup>-</sup>  $[\lambda, \theta')$ -frame. If  $\mathfrak{s}$  has bs-stability, the extended frame will also have bs-stability. If  $\mathfrak{s}$  is type-full, then the extended frame will also be type-full.

**PROOF.** By Remark 4.2.20,  $\mathfrak{s}$  is determined by  $\mathfrak{s} \upharpoonright \lambda$ . Now apply Fact 4.6.1.

 $<sup>^{8}\</sup>mathrm{This}$  is not crucial to our argument, but enables us to obtain an explicit upper bound on the amount of tameness needed.

REMARK 4.6.10. We could replace  $(\lambda, \theta')$ -tameness by  $(\lambda', \theta')$ -tameness in the above, where  $\lambda' \in \mathcal{F}$ . This turns out to be equivalent (at least if we consider tameness for basic types) since the uniqueness property of  $\mathfrak{s}$  gives us  $(\lambda, \lambda')$ -tameness for basic types.

COROLLARY 4.6.11. Let  $\mathcal{F}' := [\lambda, h(\lambda))$ . Assume  $\mathbf{K}_{\mathcal{F}'}$  has amalgamation and no maximal models, and  $\mathbf{K}$  is  $(\lambda, < h(\lambda))$ -tame. Then  $\mathfrak{s}$  has symmetry.

PROOF. Using Fact 4.6.9, we can extend  $\mathfrak{s}$  to assume without loss of generality that  $\theta \geq h(\lambda)$ . Now use Corollary 4.6.8.

# 4.7. The main theorems

We finally have our promised good frame:

THEOREM 4.7.1. Assume:

- (1) **K** is an abstract elementary class.  $\mu \ge LS(\mathbf{K})$  is a cardinal.
- (2)  $\mathbf{K}_{\mu} \neq \emptyset$  has joint embedding.
- (3)  $\triangleleft$  is an abstract universal ordering on  $\mathbf{K}_{\mu}$ . In particular (by Remark 4.2.12),  $\mathbf{K}_{\mu}$  has amalgamation, no maximal models, and is stable.
- (4) Splitting has  $\aleph_0$ -local character for  $\triangleleft$ -increasing chains in  $\mathbf{K}_{\mu}$ .
- (5)  $\lambda > \mu$  is such that:
  - (a) **K** is  $(\mu, < h(\lambda))$ -tame.
  - (b)  $\mathbf{K}_{[\mu,h(\lambda))}$  has amalgamation and no maximal models.
  - (c) All the models in  $\mathbf{K}_{\lambda}$  are  $\mu^+$ -model-homogeneous.

Then **K** has a type-full good  $[\lambda, h(\lambda))$ -frame.

PROOF. Corollary 4.5.11, gives us a good<sup>-S</sup>  $\lambda$ -frame  $\mathfrak{s}$ . By Corollary 4.6.11,  $\mathfrak{s}$  also has symmetry.

We can use categoricity to derive some of the hypotheses above. We will use:

FACT 4.7.2. Assume **K** has amalgamation and no maximal models. Assume **K** is categorical in  $\lambda$ . Then:

- (1) **K** is stable in all  $LS(\mathbf{K}) \leq \mu < \lambda$ .
- (2) For any  $LS(\mathbf{K}) \leq \mu < cf(\lambda)$  and any limit  $\delta < \mu^+$ ,  $\mu$ -splitting has  $\aleph_0$ -local character for  $\triangleleft$ -chains, where  $\triangleleft := <_{\mathbf{K}}^{\mu,\delta}$ .
- (3) Let  $h_2 := h(h(\text{LS}(\mathbf{K})))$ . Assume  $\lambda$  is a successor cardinal and  $\lambda > \lambda_0 \ge h_2$ . Then  $\mathbf{K}$  is  $(h_2, \lambda_0)$ -tame and categorical in  $\lambda_0$ . In addition, the model of size  $\lambda_0$  is saturated.

PROOF. (1) is [She99, Claim 1.7]. (2) is [She99, Lemma 6.3], and (3) were originally stated (with a lower Hanf number) in [She99, Main Claim II.2.3] and [She99, Theorem II.2.7]. A full proof (with discussion on whether it is possible to lower the  $h_2$  bound) can be found in [Bal09, Chapter 14].

THEOREM 4.7.3. Let **K** be an abstract elementary class and let  $\lambda$  be a cardinal such that  $cf(\lambda) > \mu \ge LS(\mathbf{K})$ . Let  $\mathcal{F} := [\lambda, h(\lambda)), \mathcal{F}' := [\mu, h(\lambda))$ . Assume:

- (1)  $\mathbf{K}_{\mathcal{F}'}$  has amalgamation and no maximal models.
- (2)  $\mathbf{K}_{\lambda}$  is categorical.
- (3) **K** is  $(\mu, < h(\lambda))$ -tame.

Then **K** has a type-full good  $\mathcal{F}$ -frame.
PROOF. First,  $\mathbf{K}_{\lambda} \neq \emptyset$  by categoricity. By Lemma 4.5.12,  $\mathbf{K}_{\mathcal{F}'}$  has joint embedding and all models in  $\mathbf{K}_{\lambda}$  are  $\mu^+$ -model-homogeneous. By Fact 4.7.2,  $\mu$ -splitting has  $\aleph_0$ -local character for  $\triangleleft$ -chains, where  $\triangleleft := <_{\mathbf{K}}^{\lambda,\omega}$ . This shows all the hypotheses of Theorem 4.7.1 are satisfied.

Assuming categoricity in a high-enough successor, we obtain the tameness assumption:

THEOREM 4.7.4. Let **K** be an abstract elementary class. Let  $\mu := h_2 := h(h(\text{LS}(\mathbf{K})))$ . Let  $\lambda := \mu^+$ . Assume **K** has amalgamation, joint embedding, and is categorical in some successor  $\theta \ge h(\lambda)$ .

Let  $\mathcal{F} := [\lambda, \theta)$ . Then there is a type-full good  $\mathcal{F}$ -frame with underlying AEC **K**.

PROOF. Since  $\theta \ge h(\text{LS}(\mathbf{K}))$ , **K** has arbitrarily large models and so using joint embedding **K** has no maximal models. By Fact 4.7.2, **K** is categorical in  $\lambda$  and **K** is  $(\mu, < h(\lambda))$ -tame. Apply Theorem 4.7.3.

Notice that one also obtains that categoricity (at a cardinal of high-enough cofinality) and tameness implies stability everywhere. This improves on [**BKV06**, Corollary 4.7]:

THEOREM 4.7.5. Let **K** be an abstract elementary class with amalgamation and no maximal models. Assume **K** is categorical in some  $\lambda$  such that  $cf(\lambda) > \mu \ge$  $LS(\mathbf{K})$  and **K** is  $(\mu, \mu')$ -tame. Then **K** is stable in all  $\theta \in [LS(\mathbf{K}), \mu']$ . In particular, if  $\mu' = \infty$ , then **K** is stable everywhere.

PROOF. By Fact 4.7.2,  $\mu$ -splitting has  $\aleph_0$ -local character for  $\triangleleft$ -chains, where  $\triangleleft := <_{\mathbf{K}}^{\mu,\omega}$  and  $\mathbf{K}$  is stable everywhere below and at  $\mu$ . Apply Theorem 4.5.6 to see  $\mathbf{K}$  is stable everywhere in  $(\mu, \mu']$ .

This result is much more local than the other results of this section. For example, we do not need to assume that  $\mu' \ge h(\mu)$ . Moreover, as Theorem 4.5.6 shows, the categoricity hypothesis can be replaced by  $\mu$ -splitting having  $\aleph_0$ -local character for <-chains, for some abstract universal ordering < on  $\mathbf{K}_{\mu}$ .

Assuming the generalized continuum hypothesis (GCH), we obtain a more general stability spectrum theorem:

THEOREM 4.7.6. Assume GCH. Let **K** be an abstract elementary class with amalgamation and no maximal models. Assume **K** is  $\mu$ -tame for  $\mu \geq \text{LS}(\mathbf{K})$ ,  $\triangleleft$  is an abstract universal ordering on  $\mathbf{K}_{\mu}$ , and  $\mu$ -splitting has  $\kappa$ -local character for  $\triangleleft$ -increasing chains. Then **K** is stable in all  $\lambda \geq \mu$  with  $\lambda = \lambda^{<\kappa}$ .

PROOF. **K** is stable in  $\mu$  since we have an abstract universal ordering on **K**<sub> $\mu$ </sub>. If  $\lambda > \mu$ , the result follows from Theorem 4.5.7.

REMARK 4.7.7. If **K** is the class of models of a complete first-order theory, the conditions for stability given by Corollary 4.7.6 are very close<sup>9</sup> to optimal (see [**She90**, Corollary III.3.8]).

<sup>&</sup>lt;sup>9</sup>The least regular cardinal  $\kappa$  such that splitting has  $\kappa$ -local character will be at most the successor of  $\kappa(T)$ .

REMARK 4.7.8. Let **K** be an abstract elementary class with amalgamation and no maximal models. Assume **K** is  $\chi$ -tame and stable in some  $\mu \ge h(\chi)$ . Then [**GV06b**, Theorem 4.13] shows that for some  $\kappa < h(\chi)$ ,  $\mu$ -splitting has  $\kappa$ -local character. Thus we have:

COROLLARY 4.7.9. Assume GCH. Let **K** be an abstract elementary class with amalgamation and no maximal models. Assume **K** is  $\chi$ -tame and stable in some  $\mu \geq \text{LS}(\mathbf{K})$ . Then there is  $\kappa < h(\chi)$  such that **K** is stable in all  $\lambda \geq \mu$  with  $\lambda^{<\kappa} = \lambda$ .

PROOF. If  $\mu < h(\chi)$ , then by [**GV06b**, Corollary 6.4] one can take  $\kappa := \mu^+$ , so assume  $\mu \ge h(\chi)$ . By the previous remark, there is  $\kappa < h(\chi)$  such that  $\mu$ -splitting has  $\kappa$ -local character. The result now follows from Theorem 4.7.6.

REMARK 4.7.10. In Chapter 2, we use different methods to prove Corollary 4.7.9 in ZFC. We do not know whether Corollary 4.7.6 also holds in ZFC (although it is clear from the proof that much less than GCH is needed).

We can also apply our good frame to the question of uniqueness of limit models:

THEOREM 4.7.11 (Uniqueness of limit models). Assume the hypotheses of Theorem 4.7.3 hold. Then **K** has a unique limit model in any  $\mu' \in \mathcal{F}$ . In fact, if  $M_0 \in \mathbf{K}_{\mu'}$  and  $M_\ell$  is  $(\mu', \delta_l)$ -limit over  $M_0$  for  $\ell = 1, 2$  and  $\delta_l$  a limit ordinal, then  $M_1 \cong_{M_0} M_2$ .

In particular, if **K** has amalgamation and no maximal models, is categorical in  $\lambda$  and is  $\mu$ -tame for some  $\mu < cf(\lambda)$ , then **K** has a unique limit model in any  $\mu' \geq \lambda$ .

PROOF. By Theorem 4.7.3, **K** has a good  $\mathcal{F}$ -frame  $\mathfrak{s}$ . In particular, **K** is stable in  $\mu'$ , so one can iterate Fact 4.2.8 to build a  $(\mu', \delta)$ -limit model for any desired  $\delta < (\mu')^+$ . To see uniqueness, apply [**She09a**, Lemma II.4.8] (see [**Bon14a**, Theorem 9.2] for a detailed proof of that result).

We see this theorem as an encouraging approximation to generalizing the upward categoricity transfer result of [**GV06a**] (which assumes categoricity in a successor cardinal) to categoricity in a limit cardinal.

REMARK 4.7.12. Uniqueness of limit models of cardinality  $\mu$  was asserted to follow from categoricity in some  $\lambda^+ > \mu$  already in [**SV99**]. However, an error was found by VanDieren in 1999. VanDieren [**Van06**, **Van13**] proves uniqueness with the additional assumption that unions of amalgamation bases are amalgamation bases (but does not use tameness). It is still open whether uniqueness of limit models follows from categoricity only. In [**GVV16**], it is shown that uniqueness of limit models follows from a superstability-like assumption akin to  $\aleph_0$ -local character of  $\mu$ -splitting, amalgamation, and a unidimensionality assumption (the authors initially claimed to prove the result without unidimensionality but the claim was later retracted).

REMARK 4.7.13. A variation on Theorem 4.7.11 is [**BG**, Corollary 6.18], which uses stronger locality assumptions but manages to obtain uniqueness of limit models below the categoricity cardinal without any cofinality restriction.

#### 4.8. Conclusion and further work

Assuming amalgamation, joint embedding, no maximal models, and tameness, we have given superstability-like conditions under which an abstract elementary class has a type-full good frame  $\mathfrak{s}$ , i.e. a forking-like notion for 1-types. These arguments would work just as well to get a notion of independence for all *n*-types, with  $n < \omega$ . The proof of extension breaks down, however, for types of infinite length (difficulties in obtaining the extension property in the absence of compactness is one of the reasons<sup>10</sup> it was assumed as an axiom in [**BG**]).

Shelah's approach around this in [She09a, Chapter II] is to show that if the frame is *weakly successful* (a uniqueness condition for certain kinds of amalgamations), then it has a notion of forking for types of models. In [She09a, Chapter III], Shelah has several hundreds of pages of approximations on when weak successfulness can be transferred across cardinals (many of his difficulties come from the fact he is not assuming amalgamation or no maximal models), but even assuming  $\mathfrak{s} \upharpoonright \lambda$  is weakly successful for every  $\lambda$ , it is not clear how we can get a good forking notion for models of different sizes. This is one direction further work could focus on.

Another (non-orthogonal) direction would be to find applications for such a forking notion. As mentioned in the previous section, we believe it could be useful in proving categoricity transfer theorems. Moreover, the frame built in Section 4.5 is only well-behaved for  $\mu^+$ -saturated models, and it would be interesting to know when the class of  $\mu^+$ -saturated models is an AEC. This calls for tools to deal with unions of saturated models and we plan to explore this further in future work<sup>11</sup>.

<sup>&</sup>lt;sup>10</sup>Another reason was Shelah's example (see [**HL02**, Section 4]) of an  $\aleph_0$ -stable non-simple diagram, but we have shown that we do not get into trouble as long as we restrict the base of our types to be sufficiently saturated models.

<sup>&</sup>lt;sup>11</sup>Since this chapter was first circulated, several extensions have been written. In Chapter 6, the argument here is axiomatized, the cofinality assumption on the categoricity cardinal is removed and a global independence relation (for types of all lengths) is built (assuming more hypotheses). This is used to prove an approximation to Shelah's categoricity conjecture. In Chapter 7, it is shown that it follows from  $\aleph_0$ -local character of splitting and tameness that, for all high-enough cardinals  $\lambda$ , the union of a chain of  $\lambda$ -model-homogeneous models is  $\lambda$ -model-homogeneous. All these works ultimately rely on the methods of this chapter.

# CHAPTER 5

# Tameness and frames revisited

This chapter is based on [**BVe**] and is joint work with Will Boney. The authors would like to thank the referee for their helpful report that greatly assisted the clarity and presentation of this paper.

#### Abstract

We study the problem of extending an abstract independence notion for types of singletons (what Shelah calls a good frame) to longer types. Working in the framework of tame abstract elementary classes, we show that good frames can always be extended to types of independent sequences. As an application, we show that tameness and a good frame imply Shelah's notion of dimension is well-behaved, complementing previous work of Jarden and Sitton. We also improve a result of Boney on extending a frame to larger models.

## 5.1. Introduction

Good  $\lambda$ -frames are an axiomatic notion of independence in abstract elementary classes (AECs) introduced by Shelah [She09a, Chapter II]. They are one of the main tools in the classification theory of AECs. They describe a relation "p does not fork over M" for certain types of singletons over models of size  $\lambda$ . The frame's nonforking relation is required to satisfy properties akin to those of forking in a first-order superstable theory. The definition can be generalized to that of a good ( $< \alpha, [\lambda, \theta)$ )-frame, where instead of types of singletons one allows types of sequences of less than  $\alpha$ -many elements, and instead of the models being of size  $\lambda$ , one allows their size to lie in the interval  $[\lambda, \theta)$ .

There are at least two questions one can ask about frames: first, under what hypotheses do they exist? Second, can we extend them? That is, assuming there is a frame can we extend it to give a nonforking definition for larger models or longer<sup>1</sup> types?

Shelah tackles these problems in [She09a, Chapters II and III], but the answers use strong model-theoretic hypotheses (typically categoricity in two successive cardinals  $\lambda$  and  $\lambda^+$  together with few models in  $\lambda^{++}$ ), as well as set-theoretic hypotheses (like the weak generalized continuum hypothesis,  $2^{\lambda} < 2^{\lambda^+}$ )<sup>2</sup>.

Recently, the two questions above have been studied in the framework of *tame* AECs. Tameness is a locality property of AECs isolated by Grossberg and VanDieren [**GV06b**] from an argument in [**She99**]. Grossberg and VanDieren have shown [**GV06c**, **GV06a**] that Shelah's eventual categoricity conjecture from

<sup>&</sup>lt;sup>1</sup>The length of a type is the length or indexing set of a tuple that satisfies it.

 $<sup>^{2}</sup>$ Shelah also looks at the existence problem in a more global setup and in ZFC in [She09a, Chapter IV] but does not study the extension problem there.

a successor holds in tame AECs, and Boney [**Bon14b**] (building on work of Makkai-Shelah [**MS90**]) has shown that tameness follows from a large cardinal axiom. Many examples of interest are also known to be tame.

Under tameness, we have shown that frames exist in ZFC assuming a reasonable categoricity hypothesis (Chapter 4), and Boney has shown [**Bon14a**] that frames can be extended to larger *models* under the assumption of tameness for types of length two. In this chapter, we further study the frame extension question in tame AECs. We look at the problem of *elongating* the frame: extending it to longer *types*.

Let us give discuss a natural approach to the problem and its shortcomings. In stable first-order theories, we have that  $ab \downarrow B$  if and only if  $a \downarrow B$  and  $b \downarrow Ba$ . One might think that this allows us to define forking for types of all lengths if we have a definition of forking for singleton types (as in [**GL00**]). However, this turns out not to work in full generality, as good frames only define forking over  $models^3$ . We might want to say that  $ab \downarrow N$  if and only if there are  $M \prec M' \prec N'$  with  $N \prec N'$  and  $a \in M'$  such that  $a \downarrow N$  and  $b \downarrow N'$ . This means that a choice must be made for the models N' and, especially, M' and this choice can cause problems. In particular, if M' is too big, then uniqueness of nonforking extensions can fail. This does not cause issues in the first-order context essentially because there is a prime/minimal set containing A and a (namely Aa).

There are two options to work around this issue. The first option is to assume the existence of a *unique* prime/minimal extension of Ma; Shelah says that the frame is *weakly successful* [She09a, Definition III.1.1] if this is the case. Shelah proved [She09a, Section II.6] that weakly successful frames can be elongated as desired without any assumption of tameness. Shelah has also shown [She09a, II.5] that a good  $\lambda$ -frame is weakly successful when the underlying AEC has few models in  $\lambda^{++}$  and certain set-theoretic hypotheses hold. It is not known whether being weakly successful follows from tameness<sup>4</sup>.

The second option is to strengthen the condition on nonforking, essentially setting  $ab \downarrow B$  if and only if  $a \downarrow B$  and  $b \downarrow Ba$  (the noncanonical choice of a cover for Ba is less important). This loses some information about nonforking, so only works for certain kinds of types: types of independent sequences. As we show, this has the advantage of working in a larger class of AECs, i.e. those with frames that are not weakly successful, although we do assume tameness to prove the symmetry property.

This brings us to the precise statement of the main result of this chapter.

THEOREM 5.1.1. Let **K** be an AEC with amalgamation and  $\mathcal{F} = [\lambda, \theta)$  be an interval of cardinals.

(1) Assume  $\mathfrak{s}$  satisfies the axioms of a good  $\mathcal{F}$ -frame, except possibly symmetry. Then  $\mathfrak{s}$  can be extended to a certain frame  $\mathfrak{s}^{<\theta}$  which satisfies the axioms of a good ( $< \theta, \mathcal{F}$ )-frame, except possibly for symmetry.

 $<sup>^{3}</sup>$ Note that an example of Shelah (see [**HL02**, Section 4]) shows that there exists a superstable homogeneous diagram where extension (over sets) fails for any reasonable independence notion.

 $<sup>^{4}</sup>$ After the initial circulation of the paper this chapter is based on, it has been shown that being weakly successful follows from a stronger locality property: full tameness and shortness, see Section 6.11.

#### 5.1. INTRODUCTION

(2) If **K** is  $\lambda$ -tame, then both  $\mathfrak{s}$  and  $\mathfrak{s}^{<\theta}$  also satisfy symmetry.

The following two questions are open.

QUESTION 5.1.2.

- If  $\mathfrak{s}$  is a good frame in a tame AEC, must  $\mathfrak{s}$  be weakly successful?
- Is there an example of a good λ-frame (necessarily not weakly successful) that has no better extension to longer types than independent sequences?

As has already been alluded to, in Theorem 5.1.1 the frame  $\mathfrak{s}$  is elongated by use of *independent sequences* (see Definition 5.4.1 here, or Shelah [She09a, Definition III.5.2]). Independent sequences in that context have been previously studied by Shelah [She09a, III.5] and Jarden and Sitton [JS12]. Throughout these studies, several additional assumptions have appeared–such as  $\mathfrak{s}$  being weakly successful or having continuity of serial independence<sup>5</sup>–that we are able to eliminate or replace with the hypothesis of tameness.

We present two applications of Theorem 4.7. The first involves a natural notion of dimension that Shelah introduced with the goal of building a theory of regular types for AECs [**She09a**, Definition III.5.12]: Let us define the dimension of a type p in an ambient model N, dim(p, N) to be the size of a maximal independent set of realizations of p in N. In the first-order case, Shelah [**She90**, III.4.21.(2)] shows that, under stability, every infinite maximal independent set of realizations of p has the same size. In the AEC framework, Shelah [**She09a**, III.5.14] first showed that this held when the frame is weakly successful, and Jarden and Sitton [**JS12**] have refined these hypotheses. The analysis of this chapter allows us to show that the dimension is well-behaved in any tame AEC with a good frame (see Corollary 5.6.1 and the surrounding discussion). This gives a natural nonelementary framework in which a theory of regular types could be studied.

The second application involves the project of extending a frame to larger *models* using tameness. As mentioned above, Boney has shown that this is possible if one assumes tameness for types of length two. Analyzing the elongations of frames allows us to give an aesthetic improvement: we remove this strange assumption and replace it with only tameness for types of length one (see Corollary 5.6.9 and the preceding discussions). While no example of an AEC that is tame for types of length one and not for length two is known, thinking about this statement led us to the main theorem of this chapter. Further, we are told that Rami Grossberg conjectured Corollary 5.6.9 already in 2006 (he told it to Adi Jarden and John Baldwin); our result proves Grossberg's original conjecture.

Since this chapter was first circulated (June 2014), several applications of Corollary 5.6.9 have been found. They include Shelah's eventual categoricity conjecture for universal classes (Chapters 8, 16), as well as a downward categoricity transfer for tame AECs in Chapter 14 (the latter actually uses the theory of independent sequences in good frames developped in Section 5.4). In Chapter 18, we show that a natural good frame appearing in the Hart-Shelah example is not weakly successful, and in [**Vasd**] we study an example of Shelah where a good frame cannot be extended to all types. These examples show that this chapter *strictly* generalizes Shelah's study of independent sequences in [**She09a**, Section III.5].

The chapter is structured as follows. In Section 5.2, we review background in the theory of AECs. In Section 5.3, we give the definition of good frames and

<sup>&</sup>lt;sup>5</sup>This and other variations on continuity are defined and explored in Section 5.5.1.

prove some easy general facts. In Section 5.4, we define independent sequences and show how to use them to extend a frame for types of singletons to a frame for longer types. We show all properties are preserved in the process, except perhaps symmetry. In Section 5.5, we give conditions under which symmetry also transfers and show how to use it to define a well-behaved notion of dimension. In Section 5.6, we prove the promised applications to dimension and tameness.

### 5.2. Preliminaries

If  $p \in gS^{\alpha}(M)$ , we define  $\ell(p) := \alpha$  and dom p := M. Note that  $\alpha$  is an invariant of the Galois type and is referred to as its *length*.

Say  $p = \operatorname{gtp}(\bar{a}/M; N) \in \operatorname{gS}^{\alpha}(M)$ , where  $\bar{a} = \langle a_i : i < \alpha \rangle$ . For  $X \subseteq \alpha$  and  $M_0 \leq_{\mathbf{K}} M$ , write  $p^X \upharpoonright M_0$  for  $\operatorname{gtp}(\bar{a}_X/M_0; N)$ , where  $\bar{a}_X := \langle a_i : i \in X \rangle$ . We say  $p \in \operatorname{gS}^{\alpha,\operatorname{na}}(M)$  if  $a_i \notin M$  for all  $i < \alpha$ , and similarly define  $\operatorname{gS}^{<\alpha,\operatorname{na}}(M)$  (it is easy to check these definitions do not depend on the choice of  $\bar{a}$  and N).

We briefly review the notion of tameness. Although it appears implicitly (for saturated models) in Shelah [She99], tameness as a property of AECs was first introduced in Grossberg and VanDieren [GV06b] and used to prove a stability spectrum theorem there.

DEFINITION 5.2.1 (Tameness). Let  $\theta > \lambda \geq \mathrm{LS}(\mathbf{K})$  and let  $\mathcal{G} \subseteq \bigcup_{M \in \mathbf{K}} \mathrm{gS}^{<\infty}(M)$ be a family of types. We say that  $\mathbf{K}$  is  $(\lambda, \theta)$ -tame for  $\mathcal{G}$  if for any  $M \in \mathbf{K}_{\leq \theta}$  and any  $p, q \in \mathcal{G} \cap \mathrm{gS}^{<\infty}(M)$ , if  $p \neq q$ , then there exists  $M_0 \leq_{\mathbf{K}} M$  of size  $\leq \lambda$  such that  $p \upharpoonright M_0 \neq q \upharpoonright M_0$ . We define similarly  $(\lambda, < \theta)$ -tame,  $(<\lambda, \theta)$ -tame, etc. When  $\theta = \infty$ , we omit it.  $(\lambda, \theta)$ -tame for  $\alpha$ -types means  $(\lambda, \theta)$ -tame for  $\bigcup_{M \in \mathbf{K}} \mathrm{gS}^{\alpha}(M)$ , and similarly for  $< \alpha$ -types. When  $\alpha = 1$ , we omit it and simply say  $(\lambda, \theta)$ -tame.

**5.2.1.** Commutative Diagrams. Since a picture is worth a thousand words, we make extensive use of commutative diagrams to illustrate the proofs. Most of the notation is standard. When we write

$$M_0 \xrightarrow{f} M_1 \xrightarrow{g} M_2$$

The functions f and g, typically written above arrows, are always **K**-embeddings; that is,  $f: M_0 \cong f[M_0] \leq_{\mathbf{K}} M_1$ . Writing no functions means that the **K**-embedding is the identity. The elements in square brackets a and  $\bar{b}$ , typically written below arrows, are elements that exist in the target model, but not the source model; that is,  $a \in M_1 - f[M_0]$ . Writing no element simply means that there are no elements that we wish to draw the reader's attention to in the difference. In particular, it does not mean that the two models are isomorphic. We sometimes make a distinction between embeddings appearing in the hypothesis of a statement (denoted by solid lines), and those appearing in the conclusion (denoted by dotted lines).

### 5.3. Good frames

Good frames were first defined in [She09a, Chapter II]. The idea is to provide a localized (i.e. only for base models of a given size  $\lambda$ ) axiomatization of a forking-like notion for a "nice enough" set of 1-types. These axioms are similar to the properties of first-order forking in a superstable theory. Jarden and Shelah (in [JS13]) later gave a slightly more general definition, not assuming the existence of a superlimit model and dropping some of the redundant clauses. We give a slightly more general

variation here: following Chapter 4, we assume the models come from  $\mathbf{K}_{\mathcal{F}}$ , for  $\mathcal{F}$  an interval, instead of just  $\mathbf{K}_{\lambda}$ . We also assume that the types could be longer than just types of singletons. We first adapt the definition of a pre- $\lambda$ -frame from [**She09a**, Definition III.0.2.1]:

DEFINITION 5.3.1 (Pre-frame). Let  $\alpha$  be an ordinal and let  $\mathcal{F}$  be an interval of the form  $[\lambda, \theta)$ , where  $\lambda$  is a cardinal, and  $\theta > \lambda$  is either a cardinal or  $\infty$ .

A pre-( $< \alpha, \mathcal{F}$ )-frame is a triple  $\mathfrak{s} = (\mathbf{K}, \bigcup, gS^{bs})$ , where:

- (1) **K** is an abstract elementary class with  $\lambda \geq LS(\mathbf{K}), \mathbf{K}_{\lambda} \neq \emptyset$ .
- (2)  $\mathrm{gS}^{\mathrm{bs}} \subseteq \bigcup_{M \in \mathbf{K}_{\mathcal{F}}} \mathrm{gS}^{<\alpha,\mathrm{na}}(M)$ . For  $M \in \mathbf{K}_{\mathcal{F}}$  and  $\beta$  an ordinal, we write  $\mathrm{gS}^{\beta,\mathrm{bs}}(M)$  for  $\mathrm{gS}^{\mathrm{bs}} \cap \mathrm{gS}^{\beta,\mathrm{na}}(M)$  and similarly for  $\mathrm{gS}^{<\beta,\mathrm{bs}}(M)$ .
- (3)  $\downarrow$  is a relation on quadruples of the form  $(M_0, M_1, \bar{a}, N)$ , where  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N, \bar{a} \in {}^{<\alpha}N$ , and  $M_0, M_1, N$  are all in  $\mathbf{K}_{\mathcal{F}}$ . We write  $\downarrow (M_0, M_1, \bar{a}, N)$  or  $\bar{a} \downarrow M_1$  instead of  $(M_0, M_1, a, N) \in \downarrow$ .
- (4) The following properties hold:
  - (a) <u>Invariance</u>: If  $f : N \cong N'$  and  $\bar{a} \underset{M_0}{\downarrow}^N M_1$ , then  $f(\bar{a}) \underset{f[M_0]}{\downarrow}^{N'} f[M_1]$ . If  $gtp(\bar{a}/M_1; N) \in gS^{bs}(M_1)$ , then  $gtp(f(\bar{a})/f[M_1]; N') \in gS^{bs}(f[M_1])$ .
  - (b) <u>Monotonicity</u>: If  $\bar{a} \downarrow_{M_0}^{V} M_1$ ,  $\bar{a}'$  is a subsequence of  $\bar{a}$ ,  $M_0 \leq_{\mathbf{K}} M'_0 \leq_{\mathbf{K}} M'_0 \leq_{\mathbf{K}} M'_1 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N \leq_{\mathbf{K}} N''$  with  $\bar{a}' \in N'$ , and  $N'' \in \mathbf{K}_{\mathcal{F}}$ , then  $\bar{a}' \downarrow_{M'_0}^{V} M'_1$  and  $\bar{a}' \downarrow_{M'_0}^{V} M'_1$ . If  $\operatorname{gtp}(\bar{a}/M_1; N) \in \operatorname{gS}^{\operatorname{bs}}(M_1)$  and  $\bar{a}'$  is a subsequence of  $\bar{a}$ , then  $\operatorname{gtp}(\bar{a}'/M_1; N) \in \operatorname{gS}^{\operatorname{bs}}(M_1)$ .
  - (c) <u>Nonforking types are basic</u>: If  $\bar{a} \downarrow_{M}^{N} M$ , then  $gtp(\bar{a}/M; N) \in gS^{bs}(M)$ .

A pre- $(\leq \alpha, \mathcal{F})$ -frame is a pre- $(< (\alpha + 1), \mathcal{F})$ -frame. When  $\alpha = 1$ , we drop it. We write pre- $(< \alpha, \lambda)$ -frame instead of pre- $(< \alpha, \{\lambda\})$ -frame or pre- $(< \alpha, [\lambda, \lambda^+))$ -frame; and pre- $(< \alpha, (\geq \lambda))$ -frame instead of pre- $(< \alpha, [\lambda, \infty))$ -frame. We sometimes drop the  $(< \alpha, \mathcal{F})$  when it is clear from context.

For  $\mathfrak{s}$  a pre-( $\langle \alpha, \mathcal{F}$ )-frame,  $\beta \leq \alpha$ , and  $\mathcal{F}' \subseteq \mathcal{F}$  an interval, we let  $\mathfrak{s}_{\mathcal{F}'}^{<\beta}$  denote the pre-( $\langle \beta, \mathcal{F}'$ )-frame defined in the obvious way by restricting the basic types and  $\downarrow$  to models in  $\mathbf{K}_{\mathcal{F}'}$  and elements of length  $\langle \beta$ . If  $\mathcal{F}' = \mathcal{F}$  or  $\beta = \alpha$ , we omit it. For  $\lambda' \in \mathcal{F}$ , we write  $\mathfrak{s}_{\lambda'}^{<\beta}$  instead of  $\mathfrak{s}_{\{\lambda'\}}^{<\beta}$ .

REMARK 5.3.2. Note that, following Shelah's original definition, we have defined nonforking (in the sense of frames) only for nonalgebraic types. However, this restriction is inessential: We could expand the definition of nonforking to algebraic types by saying that an algebraic  $p \in S(M)$  does not fork over  $M_0$  if and only if  $p \upharpoonright M_0$  is algebraic. This change would not affect whether or not a frame satisfies the properties given<sup>6</sup>.

REMARK 5.3.3. The reader might wonder about the reasons for having a special class of basic types. Following Shelah [She09a, Definition III.9.2], let us call a pre-frame *type-full* if the basic types are all the nonalgebraic types. It can be shown

 $<sup>^{6}</sup>$ In the statement of the extension property, we would need to require that the nonforking extension of a nonalgebraic type is nonalgebraic.

[She09a, III.9.6] that any weakly successful good frame can be extended to a type-full one. Furthermore, there are no known examples of a good  $\lambda$ -frame which which cannot be extended to a type-full one. However Shelah's initial construction [She09a, II.3] builds a non type-full good frame and it is not clear that it can be extended to a type-full one until after Shelah shows that the frame is weakly successful. Thus it can be easier to build a good frame than to build a type-full one, and most results about frame already hold in the non-type-full context. In this chapter, we will set the basic types to be the independent sequences, hence getting another natural example of a non type-full good frame.

NOTATION 5.3.4. If  $\mathfrak{s} = (\mathbf{K}, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  is a pre- $(\langle \alpha, \mathcal{F})$ -frame, then  $\alpha_{\mathfrak{s}} := \alpha$ ,  $\mathcal{F}_{\mathfrak{s}} := \mathcal{F}, \mathbf{K}_{\mathfrak{s}} := \mathbf{K}, \downarrow_{\mathfrak{s}} := \downarrow$ , and  $\mathrm{gS}^{\mathrm{bs}}_{\mathfrak{s}} := \mathrm{gS}^{\mathrm{bs}}$ . If  $\mathcal{F} = [\lambda, \theta)$ , then let  $\lambda_{\mathfrak{s}} := \lambda$ ,  $\theta_{\mathfrak{s}} := \theta$ .

By the invariance and monotonicity properties,  $\downarrow$  is really a relation on types. This justifies the next definition.

DEFINITION 5.3.5. If  $\mathfrak{s} = (\mathbf{K}, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  is a pre- $(\langle \alpha, \mathcal{F}\rangle)$ -frame,  $p \in \mathrm{gS}^{\langle \alpha}(M_1)$ is a type, we say p does not fork over  $M_0$  if  $\bar{a} \downarrow_{M_0}^N M_1$  for some (equivalently any)  $\bar{a}$ and N such that  $p = \mathrm{gtp}(\bar{a}/M_1; N)$ . If  $\mathfrak{s}$  is not clear from context, we add "with respect to  $\mathfrak{s}$ ".

REMARK 5.3.6. We could have started from  $(\mathbf{K}, \downarrow)$  and defined the basic types as those that do not fork over their own domain. Since we are sometimes interested in studying frames that only satisfy existence over a certain class of models (like the saturated models), we will not adopt this approach.

REMARK 5.3.7. We could also have specified only  $\mathbf{K}_{\mathcal{F}}$  or even only  $\mathbf{K}_{\lambda}$  instead of the full AEC **K**. This is completely equivalent since, by [She09a, Section II.2],  $\mathbf{K}_{\lambda}$  fully determines **K**.

DEFINITION 5.3.8 (Good frame). Let  $\alpha$ ,  $\mathcal{F}$  be as above.

A good  $(< \alpha, \mathcal{F})$ -frame is a pre- $(< \alpha, \mathcal{F})$ -frame  $(\mathbf{K}, \perp, gS^{bs})$  satisfying in addition:

- (1)  $\mathbf{K}_{\mathcal{F}}$  has amalgamation, joint embedding, and no maximal models.
- (2) bs-Stability:  $|gS^{1,bs}(M)| \le ||M||$  for all  $M \in \mathbf{K}_{\mathcal{F}}$ .
- (3) Density of basic types: If  $M <_{\mathbf{K}} N$  are in  $\mathbf{K}_{\mathcal{F}}$ , then there is  $a \in N$  such that  $gtp(a/M; N) \in \mathbf{gS}^{bs}(M)$ .
- (4) Existence of nonforking extension: If  $p \in gS^{bs}(M)$ ,  $N \geq_{\mathbf{K}} M$  is in  $\mathbf{K}_{\mathcal{F}}$ , and  $\beta < \alpha$  is such that  $\ell(p) \leq \beta$ , then there is some  $q \in gS^{\beta, bs}(N)$  that does not fork over M and extends p, i.e.  $q^{\beta} \upharpoonright M = p$ .
- (5) Uniqueness: If  $p, q \in gS^{<\alpha}(N)$  do not fork over M and  $p \upharpoonright M = q \upharpoonright M$ , then p = q.
- (6) <u>Symmetry</u>: If  $\bar{a}_1 \underset{M_0}{\overset{N}{\downarrow}} M_2$ ,  $\bar{a}_2 \in {}^{<\alpha}M_2$ , and  $\operatorname{gtp}(\bar{a}_2/M_0; N) \in \operatorname{gS}^{\operatorname{bs}}(M_0)$ , then

there is  $M_1$  containing  $\bar{a}_1$  and  $N' \geq_{\mathbf{K}} N$  such that  $\bar{a}_2 \bigcup_{M_0}^{N'} M_1$ .

(7) <u>Local character</u>: If  $\delta$  is a regular cardinal,  $\langle M_i \in \mathbf{K}_{\mathcal{F}} : i \leq \delta \rangle$  is increasing continuous, and  $p \in \mathrm{gS}^{\mathrm{bs}}(M_{\delta})$  is such that  $\ell(p) < \delta$ , then there exists  $i < \delta$  such that p does not fork over  $M_i$ .

#### 5.3. GOOD FRAMES

- (8) <u>Continuity</u>: If  $\delta$  is a limit ordinal,  $\langle M_i \in \mathbf{K}_{\mathcal{F}} : i \leq \delta \rangle$  and  $\langle \alpha_i < \alpha : i \leq \delta \rangle$ are increasing and continuous, and  $p_i \in \mathrm{gS}^{\alpha_i,\mathrm{bs}}(M_i)$  for  $i < \delta$  are such that  $j < i < \delta$  implies  $p_j = p_i^{\alpha_j} \upharpoonright M_j$ , then there is some  $p \in \mathrm{gS}^{\alpha_\delta,\mathrm{bs}}(M_\delta)$ such that for all  $i < \delta$ ,  $p_i = p^{\alpha_i} \upharpoonright M_i$ . Moreover, if each  $p_i$  does not fork over  $M_0$ , then neither does p.
- (9) <u>Transitivity</u>: If  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ ,  $p \in S(M_2)$  does not fork over  $M_1$ and  $p \upharpoonright M_1$  does not fork over  $M_0$ , then p does not fork over  $M_0$ .

We will sometimes refer to "existence of nonforking extension" as simply "existence".

For  $\mathbb{L}$  a list of properties<sup>7</sup>, a good<sup> $-\mathbb{L}$ </sup> ( $< \alpha, \mathcal{F}$ )-frame is a pre-( $< \alpha, \mathcal{F}$ )-frame that satisfies all the properties of good frames except possibly the ones in  $\mathbb{L}$ . In this chapter,  $\mathbb{L}$  will only contain symmetry and/or bs-stability. We abbreviate symmetry by S, bs-stability by St, and write good<sup>-</sup> for good<sup>-(S,St)</sup>.

We say that **K** has a good ( $< \alpha, \mathcal{F}$ )-frame if there is a good ( $< \alpha, \mathcal{F}$ )-frame where **K** is the underlying AEC (and similarly for good<sup>-</sup>).

REMARK 5.3.9. Transitivity follows directly from existence and uniqueness by [She09a, Claim II.2.18].

REMARK 5.3.10. The obvious monotonicity properties hold: If  $\mathfrak{s}$  is a good  $(< \alpha, \mathcal{F})$ -frame,  $\beta \leq \alpha$ , and  $\mathcal{F}'$  is a subinterval of  $\mathcal{F}$ , then  $\mathfrak{s}_{\mathcal{F}'}^{<\beta}$  is a good  $(< \beta, \mathcal{F}')$  frame (and similarly for good<sup>-</sup>).

REMARK 5.3.11. If T is a superstable first-order theory, then forking induces a good ( $\geq |T|$ )-frame on the class of models of T ordered by elementary submodel. In the non-elementary context, Shelah showed in [She09a, Theorem II.3.7] how to build a good frame from local categoricity hypotheses and GCH-like assumptions, while we (Chapter 4) showed how to build a good frame in ZFC from categoricity, tameness, and a monster model. Note that a family of examples due to Hart and Shelah [HS90] demonstrates that, in the absence of tameness, an AEC could have a good  $\lambda$ -frame but no good ( $\geq \lambda$ )-frame (see [Bon14a, Section 10] for a detailed writeup).

Note that for types of finite length, local character implies that nonforking is witnessed by a model of small size:

PROPOSITION 5.3.12. Let  $\alpha \leq \omega$ . Assume  $\mathfrak{s} = (\mathbf{K}, \bigcup, \mathrm{gS}^{\mathrm{bs}})$  is a pre- $(\langle \alpha, \mathcal{F})$ -frame satisfying local character and transitivity. If  $M \in \mathbf{K}_{\mathcal{F}}$  and  $p \in \mathrm{gS}^{\mathrm{bs}}(M)$ , then there exists  $M' \in \mathbf{K}_{\lambda}$  such that p does not fork over M'.

PROOF. Same proof as Proposition 4.2.21 (there  $\alpha = 1$  but this does not change the proof).

We conclude this section with an easy variation on the existence property that will be used later.

LEMMA 5.3.13. Assume  $\mathfrak{s} = (\mathbf{K}, \, \bigcup, \mathrm{gS}^{\mathrm{bs}})$  is a pre- $(< \alpha, \mathcal{F})$ -frame with amalgamation, existence, and transitivity. Suppose  $M \leq_{\mathbf{K}} M_0 \leq_{\mathbf{K}} M_1$  are in  $\mathbf{K}_{\mathcal{F}}$  and  $f: M_0 \to M_2$  is given with  $M_2 \in \mathbf{K}_{\mathcal{F}}$ . Assume also that we have  $\bar{a} \in M_1$  such that  $\overset{M_1}{\bar{a}} \underset{M}{\bigcup} M_0$ .

<sup>&</sup>lt;sup>7</sup>This notation was already used in Definition 4.2.19.

There is  $N \geq_{\mathbf{K}} M_2$  and  $g: M_1 \to N$  extending f such that  $g(\bar{a}) \stackrel{N}{\underset{\longrightarrow}{\downarrow}} M_2$ . A diagram is below.



**PROOF.** Extend f to an  $L(\mathbf{K})$ -isomorphism  $\hat{f}$  with range  $M_2$ . By existence, there is some  $q \in \mathrm{gS}^{\mathrm{bs}}(\widehat{f}^{-1}[M_2])$  that extends  $\mathrm{gtp}(\overline{a}/M_0; M_1)$  and does not fork over  $M_0$ . Realize q as  $\operatorname{gtp}(\bar{b}/\hat{f}^{-1}[M_2]; N^+)$ . Since  $\operatorname{gtp}(\bar{a}/M_0; M_1) = \operatorname{gtp}(\bar{b}/M_0; N^+)$ , there is  $N^{++} \geq_{\mathbf{K}} N^+$  and  $h: M_1 \xrightarrow{M_0} N^{++}$  such that  $h(\bar{a}) = \bar{b}$ . Then, since  $N^{++}$ extends  $\widehat{f}^{-1}[M_2]$ , we can find an  $L(\mathbf{K})$ -isomorphism  $\widehat{f}^+$  that extends  $\widehat{f}$  such that  $N^{++}$  is the domain of  $\widehat{f}^+$ . Set  $N := \widehat{f}^+[N^{++}]$  and  $g := \widehat{f}^+ \circ h$ . Some nonforking calculus shows that this works.  $\square$ 

# 5.4. Independent sequences form a $good^-$ frame

In this section, we show how to make a good  $^{-S}$   $\mathcal{F}$ -frame longer (i.e. extend the nonforking relation to longer sequences). This is done by using independent sequences, introduced by Shelah [She09a, Definition III.5.2] and also studied by Jarden and Sitton [JS12], to define basic types and nonforking. Preservation of the symmetry property will be studied in Section 5.5, and in Section 5.6 we will review how to make the frame larger (i.e. extend the nonforking relation to larger models).

Note that Shelah already claims many of the results of this section (for finite tuples) in [She09a, Exercise III.9.4.1] but the proofs have never appeared anywhere.

DEFINITION 5.4.1 (Independent sequence). Let  $\alpha$  be an ordinal and let  $\mathfrak{s}$  be a pre- $\mathcal{F}$ -frame.

- (1)  $\langle a_i : i < \alpha \rangle$ ,  $\langle M_i : i < \alpha \rangle$  is said to be *independent in* (M, M', N) when:
  - (a)  $(M_i)_{i < \alpha}$  is increasing continuous in  $\mathbf{K}_{\mathcal{F}}$

(a)  $(M_i)_{i \leq \alpha}$  is increasing continuous in  $\mathbf{K}_{\mathcal{F}}$ . (b)  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M_0$  and  $M, M' \in \mathbf{K}_{\mathcal{F}}$ . (c)  $M_{\alpha} \leq_{\mathbf{K}} N$  is in  $\mathbf{K}_{\mathcal{F}}$ . (d) For every  $i < \alpha$ ,  $a_i \downarrow_{M}^{M_{i+1}} M_i$ .  $\langle a_i : i < \alpha \rangle, \langle M_i : i \leq \alpha \rangle$  is said to be *independent over* M when it is independent in  $(M, M_0, M_\alpha)$ .

- (2)  $\bar{a} := \langle a_i : i < \alpha \rangle$  is said to be *independent in* (M, M', N) when for some  $\langle M_i : i \leq \alpha \rangle$  we have that  $\langle a_i : i < \alpha \rangle$ ,  $\langle M_i : i \leq \alpha \rangle$  is independent in (M, M', N).
- (3) We say that  $\langle a_i : i < \alpha \rangle$ ,  $\langle M_i : i \leq \alpha \rangle$  is independent from M' over M in N if it is independent in (M, M', N). We similarly define  $\bar{a}$  is independent from M' over M in N. When N is clear from context, we drop it.

REMARK 5.4.2. If  $\alpha = 1$ , then  $\bar{a} := \langle a_0 \rangle$  is independent from M' over M in N if and only if  $gtp(a_0/M'; N)$  does not fork over M.

This motivates the next definition:

DEFINITION 5.4.3. Let  $\mathfrak{s} := (\mathbf{K}, \bigcup, gS^{bs})$  be a pre- $\mathcal{F}$ -frame, where  $\mathcal{F} = [\lambda, \theta)$ . Let  $\alpha \leq \theta$ . Define  $\mathfrak{s}^{<\alpha} := (\mathbf{K}, \stackrel{<\alpha}{\downarrow}, \mathrm{gS}^{<\alpha,\mathrm{bs}})$  as follows:

- For  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{\mathcal{F}}$  and  $\bar{a} := (a_i)_{i < \beta}$  in N with  $\beta < \alpha$ ,
- $\overset{<\alpha}{\downarrow}(M_0, M_1, \bar{a}, N) \text{ if and only if } \bar{a} \text{ is independent from } M_1 \text{ over } M_0 \text{ in } N.$  For  $M \in \mathbf{K}_{\mathcal{F}}$  and  $p \in \mathrm{gS}^{<\alpha}(M), p \in \mathrm{gS}^{<\alpha,\mathrm{bs}}(M)$  if and only if there exists  $N \geq_{\mathbf{K}} M$  and  $\bar{a} \in N$  such that  $p = \operatorname{gtp}(\bar{a}/M; N)$  and  $\stackrel{<\alpha}{\downarrow}(M, M, \bar{a}, N)$ .

LEMMA 5.4.4 (Invariance). Let  $\mathfrak{s} := (\mathbf{K}, \bigcup, \mathrm{gS}^{\mathrm{bs}})$  be a pre- $\mathcal{F}$ -frame, where  $\mathcal{F} = [\lambda, \theta)$ . Let  $\alpha \leq \theta$ . Assume  $\mathbf{K}_{\mathcal{F}}$  has amalgamation. Given  $\bar{a} = \langle a_i : i \rangle$  $\alpha$  independent from  $M_0$  over M in  $M_1$  and  $M_2 \geq_{\mathbf{K}} M_0$  containing  $\bar{b}$  such that  $gtp(\bar{a}/M_0; M_1) = gtp(\bar{b}/M_0; M_2)$ , we have that  $\bar{b}$  is independent from  $M_0$  over Min  $M_2$ .

**PROOF.** Straightforward.

REMARK 5.4.5. When dealing with types rather than sequences, the  $N^+$  in the definition can be avoided. That is, given  $p \in gS^{\beta, bs}(N)$  that does not fork over M, there is some  $\langle a_i : i < \beta \rangle$ ,  $\langle N^i : i \leq \beta \rangle$  such that  $p = \operatorname{gtp}(\langle a_i : i < \beta \rangle / N; N^\beta)$  that witnesses that  $\langle a_i : i < \beta \rangle$  is independent from N over M in  $N^{\beta}$ .

LEMMA 5.4.6. Let  $\mathfrak{s} := (\mathbf{K}, \downarrow, gS^{bs})$  be a pre- $\mathcal{F}$ -frame, where  $\mathcal{F} = [\lambda, \theta)$ . Let  $\alpha < \theta$ . If  $\mathbf{K}_{\mathcal{F}}$  has amalgamation, then  $\mathfrak{s}^{<\alpha}$  is a pre-( $< \alpha, \mathcal{F}$ )-frame.

PROOF. Invariance is Lemma 5.4.4. For monotonicity, one can also use invariance to see that if  $\bar{a}$  is independent from  $M_1$  over  $M_0$  in N and  $N' \geq_{\mathbf{K}} N$ , then  $\bar{a}$ is independent from  $M_1$  over  $M_0$  in N'. The rest is straightforward.  $\square$ 

The next result shows that local character and existence are preserved when elongating a frame:

THEOREM 5.4.7. Assume  $\mathfrak{s} := (\mathbf{K}, \bigcup, \mathrm{gS}^{\mathrm{bs}})$  is a good  $\mathcal{F}$ -frame, where  $\mathcal{F} =$  $[\lambda, \theta)$ . Then:

- (1)  $\mathfrak{s}^{<\theta}$  has local character. Moreover, if  $p \in \mathrm{gS}^{\alpha,\mathrm{bs}}(N)$  with  $\alpha < \theta$ , then there exists  $M \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{<\lambda+|\alpha|}$  such that p does not fork over M.
- (2)  $\mathfrak{s}^{<\theta}$  has existence.

Proof.

- (1) Assume  $p \in gS^{\alpha, bs}(N)$  and  $N = \bigcup_{i < \delta} N_i$  with  $\alpha < \delta < \theta$ ,  $\delta$  a regular cardinal. Then, there is some  $\bar{a} = \langle a_i : i < \alpha \rangle$  and increasing, continuous  $\langle N^i : i \leq \alpha \rangle$  such that  $\alpha < \delta$ ,  $p = \operatorname{gtp}(\bar{a}/N; N^{\alpha})$ , and, for all  $i < \alpha$ ,  $a_i \bigcup_{N \to N} N^i$ . By Monotonicity for  $\mathfrak{s}$ ,  $gtp(a_i/N; N^{i+1}) \in gS^{bs}(N)$ . By Local

Character for  $\mathfrak{s}$ , for all  $i < \alpha$  there is some  $j_i < \delta$  such that  $a_i \bigcup_{N_i} N$ . By

Transitivity for  $\mathfrak{s}$ ,  $a_i \bigcup_{N_{j_i}}^{N^{i+1}} N^i$ . Set  $j_* := \sup_{i < \alpha} j_i$ ; since  $\delta = \operatorname{cf} \delta > \alpha$ , we

have that  $j_* < \alpha$ . By Monotonicity for  $\mathfrak{s}$ ,  $a_i \bigcup_{N_{j_*}}^{N^{i+1}} N^i$  for all  $i < \alpha$ . This is exactly what we need to conclude that  $\bar{a}$  is independent from N over  $N_{j_*}$ 

in  $N^{\alpha}$ . Thus  $p = \text{gtp}(\bar{a}/N; N^{\alpha})$  does not fork over  $N_{j_*}$ . The moreover part is proved similarly: By Proposition 5.3.12, for each

 $i < \alpha$  there exists  $M^i \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{\lambda}$  such that  $N \bigcup_{M^i}^{N^{\alpha}} a_i$ . By Transitivity,

 $N^i \underset{M^i}{\overset{N^\alpha}{\downarrow}} a_i.$  Now by the Löwenheim-Skolem axiom, there exists  $M \leq_{\mathbf{K}} N$ 

in  $\mathbf{K}_{\leq \lambda+|\alpha|}$  such that  $\bigcup_{i<\alpha} M^i \leq_{\mathbf{K}} M$ . By Monotonicity,  $N^i \bigcup_{M}^{N^{\alpha}} a_i$ , so  $\bar{a}$  is independent from N over M in  $N^{\alpha}$ , so p does not fork over M, as needed.

(2) We prove two extension results separately: extending the domain and extending the length. Combining these two results shows that  $\mathfrak{s}^{<\theta}$  has existence.

For extending the domain, let  $p \in \mathrm{gS}^{<\theta,\mathrm{bs}}(M)$  and  $N \geq_{\mathbf{K}} M$ . By definition of this frame, there is some  $\bar{a} = \langle a_i : i < \beta \rangle$  and increasing,

continuous  $\langle M^i : i \leq \beta \rangle$  such that  $a_i \downarrow M^i$  for all  $i < \beta$ . We wish to construct increasing and continuous  $\langle N^i : i \leq \beta \rangle$  and  $\langle f_i : M^i \to N^i : i \leq \beta \rangle$  such that

(a)  $f_0 \upharpoonright M = \text{id};$  and M = M = M

(b) 
$$f_i(a_i) \stackrel{N^+}{\underset{M}{\bigcup}} N^i$$
.

This is done by induction by taking unions at limits and by using Lemma 5.3.13 at all successor steps. Since  $\beta < \theta$ ,  $N^i$  is in  $\mathbf{K}_{\mathcal{F}}$  at all steps and the induction can continue. Then  $\operatorname{gtp}(\bar{a}/M; M^{\beta}) = \operatorname{gtp}(f(\bar{a})/M; N^{\beta})$  as witnessed by f and  $f(\bar{a})$  is independent in  $(M, N, N^{\beta})$ . Thus,  $q = \operatorname{gtp}(f(\bar{a})/N, N^{\beta})$  is as desired.



To extend the length, suppose that  $\beta < \alpha < \theta$  and  $p \in gS^{\beta,bs}(N)$ does not fork over M. This means that there is  $\langle a_i : i < \beta \rangle$ ,  $\langle N^i : i \leq \beta \rangle$ independent in  $(M, N, N^\beta)$  such that  $p = gtp(\langle a_i : i < \beta \rangle / N; N^\beta)$ . We will extend this sequence to be of length  $\alpha$  by induction. At limit steps, simply taking the union of the extensions works. If we have  $\beta \leq \gamma < \alpha$ and have already extended to  $\gamma$  (i.e.,  $\langle a_i : i < \gamma \rangle$ ,  $\langle N^i : i \leq \gamma \rangle$  is defined), then let  $r \in gS^{bs}(M)$  be arbitrary (use no maximal models and density of basic types). Let  $r^+ \in gS^{bs}(N^\gamma)$  be its nonforking extension. Thus, there

is 
$$a_{\gamma} \in N^{\gamma+1}$$
 that realizes  $r^+$  such that  $a_{\gamma} \stackrel{N^{\gamma+1}}{\underset{M}{\downarrow}} N^{\gamma}$ . Then  $\langle a_i : i < \gamma + 1 \rangle$ ,  
 $\langle N^i : i \leq \gamma + 1 \rangle$  is independent from N over M in  $N^{\gamma+1}$ , as desired.

The next technical lemma is key in showing that uniqueness and continuity are preserved when making a frame longer. This allows us to put together two independent sequences into one.

LEMMA 5.4.8 (Amalgamation of independent sequences). Let  $\mathfrak{s}$  be a good<sup>-</sup>  $\mathcal{F}$ -frame, and  $\beta < \theta_{\mathfrak{s}}$ . Suppose that  $p, q \in \mathrm{gS}^{\beta,\mathrm{bs}}(N)$  do not fork over M, that  $p \upharpoonright M = q \upharpoonright M$ , and that there are witnessing sequences  $\bar{a}_{\ell} = \langle a_{\ell}^i : i < \beta \rangle$ ,  $\langle N_{\ell}^{i} : i \leq \beta \rangle$  independent from N over M in  $N_{\ell}^{\beta}$  for  $\ell = 0, 1$  with  $\bar{a}_{0} \models p$  and  $\bar{a}_{1} \models q$ . Then, there are coherent, continuous, increasing  $(N_i, f_{j,i})_{j < i \leq \beta}$  and  $g^i_{\ell} : N^i_{\ell} \to N_i$ such that, for all  $j < i < \beta$ ,



commutes,  $g_0^{i+1}(a_0^i) = g_1^{i+1}(a_1^i)$ , and  $g_0^{i+1}(a_0^i) \bigcup_{a_0^{i+1} \in M^1} f_{i,i+1}[N_i]$ .

PROOF. We will build:

(1) models  $\{N_i, M_\ell^i : i \leq \beta, \ell = 0, 1\};$ (2) embeddings  $\{h_\ell^i : N_\ell^i \to M_\ell^i, r_\ell^i : M_\ell^i \to N_i : i \leq \beta, \ell = 0, 1\};$  and (3) coherent embeddings  $\{f_{j,i} : N_j \to N_i, \widehat{r}_\ell^{j,i} : M_\ell^j \to M_\ell^i : i \leq \beta, \ell = 0, 1\}$ such that, for  $i < \beta$ :

(1)



commutes;





commutes:

<sup>8</sup>Note that  $g_0^i[M] = g_1^i[M]$  by commutativity of the diagram.

(3)  $M_{\ell}^0 = N_0, r_{\ell}^0 = \mathrm{id}_{N_0}$  for  $\ell = 0, 1$ , and



- commutes; (4)  $h_{\ell}^{i+1}(a_{\ell}^{i}) \stackrel{M_{\ell}^{i+1}}{\downarrow} N_{i};$ (5)  $r_{0}^{i+1} \circ h_{0}^{i+1}(a_{0}^{i}) = r_{1}^{i+1} \circ h_{1}^{i+1}(a_{1}^{i});$  and (6)  $(N_{i}, f_{j,i})_{j < i \leq \beta}$  and  $(M_{\ell}^{i}, \hat{r}_{\ell}^{j,i})_{j < i \leq \beta}$  are continuous, coherent systems generated by  $\hat{r}_{\ell}^{i,i+1} = r_{\ell}^{i}$  and  $f_{i,i+1} = r_{0}^{i} \upharpoonright N_{i} = r_{1}^{i} \upharpoonright N_{i}.$

Once these objects have been constructed we will have the following commutative diagram for  $j < i \leq \beta$ :



We can then take  $g_{\ell}^i := r_{\ell}^i \circ h_{\ell}^i$ . This gives the desired diagram by removing the  $M_{\ell}^{i}$ 's. The function equality is given by (5) and the nonforking is given by applying  $f_{i,i+1}$  to (4).

The construction proceeds by induction. At stage *i*, we will construct  $h_{\ell}^{i}, r_{\ell}^{i}, M_{\ell}^{i}$ and  $N_i$  for  $\ell = 0, 1$ . Also, at each stage, we implicitly extend the coherent system by the rule given in (6) above (at successor steps) or by taking direct limits (at limit steps).

<u>*i* = 0</u>: Amalgamate  $N_0^0, N_1^0$  over N to get  $N_0$ . Also set  $M_\ell^0 := N_0$  and  $r_\ell^0 := \mathrm{id}_{N_0}$ for  $\ell = 0, 1$ .

 $\underline{i \text{ limit:}}$  Take direct limits and use continuity to see everything is preserved.

 $\underline{i = j + 1}: \text{Use Lemma 5.3.13-replace } (M, M_0, M_1, \bar{a}, f, M_2) \text{ there with } (M, N_\ell^j, N_\ell^{j+1}, a_\ell^j, r_\ell^j \circ M_\ell^j) = 0$  $\overline{h_{\ell}^{j}, N_{j}}$  here-to get  $(h_{\ell}^{j+1}, M_{\ell}^{j+1})$  here, written as (g, N) there; this gives (4):

$$h_{\ell}^{j+1}(a_{\ell}^j) \bigcup_{\substack{M_{\ell}^{j+1}[M]\\h_{\ell}^{j+1}[M]}}^{M_{\ell}^{j+1}} N_j$$

By the commutative diagrams,  $h_0^{j+1} \upharpoonright M = h_1^{j+1} \upharpoonright M$ , so, since  $a_0^j$  and  $a_1^j$  have the same type over M, we have that:

$$\operatorname{gtp}(h_0^{j+1}(a_0^j)/h_0^{j+1}[M]; M_0^{j+1}) = \operatorname{gtp}(h_1^{j+1}(a_1^j)/h_1^{j+1}[M]; M_1^{j+1})$$

By Uniqueness for  $\mathfrak{s}$ , these imply that:

$$gtp(h_0^{j+1}(a_0^j)/N_j; M_0^{j+1}) = gtp(h_1^{j+1}(a_1^j)/N_j; M_1^{j+1})$$

We can witness this with  $r_{\ell}^{j+1} : M_{\ell}^{j+1} \to N_{j+1}$  for  $\ell = 0, 1$ ; that is,  $r_0^{j+1} \upharpoonright N_j = r_1^{j+1} \upharpoonright N_j$  and  $r_0^{j+1} \circ h_0^{j+1} (a_0^j) = r_1^{j+1} \circ h_1^{j+1} (a_1^j)$ .

COROLLARY 5.4.9. Let  $\mathfrak{s} = (\mathbf{K}, \bigcup, \mathrm{gS}^{\mathrm{bs}})$  be a good<sup>-</sup>  $\mathcal{F}$ -frame. Suppose  $M_0 \leq_{\mathbf{K}}$  $M \leq_{\mathbf{K}} N$  are in  $\mathbf{K}_{\mathcal{F}}$  and  $\alpha \leq \beta < \theta_{\mathfrak{s}}$  are such that there are  $p \in \mathrm{gS}^{\alpha,\mathrm{bs}}(M)$  and  $q \in \mathrm{gS}^{\beta,\mathrm{bs}}(N)$  such that  $q^{\alpha} \upharpoonright M = p$  and p, q do not fork over  $M_0$ . If  $\bar{a}_p = \langle a_p^i \rangle$  $i < \alpha \rangle, \langle N_p^i : i \le \alpha \rangle$  is independent from M over  $M_0$  in  $N_p^{\alpha}$  such that  $\bar{a}_p \vDash p$  and  $\bar{a}_q = \langle a_q^i : i < \beta \rangle, \ \langle N_q^i : i \le \beta \rangle$  is independent from N over  $M_0$  in  $N_q^\beta$  such that  $\bar{a}_q \models q$ , then there is  $\langle M_q^i : i \le \beta \rangle$  and  $h_i : N_p^i \to M_q^i$  for  $i \le \alpha$  such that:

- (1)  $\bar{a}_q$ ,  $\langle M_q^i : i \leq \beta \rangle$  is independent from N over  $M_0$  in  $M_q^\beta$ ;
- (2)  $N_q^i \leq_{\mathbf{K}} M_q^i$  for all  $i \leq \beta$ ; and
- (3)  $h_{i+1}(a_n^i) = a_n^i$  and  $\mathrm{id}_M \subseteq h_i \subseteq h_{i+1}$ .

PROOF. First, extend the *p*-sequence to  $\langle a_p^i : i < \beta \rangle$ ,  $\langle N_p^i : i \le \beta \rangle$  independent from M over  $M_0$  in  $N_p^{\beta}$  (use that  $\mathfrak{s}^{<\theta_s}$  has existence). We can then amalgamate these sequences over M using Lemma 5.4.8: there is  $(N_i, f_{j,i})_{j \leq i \leq \beta}$  and  $g_x^i : N_x^i \to N_i$  for x = p, q and  $i \leq \beta$  as above. Since we have  $g_q^\beta : N_q^\beta \cong g_q^\beta [N_q^\beta] \leq_{\mathbf{K}} N_\beta$ , we can extend  $g_q^{\beta}$  to an  $L(\mathbf{K})$ -isomorphism h with  $N_{\beta}$  in its range. Set  $M_q^i := h^{-1}[N_i]$  for  $i \leq \beta$ . Note that  $h_i := h^{-1} \circ g_q^i : N_q^i \to M_q^i$  is the identity.  $\Box$ 

COROLLARY 5.4.10. Assume  $\mathfrak{s} := (\mathbf{K}, \downarrow, \mathbf{gS}^{\mathrm{bs}})$  is a good  $\mathcal{F}$ -frame, where  $\mathcal{F} = [\lambda, \theta)$ . Then:

- (1)  $\mathfrak{s}^{<\theta}$  has uniqueness.
- (2)  $\mathfrak{s}^{<\theta}$  has continuity.

Proof.

- (1) This follows directly from Lemma 5.4.8.
- (2) We prove the moreover clause in the definition of continuity. For the main clause, the  $M_0$ 's appearing in this proof can be replaced by  $M_i$  or  $M_\delta$  as appropriate.

For all  $i < \delta$ , there is some  $\bar{a}_i = \langle a_i^k : k < \alpha_i \rangle$ ,  $\langle N_i^k : k \le \alpha_i \rangle$ independent from  $M_i$  over  $M_0$  in  $N_i^{\alpha_i}$  such that  $p_i = \operatorname{gr}(\bar{a}_i/M_i; N_i^{\alpha_i})$ . We will construct  $\langle M_i^k : i < \delta, k \le \alpha_i \rangle$  and  $\{f_{j,i}^k : M_j^k \to M_i^k : k \le \alpha_i \rangle$ 

 $\alpha_j, j < i < \alpha_{\delta}$  such that

(a)  $N_i^k \leq_{\mathbf{K}} M_i^k$  and  $\bar{a}_i, \langle M_i^k : k < \alpha_i \rangle$  is independent from  $M_i$  over  $M_0$ in  $M_i^{\alpha_i}$ ;

(b) for each  $k \leq \alpha_j$ ,  $(M_i^k, f_{l,i}^k)_{j \leq l \leq i < \alpha_{\delta}}$  is a coherent, direct system such that



commutes; and

(c)  $f_{j,i}^k(a_j^k) = a_i^k$ .

This is possible: just apply Corollary 5.4.9 at successors and take direct limits at limits.

This is enough. For each  $k < \alpha_{\delta}$ , set  $(M_{\delta}^{k}, f_{i,\delta}^{k})_{i < \delta, k \leq \alpha_{i}} = \varinjlim(M_{i}^{k}, f_{j,i}^{k})_{i < \delta, k \leq \alpha_{i}}$ Then  $\langle M_{\delta}^{k} : k < \alpha_{\delta} \rangle$  is increasing and continuous because each  $\langle M_{i}^{k} : k < \alpha_{i} \rangle$  is. Set  $M_{\delta}^{\alpha_{\delta}} := \bigcup_{k < \alpha_{\delta}} M_{\delta}^{k}$ . For  $k < \alpha_{i}, \alpha_{j}$ , we have that  $f_{i,\delta}^{k+1}(a_{i}^{k}) = f_{j,\delta}^{k+1}(a_{j}^{k})$ . Thus, there is no confusion in setting  $a_{\delta}^{k} = f_{i,\delta}^{k+1}(a_{i}^{k})$  for some/any  $k < \alpha_{i}$ . Set  $p = \operatorname{gtp}(\bar{a}_{\delta}/M_{\delta}, M_{\delta}^{\alpha_{\delta}})$ .

Note that  $M_{\delta} \leq_{\mathbf{K}} M_{\delta}^{0}$ ; indeed  $f_{i,\delta}^{k} \upharpoonright M_{i}$  is the identity for all  $k \leq \alpha_{i}$ . Thus, we have that

$$p_{i} = \operatorname{gtp}(\bar{a}_{i}/M_{i}; M_{i}^{\alpha_{i}}) = \operatorname{gtp}(\langle a_{\delta}^{k} : k < \alpha_{i} \rangle / M_{i}; M_{\delta}^{\alpha_{\delta}}) = p^{\alpha_{i}} \upharpoonright M_{i}$$
  
Claim: For all  $k < \alpha_{\delta}, a_{\delta}^{k} \bigcup_{M_{\delta}}^{M_{\delta}^{k+1}} M_{\delta}^{k}.$ 

**Proof of Claim:** Given  $i < \delta$  and  $k < \alpha_i$ , we have by construction  $M_i^{k+1}$  that  $a_i^k \underset{M_0}{\downarrow} M_i^k$ . Applying  $f_{i,\delta}^k$  to this, we get  $a_{\delta}^k \underset{M_0}{\downarrow} f_{i,\delta}^k(M_i^k)$ . By construction,

$$M^k_\delta = \bigcup_{i < \delta} f^k_{i,\delta}(M^k_i) \text{ and } M^{k+1}_\delta = \bigcup_{i < \delta} f^{k+1}_{i,\delta}(M^{k+1}_i)$$

Thus, by Continuity for  $\mathfrak{s}$ , we have, for all  $i < \delta$ ,  $a_{\delta}^k \stackrel{M_{\delta}^{k+1}}{\underset{M_0}{\cup}} M_{\delta}^k$ .

Thus,  $\bar{a}_{\delta}, \langle M_{\delta}^k : k \leq \alpha_{\delta} \rangle$  is independent from  $M_{\delta}$  over  $M_0$  in  $M_{\delta}^{\alpha_{\delta}}$ . So  $p \in gS^{\alpha_{\delta}, bs}(M_{\delta})$  and extends each  $p_i$  as desired.

REMARK 5.4.11. Note that a special case (when  $\mathcal{F} = [\lambda, \lambda^+]$ ) of the continuity property above is Jarden's  $\lambda^+$ -continuity of serial independence (see [Jar16, Definition 5.3]). This allows Jarden's proof that symmetry transfers up ([Jar16, Theorem 5.4]) to go through without any extra hypotheses. Another corollary of continuity is what Jarden and Sitton call the *finite continuity property* (see [JS12, Definition 8.2]). This is discussed in detail in Section 5.5.1.

Putting everything together, we obtain that all the property of a good<sup>-</sup> frame transfer to the elongation; recall that good<sup>-</sup> frames are good frames except they

might fail stability and/or symmetry. We will later see that symmetry transfers to finite sequences and give conditions under which it transfers to all sequences.

COROLLARY 5.4.12. Assume  $\mathfrak{s}$  is a good<sup>-</sup>  $\mathcal{F}$ -frame. Then  $\mathfrak{s}^{<\theta_{\mathfrak{s}}}$  is a good<sup>-</sup>  $(<\theta_{\mathfrak{s}},\mathcal{F})$ -frame.

PROOF. Set  $\theta := \theta_{\mathfrak{s}}$ .  $\mathfrak{s}^{<\theta_{\mathfrak{s}}}$  is a pre-(<  $\theta, \mathcal{F}$ )-frame by Lemma 5.4.6. Amalgamation, joint-embedding, no maximal models, and density of basic types hold since they hold in  $\mathfrak{s}$ . Existence and local character hold by Theorem 5.4.7, uniqueness and continuity hold by Corollary 5.4.10. Finally, transitivity follows from Remark 5.3.9.

Note that bs-stability only mentions basic 1-types, so it transfers immediately. Thus, the only property left is symmetry, which is discussed in the next two sections.

We conclude by proving a concatenation lemma for independent sequences. This is already proved for good frames in [JS12, Proposition 4.1], but the proof relies on [JS12, Proposition 2.6], which is proved as [JS13, Proposition 3.1.8] and uses symmetry in an essential way. Here, we improve this to just requiring that  $\mathfrak{s}$ is a pre-frame that also satisfies amalgamation, existence, continuity, and transitivity. In particular, we avoid any use of symmetry or nonforking amalgamation. This shows that the situation is somewhat similar to the first-order context, where concatenation holds in any theory (see, e.g., [GIL02, Lemma 1.6]).

THEOREM 5.4.13 (Concatenation). Assume  $\mathfrak{s}$  is a pre- $\mathcal{F}$ -frame with amalgamation, existence, transitivity, and continuity. Let  $M \leq_{\mathbf{K}} M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$  be such that  $\bar{a} = \langle a_i : i < \alpha \rangle$  is independent from  $M_0$  over M in  $M_1$  and  $\bar{b} = \langle b_i : i < \beta \rangle$  is independent from  $M_1$  over M in  $M_2$ . Then  $\bar{a}b$  is independent from  $M_0$  over M in  $M_2$ .

**PROOF.** From the independence of  $\bar{a}$  from  $M_0$  over M in  $M_1$ , there is a continuous, increasing  $\langle M_0^i:i\leq\alpha\rangle$  and  $N_0^+$  such that

- $M_0 \leq_{\mathbf{K}} M_0^i \leq_{\mathbf{K}} N_0^+;$   $M_1 \leq_{\mathbf{K}} N_0^+;$  and  $a_i \downarrow_M M_0^i.$

From the independence of  $\bar{b}$  from  $M_1$  over M in  $M_2$ , there is a continuous, increasing  $\langle M_1^i:i\leq\beta\rangle$  and  $N_1^+$  such that

- $M_1 \leq_{\mathbf{K}} M_1^i \leq_{\mathbf{K}} N_1^+;$   $M_2 \leq_{\mathbf{K}} N_1^+;$  and  $b_i \downarrow_M M_1^i.$

Define increasing and continuous  $\langle N_1^i : i \leq \beta \rangle$  and  $\langle g_i : M_1^i \to N_i : i \leq \beta \rangle$  such that:

- $N_0^+ \prec N_1^0$  and  $g_0 \upharpoonright M_1 = \operatorname{id}_{M_1}$ ; and For all  $i < \beta$ ,  $g_{i+1}(b_i) \bigcup_{M}^{N_1^{i+1}} N_1^i$ .

This can easily be constructed by inductions: amalgamate  $M_1^0$  and  $N_0^+$  over  $M_1$  to get  $N_1^0$  and  $g_0$ . At successor steps, apply Lemma 5.3.13 and take unions at limit stages.

After this construction, amalgamate  $N_1^+$  and  $N_1^\beta$  over  $M_1^\beta$  to get  $N^{++}$  and g so the following diagram commutes for  $j < \beta$ :



Define the sequence  $\langle N^i : i \leq \alpha + \beta \rangle$  by

$$N^{i} := \begin{cases} M_{0}^{i} & \text{if } i < \alpha \\ N_{1}^{j} & \text{if } i = \alpha + j \in [\alpha, \beta] \end{cases}$$

**Claim:** This sequences witnesses that  $\bar{c} := \bar{a} \cap g(\bar{b})$  is independent from  $M_0$ over M in  $N^{++}$ .

**Proof of Claim:** It is easy to see that this sequence is of the proper type, i.e.

it is increasing and continuous and  $M_0 \leq_{\mathbf{K}} N^i \leq_{\mathbf{K}} N^{++}$  for all  $i \leq \alpha + \beta$ . If  $i < \alpha$ , then we need to show that  $c_i \downarrow_{M}^{N^{i+1}} N^i$ , which is the same as  $a_i \downarrow_{M}^{M^{i+1}} M_0^i$ . This just follows from independence of  $\bar{a}$ .

If  $i = \alpha + j \ge \alpha$ , then we need to show that  $c_i \bigcup_{M}^{N^{i+1}} N^i$ , which is the same as  $N_{i}^{j+1}$ 

$$g_{j+1}(b_j) \stackrel{M}{\underset{M}{\longrightarrow}} N_1^j$$
. This holds directly by the construction.  $\dagger_{Claim}$ 

Notice that the map g shows that  $gtp(\bar{a}g(\bar{b})/M_0; N_1^\beta) = gtp(\bar{a}\bar{b}/M_0; M_2)$ . Thus, by Invariance (Lemma 5.4.4), we have that  $\bar{a}\bar{b}$  is independent from  $M_0$  over M In  $M_2$ . 

# 5.5. Symmetry in long frames

In this section, we discuss when symmetry transfers from a good frame to its elongation. We do so by studying the following unordered version of independence:

DEFINITION 5.5.1. A set I is said to be independent in  $(M, M_0, N)$  if some enumeration of I is independent in  $(M, M_0, N)$ . As usual, we say instead that I is independent from  $M_0$  over M in N.

5.5.1. Several versions of continuity. The notion of a set being independent gives rise to several notions of continuity. We gave a definition of continuity for a pre-frame  $\mathfrak{s}$ , as well as continuity for the corresponding frame of independent sequences  $\mathfrak{s}^{<\theta_{\mathfrak{s}}}$  (what Jarden calls the continuity of serial independence [Jar16, Definition 5.3, see Remark 5.4.11). We can now study the corresponding continuity properties for sets rather than sequences: for  $\mathfrak{s}$  a pre- $\mathcal{F}$ -frame, let us say that  $\mathfrak{s}^{<\theta_{\mathfrak{s}}}$  has the unordered continuity property if for every increasing chain  $\langle M_{\alpha} : \alpha < \delta \rangle$ every N containing  $\bigcup_{\alpha < \delta} M_{\alpha}$  and every  $I \subseteq |N|$ , I is independent from  $\bigcup_{\alpha < \delta} M_{\alpha}$ over  $M_0$  if I is independent from  $M_\alpha$  over  $M_0$  for all  $\alpha < \delta$  (so the enumeration witnessing the independence is allowed to change each time). Confusingly, Jarden and Sitton [**JS12**, Definition 5.5] call this the continuity property.

Another notion of continuity was also introduced by Jarden and Sitton. Let us say that a set I is *finitely independent* (from  $M_0$  over M in N) if every finite subset of I is. Jarden and Sitton [**JS12**, Definition 8.2] say that the finite continuity property holds when unordered continuity holds for the notion of finite independence. We will refer to this as *unordered finite continuity*.

Jarden and Sitton show [**JS12**, Proposition 8.4] that unordered finite continuity holds in good<sup>-St</sup> frames which satisfy the additional assumption of the conjugation property and being weakly successful. Using the (ordered) continuity property for independent sequences (Corollary 5.4.10), together with Fact 5.5.2 below, we immediately obtain that the unordered finite continuity holds in any good<sup>-St</sup> frame.

FACT 5.5.2 (Theorem 4.2.(a) in [**JS12**]). Let  $\mathfrak{s}$  be a good<sup>-St</sup>  $\mathcal{F}$ -frame. If  $\bar{a}$  is a finite tuple independent from M' over M in N, then any permutation of  $\bar{a}$  is independent from M' over M in N.

Implicit in this notion is a notion of independence being *finitely witnessed* [**JS12**, Definition 3.4] which says that a set I is independent if and only if all its finite subsets are. We give a more general parametrized definition here:

DEFINITION 5.5.3. Let  $\mathfrak{s}$  be a pre- $\mathcal{F}$ -frame and  $\mu \leq \theta_{\mathfrak{s}}$  be a cardinal. We say that  $\mu$ -independence in  $\mathfrak{s}$  is finitely witnessed if for any  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{\mathcal{F}}$ and any  $I \subseteq N$  with  $|I| < \mu$ , I is independent from M over  $M_0$  in N if and only if all its finite subsets are independent from M over  $M_0$  in N.

If  $\mu = \theta_{\mathfrak{s}}$ , we omit it.

REMARK 5.5.4. In [**JS12**, Theorem 9.3] shows that independence is finitely witnessed in a good  $\lambda$ -frame assuming the conjugation property, categoricity in  $\lambda$ , and density of uniqueness triples. Earlier, Shelah had proven the same result under stronger hypotheses [**She09a**, Theorem III.5.4].

REMARK 5.5.5. It is straightforward to see that if independence is finitely witnessed and the finite unordered continuity property holds, then the unordered continuity property holds. Recall from the discussion above that the finite unordered continuity property holds in any good<sup>-St</sup>-frame.

Our next goal is to show that if  $\mathfrak{s}^{<\mu}$  has symmetry then  $\mu$ -independence is finitely witnessed (Theorem 5.5.9). Together with Lemma 5.5.11 deducing symmetry from the frame being sufficiently global, this will show (Corollary 5.6.10) that tameness implies independence is finitely witnessed.

**5.5.2.** Symmetry implies being finitely witnessed. First we show that symmetry is equivalent to showing that the order of enumeration does not matter. The finite case is essentially Fact 5.5.2. To state the infinite case precisely, we introduce new terminology:

DEFINITION 5.5.6. Let  $\mathfrak{s}$  be a pre- $\mathcal{F}$ -frame and  $\mu \leq \theta_{\mathfrak{s}}$  be a cardinal. We say that  $\mathfrak{s}$  has  $\mu$ -symmetry of independence if for any  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{\mathcal{F}}$  and any  $I \subseteq N$  with  $|I| < \mu$ , I is independent from M over  $M_0$  in N if and only if every enumeration of I is independent from M over  $M_0$  in N.

If  $\mu = \theta_{\mathfrak{s}}$ , we omit it.

Thus a restatement of Fact 5.5.2 is that any good<sup>-St</sup> frame has  $\aleph_0$ -symmetry of independence. The next theorem says that  $\mu$ -symmetry of independence is equivalent to  $\mathfrak{s}^{<\mu}$  having symmetry.

THEOREM 5.5.7. Let  $\mathfrak{s}$  be a good<sup>-</sup>  $\mathcal{F}$ -frame and let  $\mu \leq \theta_{\mathfrak{s}}$  be a cardinal. The following are equivalent:

- (1)  $\mathfrak{s}^{<\mu}$  has symmetry.
- (2) For any  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{\mathcal{F}}$  and  $\bar{a}\bar{b} \in N$  such that  $\ell(\bar{a}\bar{b}) < \mu$ ,  $\bar{a}\bar{b}$  is independent from M over  $M_0$  in N if and only if  $\bar{b}\bar{a}$  is independent in from M over  $M_0$  in N.
- (3)  $\mathfrak{s}$  has  $\mu$ -symmetry of independence.

PROOF. We first show (1) is equivalent to (2). Assume  $\mathfrak{s}^{<\mu}$  has symmetry, and let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{\mathcal{F}}$  and  $\bar{a}\bar{b} \in N$  be such that  $\ell(\bar{a}\bar{b}) < \mu$  and  $\bar{a}\bar{b}$ is independent from M over  $M_0$  in N. Then there exists  $\langle M^i : i \leq \ell(\bar{a}\bar{b}) \rangle$  and  $N^+ \geq_{\mathbf{K}} N$  witnessing it. Say  $\alpha := \ell(\bar{a})$ . Then  $\bar{a} \in M^{\alpha}$ ,  $tp(\bar{a}/M; M^{\alpha}) \in \mathrm{gS}^{\alpha,\mathrm{bs}}(M_0)$ , and  $\bar{b}$  is independent from  $M^{\alpha}$  over M in  $N^+$ , i.e.  $\bar{b} \bigcup_{M}^{N+} M^{\alpha}$ . By Symmetry, there must exist a model M' containing  $\bar{b}$  and  $N^{++} \geq_{\mathbf{K}} N^+$  such that  $\bar{a} \bigcup_{M}^{N++} M'$ . By Monotonicity,  $\bar{a} \bigcup_{M_0}^{N++} M$ , so by Transitivity,  $\bar{a} \bigcup_{M_0}^{N++} M'$ . By Monotonicity,  $\bar{b} \bigcup_{M}^{M} M$ . By concatenation (Theorem 5.4.13),  $\bar{b}\bar{a} \bigcup_{M_0}^{N++} M$  and so by Monotonicity,  $\bar{b}\bar{a} \bigcup_{M_0}^{N} M$ , as needed. Conversely, assume (2). Assume  $\bar{a}_1 \bigoplus_{M_0}^{N} M_2$  with  $\bar{a}_1 \in {}^{<\mu}N$ , and  $\bar{a}_2 \in$  ${}^{<\mu}M_2$  is such that  $\mathrm{gtp}(\bar{a}_2/M_0; N) \in \mathrm{gS}^{<\mu,\mathrm{bs}}(M_0)$ . By existence,  $\bar{a}_2 \bigoplus_{M_0}^{M_2} M_0$ . By concatenation,  $\bar{a}_1 \bar{a}_2 \bigoplus_{M_0}^{N} M_0$ . By (2),  $\bar{a}_2 \bar{a}_1 \bigoplus_{M_0}^{N} M_0$ . By definition of independent,

there exists  $M_1$  containing  $\bar{a}_1$  and  $N' \geq_{\mathbf{K}} N$  such that  $\bar{a}_2 \bigcup_{M_0}^{N'} M_1$ , as needed. Next, we show that (2) is equivalent to (3). It is clear that (3) implies (2), so

Next, we show that (2) is equivalent to (3). It is clear that (3) implies (2), so we assume (2) and we prove (3) as follows: we prove the following by induction on  $\alpha < \mu$ :

 $(*)_{\alpha}$  Let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  be in  $\mathbf{K}_{\mathcal{F}}$  and let  $I \subseteq |N|$  have size less than  $\mu$ . If I is independent from M over  $M_0$  in N, then every enumeration of I of order type  $\alpha$  is independent from M over  $M_0$  in N.

So let  $\alpha < \mu$  and assume  $(*)_{\beta}$  holds for all  $\beta < \alpha$ . Suppose *I* as above is independent from *M* over  $M_0$  in *N* and let  $\langle a_i : i < \alpha \rangle$  be an enumeration of *I* of type  $\alpha$ .

First, suppose  $\alpha$  is finite. Then I is finite so Fact 5.5.2 gives the result.

Second, suppose  $\alpha = \beta + 1$  is an infinite successor. Then  $\langle a_{\beta} \rangle \widehat{\ } \langle a_i : i < \beta \rangle$  has order type  $\beta$  and so (by  $(*)_{\beta}$ ) is independent from M over  $M_0$  in N. Since (2) implies (1), the original sequence must also be independent.

Finally, suppose that  $\alpha$  is limit. By monotonicity, every subset of I is independent from M over  $M_0$  in N. In particular, for each  $\beta < \alpha \{a_i : i < \beta\}$  is

independent from M over  $M_0$  in N, and so by  $(*)_\beta \langle a_i : i < \beta \rangle$  is also independent from M over  $M_0$  in N. Thus by continuity (Corollary 5.4.10)  $\langle a_i : i < \alpha \rangle$  is independent from M over  $M_0$  in N.

As a corollary, we manage to solve Exercise III.9.4.1 in [She09a]:

COROLLARY 5.5.8. Let  $\mathfrak{s}$  be a good  $[\operatorname{good}^{-St}] \mathcal{F}$ -frame. Then  $\mathfrak{s}^{<\omega}$  is a good  $[\operatorname{good}^{-St}] \mathcal{F}$ -frame.

PROOF. By Corollary 5.4.12,  $\mathfrak{s}^{<\omega}$  is a good<sup>-</sup>  $\mathcal{F}$ -frame. By Fact 5.5.2,  $\mathfrak{s}$  has  $\aleph_0$ -symmetry of independence. By Theorem 5.5.7,  $\mathfrak{s}^{<\omega}$  has symmetry, as needed. Since bs-stability only refers to basic 1-types,  $\mathfrak{s}$  satisfies it if and only if  $\mathfrak{s}^{<\omega}$  does.

Unfortunately, we do not know whether in general  $\omega$  above can be replaced by a larger ordinal. To give a criteria on when this is possible, we show that independence being finitely witnessed (see Definition 5.5.3) follows from symmetry.

THEOREM 5.5.9. Let  $\mathfrak{s}$  be a good<sup>-St</sup>  $\mathcal{F}$ -frame and let  $\mu \leq \theta_{\mathfrak{s}}$  be a cardinal. If  $\mathfrak{s}^{<\mu}$  has symmetry, then  $\mu$ -independence in  $\mathfrak{s}$  is finitely witnessed.

PROOF. By Theorem 5.5.7  $\mathfrak{s}$  has  $\mu$ -symmetry of independence, and by Corollary 5.4.10  $\mathfrak{s}^{<\mu}$  has continuity. Let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  be in  $\mathbf{K}_{\mathcal{F}}$  and let  $I \subseteq N$  be such that  $|I| < \mu$ . Assume that every finite subset of I is independent in from Mover  $M_0$  in N. Assume inductively that  $\mu_0$ -independence is finitely witnessed for all  $\mu_0 < \mu$ . Let  $\mu_0 := |I|$  and write  $\{a_i : i < \mu_0\}$ . Let  $I_i := \{a_i : j < i\}$ . By the induction hypothesis,  $I_i$  is independent from M over  $M_0$  in N for all  $i < \mu_0$ . By  $\mu$ -symmetry of independence, the ordered sequence  $\langle a_j : j < i \rangle$  is independent from M over  $M_0$  in N. By continuity of  $\mathfrak{s}^{<\mu}$ ,  $\langle a_i : i < \mu_0 \rangle$  is independent from M over  $M_0$  in N. Thus I is independent from M over  $M_0$  in N, as desired.

REMARK 5.5.10. A similar proof shows that the *ordered* version of  $\mu$ -independence being finitely witnessed (that is, a sequence is independent if and only if all of its finite subsequences are) is *equivalent* to symmetry in  $\mathfrak{s}^{<\mu}$ .

Next, we show symmetry indeed holds in the elongation if the original frame is "sufficiently global" (this does not even use that  $\mathfrak{s}$  has symmetry):

LEMMA 5.5.11. Assume  $\mathfrak{s}$  is a good<sup>-</sup>  $\mathcal{F}$ -frame and  $\mathcal{F} = [\lambda, \theta)$ . If  $\theta \geq \beth_{(2^{\lambda})^+}$ , then  $\mathfrak{s}_{\lambda}^{<\lambda^+}$  has symmetry.

PROOF. Using uniqueness and local character, it is straightforward to see that  $\mathbf{K}_{\mathcal{F}}$  is stable in  $2^{\lambda}$  (for 1-types), see e.g. Proposition 4.6.4. By Fact 4.2.5 this means  $\mathbf{K}_{\mathcal{F}}$  is stable in  $2^{\lambda}$  for  $\lambda$ -types. Then the same nonstructure proof as Corollary 4.6.11 generalizes: if  $\mathfrak{s}$  does not have symmetry, then the same proof as Theorem 3.5.13 shows that  $\mathbf{K}_{\mathcal{F}}$  has an order property, and this order property is enough to deduce instability in  $2^{\lambda}$  for  $\lambda$ -types (see [She99, Section 4] or Fact 3.5.12 for a sketch).  $\Box$ 

Note, by uniqueness and local character, if  $\chi := \mathfrak{tb}_{\lambda}^{1} := \sup_{M \in \mathbf{K}_{\lambda}} |gS(M)|$ , and  $\mathfrak{s}$  is a good<sup>-</sup>  $[\lambda, \chi]$ -frame, then  $\mathfrak{s}_{\chi}$  will satisfy bs-stability (and hence be a good<sup>-S</sup>-frame); see Proposition 4.6.3.

We now apply the lemma to the maximal elongation of a  $(\geq \lambda)$ -frame  $\mathfrak{s}$ , namely  $\mathfrak{s}^{<\infty} := \bigcup_{\alpha \in OR} \mathfrak{s}^{<\alpha}$ .

COROLLARY 5.5.12. Assume  $\mathfrak{s}$  is a good<sup>-</sup> ( $\geq \lambda$ )-frame. Then  $\mathfrak{s}^{<\infty}$  has symmetry.

PROOF. Use Lemma 5.5.11 with each 
$$\lambda' \in [\lambda, \infty)$$
.

COROLLARY 5.5.13. Assume  $\mathfrak{s}$  is a good<sup>-S</sup> [good<sup>-</sup>] ( $\geq \lambda$ )-frame. Then  $\mathfrak{s}^{<\infty}$  is a good [good<sup>-St</sup>] ( $< \infty, \geq \lambda$ )-frame.

PROOF. Combine Corollary 5.4.12 and Corollary 5.5.12.

# 5.6. Applications

This section gives some applications of these results.

**5.6.1. Dimension.** In [She09a, Definition III.5.12], Shelah introduced a notion of dimension based on a frame. In [She09a, Conclusion III.5.14], he shows that this notion is well-behaved (in the sense of Corollary 5.6.1) from an assumption that is a little stronger than  $\mathfrak{s}$  being weakly successful and Jarden and Sitton [JS12, Theorem 1.1] reduce this assumption to just assuming the good<sup>-St</sup>  $\lambda$ -frame has the unordered continuity property. A corollary of our results on symmetry and independence being finitely witnessed is that we can remove any extra hypothesis.

COROLLARY 5.6.1. Let  $\mathfrak{s}$  be a good<sup>-St</sup>  $\lambda$ -frame and assume  $\mathfrak{s}^{<\lambda^+}$  has symmetry. Let  $M \leq_{\mathbf{K}} M_0 \leq_{\mathbf{K}} N$  be in  $\mathbf{K}_{\lambda}$ . If:

(1)  $P \subseteq gS^{bs}(M_0)$ 

(2)  $I_1, I_2$  are each  $\subseteq$ -maximal sets in

 $\{I: I \text{ is independent from } M_0 \text{ over } M \text{ in } N \text{ and } a \in I \Rightarrow gtp(a/M_0; N) \in P\}$ 

(3) One of  $I_1$ ,  $I_2$  is infinite.

Then  $I_1$  and  $I_2$  are both infinite and  $|I_1| = |I_2|$ .

PROOF. Since Symmetry holds, independence in  $\mathfrak{s}$  is finitely witnessed by Theorem 5.5.9. Recalling Remark 5.5.5, the hypotheses of [**JS12**, Theorem 1.1] hold, and the conclusion is this result.

This dimension-defining dim(P, N) to be the single infinite size of a  $I_1$  from Corollary 5.6.1-is used to develop the theory of regular types in [She09a, Section III.10]. As it stands, there is no known example showing that symmetry is necessary to develop a dimension theory (or a theory of regular types). In fact, there is no known example of a good<sup>-</sup>-frame which fails to have symmetry (i.e. it is not known whether symmetry follows from the other axioms of a good frame, although we suspect it does not). However, the fact that this definition compares independent sets rather than sequences implicitly assumes the symmetry of independence (see Theorem 5.5.7).

**5.6.2. Tameness and extending frames revisited.** Recall the definition of tameness from Definition 5.2.1. Boney [Bon14a] first studied the connection between tameness and frames. As in [Bon14a, Theorem 3.2], having a frame that spans multiple cardinals already gives some tameness.

PROPOSITION 5.6.2. Assume  $\mathfrak{s} := (\mathbf{K}, \, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  is a good<sup>-</sup>  $\mathcal{F}$ -frame. Let  $\mathcal{F} := [\lambda, \theta)$ .

For each  $\alpha < \theta$ , **K** is  $(\lambda + |\alpha|, < \theta)$ -tame for the basic types of  $\mathfrak{s}^{<\theta}$  of length  $\leq \alpha$ .

PROOF. Let  $\alpha < \theta$ , and let  $p, q \in gS^{\leq \alpha, bs}(M)$  be distinct. By the moreover part of Theorem 5.4.7.(1), one can find  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{\leq \lambda + |\alpha|}$  such that both pand q do not fork over  $M_0$ . By uniqueness, we must have  $p \upharpoonright M_0 \neq q \upharpoonright M_0$ , as needed.

In [Bon14a], the main concern was using  $\lambda$ -tameness to extend a  $\lambda$ -frame to a  $(\geq \lambda)$ -frame. The definition of the extension and the preservation of several properties were already done by Shelah.

DEFINITION 5.6.3 (Going up, Definitions II.2.4 and II.2.5 of [She09a]). Let  $\mathfrak{s} := (\mathbf{K}, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  be a pre-( $\langle \alpha, \lambda \rangle$ -frame, and let  $\mathcal{F} = [\lambda, \theta)$  be an interval of cardinals as usual. Define  $\mathfrak{s}_{\mathcal{F}} := (\mathbf{K}, \downarrow, \mathrm{gS}_{\mathcal{F}}^{\mathrm{bs}})$  as follows:

- For  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{\mathcal{F}}$  and  $\bar{a} \in {}^{<\alpha}N, \bigcup_{\mathcal{F}} (M_0, M_1, \bar{a}, N)$  if and only if there exists  $M'_0 \leq_{\mathbf{K}} M_0$  in  $\mathbf{K}_{\lambda}$  such that for all  $M'_0 \leq_{\mathbf{K}} M'_1 \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ with  $\bar{a} \in N'$ , and  $M'_1, N'$  in  $\mathbf{K}_{\lambda}$ , we have  $\bar{a} \bigcup_{M'_0}^{N'} M'_1$ .
- For  $M \in \mathbf{K}_{\mathcal{F}}$  and  $p \in \mathrm{gS}^{<\alpha}(M)$ ,  $p \in \mathrm{gS}^{\mathrm{bs}}_{\mathcal{F}}(M)$  if and only if there exists  $N \geq_{\mathbf{K}} M$  and  $\bar{a} \in N$  such that  $p = \mathrm{gtp}(\bar{a}/M; N)$  and  $\bigcup_{\mathcal{F}} (M, M, \bar{a}, N)$ .

FACT 5.6.4. Let  $\mathfrak{s}$  be a good<sup>-</sup>  $\lambda$ -frame, and let  $\mathcal{F} = [\lambda, \theta)$  be an interval of cardinals as usual. Then  $\mathfrak{s}_{\mathcal{F}}$  satisfies all the properties of a good  $\mathcal{F}$ -frame except perhaps bs-stability, existence, uniqueness, and symmetry.

PROOF. See [She09a, Section II.2].

Transferring the rest of the properties from a good  $\lambda$ -frame to a good  $[\lambda, \lambda^+]$ -frame was the project of the rest of [**She09a**, Section II] and involved combinatorial set-theoretic hypotheses and shrinking the AEC under consideration. [**Bon14a**] replaced these assumptions with tameness.

FACT 5.6.5 (Theorem 8.1 in [Bon14a]). Let  $\mathfrak{s}$  be a good<sup>-</sup> [good<sup>-S</sup>]  $\lambda$ -frame, and let  $\mathcal{F} = [\lambda, \theta)$  be an interval of cardinals where  $\theta > \lambda$  can be  $\infty$ . If  $\mathbf{K}_{\mathcal{F}}$  has amalgamation and no maximal models, the following are equivalent:

(1) **K** is  $\lambda$ -tame for the basic types of  $\mathfrak{s}_{\mathcal{F}}$ .

(2)  $\mathfrak{s}_{\mathcal{F}}$  is a good<sup>-</sup> [good<sup>-S</sup>]  $\mathcal{F}$ -frame.

Moreover, if  $\mathfrak{s}$  has symmetry and **K** is  $(\lambda, \theta)$ -tame for 2-length types, then  $\mathfrak{s}_{\mathcal{F}}$  has symmetry. In this case, the no maximal models hypothesis is not needed.

A surprising feature of this result is that, although the frames involved only 1types, the proof required tameness for longer types. This is connected to an emerging divide in the literature on tameness: although Grossberg and VanDieren's initial definition for tameness [**GV06b**] included the length of types, their categoricity transfer [**GV06c**, **GV06a**] and several subsequent works (e.g. [**BKV06**, **Lie13**])) required only tameness for 1-types. However, later works, beginning with Boney and Grossberg [**BG**] and Vasey (see Chapter 6), began to use tameness for longer types (and stronger locality properties like type shortness) in essential ways. It remains to be seen which version of tameness is the "proper one" for developing classification theory (or indeed if they are the same under some reasonable hypothesis). However, Fact 5.6.5 seemed to straddle this divide: it used more than tameness for 1-types, but not much more and it was unclear if the use was essential.

By using the results of this chapter, we are able to remove the assumption of tameness for 2-types in the proof of symmetry and show that the use was unnecessary. We know that the tameness for 1-types gives uniqueness for the extension  $\mathfrak{s}_{\mathcal{F}}$ , and that this uniqueness transfers to uniqueness for the elongation of  $\mathfrak{s}_{\mathcal{F}}$ . Thus, it suffices to show that the 2-types considered in the proof of symmetry are basic in this sense, which we do in Theorem 5.6.8. Before we do this, we must be careful that the order does not matter, i. e., that extending and then elongating a frame gives you the same result as elongating and then extending it. One direction is easy.

PROPOSITION 5.6.6. Let  $\mathfrak{s} := (\mathbf{K}, \downarrow, gS^{bs})$  be a pre- $\lambda$ -frame, and let  $\mathcal{F} := [\lambda, \theta)$  be an interval of cardinals as usual. Assume  $\mathbf{K}_{\mathcal{F}}$  has amalgamation. Then:

$$\left(\mathfrak{s}_{\mathcal{F}}\right)^{<\lambda^+} \subseteq \left(\mathfrak{s}^{<\lambda^+}\right)_{\mathcal{F}}$$

Where  $\subseteq$  is taken componentwise.

PROOF. Assume we know that  $\bigcup_{(\mathfrak{s}_{\mathcal{F}})^{<\lambda^+}} (M_0, M, \bar{a}, N)$ . We show that  $\bigcup_{(\mathfrak{s}^{<\lambda^+})_{\mathcal{F}}} (M_0, M, \bar{a}, N)$ . The proof of inclusion of the basic types is completely similar.

Let  $\bar{a} := \langle a_i : i < \beta \rangle$ , for  $\beta < \lambda^+$ . By assumption,  $\bar{a}$  is independent (with respect to  $\underset{\mathcal{F}}{\downarrow}$ ) from M over  $M_0$  in N. Fix  $\langle M^i : i \leq \beta \rangle$  and  $N^+$  witnessing the independence. In particular, for every  $i < \beta$ ,  $\underset{\mathcal{F}}{\downarrow}(M_0, M^i, a_i, N^+)$ . By definition of  $\underset{\mathcal{F}}{\downarrow}$ , this implies in particular that for each  $i < \beta$ , there exists  $M_i^0 \leq_{\mathbf{K}} M_0$  in  $\mathbf{K}_{\lambda}$  such that  $\underset{\mathcal{F}}{\downarrow}(M_i^0, M^i, a_i, N^+)$ . Using the Löwenheim-Skolem axiom and the fact that  $|\beta| \leq \lambda$ , we can choose  $M^* \leq_{\mathbf{K}} M_0$  in  $\mathbf{K}_{\lambda}$  such that for all  $i < \beta$ , we have  $M_i^0 \leq_{\mathbf{K}} M^*$ . Thus,  $\underset{\mathcal{F}}{\downarrow}(M^0, M_i, a_i, N^+)$  for all  $i < \beta$ . In particular,  $\bar{a}$  is independent (with respect to  $\underset{\mathcal{F}}{\downarrow}$ ) from M over  $M^*$  in N.

Now fix any  $M'_0, N' \in \mathbf{K}_{\lambda}$  such that  $\bar{a} \in N', M^* \leq_{\mathbf{K}} M'_0 \leq_{\mathbf{K}} M$ , and  $M'_0 \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ . We claim that  $\bar{a}$  is independent (with respect to  $\downarrow$ ) from  $M'_0$  over  $M^* \leq_{\mathbf{K}} N'$ , i.e.  $\downarrow (M^*, M'_0, \bar{a}, N')$ . To see this, construct  $\langle M'_i \in \mathbf{K}_{\lambda} : i \leq \beta \rangle$  increasing continuous such that for all  $i \leq \beta, M^* \leq_{\mathbf{K}} M'_i \leq_{\mathbf{K}} M^i$  and  $a_i \in M'_{i+1}$ . Finally, pick  $(N^+)' \in \mathbf{K}_{\lambda}$  such that  $M'_{\beta}, N' \leq_{\mathbf{K}} (N^+)' \leq_{\mathbf{K}} N^+$ . Then  $\langle M'_i : i \leq \beta \rangle$  and  $(N^+)'$  witness our claim. By definition, this means exactly that  $\downarrow (M_0, M, \bar{a}, N)$ , as needed.

The converse needs more hypotheses and relies on Corollary 5.4.12:

THEOREM 5.6.7. Let  $\mathfrak{s} := (\mathbf{K}, \bigcup, \mathrm{gS}^{\mathrm{bs}})$  be a good<sup>-</sup>  $\lambda$ -frame, and let  $\mathcal{F} := [\lambda, \theta)$  be an interval of cardinals as usual. Assume that  $\mathfrak{s}_{\mathcal{F}}$  is a good<sup>-</sup>  $\mathcal{F}$ -frame. Then:

$$\left(\mathfrak{s}_{\mathcal{F}}\right)^{<\lambda^{+}} = \left(\mathfrak{s}^{<\lambda^{+}}\right)_{\mathcal{F}}$$

PROOF. By Proposition 5.6.6 and existence, it is enough to show  $\bigcup_{(\mathfrak{s}^{<\lambda^+})_{\mathcal{F}}} \subseteq$ 

 $\bigcup_{(\mathfrak{s}^{\mathcal{F}})^{\leq \lambda^+}} . \text{ Assume } \bigcup_{(\mathfrak{s}^{\leq \lambda^+})_{\mathcal{F}}} (M, N, \bar{a}, \widehat{N}). \text{ By definition of } \bigcup_{(\mathfrak{s}^{\leq \alpha})_{\mathcal{F}}} \text{ and monotonicity, we}$  can assume without loss of generality that  $M \in \mathbf{K}_{\lambda}$ . We know that for all  $N' \leq_{\mathbf{K}} N$  and  $\widehat{N}' \leq_{\mathbf{K}} \widehat{N}$  in  $\mathbf{K}_{\lambda}$  with  $M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} \widehat{N}'$  and  $\bar{a} \in \widehat{N}', \bar{a}$  is independent (with respect to  $\bigcup$ ) from N' over M in  $\widehat{N}'$ . We want to see that  $\bar{a}$  is independent (with respect to  $\bigcup_{\mathcal{F}}$ ) from N Over M in  $\widehat{N}$ .

Let  $\mu \geq \lambda$  be such that  $N, \widehat{N} \in \mathbf{K}_{\leq \mu}$ . Work by induction on  $\mu$ . We already have what we want if  $\mu = \lambda$ , so assume  $\mu > \lambda$ . Let  $(N_i)_{i \leq \mu}$  be an increasing continuous resolution of N such that  $N_{\mu} = N$ ,  $N_0 = M$ ,  $||N_i|| = \lambda + |i|$ .

By the induction hypothesis and monotonicity,  $\bar{a}$  is independent (with respect to  $\bigcup_{\mathcal{F}}$ ) from  $N_i$  over M in  $\hat{N}$  for all  $i < \mu$ . In other words, for any  $i < \mu$ ,  $gtp(\bar{a}/N_i; \hat{N})$ ) does not fork (in the sense of  $(\mathfrak{s}_{\mathcal{F}})^{<\lambda^+}$ ) over M. By Corollary 5.4.12, we know that  $(\mathfrak{s}_{\mathcal{F}})^{<\lambda^+}$  has continuity. Thus  $gtp(\bar{a}/N; \hat{N})$  also does not fork (in the sense of  $(\mathfrak{s}_{\mathcal{F}})^{<\lambda^+}$ ) over M. This is exactly what we needed to prove.

We can now prove an abstract symmetry transfer that does not mention tameness.

THEOREM 5.6.8. Assume  $\mathfrak{s}$  is a good  $\mathcal{F}$ -frame. Let  $\mathcal{F} := [\lambda, \theta]$ .

Then  $\mathfrak{s}$  has symmetry if and only if  $\mathfrak{s}_{\lambda}$  has symmetry.

PROOF. Of course, symmetry for  $\mathfrak{s}$  implies in particular symmetry for  $\mathfrak{s}_{\lambda}$ . Now assume symmetry for  $\mathfrak{s}_{\lambda}$ .

First note that  $\mathfrak{s} = (\mathfrak{s}_{\lambda})_{\mathcal{F}}$ . This is because by the methods of [**She09a**, Section II.2] (see especially Claim 2.14 and the remark preceding it), there is at most one good<sup>-</sup>  $\mathcal{F}$ -frame extending  $\mathfrak{s}_{\lambda}$ , and it is given by  $(\mathfrak{s}_{\lambda})_{\mathcal{F}}$  if it exists.

Let  $\mathfrak{t} := \mathfrak{s}_{\lambda} := (\mathbf{K}, \bigcup, \mathrm{gS}^{\mathrm{bs}})$ . Thus  $\mathfrak{s} = \mathfrak{t}_{\mathcal{F}}$ . Recall that [**Bon14a**, Theorem 6.1] proves symmetry for  $\mathfrak{s}$  assuming  $(\lambda, < \theta)$ -tameness for 2-types. We revisit this proof and use the same notation.

Suppose  $\bigcup_{\mathcal{F}} (M_0, M_2, a_1, M_3)$ ,  $a_2 \in M_2$  with  $\operatorname{gtp}(a_2/M_0; M_3) \in \operatorname{gS}_{\mathcal{F}}^{\operatorname{bs}}(M_0)$ . Let  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_3$  be a model containing  $a_1$ . By existence, there is  $M'_3 \geq_{\mathbf{K}} M_3$  and  $a' \in M'_3$  such that  $\bigcup_{\mathcal{F}} (M_0, M_1, a', M'_3)$  and  $\operatorname{gtp}(a'/M_0; M'_3) = \operatorname{gtp}(a_2/M_0; M_3)$ . Boney argues it is enough to see that  $p := \operatorname{gtp}(a_1a_2/M_0; M_3) = \operatorname{gtp}(a_1a'/M_0; M'_3) =:$ p', shows that this equality holds for all restrictions to models of size  $\lambda$ , and then uses tameness for 2-types. This is not part of our hypotheses, but by Proposition 5.6.2, it is enough to see that p, p' are basic types of  $\mathfrak{s}^{\leq 2}$ .

First, let us see that  $a_1a_2$  is independent (with respect to  $\downarrow_{\mathcal{F}}$ ) from  $M_0$  over  $M_0$ in  $M_3$ . The increasing chain  $(M_0, M_2, M_3)$  witnesses that  $a_2a_1$  is independent (with respect to  $\downarrow_{\mathcal{F}}$  again) from  $M_0$  over  $M_0$  in  $M_3$ . Thus  $\operatorname{gtp}(a_2a_1/M_0; M_3) \in \operatorname{gS}_{\mathfrak{s}^{\leq 2}}^{\operatorname{bs}}(M_0)$ , and  $\mathfrak{s}^{\leq 2} = (\mathfrak{t}_{\mathcal{F}})^{\leq 2} = (\mathfrak{t}^{\leq 2})_{\mathcal{F}}$  by Theorem 5.6.7. Thus there exists  $M'_0 \leq_{\mathbf{K}} M_0$  in  $\mathbf{K}_{\lambda}$  such that for all  $M''_0 \geq_{\mathbf{K}} M'_0$  in  $\mathbf{K}_{\lambda}$  with  $M''_0 \leq_{\mathbf{K}} M_3$ ,  $\operatorname{gtp}(a_2a_1/M''_0; M_3)$ does not fork (in the sense of  $\mathfrak{t}^{\leq 2}$ ) over  $M'_0$ . Since we have symmetry in  $\mathfrak{t}$ , we have (by Fact 5.5.2) that also  $\operatorname{gtp}(a_1a_2/M''_0; M_3)$  does not fork over  $M'_0$  for all  $M''_0 \geq_{\mathbf{K}} M'_0, M''_0 \leq_{\mathbf{K}} M_3$  in  $\mathbf{K}_{\lambda}$ . Thus by definition and Theorem 5.6.7 again,  $a_1a_2$  is independent (with respect to  $\downarrow_{\mathcal{F}}$ ) from  $M_0$  over  $M_0$  in  $M_3$ , as needed. Similarly,  $(M_0, M_1, M'_3)$  witnesses that  $a_1a'$  is independent from  $M_0$  over  $M_0$  in  $M'_3$ . Thus pand p' are basic types of  $\mathfrak{s}^{\leq 2}$ , as needed.

We can now prove the desired improvement.

COROLLARY 5.6.9. Let  $\mathfrak{s} := (\mathbf{K}, \downarrow, gS^{bs})$  be a good  $\lambda$ -frame, and let  $\mathcal{F} := [\lambda, \theta)$  be an interval of cardinals, where  $\theta > \lambda$  is either a cardinal or  $\infty$ . Assume  $\mathbf{K}_{\mathcal{F}}$  has amalgamation and  $\mathbf{K}$  is  $(\lambda, < \theta)$ -tame. Then  $\mathfrak{s}_{\mathcal{F}}$  is a good  $\mathcal{F}$ -frame.

PROOF. By the proof of Fact 5.6.5,  $\mathfrak{s}_{\mathcal{F}}$  has all the properties of a good frame, except perhaps no maximal models and symmetry. Symmetry follows from the previous theorem and [**Bon14a**, Theorem 7.1] now gives us no maximal models.

While we were writing up this chapter, Adi Jarden [**Jar16**] independently gave this improvement, with the additional hypothesis that the frame was weakly successful (which he used to get the  $\lambda^+$ -continuity of serial independence property; see Remark 5.4.11).

**5.6.3.** Conclusion. We conclude by summarizing what our results give from a good frame, amalgamation, and tameness:

COROLLARY 5.6.10. Let  $\mathfrak{s} := (\mathbf{K}, \bigcup, gS^{bs})$  be a good  $\lambda$ -frame. If  $\mathbf{K}_{\geq \lambda}$  has amalgamation and is  $\lambda$ -tame, then:

- (1)  $\mathfrak{s}_{\geq\lambda}$  is a good  $(\geq \lambda)$ -frame, and in fact even  $\mathfrak{t} := (s_{\geq\lambda})^{<\infty}$  is a good  $(<\infty,\geq\lambda)$ -frame.
- (2) For all  $\alpha$ , **K** is  $(\lambda + |\alpha|)$ -tame for the basic types of t of length  $\leq \alpha$ .
- (3)  $\left(\mathfrak{s}^{<\lambda^{+}}\right)_{>\lambda} = \left(\mathfrak{s}_{\geq\lambda}\right)^{<\lambda^{+}}$ .
- (4)  $\mathfrak{t}$  has symmetry of independence and independence in  $\mathfrak{s}_{\geq \lambda}$  is finitely witnessed.
- (5) We have a well-behaved notion of dimension: For  $M \leq_{\mathbf{K}} M_0 \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{\lambda}$ , if:
  - (a)  $P \subseteq gS^{bs}(M_0)$
  - (b)  $I_1, I_2$  are  $\subseteq$ -maximal sets in

 $\{I: I \text{ is independent from } M_0 \text{ over } M \text{ in } N \text{ and } a \in I \Rightarrow gtp(a/M_0; N) \in P\}$ 

(c) One of  $I_1$ ,  $I_2$  is infinite.

Then  $I_1$  and  $I_2$  are both infinite and  $|I_1| = |I_2|$ .

# Proof.

- (1)  $\mathfrak{s}_{\geq \lambda}$  is a good  $(\geq \lambda)$ -frame by Corollary 5.6.9.  $\mathfrak{t}$  is a good  $(< \infty, \geq \lambda)$ -frame by Corollary 5.5.13.
- (2) By Proposition 5.6.2.
- (3) By Theorem 5.6.7.
- (4) By Theorem 5.5.7, Proposition 5.5.9, and Corollary 5.5.12.
- (5) By Corollary 5.6.1.

# CHAPTER 6

# Building independence relations in abstract elementary classes

This chapter is based on [Vas16a]. I thank Andrés Villaveces for sending his thoughts on my results and Will Boney for carefully reading this chapter and giving invaluable feedback. I thank the referee for a thorough report that greatly helped to improve the presentation of this chapter.

#### Abstract

We study general methods to build forking-like notions in the framework of tame abstract elementary classes (AECs) with amalgamation. We show that whenever such classes are categorical in a high-enough cardinal, they admit a good frame: a forking-like notion for types of singleton elements.

THEOREM 6.0.11 (Superstability from categoricity). Let **K** be a ( $< \kappa$ )-tame AEC with amalgamation. If  $\kappa = \beth_{\kappa} > LS(\mathbf{K})$  and **K** is categorical in a  $\lambda > \kappa$ , then:

- **K** is stable in any cardinal  $\mu$  with  $\mu \geq \kappa$ .
- **K** is categorical in  $\kappa$ .
- There is a type-full good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}$ .

Under more locality conditions, we prove that the frame extends to a global independence notion (for types of arbitrary length).

THEOREM 6.0.12 (A global independence notion from categoricity). Let **K** be a densely type-local, fully tame and type short AEC with amalgamation. If **K** is categorical in unboundedly many cardinals, then there exists  $\lambda \geq \text{LS}(\mathbf{K})$  such that  $\mathbf{K}_{\geq\lambda}$  admits a global independence relation with the properties of forking in a superstable first-order theory.

As an application, we deduce (modulo an unproven claim of Shelah) that Shelah's eventual categoricity conjecture for AECs (without assuming categoricity in a successor cardinal) follows from the weak generalized continuum hypothesis and a large cardinal axiom.

COROLLARY 6.0.13. Assume  $2^{\lambda} < 2^{\lambda^+}$  for all cardinals  $\lambda$ , as well as an unpublished claim of Shelah. If there exists a proper class of strongly compact cardinals, then any AEC categorical in *some* high-enough cardinal is categorical in *all* highenough cardinals.

## 6.1. Introduction

Independence (or forking) is a central notion of model theory. In the first-order setup, it was introduced by Shelah [She78] and is one of the main devices of his

book. One can ask whether there is such a notion in the nonelementary context. In homogeneous model theory, this was investigated in [**HL02**] for the superstable case and [**BL03**] for the simple and stable cases. Some of their results were later generalized by Hyttinen and Kesälä [**HK06**] to tame and  $\aleph_0$ -stable finitary abstract elementary classes (AECs). For general<sup>1</sup> AECs, the answer is still a work in progress.

In [She99, Remark 4.9.1] it was asked whether there is such a notion as forking in AECs. In his book on AECs [She09a], Shelah introduced the concept of good  $\lambda$ -frames (a local independence notion for types of singletons) and some conditions are given for their existence. Shelah's main construction (see [She09a, Theorem II.3.7]) uses model-theoretic and set-theoretic assumptions: categoricity in two successive cardinals and principles like the weak diamond<sup>2</sup>. It has been suggested<sup>3</sup> that replacing Shelah's strong local model-theoretic hypotheses by the global hypotheses of amalgamation and tameness (a locality property for types introduced by Grossberg and VanDieren [GV06b]) should lead to better results with simpler proofs. Furthermore, one can argue that any "reasonable" AEC should be tame and have amalgamation, see for example the discussion in Section 5 of [BG], and the introductions of [Bon14b] or [GV06b]. In particular, they follow from a large cardinal axiom and categoricity:

FACT 6.1.1. Let **K** be an AEC and let  $\kappa > LS(\mathbf{K})$  be a strongly compact cardinal. Then:

- (1) **[Bon14b]** K is  $(< \kappa)$ -tame (in fact fully  $(< \kappa)$ -tame and short).
- (2) [MS90, Proposition 1.13]<sup>4</sup> If  $\lambda > \beth_{\kappa+1}$  is such that **K** is categorical in  $\lambda$ , then  $\mathbf{K}_{>\kappa}$  has amalgamation.

Examples of the use of tameness and amalgamation include [**BKV06**] (an upward stability transfer), [**Lie11b**] (showing that tameness is equivalent to a natural topology on Galois types being Hausdorff), [**GV06c**] (an upward categoricity transfer theorem, which can be combined with Fact 6.1.1 and the downward transfer of Shelah [**She99**] to prove that Shelah's eventual categoricity conjecture for a successor follows from the existence of a proper class of strongly compact cardinals) and [**Bon14a**, **Jar16**], Chapter 5, showing that good frames behave well in tame classes.

Chapter 4 constructed good frames in ZFC using global model-theoretic hypotheses: tameness, amalgamation, and categoricity in a cardinal of high-enough cofinality. However we were unable to remove the assumption on the cofinality of the cardinal or to show that the frame was  $\omega$ -successful, a key technical property of frames. Both in Shelah's book and in Chapter 4, the question of whether there exists a global independence notion (for longer types) was left open. In this chapter, we continue working in ZFC with tameness and amalgamation, and make progress toward these problems. Regarding the cofinality of the categoricity cardinal, we

 $<sup>^{1}</sup>$ For a discussion of how the framework of tame AECs compare to other non first-order frameworks, see the introduction of Chapter 2.

<sup>&</sup>lt;sup>2</sup>Shelah claims to construct a good frame in ZFC in [She09a, Theorem IV.4.10] but he has to change the class and still uses the weak diamond to show his frame is  $\omega$ -successful.

 $<sup>^{3}</sup>$ The program of using tameness and amalgamation to prove Shelah's results in ZFC is due to Rami Grossberg and dates back to at least [**GV06b**], see the introduction there.

<sup>&</sup>lt;sup>4</sup>This is stated there for the class of models of an  $\mathbb{L}_{\kappa,\omega}$  theory but Boney [Bon14b] argues that the argument generalizes to any AEC K with  $\mathrm{LS}(\mathbf{K}) < \kappa$ .

show that it is possible to take the categoricity cardinal to be high-enough: (Theorem 6.10.16):

THEOREM 6.10.16. Let **K** be a (<  $\kappa$ )-tame AEC with amalgamation. If  $\kappa = \beth_{\kappa} > \text{LS}(\mathbf{K})$  and **K** is categorical in a  $\lambda > \kappa$ , then there is a type-full good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}$ .

As a consequence, the class **K** above has several superstable-like properties: for all  $\mu \geq \lambda$ , **K** is stable<sup>5</sup> in  $\mu$  (this is also part of Theorem 6.10.16) and has a unique limit model of cardinality  $\mu$  (by e.g. Corollary 5.6.9 and Remark 6.2.15). Since **K** is stable in  $\lambda$ , the model of size  $\lambda$  is saturated. Hence using Morley's omitting type theorem for AECs (see the proof of Theorem 6.10.16 for the details), we deduce a downward categoricity transfer<sup>6</sup>:

COROLLARY 6.1.2. Let **K** be a (<  $\kappa$ )-tame AEC with amalgamation. If  $\kappa = \beth_{\kappa} > LS(\mathbf{K})$  and **K** is categorical in a  $\lambda > \kappa$ , then **K** is categorical in  $\kappa$ .

We emphasize that already [She99] deduced such results assuming that the model of size  $\lambda$  is saturated (or just  $\kappa$ -saturated so when cf  $\lambda \geq \kappa$  this follows). The new part here is showing that it *is* saturated, even when cf  $\lambda < \kappa$ .

The construction of the good frame in the proof of Theorem 6.10.16 is similar to that in Chapter 4 but uses local character of coheir (or ( $< \kappa$ )-satisfiability) rather than splitting. A milestone study of coheir in the nonelementary context is [**MS90**], working in classes of models of an  $\mathbb{L}_{\kappa,\omega}$ -sentence,  $\kappa$  a strongly compact cardinal. Makkai and Shelah's work was generalized to fully tame and short AECs in [**BG**], and some results were improved in Chapter 2. Building on these works, we are able to show that under the assumptions above, coheir has enough superstability-like properties to apply the arguments of Chapter 4, and obtain that coheir restricted to types of length one in fact induces a good frame.

Note that coheir is a candidate for a global independence relation. In fact, one of the main result of Chapter 3 is that it is canonical: if there is a global forking-like notion, it must be coheir. The chapter assumes additionally that coheir has the extension property. Here, we prove that coheir is canonical without this assumption (Theorem 6.9.3). We also obtain results on the canonicity of good frames. For example, any two type-full good  $\lambda$ -frames with the same *categorical* underlying AEC must be the same (Theorem 6.9.7). This answers several questions from Chapter 3.

Using that coheir is global and (under categoricity) induces a good frame, we can use more locality assumptions to get that the good frame is  $\omega$ -successful:

THEOREM 6.15.6. Let **K** be a fully  $(< \kappa)$ -tame and short AEC. If  $LS(\mathbf{K}) < \kappa = \beth_{\kappa} < \lambda = \beth_{\lambda}$ , cf  $\lambda \ge \kappa$ , and **K** is categorical in a  $\mu \ge \lambda$ , then there exists an  $\omega$ -successful type-full good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}$ .

<sup>&</sup>lt;sup>5</sup>The downward stability transfer from categoricity is an early result of Shelah [She99, Claim 1.7], but the upward transfer is new and improves on Theorem 4.7.5. In fact, the proof here is new even when  $\mathbf{K}$  is the class of models of a first-order theory.

<sup>&</sup>lt;sup>6</sup>[**MS90**, Conclusion 5.1] proves a stronger conclusion under stronger assumptions (namely that **K** is the class of models of an  $\mathbb{L}_{\kappa,\omega}$  sentence,  $\kappa$  a strongly compact cardinal).

We believe that the locality hypotheses in Theorem 6.15.6 are reasonable: they follow from large cardinals (Fact 6.1.1) and slightly weaker assumptions can be derived from the existence of a global forking-like notion, see the discussion in Section 6.15.

Theorem 6.15.6 can be used to build a global independence notion (Theorem 6.15.1 formalizes Theorem 6.0.12 from the abstract). We assume one more locality hypothesis (dense type-locality) there. We suspect it can be removed, see the discussion in Section 6.15. Without dense type-locality, one still obtains an independence relation for types of length less than or equal to  $\lambda$  (see Theorem 6.15.6). This improves several results from [**BG**] (see Section 6.16 for a more thorough comparison).

These results bring us closer to solving one of the main test questions in the classification theory of abstract elementary classes<sup>7</sup>:

CONJECTURE 6.1.3 (Shelah's eventual categoricity conjecture). An AEC that is categorical in a high-enough cardinal is categorical on a tail of cardinals.

The power of  $\omega$ -successful frames comes from Shelah's analysis in Chapter III of his book. Unfortunately, Shelah could not prove the stronger results he had hoped for. Still, in [She09a, Discussion III.12.40], he claims the following (a proof should appear in a future publication [Sheb]):

CLAIM 6.1.4. Assume the weak generalized continuum hypothesis<sup>8</sup> (WGCH). Let **K** be an AEC such that there is an  $\omega$ -successful good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}$ . Write  $\mathbf{K}^{\lambda^{+\omega}\text{-sat}}$  for the class of  $\lambda^{+\omega}$ -saturated models in **K**. Then  $\mathbf{K}^{\lambda^{+\omega}\text{-sat}}$  is categorical in some  $\mu > \lambda^{+\omega}$  if and only if it is categorical in all  $\mu > \lambda^{+\omega}$ .

Modulo this claim, we obtain the consistency of Shelah's eventual categoricity conjecture from large cardinals. This partially answers [She00, Question 6.14]:

THEOREM 6.1.5. Assume Claim 6.1.4 and WGCH.

- (1) Shelah's categoricity conjecture holds in fully tame and short AECs with amalgamation.
- (2) If there exists a proper class of strongly compact cardinals, then Shelah's categoricity conjecture holds.

PROOF. Let  $\mathbf{K}$  be an AEC.

- (1) Assume **K** is fully LS(**K**)-tame and short and has amalgamation. Pick  $\kappa$  and  $\lambda$  such that LS(**K**)  $< \kappa = \beth_{\kappa} < \lambda = \beth_{\lambda}$  and cf  $\lambda \ge \kappa$ . By Theorem 6.15.6, there is an  $\omega$ -successful good  $\lambda$ -frame on **K**<sub> $\lambda$ </sub>. By Claim 6.1.4, **K**<sup> $\lambda^{+\omega}$ -sat</sup> is categorical in all  $\mu > \lambda^{+\omega}$ . By Morley's omitting type theorem for AECs (see [**She99**, II.1.10]), **K** is categorical in all  $\mu \ge \beth_{(2^{\lambda+\omega})^+}$ .
- (2) Let  $\kappa > \text{LS}(\mathbf{K})$  be strongly compact. By [Bon14b], K is fully  $(< \kappa)$ -tame and short. By the methods of [MS90, Proposition 1.13],  $\mathbf{K}_{\geq\kappa}$  has amalgamation. Now apply the previous part to  $\mathbf{K}_{\geq\kappa}$ .

<sup>&</sup>lt;sup>7</sup>A version of Shelah's categoricity conjecture already appears as [She90, Open problem D.3(a)] and the statement here appears in [She09a, Conjecture N.4.2], see [Gro02] or the introduction to [She09a] for history and motivation.

 $<sup>^{8}2^{\</sup>lambda} < 2^{\lambda^{+}}$  for all cardinals  $\lambda$ .

#### 6.2. PRELIMINARIES

REMARK 6.1.6. Previous works (e.g. [MS90, She99, GV06c, Bon14b]) all assume categoricity in a successor cardinal, and this was thought to be hard to remove. Here, we do *not* need to assume categoricity in a successor.

Note that [She09a, Theorem IV.7.12] is stronger than Theorem 6.1.5 (since Shelah assumes only Claim 6.1.4, WGCH, and amalgamation): unfortunately we were unable to verify Shelah's proof. The statement contains an error as it contradicts Morley's categoricity theorem.

This chapter is organized as follows. In Section 6.2 we review some of the background. In Sections 6.3-6.4, we introduce the framework with which we will study independence. In Sections 6.5-6.8, we introduce the definition of a generator for an independence relation and show how to use it to build good frames. In Section 6.9, we use the theory of generators to prove results on the canonicity of coheir and good frames. In Section 6.10, we use generators to study the definition of superstability implicit in [SV99] (and further studied in [GVV16] and Chapter 4). We derive superstability from categoricity and use it to construct good frames. In Section 6.11, we show how to prove a good frame is  $\omega$ -successful provided it is induced by coheir. In Sections 6.12-6.14, we show how to extend such a frame to a global independence relation. In Section 6.15, some of the main theorems are established. In Section 6.16, we give examples (existence of large cardinals, totally categorical classes, and fully ( $< \aleph_0$ )-tame and short AECs) where Theorem 6.0.12 can be applied to derive the existence of a global independence relation.

Since this chapter was first circulated (in December 2014), several improvements and applications were discovered. Threshold cardinals for the construction of a good frame are improved in Chapter 10. Global independence relations are studied in the framework of universal classes in Chapter 8 and a categoricity transfer is obtained there (later improved to the full eventual categoricity conjecture in Chapter 16). Global independence can also be used to build prime models over sets of the form Ma, for M a saturated models (Chapter 12). Several of the results of this chapter are exposed in [**BVd**].

# 6.2. Preliminaries

We recall the definition of an abstract elementary class (AEC) in  $\mathcal{F}$ , for  $\mathcal{F}$  an interval of cardinal. Localizing to an interval is convenient when dealing with good frames and appears already (for  $\mathcal{F} = \{\lambda\}$ ) in [**JS13**, Definition 1.0.3.2]. Confusingly, Shelah earlier on called an AEC in  $\lambda$  a  $\lambda$ -AEC (in [**She09a**, Definition II.1.18]).

DEFINITION 6.2.1. For  $\mathcal{F} = [\lambda, \theta)$  an interval of cardinals, we say an abstract class **K** in  $\mathcal{F}$  is an *abstract elementary class* (AEC for short) in  $\mathcal{F}$  if it satisfies:

- (1) Coherence: If  $M_0, M_1, M_2$  are in  $\mathbf{K}, M_0 \leq_{\mathbf{K}} M_2, M_1 \leq_{\mathbf{K}} M_2$ , and  $|M_0| \subseteq |M_1|$ , then  $M_0 \leq_{\mathbf{K}} M_1$ .
- (2)  $L(\mathbf{K})$  is finitary.
- (3) Tarski-Vaught axioms: If  $\langle M_i : i < \delta \rangle$  is an increasing chain in **K** and  $\delta < \theta$ , then  $M_{\delta} := \bigcup_{i < \delta} M_i$  is such that:
  - (a)  $M_{\delta} \in \mathbf{K}$ .
  - (b)  $M_0 \leq_{\mathbf{K}} M_{\delta}$ .
  - (c) If  $M_i \leq_{\mathbf{K}} N$  for all  $i < \delta$ , then  $M_\delta \leq_{\mathbf{K}} N$ .

(4) Löwenheim-Skolem-Tarski axiom: There exists a cardinal  $\mu \geq |L(\mathbf{K})| + \aleph_0$ such that for any  $M \in \mathbf{K}$  and any  $A \subseteq |M|$ , there exists  $M_0 \leq_{\mathbf{K}} M$ containing A with  $||M_0|| \leq |A| + \mu$ . We write LS(**K**) (the Löwenheim-Skolem-Tarski number of **K**) for the least such cardinal.

When  $\mathcal{F} = [0, \infty)$ , we omit it. We say **K** is an AEC in  $\lambda$  if it is an AEC in  $\{\lambda\}$ .

Recall that an AEC in  $\mathcal{F}$  can be made into an AEC:

FACT 6.2.2 (Lemma II.1.23 in [She09a]). If **K** is an AEC in  $\lambda := \text{LS}(\mathbf{K})$ , then there exists a unique AEC **K'** such that  $(\mathbf{K}')_{\lambda} = \mathbf{K}$  and  $\text{LS}(\mathbf{K}') = \lambda$ . The same holds if **K** is an AEC in  $\mathcal{F}, \mathcal{F} = [\lambda, \theta)$  (apply the previous sentence to  $\mathbf{K}_{\lambda}$ ).

NOTATION 6.2.3. Let **K** be an AEC in  $\mathcal{F}$  with  $\mathcal{F} = [\lambda, \theta), \lambda = \text{LS}(\mathbf{K})$ . Write **K**<sup>up</sup> for the unique AEC **K**' described by Fact 6.2.2.

When studying independence, the following definition will be useful:

DEFINITION 6.2.4. A coherent abstract class in  $\mathcal{F}$  is an abstract class in  $\mathcal{F}$  satisfying the coherence property (see Definition 6.2.1).

We also define the following weakening of the existence of a Löwenheim-Skolem-Tarski number:

DEFINITION 6.2.5. An abstract class **K** is  $(\langle \lambda \rangle)$ -closed if for any  $M \in \mathbf{K}$  and  $A \subseteq |M|$  with  $|A| < \lambda$ , there exists  $M_0 \leq_{\mathbf{K}} M$  which contains A and has size less than  $\lambda$ .  $\lambda$ -closed means  $(\langle \lambda^+)$ -closed.

REMARK 6.2.6. An AEC K is  $(< \lambda)$ -closed in every  $\lambda > LS(K)$ .

We will sometimes use the following consequence of Shelah's presentation theorem:

FACT 6.2.7 (Conclusion I.1.11 in [She09a]). Let **K** be an AEC. If  $\mathbf{K}_{\geq \lambda} \neq \emptyset$  for every  $\lambda < h(\mathrm{LS}(\mathbf{K}))$ , then **K** has arbitrarily large models.

As in the preliminaries of Chapter 2, we can define a notion of embedding for abstract classes and go on to define amalgamation, joint embedding, no maximal models, Galois types, tameness, and type-shortness (that we will just call shortness). Recall also Fact 6.1.1 which says that under a large cardinal axiom any AEC is fully tame and short.

The following fact tells us that an AEC with amalgamation is a union of AECs with amalgamation and joint embedding. This a trivial observation from the definition of the *diagram* of an AEC [She09a, Definition I.2.2].

FACT 6.2.8 (Lemma 16.14 in [Bal09]). Let **K** be an AEC with amalgamation. Then we can write  $\mathbf{K} = \bigcup_{i \in I} \mathbf{K}^i$  where the  $\mathbf{K}^i$ 's are disjoint AECs with  $\mathrm{LS}(\mathbf{K}^i) = \mathrm{LS}(\mathbf{K})$  and each  $\mathbf{K}^i$  has joint embedding and amalgamation.

The following sums up all the results we will use about stability and the order property:

FACT 6.2.9. Let  $\mathbf{K}$  be an AEC.

- (1) (Corollary 2.4.8) Let  $\kappa = \beth_{\kappa} > LS(\mathbf{K})$ . The following are equivalent: (a) **K** has the weak  $\kappa$ -order property.
  - (b) **K** has the  $(< \kappa)$ -order property of length  $\kappa$ .

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- (c) **K** has the  $(< \kappa)$ -order property.
- (2) (Theorem 2.4.15) Assume **K** is  $(< \kappa)$ -tame and has amalgamation. The following are equivalent:
  - (a) **K** is stable in some  $\lambda \ge \kappa + LS(\mathbf{K})$ .
  - (b) There exists  $\mu \leq \lambda_0 < h(\kappa + LS(\mathbf{K}))$  such that **K** is stable in any  $\lambda \geq \lambda_0$  with  $\lambda = \lambda^{<\mu}$ .
  - (c) **K** does not have the order property.
  - (d) **K** does not have the  $(< \kappa)$ -order property.
- (3)  $[\mathbf{BKV06}, \text{Theorem 4.5}]$  If K is  $LS(\mathbf{K})$ -tame, has amalgamation, and is stable in  $LS(\mathbf{K})$ , then it is stable in  $LS(\mathbf{K})^+$ .

# 6.2.1. Universal and limit extensions.

DEFINITION 6.2.10. Let **K** be an abstract class,  $\lambda$  be a cardinal.

- (1) For  $M, N \in \mathbf{K}$ , say  $M <_{\mathbf{K}}^{\text{univ}} N$  (N is universal over M) if and only if  $M <_{\mathbf{K}} N$  and whenever we have  $M' \ge_{\mathbf{K}} M$  such that  $||M'|| \le ||N||$ , then there exists  $f : M' \xrightarrow{M} N$ . Say  $M \le_{\mathbf{K}}^{\mathrm{univ}} N$  if and only if M = N or  $M <^{\text{univ}}_{\mathbf{K}} N.$
- (2) For  $M, N \in \mathbf{K}$ ,  $\lambda$  a cardinal and  $\delta \leq \lambda^+$ , say  $M <_{\mathbf{K}}^{\lambda,\delta} N$  (N is  $(\lambda, \delta)$ -limit over M) if and only if  $M \in \mathbf{K}_{\lambda}$ ,  $N \in \mathbf{K}_{\lambda+|\delta|}$ ,  $M <_{\mathbf{K}} N$ , and there exists  $\langle M_i : i \leq \delta \rangle$  increasing continuous such that  $M_0 = M$ ,  $M_i <_{\mathbf{K}}^{\min} M_{i+1}$  for all  $i < \delta$ , and  $M_{\delta} = N$  if  $\delta > 0$ . Say  $M \leq_{\mathbf{K}}^{\lambda,\delta}$  if M = N or  $M <_{\mathbf{K}}^{\lambda,\delta} N$ . We say  $N \in \mathbf{K}$  is a  $(\lambda, \delta)$ -limit model if  $M <_{\mathbf{K}}^{\lambda,\delta} N$  for some M. We say Nis  $\lambda$ -limit if it is  $(\lambda, \delta)$ -limit for some limit  $\delta < \lambda^+$ . When  $\lambda$  is clear from context, we omit it.

REMARK 6.2.11. So for  $M, N \in \mathbf{K}_{\lambda}, M <^{\lambda,0}_{\mathbf{K}} N$  if and only if  $M <_{\mathbf{K}} N$ , while  $M <^{\lambda,1}_{\mathbf{K}}$  if and only if  $M <^{\text{univ}}_{\mathbf{K}} N$ .

REMARK 6.2.12. Variations on  $<_{\mathbf{K}}^{\lambda,\delta}$  already appear as [She99, Definition 2.1]. This chapter definition of being universal is different from the usual one (see e.g. **[Van06**, Definition I.2.1.2]) because we ask only for  $||M'|| \leq ||N||$  rather than ||M'|| = ||M||.

The next fact is folklore.

FACT 6.2.13. Let **K** be an AC with amalgamation,  $\lambda$  be an infinite cardinal, and  $\delta \leq \lambda^+$ . Then:

- (1)  $M_0 <_{\mathbf{K}}^{\mathrm{univ}} M_1 \leq_{\mathbf{K}} M_2$  and  $||M_1|| = ||M_2||$  imply  $M_0 <_{\mathbf{K}}^{\mathrm{univ}} M_2$ . (2)  $M_0 \leq_{\mathbf{K}} M_1 <_{\mathbf{K}}^{\mathrm{univ}} M_2$  implies  $M_0 <_{\mathbf{K}}^{\mathrm{univ}} M_2$ . (3) If  $M_0 \in \mathbf{K}_{\lambda}$ , then  $M_0 \leq_{\mathbf{K}} M_1 <_{\mathbf{K}}^{\lambda\delta} M_2$  implies  $M_0 <_{\mathbf{K}}^{\lambda,\delta} M_2$ . (4) If  $\delta < \lambda^+$ , **K** is an AEC in  $\lambda = \mathrm{LS}(\mathbf{K})$  with no maximal models and stability in  $\lambda$ , then for any  $M_0 \in \mathbf{K}$  there exists  $M'_0$  such that  $M_0 <_{\mathbf{K}}^{\lambda,\delta} M'_0$ .

PROOF. All are straightforward, except perhaps the last which is due to Shelah. For proofs and references see Proposition 4.2.10.  $\square$ 

By a routine back and forth argument, we have:

FACT 6.2.14 (Fact 1.3.6 in [SV99]). Let K be an AEC in  $\lambda := LS(K)$  with amalgamation. Let  $\delta \leq \lambda^+$  be a limit ordinal and for  $\ell = 1, 2$ , let  $\langle M_i^{\ell} : i \leq \delta \rangle$  be increasing continuous with  $M_0 := M_0^1 = M_0^2$  and  $M_i^{\ell} <_{\mathbf{K}}^{\mathrm{univ}} M_{i+1}^{\ell}$  for all  $i < \delta$  (so

they witness  $M_0^{\ell} <_{\mathbf{K}}^{\lambda,\delta} M_{\delta}^{\ell}$ ). Then there exists  $f: M_{\delta}^1 \cong_{M_0} M_{\delta}^2$  such that for all  $i < \delta$ , there exists  $j < \delta$  such that  $f[M_i^1] \leq_{\mathbf{K}} M_j^2$  and  $f^{-1}[M_i^2] \leq_{\mathbf{K}} M_j^1$ .

REMARK 6.2.15. Uniqueness of limit models that are not of the same cofinality (i.e. the statement  $M_0 <_{\mathbf{K}}^{\hat{\lambda},\delta} M_1$ ,  $M_0 <_{\mathbf{K}}^{\lambda,\delta'} M_2$  implies  $M_1 \cong_{M_0} M_2$  for any limit  $\delta, \delta' < \lambda^+$ ) has been argued to be an important dividing line, akin to superstability in the first-order theory. See for example [SV99, Van06, Van13, GVV16]. It is known to follow from the existence of a good  $\lambda$ -frame (see [She09a, Lemma II.4.8], or [Bon14a, Theorem 9.2] for a detailed proof).

We could not find a proof of the next result in the literature, so we included one here.

LEMMA 6.2.16. Let **K** be an AEC with amalgamation. Let  $\delta$  be a (not necessarily limit) ordinal and assume  $(M_i)_{i \leq \delta}$  is increasing continuous with  $M_i <_{\mathbf{K}}^{\text{univ}} M_{i+1}$ for all  $i < \delta$ . Then  $M_i <_{\mathbf{K}}^{\text{univ}} M_{\delta}$  for all  $i < \delta$ .

**PROOF.** By induction on  $\delta$ . If  $\delta = 0$ , there is nothing to do. If  $\delta = \alpha + 1$  is a successor, let  $i < \delta$ . We know  $M_i \leq_{\mathbf{K}} M_{\alpha}$ . By hypothesis,  $M_{\alpha} <_{\mathbf{K}}^{\text{univ}} M_{\delta}$ . By Fact 6.2.13.(2),  $M_i <_{\mathbf{K}}^{\text{univ}} M_{\delta}$ . Assume now  $\delta$  is limit. In that case it is enough to show  $M_0 <_{\mathbf{K}}^{\text{univ}} M_{\delta}$ . By the induction hypothesis, we can further assume that  $\delta = \operatorname{cf} \delta$ . Let  $N \geq_{\mathbf{K}} M_0$  be given such that  $\mu := \|N\| \leq \|M_{\delta}\|$ , and  $N, M_{\delta}$  are inside a common model N. If  $\mu < ||M_{\delta}||$ , then there exists  $i < \delta$  such that  $\mu \leq ||M_i||$ , and we can use the induction hypothesis, so assume  $\mu = ||M_{\delta}||$ . We can further assume  $\mu > ||M_0||$ , for otherwise N directly embeds into  $M_1$  over  $M_0$ . The  $M_i$ s show that  $\gamma := \operatorname{cf} \mu \leq \delta$ . Let  $\langle N_i : i \leq \gamma \rangle$  be increasing continuous such that for all  $i < \gamma$ .

- (1)  $N_0 = M_0$ .
- (2)  $N_{\gamma} = N$ .
- (3)  $||N_i|| < \mu$ .

This exists since  $\gamma = \operatorname{cf} \mu$ .

Build  $\langle f_i : i \leq \gamma \rangle$ , increasing continuous such that for all  $i < \gamma$ ,  $f_i : N_i \xrightarrow{M_0} M_{k_i}$ for some  $k_i < \delta$ . This is enough, since then  $f_{\gamma}$  will be the desired embedding. This is possible: For i = 0, take  $f_0 := id_{M_0}$ . At limits, take unions: since  $\delta$  is regular and  $\gamma \leq \delta$ ,  $k_j < \delta$  for all  $j < i < \gamma$  implies  $k_i := \sup_{j < i} k_j < \delta$ .

Now given i = j + 1, first pick  $k = k_j < \delta$  such that  $f_j[N_j] \leq_{\mathbf{K}} M_k$ . Such a k exists by the induction hypothesis. Find k' > k such that  $||N_i|| \le ||M_{k'}||$ . This exists since  $||N_i|| < \mu = ||M_{\delta}||$ . Now by the induction hypothesis,  $M_k <_{\mathbf{K}}^{\text{univ}} M_{k'}$ , so by Fact 6.2.13.(2),  $f_j[N_j] <_{\mathbf{K}}^{\text{univ}} M_{k'}$ . Hence by some renaming, we can extend  $f_i$  as desired.

REMARK 6.2.17.  $(K, \leq_{\mathbf{K}}^{\text{univ}})$  is in general not an AEC as it may fail the Löwenheim-Skolem-Tarski axiom, the coherence axiom, and (3c) in the Tarski-Vaught axioms of Definition 6.2.1.

### 6.3. Independence relations

Since this section mostly lists definitions, the reader already familiar with independence (in the first-order context) may want to skip it and refer to it as needed.
We would like a general framework in which to study independence in abstract elementary classes. One such framework is Shelah's good  $\lambda$ -frames [She09a, Section II.6]. Another is given by the definition of independence relation in Definition 3.3.1 (itself adapted from [BG, Definition 3.3] which can be traced back to the work of Makkai and Shelah [MS90]). Both definitions describe a relation "p does not fork over M" for p a Galois type over N and  $M \leq_{\mathbf{K}} N$  and require it to satisfy some properties.

In Chapter 3, it is also shown how to "close" such a relation to obtain a relation "p does not fork over M" when p is a type over an arbitrary set. We find that starting with such a relation makes the statement of symmetry transparent, and hence makes several proofs easier. Perhaps even more importantly, we can be very precise<sup>9</sup> when dealing with chain local character properties (see Definition 6.3.16).

The definition in Chapter 3 is not completely adequate for our purpose, however. There it is assumed that everything is contained inside a big homogeneous monster model. While we will always assume amalgamation, assuming the existence of a monster model is still problematic when for example we want to study independence over models of size  $\lambda$  only (the motivation for good  $\lambda$ -frames, note that Shelah's definition does not assume the existence of a monster model). We also allow working inside more general classes than AECs: coherent abstract classes (recall Definition 6.2.4). This is convenient when working with classes of saturated models (see for example the study of weakly good independence relation in Section 6.7), but note that in general we may not be able to build a monster model there.

We also give a more general definition than in Chapter 3, as we do not assume that everything happens in a big homogeneous monster model, and we allow working inside coherent abstract classes (recall Definition 6.2.4) rather than only abstract elementary classes. The later feature is convenient when working with classes of saturated models.

This means that we always have to carry over an ambient model N that may shrink or be extended as needed. Although this makes the notation slightly heavier, it does not cause any serious technical difficulties. At first reading, the reader may simply want to ignore N and assume everything takes place inside a monster model.

Because we quote extensively from [She09a], which deals with frames, and also because it is sometimes convenient to "forget" the extension of the relation to arbitrary sets, we will still define frames and recall their relationship with independence relations over sets.

**6.3.1. Frames.** Shelah's definition of a pre-frame appears in [She09a, Definition III.0.2.1] and is meant to axiomatize the bare minimum of properties a relation must satisfy in order to be a meaningful independence notions.

We make several changes: we do not mention basic types (we have no use for them), so in Shelah's terminology our pre-frames will be type-full. In fact, it is notationally convenient for us to define our frame on every type, not just the nonalgebraic ones. The *disjointness* property (see Definition 6.3.12) tells us that

<sup>&</sup>lt;sup>9</sup>Assume for example that  $\mathfrak{s}$  is a good-frame on a class of saturated models of an AEC **K**. Let  $\langle M_i : i < \delta \rangle$  be an increasing chain of saturated models. Let  $M_\delta := \bigcup_{i < \delta} M_i$  and let  $p \in \mathrm{gS}(M_\delta)$ . We would like to say that there is  $i < \delta$  such that p does not fork over  $M_i$  but we may not know that  $M_\delta$  is saturated, so maybe forking is not even defined for types over  $M_\delta$ . However if the forking relation were defined for types over sets, there would be no problem.

the frame behaves trivially on the algebraic types. We do not require it (as it is not required in Definition 3.3.1) but it will hold of all frames we consider.

We require that the class on which the independence relation operates has amalgamation<sup>10</sup>, and we do not require that the base monotonicity property holds (this is to preserve the symmetry between right and left properties in the definition. All the frames we consider will have base monotonicity). Finally, we allow the size of the models to lie in an interval rather than just be restricted to a single cardinal as Shelah does. We also parametrize on the length of the types. This allows more flexibility and was already the approach favored in Chapters 4 and 5.

DEFINITION 6.3.1. Let  $\mathcal{F} = [\lambda, \theta)$  be an interval of cardinals with  $\aleph_0 \leq \lambda < \theta$ ,  $\alpha \leq \theta$  be a cardinal or  $\infty$ .

A type-full pre-( $\langle \alpha, \mathcal{F} \rangle$ -frame is a pair  $\mathfrak{s} = (\mathbf{K}, \downarrow)$ , where:

- (1) **K** is a coherent abstract class in  $\mathcal{F}$  (see Definition 6.2.4) with amalgamation.
- (2)  $\downarrow$  is a relation on quadruples of the form  $(M_0, A, M, N)$ , where  $M_0 \leq_{\mathbf{K}}$ (2)  $\subseteq$  is a relation on quadraphics of the form  $(M_0, A, M, V)$ , where  $M_0 \leq \mathbf{K}$   $M \leq_{\mathbf{K}} N$  are all in  $\mathbf{K}, A \subseteq |N|$  is such that  $|A \setminus |M_0|| < \alpha$ . We write  $\downarrow (M_0, A, M, N)$  or  $A \downarrow_{M_0}^N M$  instead of  $(M_0, A, M, N) \in \downarrow$ . (3) The following properties hold:
- - (a) <u>Invariance</u>: If  $f : N \cong N'$  and  $A \underset{M_0}{\overset{N}{\downarrow}} M$ , then  $f[A] \underset{f[M_0]}{\overset{N'}{\downarrow}} f[M]$ . (b) <u>Monotonicity</u>: Assume  $A \underset{M_0}{\overset{N}{\downarrow}} M$ . Then: (i) Ambient monotonicity: If  $N' \geq_{\mathbf{K}} N$ , then  $A \underset{M_0}{\stackrel{N'}{\downarrow}} M$ . If  $M \leq_{\mathbf{K}} M$  $N_0 \leq_{\mathbf{K}} N \text{ and } A \subseteq |N_0|, \text{ then } A \underset{M_0}{\overset{N_0}{\downarrow}} M.$ (ii) Left and right monotonicity: If  $A_0 \subseteq A, M_0 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M,$ then  $A_0 \underset{M_0}{\overset{N}{\downarrow}} M'.$

(c) Left normality: If 
$$A \underset{M_0}{\overset{N}{\downarrow}} M$$
, then<sup>11</sup>  $AM_0 \underset{M_0}{\overset{N}{\downarrow}} M$ .

When  $\alpha$  or  $\mathcal{F}$  are clear from context or irrelevant, we omit them and just say that  $\mathfrak{s}$  is a pre-frame (or just a frame). We may omit the "type-full". A ( $\leq \alpha$ )-frame is just a  $(<\alpha^+)$ -frame. We might omit  $\alpha$  when  $\alpha = 2$  (i.e.  $\mathfrak{s}$  is a (<1)-frame) and we might talk of a  $\lambda$ -frame or a (>  $\lambda$ )-frame instead of a { $\lambda$ }-frame or a [ $\lambda, \infty$ )-frame.

NOTATION 6.3.2. For  $\mathfrak{s} = (\mathbf{K}, \downarrow)$  a pre- $(\langle \alpha, \mathcal{F})$ -frame with  $\mathcal{F} = [\lambda, \theta)$ , write  $\mathbf{K}_{\mathfrak{s}} := \mathbf{K}, \downarrow := \downarrow, \alpha_{\mathfrak{s}} := \alpha, \mathcal{F}_{\mathfrak{s}} = \mathcal{F}, \lambda_{\mathfrak{s}} := \lambda, \theta_{\mathfrak{s}} := \theta$ . Note that pedantically,  $\alpha$ ,  $\mathcal{F}$ , and  $\theta$  should be part of the data of the frame in order for this notation to be well-defined but we ignore this detail.

<sup>&</sup>lt;sup>10</sup>This is required in Shelah's definition of good frames, but not in his definition of pre-frames. <sup>11</sup>For sets A and B, we sometimes write AB instead of  $A \cup B$ .

NOTATION 6.3.3. For  $\mathfrak{s} = (\mathbf{K}, \downarrow)$  a pre-frame, we write  $\downarrow (M_0, \bar{a}, M, N)$  or  $\bar{a} \underset{M_0}{\downarrow} M$  for ran $(\bar{a}) \underset{M_0}{\downarrow} M$  (similarly when other parameters are sequences). When  $p \in \mathrm{gS}^{<\infty}(M)$ , we say p does not  $\mathfrak{s}$ -fork over  $M_0$  (or just does not fork over  $M_0$  if  $\mathfrak{s}$  is clear from context) if  $\bar{a} \underset{M_0}{\downarrow} M$  whenever  $p = \mathrm{gtp}(\bar{a}/M; N)$  (using monotonicity and invariance, it is easy to check that this does not depend on the choice of representatives).

REMARK 6.3.4. In the definition of a pre-frame given in Definition 5.3.1, the left hand side of the relation  $\downarrow$  is a sequence, not just a set. Here, we simply assume outright that the relation is defined so that order does not matter.

REMARK 6.3.5. We can go back and forth from this chapter's definition of preframe to Shelah's. We sketch how. From a pre-frame  $\mathfrak{s}$  in our sense (with  $\mathbf{K}_{\mathfrak{s}}$  an AEC), we can let  $\mathrm{gS}^{\mathrm{bs}}(M)$  be the set of nonalgebraic  $p \in \mathrm{gS}(M)$  that do not  $\mathfrak{s}$ -fork over M. Then restricting  $\downarrow_{\mathfrak{s}}$  to the basic types we obtain (assuming that  $\mathfrak{s}$  has base monotonicity, see Definition 6.3.12) a pre-frame in Shelah's sense. From a pre-frame  $(\mathbf{K}, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  in Shelah's sense (where  $\mathbf{K}$  has amalgamation), we can extend  $\downarrow$  by specifying that algebraic and basic types do not fork over their domains. We then get a pre-frame  $\mathfrak{s}$  in our sense with base monotonicity and disjointness.

**6.3.2.** Independence relations. We now give a definition for an independence notion that also takes sets on the right hand side.

DEFINITION 6.3.6 (Independence relation). Let  $\mathcal{F} = [\lambda, \theta)$  be an interval of cardinals with  $\aleph_0 \leq \lambda < \theta$ ,  $\alpha, \beta \leq \theta$  be cardinals or  $\infty$ . A ( $< \alpha, \mathcal{F}, < \beta$ )-independence relation is a pair  $\mathfrak{i} = (\mathbf{K}, \downarrow)$ , where:

- (1) **K** is a coherent abstract class in  $\mathcal{F}$  with amalgamation.
- (2)  $\downarrow$  is a relation on quadruples of the form (M, A, B, N), where  $M \leq_{\mathbf{K}} N$  are all in  $\mathbf{K}, A \subseteq |N|$  is such that  $|A \setminus |M|| < \alpha$  and  $B \subseteq |N|$  is such that  $|B \setminus |M|| < \beta$ . We write  $\downarrow (M, A, B, N)$  or  $A \downarrow_M B$  instead of  $(M, A, B, N) \in \downarrow$ .
- (3) The following properties hold:
  - (a) <u>Invariance</u>: If  $f : N \cong N'$  and  $A \bigcup_{M}^{N} B$ , then  $f[A] \bigcup_{f[M]}^{N'} f[B]$ .
  - (b) <u>Monotonicity</u>: Assume  $A \bigcup_{M}^{N} B$ . Then:

(i) Ambient monotonicity: If  $N' \geq_{\mathbf{K}} N$ , then  $A \bigcup_{M}^{N'} B$ . If  $M \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$  and  $A \cup B \subseteq |N_0|$ , then  $A \bigcup_{M}^{N_0} B$ .

(ii) Left and right monotonicity: If  $A_0 \subseteq A$ ,  $B_0 \subseteq B$ , then  $A_0 \stackrel{N}{\underset{M}{\downarrow}} B_0$ . (c) Left and right normality: If  $A \stackrel{N}{\underset{M}{\downarrow}} B$ , then  $AM \stackrel{N}{\underset{M}{\downarrow}} BM$ . We adopt the conventions described at the end of Definition 6.3.1. For example, a  $(\leq \alpha, \mathcal{F}, < \beta)$ -independence relation is just a  $(< \alpha^+, \mathcal{F}, < \beta)$ -independence relation.

When  $\beta = \theta$ , we omit it. More generally, when  $\alpha$ ,  $\beta$  are clear from context or irrelevant, we omit them and just say that i is an independence relation.

NOTATION 6.3.7. We adopt the same notational conventions as for pre-frames:  $\mathbf{K}_i, \, \underset{i}{\cup}, \, \alpha_i, \, \beta_i, \, \mathcal{F}_i, \, \lambda_i, \, \theta_i$  are defined as in Notation 6.3.2 and p does not i-fork over  $M_0$  is defined as in 6.3.3.

REMARK 6.3.8. It seems that in every case of interest  $\beta = \theta$  (this will always be the case in the next sections of this chapter). We did not make it part of the definition to avoid breaking the symmetry between  $\alpha$  and  $\beta$  (and hence make it possible to define the dual independence relation and the left version of a property, see Definitions 6.3.13 and 6.3.15). Note also that the case  $\alpha = \theta = \infty$  is of particular interest in Section 6.14.

Before listing the properties independence relations and frames could have, we discuss how to go from one to the other. The cl operation is called the *minimal closure* in Definition 3.3.4.

DEFINITION 6.3.9.

- (1) Given a pre-frame  $\mathbf{s} := (\mathbf{K}, \downarrow)$ , let  $cl(\mathbf{s}) := (\mathbf{K}, \downarrow)$ , where  $\downarrow^{cl}(M, A, B, N)$ if and only if  $M \leq_{\mathbf{K}} N$ ,  $|B| < \theta_{\mathbf{s}}$ , and there exists  $N' \geq_{\mathbf{K}} N$ ,  $M' \geq_{\mathbf{K}} M$ containing B such that  $\downarrow(M, A, M', N')$ .
- (2) Given a  $(< \alpha, \mathcal{F})$ -independence relation  $\mathbf{i} = (\mathbf{K}, \bot)$  let  $\operatorname{pre}(\mathbf{i}) := (\mathbf{K}, \bot)$ , where  $\bigcup (M, A, M', N)$  if and only if  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N$  and  $\bigcup (M, A, M', N)$ .

Remark 6.3.10.

- (1) If  $\mathfrak{i}$  is a  $(< \alpha, \mathcal{F})$ -independence relation, then  $\operatorname{pre}(\mathfrak{i})$  is a  $\operatorname{pre-}(< \alpha, \mathcal{F})$ -frame.
- (2) If  $\mathfrak{s}$  is a pre-( $< \alpha, \mathcal{F}$ )-frame, then  $cl(\mathfrak{s})$  is a ( $< \alpha, \mathcal{F}$ )-independence relation and  $pre(cl(\mathfrak{s})) = \mathfrak{s}$ .

Other properties of cl and pre are given by Proposition 6.4.1.

REMARK 6.3.11. The reader may wonder why we do not assume that every independence relation is the closure of a pre-frame, i.e. why we do not assume that for any independence relation  $\mathbf{i} = (\mathbf{K}, \downarrow)$ , if  $A \underset{M}{\downarrow} B$ , there exists  $N' \ge_{\mathbf{K}} N$  and  $M' \le_{\mathbf{K}} N'$  with  $M \le_{\mathbf{K}} M'$  such that  $B \subseteq |M'|$  and  $A \underset{M}{\downarrow} M'$  (this can be written abstractly as  $\mathbf{i} = \operatorname{cl}(\operatorname{pre}(\mathbf{i}))$ ? This would allow us to avoid the redundancies between the definition of an independence relation and that of a pre-frame. However, several interesting independence notions do not satisfy that property (see Section 3.3.2). Further, it is not clear that the property  $\mathbf{i} = \operatorname{cl}(\operatorname{pre}(\mathbf{i}))$  transfers upward (see Definition 6.6.3). Therefore we prefer to be agnostic and not require it.

Next, we give a long list of properties that an independence relation may or may not have. Most are classical and already appear for example in Chapter 3. We give

them here again both for the convenience of the reader and because their definition is sometimes slightly modified compared to Chapter 3 (for example, symmetry there is called right full symmetry here, and some properties like uniqueness and extensions are complicated by the fact we do not work in a monster model). They will be used throughout this chapter (for example, Section 6.4 discusses implications between the properties).

DEFINITION 6.3.12 (Properties of independence relations). Let  $i := (\mathbf{K}, \downarrow)$  be a ( $< \alpha, \mathcal{F}, < \beta$ )-independence relation.

- (1) i has disjointness if  $A \stackrel{N}{\underset{M}{\longrightarrow}} B$  implies  $A \cap B \subseteq |M|$ . (2) i has symmetry if  $A \stackrel{N}{\underset{M}{\longrightarrow}} B$  implies that for all<sup>12</sup>  $B_0 \subseteq B$  of size less than  $\alpha$ and all  $A_0 \subseteq A$  of size less than  $\beta$ ,  $B_0 \bigcup_M^N A_0$ .
- (3) i has right full symmetry if  $A \stackrel{N}{\downarrow} B$  implies that for all  $B_0 \subseteq B$  of size less than  $\alpha$  and all  $A_0 \subseteq A$  of size less than  $\beta$ , there exists  $N' \geq_{\mathbf{K}} N$ ,  $M' \geq_{\mathbf{K}} M$  containing  $A_0$  such that  $B_0 \bigcup_{M}^{N'} M'$ .
- (4) i has right base monotonicity if  $A \underset{M}{\stackrel{N}{\downarrow}} B$  and  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N$ ,  $|M'| \subseteq$  $B \cup |M| \text{ implies } A \underset{M'}{\stackrel{N}{\downarrow}} B.$ (5) i has right existence if  $A \underset{M}{\stackrel{N}{\downarrow}} M$  for any  $A \subseteq |N|$  with  $|A| < \alpha$ .

- (6) i has right uniqueness if whenever  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N_\ell$ ,  $\ell = 1, 2, |M_0| \subseteq B \subseteq |M|, q_\ell \in gS^{<\alpha}(B; N_\ell), q_1 \upharpoonright M_0 = q_2 \upharpoonright M_0$ , and  $q_\ell$  does not fork over  $M_0$ , then  $q_1 = q_2$ .
- (7) i has right extension if whenever  $p \in gS^{<\alpha}(MB; N)$  does not fork over M and  $B \subseteq C \subseteq |N|$  with  $|C| < \beta$ , there exists  $N' \geq_{\mathbf{K}} N$  and  $q \in$  $gS^{<\alpha}(MC; N')$  extending p such that q does not fork over M.
- (8) i has right independent amalgamation if  $\alpha > \lambda$ ,  $\beta = \theta$ , and  $^{13}$  whenever  $M_0 \leq_{\mathbf{K}} M_\ell$  are in  $\mathbf{K}, \ell = 1, 2$ , there exists  $N \in \mathbf{K}$  and  $f_\ell : M_\ell \xrightarrow{M_0} N$

such that  $f_1[M_1] \underset{M_0}{\downarrow}^N f_2[M_2]$ . (9) i has the right  $(<\kappa)$ -model-witness property if whenever  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N$ ,  $||M'| \setminus |M|| < \beta, A \subseteq |N|$ , and  $A \underset{M}{\downarrow}^N B_0$  for all  $B_0 \subseteq |M'|$  of size less than

$$\kappa$$
, then  $A \stackrel{N}{\underset{M}{\downarrow}} M'$ . i has the right  $(< \kappa)$ -witness property if this is true when

<sup>&</sup>lt;sup>12</sup>Why not just take  $B_0 = B$ ? Because the definition of  $\bigcup$  requires that the left hand side has size less than  $\alpha$ . Similarly for right full symmetry.

<sup>&</sup>lt;sup>13</sup>Note that even though the next condition is symmetric, the condition on  $\alpha$  and  $\beta$  make the left version of the property different from the right.

M' is allowed to be an arbitrary set. The  $\lambda$ -[model-]witness property is the  $(<\lambda^+)$ -[model-]witness property.

(10) i has right transitivity if whenever  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N$ ,  $A \underset{M_0}{\downarrow}^N M_1$  and  $A \underset{M_1}{\downarrow}^N B$ implies  $A \underset{M_0}{\downarrow}^N B$ . Strong right transitivity is the same property when we do

not require  $M_0 \leq_{\mathbf{K}} M_1$ .

(11) i has right full model-continuity if **K** is an AEC in  $\mathcal{F}$ ,  $\alpha > \lambda$ ,  $\beta = \theta$ , and whenever  $\langle M_i^{\ell} : i \leq \delta \rangle$  is increasing continuous with  $\delta$  limit,  $\ell \leq 3$ , for all

$$i < \delta, M_i^0 \leq_{\mathbf{K}} M_i^t \leq_{\mathbf{K}} M_i^3, \ell = 1, 2, \|M_{\delta}^1\| < \alpha, \text{ and } M_i^1 \downarrow M_i^2 \text{ for all}$$

$$i < \delta$$
, then  $M_{\delta}^1 \downarrow M_{\delta}^2$ .

(12) Weak chain local character is a technical property used to generate weakly good independence relations, see Definition 6.6.6.

Whenever this makes sense, we similarly define the same properties for preframes.

Note that we have defined the right version of the asymmetric properties. One can define a left version by looking at the *dual independence relation*.

DEFINITION 6.3.13. Let  $\mathbf{i} := (\mathbf{K}, \downarrow)$  be a  $(< \alpha, \mathcal{F}, < \beta)$ -independence relation. Define the dual independence relation  $\mathbf{i}^d := (\mathbf{K}, \downarrow)$  by  $\downarrow (M, A, B, N)$  if and only if  $\downarrow (M, B, A, N)$ .

Remark 6.3.14.

- (1) If  $\mathbf{i}$  is a  $(< \alpha, \mathcal{F}, < \beta)$ -independence relation, then  $\mathbf{i}^d$  is a  $(< \beta, \mathcal{F}, < \alpha)$ independence relation and  $(\mathbf{i}^d)^d = \mathbf{i}$ .
- (2) Let i be a  $(< \alpha, \mathcal{F}, < \alpha)$ -independence relation. Then i has symmetry if and only if  $i = i^d$ .

DEFINITION 6.3.15. For P a property, we will say i has left P if  $i^d$  has right P. When we omit left or right, we mean the right version of the property.

DEFINITION 6.3.16 (Locality cardinals). Let  $i = (\mathbf{K}, \downarrow)$  be a  $(< \alpha, \mathcal{F})$ -independence relation,  $\mathcal{F} = [\lambda, \theta)$ . Let  $\alpha_0 < \alpha$  be such that  $|\alpha_0|^+ < \theta$ .

- (1) Let  $\bar{\kappa}_{\alpha_0}(\mathbf{i})$  be the minimal cardinal  $\mu \geq |\alpha_0|^+ + \lambda^+$  such that for any  $M \leq_{\mathbf{K}} N$  in  $\mathbf{K}$ , any  $A \subseteq |N|$  with  $|A| \leq \alpha_0$ , there exists  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{<\mu}$  with  $A \underset{M_0}{\downarrow} M$ . When  $\mu$  does not exist, we set  $\bar{\kappa}_{\alpha_0}(\mathbf{i}) = \infty$ .
- (2) For R a binary relation on  $\mathbf{K}$ , Let  $\kappa_{\alpha_0}(\mathbf{i}, R)$  be the minimal cardinal  $\mu \geq |\alpha_0|^+ + \aleph_0$  such that for any regular  $\delta \geq \mu$ , any R-increasing (recall Definition 2.2.11)  $\langle M_i : i < \delta \rangle$  in  $\mathbf{K}$ , any  $N \in \mathbf{K}$  extending all the M's and any  $A \subseteq |N|$  of size  $\leq \alpha_i$ , there exists  $i \leq \delta$  such that  $A \stackrel{N}{=} M$ .

 $M_i$ 's, and any  $A \subseteq |N|$  of size  $\leq \alpha_0$ , there exists  $i < \delta$  such that  $A \bigcup_{M_i} M_{\delta}$ .

Here, we have set<sup>14</sup>  $M_{\delta} := \bigcup_{i < \delta} M_i$ . When  $R = \leq_{\mathbf{K}}$ , we omit it. When  $\mu$  does not exist or  $\mu_r \geq \theta$ , we set  $\kappa_{\alpha_0}(\mathfrak{i}) = \infty$ .

When **K** is clear from context, we may write  $\bar{\kappa}_{\alpha_0}(\downarrow)$ . For  $\alpha_0 \leq \alpha$ , we also let  $\bar{\kappa}_{<\alpha_0}(\mathfrak{i}) := \sup_{\alpha'_0 < \alpha_0} \bar{\kappa}_{\alpha'_0}(\mathfrak{i})$ . Similarly define  $\kappa_{<\alpha_0}$ .

We similarly define  $\bar{\kappa}_{\alpha_0}(\mathfrak{s})$  and  $\kappa_{\alpha_0}(\mathfrak{s})$  for  $\mathfrak{s}$  a pre-frame (in the definition of  $\kappa_{\alpha_0}(\mathfrak{s})$ , we require in addition that  $M_{\delta}$  be a member of **K**).

We will use the following notation to restrict independence relations to smaller domains:

NOTATION 6.3.17. Let i be a  $(< \alpha, \mathcal{F}, < \beta)$ -independence relation.

- (1) For  $\alpha_0 \leq \alpha, \beta_0 \leq \beta$ , let  $i^{<\alpha_0,<\beta_0}$  denotes the  $(<\alpha_0, \mathcal{F}, <\beta_0)$ -independence relation obtained by restricting the types to have length less than  $\alpha_0$  and the right hand side to have size less than  $\beta_0$  (in the natural way). When  $\beta_0 = \beta$ , we omit it.
- (2) For  $\mathbf{K}'$  a coherent sub-AC of  $\mathbf{K}_i$ , let  $\mathbf{i} \upharpoonright \mathbf{K}'$  be the  $(< \alpha, \mathcal{F}, < \beta)$ independence relation obtained by restricting the underlying class to  $\mathbf{K}'$ .
  When  $\mathbf{i}$  is a  $(< \alpha, \mathcal{F})$ -independence relation and  $\mathcal{F}_0 \subseteq \mathcal{F}$  is an interval of
  cardinals,  $\mathcal{F}_0 = [\lambda_i, \theta_0)$ , we let  $\mathbf{i}_{\mathcal{F}_0} := \mathbf{i}^{<\min(\alpha, \theta_0)} \upharpoonright (\mathbf{K}_i)_{\mathcal{F}_0}$  be the restriction of  $\mathbf{i}$  to models of size in  $\mathcal{F}_0$  and types of appropriate length.

We end this section with two examples of independence relations. The first is coheir. In first-order logic, coheir was first defined in  $[\mathbf{LP79}]^{15}$ . A definition of coheir for classes of models of an  $\mathbb{L}_{\kappa,\omega}$  sentence appears in  $[\mathbf{MS90}]$  and was later adapted to general AECs in  $[\mathbf{BG}]$ . In Chapter 2, we gave a more conceptual (but equivalent) definition and improved some of the results of Boney and Grossberg. Here, we use Boney and Grossberg's definition but rely on Chapter 2.

DEFINITION 6.3.18 (Coheir). Let **K** be an AEC with amalgamation and let  $\kappa > LS(\mathbf{K})$ .

Define  $\mathbf{i}_{\kappa-\mathrm{ch}}(\mathbf{K}) := (\mathbf{K}^{\kappa-\mathrm{sat}}, \downarrow)$  by  $\bigcup (M, A, B, N)$  if and only if  $M \leq_{\mathbf{K}} N$  are in  $\mathbf{K}^{\kappa-\mathrm{sat}}, A \cup B \subseteq |N|$ , and for any  $\bar{a} \in {}^{<\kappa}A$  and  $B_0 \subseteq |M| \cup B$  of size less than  $\kappa$ , there exists  $\bar{a}' \in {}^{<\kappa}|M|$  such that  $\operatorname{gtp}(\bar{a}/B_0; N) = \operatorname{gtp}(\bar{a}'/B_0; M)$ .

FACT 6.3.19 (Theorem 2.5.15). Let **K** be an AEC with amalgamation and let  $\kappa > LS(\mathbf{K})$ . Let  $\mathfrak{i} := \mathfrak{i}_{\kappa-ch}(\mathbf{K})$ .

- (1) i is a  $(< \infty, [\kappa, \infty))$ -independence relation with disjointness, base monotonicity, left and right existence, left and right  $(< \kappa)$ -witness property, and strong left transitivity.
- (2) If **K** does not have the  $(< \kappa)$ -order property of length  $\kappa$ , then:
  - (a) *i* has symmetry and strong right transitivity.
  - (b) For all  $\alpha$ ,  $\bar{\kappa}_{\alpha}(\mathfrak{i}) \leq \left( (\alpha + 2)^{<\kappa_r} \right)^+$ .
  - (c) If  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N_\ell$  for  $\ell = 1, 2, |M_0| \subseteq B \subseteq |M|$ .  $q_\ell \in \mathrm{gS}^{<\infty}(B; N_\ell)$ ,  $q_1 \upharpoonright M_0 = q_2 \upharpoonright M_0, q_\ell$  does not i-fork over  $M_0$  for  $\ell = 1, 2$ , and  $\mathbf{K}$  is  $(<\kappa)$ -tame and short for  $\{q_1, q_2\}$ , then  $q_1 = q_2$ .

<sup>&</sup>lt;sup>14</sup>Recall that **K** is only a coherent abstract class, so may not be closed under unions of chains of length  $\delta$ . Thus we think of  $M_{\delta}$  as a set.

<sup>&</sup>lt;sup>15</sup>The equivalence of nonforking with coheir (for stable theories) was already established by Shelah in the early seventies and appears in Section III.4 of [She78], see also [She90, Corollary III.4.10].

(d) If **K** is  $(< \kappa)$ -tame and short for types of length less than  $\alpha$ , then  $\operatorname{pre}(\mathfrak{i}^{<\alpha})$  has uniqueness. Moreover  $\mathfrak{i}_{[\kappa,\alpha)}^{\circ}$  has uniqueness.

REMARK 6.3.20. The extension property<sup>17</sup> seems to be more problematic. In **[BG]**, Boney and Grossberg simply assumed it (they also showed that it followed from  $\kappa$  being strongly compact [**BG**, Theorem 8.2(1)]). From superstability-like hypotheses, we will obtain more results on it (see Theorem 6.10.16, Theorem 6.15.1, and Theorem 6.15.6).

We now consider another independence notion: splitting. This was first defined for AECs in [She99, Definition 3.2]. Here we define the negative property (nonsplitting), as it is the one we use the most.

DEFINITION 6.3.21 ( $\lambda$ -nonsplitting). Let **K** be a coherent abstract class with amalgamation.

- (1) For  $\lambda$  an infinite cardinal, define  $\mathfrak{s}_{\lambda-ns}(\mathbf{K}) := (\mathbf{K}, \downarrow)$  by  $\bar{a} \underset{M_0}{\downarrow}^N M$  if and only if  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ ,  $A \subseteq |N|$ , and whenever  $M_0 \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} M$ ,  $N_\ell \in$  $\mathbf{K}_{<\lambda}, \ell = 1, 2, \text{ and } f : N_1 \cong_{M_0} N_2, \text{ then } f(\operatorname{gtp}(\bar{a}/N_1; N)) = \operatorname{gtp}(\bar{a}/N_2; N).$
- (2) Define  $\mathfrak{s}_{ns}(\mathbf{K})$  to have underlying AEC **K** and forking relation defined such that  $p \in gS^{<\infty}(M)$  does not  $\mathfrak{s}_{ns}(\mathbf{K})$ -fork over  $M_0 \leq_{\mathbf{K}} M$  if and only if p does not  $\mathfrak{s}_{\lambda-ns}(\mathbf{K})$ -fork over  $M_0$  for all infinite  $\lambda$ .
- (3) Let  $\mathfrak{i}_{\lambda-\mathrm{ns}}(\mathbf{K}) := \mathrm{cl}(\mathfrak{s}_{\lambda-\mathrm{ns}}(\mathbf{K})), \, \mathfrak{i}_{\mathrm{ns}}(\mathbf{K}) := \mathrm{cl}(\mathfrak{s}_{\mathrm{ns}}(\mathbf{K})).$

FACT 6.3.22. Assume **K** is a coherent AC in  $\mathcal{F} = [\lambda, \theta)$  with amalgamation. Let  $\mathfrak{s} := \mathfrak{s}_{ns}(\mathbf{K}), \mathfrak{s}' := \mathfrak{s}_{\lambda-ns}(\mathbf{K}).$ 

- (1)  $\mathfrak{s}$  and  $\mathfrak{s}'$  are pre-( $< \infty, \mathcal{F}$ )-frame with base monotonicity, left and right existence. If **K** is  $\lambda$ -closed,  $\mathfrak{s}'$  has the right  $\lambda$ -model-witness property.
- (2) If **K** is an AEC in  $\mathcal{F}$  and is stable in  $\lambda$ , then  $\bar{\kappa}_{<\omega}(\mathfrak{s}') = \lambda^+$ .
- (3) If t is a pre-( $< \infty, \mathcal{F}$ )-frame with uniqueness and  $\mathbf{K}_t = \mathbf{K}$ , then  $\bigcup_t \subseteq \bigcup_s$ .
- (4) Always,  $\downarrow \subseteq \downarrow$ . Moreover if **K** is  $\lambda$ -tame for types of length less than  $\alpha$ , then  $\mathfrak{s}^{<\alpha} = (\mathfrak{s}')^{<\alpha}$ .
- (5) Let  $M_0 <_{\mathbf{K}}^{\text{univ}} M \leq_{\mathbf{K}} N$  with ||M|| = ||N||.
  - (a) Weak uniqueness: If  $p_{\ell} \in gS^{\alpha}(N)$ ,  $\ell = 1, 2$ , do not  $\mathfrak{s}$ -fork over  $M_0$
  - (b) Weak extension: If  $p \in \mathrm{gS}^{<\infty}(M)$  does not  $\mathfrak{s}$ -fork over  $M_0$  and  $f : N \xrightarrow{M_0} M$ , then  $q := f^{-1}(p)$  is an extension of p to  $\mathrm{gS}^{<\infty}(N)$  that does not  $\mathfrak{s}$ -fork over  $M_0$ . Moreover q is algebraic if and only if p is algebraic.

Proof.

- (1) Easy.
- (2) By [She99, Claim 3.3.1] (see also [GV06b, Fact 4.6]).
- (3) By Lemma 3.4.2.
- (4) By Proposition 3.3.12.

<sup>&</sup>lt;sup>16</sup>Of course, this is only interesting if  $\alpha \leq \kappa$ .

 $<sup>^{17}\</sup>mathrm{A}$  word of caution: In [HL02, Section 4], the authors give Shelah's example of an  $\omega\text{-stable}$ class that does not have extension. However, the extension property they consider is over all sets, not only over models.

(5) By [Van06, Theorem I.4.10, Theorem I.4.12] (the moreover part is easy to see from the definition of q).

REMARK 6.3.23. Fact 6.3.22.(3) tells us that any reasonable independence relation will be extended by nonsplitting. In this sense, nonsplitting is a *maximal* candidate for an independence relation<sup>18</sup>.

#### 6.4. Some independence calculus

We investigate relationships between properties and how to go from a frame to an independence relation. Most of it appears already in Chapter 3 and has a much longer history, described there. The following are new: Lemma 6.4.5 gives a way to get the witness properties from tameness, partially answering Question 3.5.5. Lemmas 6.4.8 and 6.4.7 are technical results used in the last sections.

The following proposition investigates what properties are preserved by the operations cl and pre (recall Definition 6.3.9). This was done already in Section 3.5.1, so we cite from there.

PROPOSITION 6.4.1. Let  $\mathfrak{s}$  be a pre-( $< \alpha, \mathcal{F}$ )-frame and let  $\mathfrak{i}$  be a ( $< \alpha, \mathcal{F}$ )-independence relation.

- For P in the list of properties of Definition 6.3.12, if i has P, then pre(i) has P.
- (2) For P a property in the following list, i has P if (and only if) pre(i) has P: existence, independent amalgamation, full model-continuity.
- (3) For P a property in the following list,  $cl(\mathfrak{s})$  has P if (and only if)  $\mathfrak{s}$  has P: disjointness, full symmetry, base monotonicity, extension, transitivity.
- (4) If pre(i) has extension, then cl(pre(i)) = i if and only if i has extension.
- (5) The following are equivalent:
  - (a)  $\mathfrak{s}$  has full symmetry.
  - (b)  $cl(\mathfrak{s})$  has symmetry.
  - (c)  $cl(\mathfrak{s})$  has full symmetry.
- (6) If pre(i) has uniqueness and i has extension, then i has uniqueness.
- (7) If pre(i) has extension and i has uniqueness, then i has extension.
- (8) <sup>19</sup>  $\bar{\kappa}_{<\alpha}(\mathfrak{i}) = \bar{\kappa}_{<\alpha}(\operatorname{pre}(\mathfrak{i})).$
- (9)  $\kappa_{<\alpha}(\text{pre}(\mathfrak{i})) \leq \kappa_{<\alpha}(\mathfrak{i})$ . If  $\mathbf{K}_{\mathfrak{i}}$  is an AEC, then this is an equality.

PROOF. All are straightforward. See Lemmas 3.5.3 and 3.5.4.

To what extent is an independence relation determined by its corresponding frame? There is an easy answer:

<sup>18</sup>Moreover,  $(< \kappa)$ -coheir is a minimal candidate in the following sense: Let us say an independence relation  $\mathbf{i} = (\mathbf{K}, \downarrow)$  has the strong  $(<\kappa)$ -witness property if whenever  $A \not\downarrow B$ , there exists  $\bar{a}_0 \in {}^{<\kappa}A$  and  $B_0 \subseteq |M| \cup B$  of size less than  $\kappa$  such that  $\operatorname{gtp}(\bar{a}'_0/B_0; N) = \operatorname{gtp}(\bar{a}_0/B_0; N)$  implies  $\bar{a}_0 \not\downarrow B$ . Intuitively, this says that forking is witnessed by a formula (and this could be made precise using the notion of Galois Morleyization, see Chapter 2). It is easy to check that  $(<\kappa)$ -coheir has this property, and any independence relation with strong  $(<\kappa)$ -witness and left existence must extend  $(<\kappa)$ -coheir.

<sup>&</sup>lt;sup>19</sup>Note that maybe  $\alpha = \infty$ . However we can always apply the proposition to  $\mathfrak{s}^{<\alpha_0}$  for an appropriate  $\alpha_0 \leq \alpha$ .

LEMMA 6.4.2. Let i and i' be independence relations with pre(i) = pre(i'). If i and i' both have extension, then i = i'.

**PROOF.** By Proposition 6.4.1.(4),  $\mathbf{i} = cl(pre(\mathbf{i}))$  and  $\mathbf{i}' = cl(pre(\mathbf{i}')) = cl(pre(\mathbf{i})) = cl(pre(\mathbf{i}))$ i.

The next proposition gives relationships between the properties. We state most results for frames, but they usually have an analog for independence relations that can be obtained using Proposition 6.4.1.

**PROPOSITION 6.4.3.** Let i be a  $(< \alpha, \mathcal{F})$ -independence relation with base monotonicity. Let  $\mathfrak{s}$  be a pre-( $< \alpha, \mathcal{F}$ )-frame with base monotonicity.

- (1) If i has full symmetry, then it has symmetry. If i has the  $(< \kappa)$ -witness property, then it has the  $(<\kappa)$ -model-witness property. If  $\mathfrak{i}[\mathfrak{s}]$  has strong transitivity, then it has transitivity.
- (2) If  $\mathfrak{s}$  has uniqueness and extension, then it has transitivity.
- (3) For  $\alpha > \lambda$ , if  $\mathfrak{s}$  has extension and existence, then  $\mathfrak{s}$  has independent amalgamation. Conversely, if  $\mathfrak{s}$  has transitivity and independent amalgamation, then  $\mathfrak{s}$  has extension and existence. Moreover if  $\mathfrak{s}$  has uniqueness and independent amalgamation, then it has transitivity.
- (4) If  $\min(\kappa_{<\alpha}(\mathfrak{s}), \bar{\kappa}_{<\alpha}(\mathfrak{s})) < \infty$ , then  $\mathfrak{s}$  has existence.
- (5)  $\kappa_{<\alpha}(\mathfrak{s}) \leq \bar{\kappa}_{<\alpha}(\mathfrak{s}).$
- (6) If  $\mathbf{K}_{\mathfrak{s}}$  is  $\lambda_{\mathfrak{s}}$ -closed,  $\bar{\kappa}_{\leq \alpha}(\mathfrak{s}) = \lambda_{\mathfrak{s}}^+$  and  $\mathfrak{s}$  has transitivity, then  $\mathfrak{s}$  has the right  $\lambda$ -model-witness property.
- (7) If  $\mathbf{K}_{\mathfrak{s}}$  does not have the order property (Definition 2.4.3), any chain in  $\mathbf{K}_{\mathfrak{s}}$  has an upper bound,  $\theta = \infty$ , and  $\mathfrak{s}$  has uniqueness, existence, and extension, then  $\mathfrak{s}$  has full symmetry.

## Proof.

- (1) Easy.
- (2) As in the proof of [She09a, Claim II.2.18].
- (3) The first sentence is easy, since independent amalgamation is a particular case of extension and existence. Moreover to show existence it is enough by monotonicity to show it for types of models. The proof of transitivity from uniqueness and independent amalgamation is as in (2).
- (4) By definition of the local character cardinals.
- (5) Let  $\delta = \operatorname{cf} \delta \geq \overline{\kappa}_{<\alpha}(\mathfrak{s})$  and  $\langle M_i : i < \delta \rangle$  be increasing in  $\mathbf{K}, N \geq_{\mathbf{K}} M_i$ for all  $i < \delta$  and  $A \subseteq |N|$  with  $|A| < \alpha$ . Assume  $M_{\delta} := \bigcup_{i < \delta} M_i$  is in **K**. By definition of  $\bar{\kappa}_{<\alpha}$  there exists  $N \leq_{\mathbf{K}} M_{\delta}$  of size less than  $\bar{\kappa}_{\alpha_0}(\mathfrak{i})$  such that p does not fork over N. Now use regularity of  $\delta$  to find  $i < \delta$  with  $N \leq_{\mathbf{K}} M_i$ .
- (6) Let  $\lambda := \lambda_{\mathfrak{s}}$ , say  $\mathfrak{s} = (\mathbf{K}, \downarrow)$ . Let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  and assume  $A \underset{M_0}{\downarrow} B$  for all  $B \subseteq |M|$  with  $|B| \leq \lambda$ . By definition of  $\bar{\kappa}_{\leq \alpha}(\mathfrak{s})$ , there exists  $M'_0 \leq_{\mathbf{K}} M$ of size  $\lambda$  such that  $A \underset{M'_0}{\downarrow} M$ . By  $\lambda$ -closure and base monotonicity, we can assume without loss of some N is a state of a set of the set of t can assume without loss of generality that  $M_0 \leq_{\mathbf{K}} M'_0$ . By assumption,  $A \stackrel{N}{\underset{M_0}{\downarrow}} M'_0$ , so by transitivity,  $A \stackrel{N}{\underset{M_0}{\downarrow}} M$ .
- (7) As in Corollary 3.5.17.

REMARK 6.4.4. The precise statement of Corollary 3.5.17 shows that Proposition 6.4.3.(7) is local in the sense that to prove symmetry over the base model M, it is enough to require uniqueness and extension over this base model (i.e. any two types that do not fork over M, have the same domain, and are equal over M are equal over their domain, and any type over M can be extended to an arbitrary domain so that it does not fork over M).

LEMMA 6.4.5. Let  $i = (\mathbf{K}, \downarrow)$  be a  $(< \alpha, \mathcal{F})$ -independence relation. If i has extension and uniqueness, then:

- (1) If **K** is  $(< \kappa)$ -tame for types of length less than  $\alpha$ , then **K** has the right  $(< \kappa)$ -model-witness property.
- (2) If **K** is  $(< \kappa)$ -tame and short for types of length less than  $\theta_i$ , then **K** has the right  $(< \kappa)$ -witness property.
- (3) If **K** is  $(< \kappa)$ -tame and short for types of length less than  $\kappa + \alpha$  and i has symmetry, then **K** has the left  $(< \kappa)$ -witness property.

Proof.

(1) Let  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N$  be in  $\mathbf{K}$ ,  $A \subseteq |N|$  have size less than  $\alpha$ . Assume  $A \downarrow_{M}^{N} B_{0}$  for all  $B_{0} \subseteq |M'|$  of size less than  $\kappa$ . We want to show that

 $A \underset{M}{\stackrel{N}{\downarrow}} M'$ . Let  $\bar{a}$  be an enumeration of  $A, p := \operatorname{gtp}(\bar{a}/M; N)$ . Note that

(taking  $B_0 = \emptyset$  above) normality implies p does not fork over M. By extension, let  $q \in gS^{<\alpha}(M')$  be an extension of p that does not fork over M. Using amalgamation and some renaming, we can assume without loss of generality that q is realized in N. Let  $p' := gtp(\bar{a}/M'; N)$ . We claim that p' = q, which is enough by invariance. By the tameness assumption, it is enough to check that  $p' \upharpoonright B_0 = q \upharpoonright B_0$  for all  $B_0 \subseteq |M'|$  of size less than  $\kappa$ . Fix such a  $B_0$ . By assumption,  $p' \upharpoonright B_0$  does not fork over M. By monotonicity,  $q \upharpoonright B_0$  does not fork over M. By uniqueness,  $p' \upharpoonright B_0 = q \upharpoonright B_0$ , as desired.

- (2) Similar to before, noting that for  $M \leq_{\mathbf{K}} N$ ,  $\operatorname{gtp}(\bar{a}/M\bar{b}; N) = \operatorname{gtp}(\bar{a}'/M\bar{b}; N)$ if and only if  $\operatorname{gtp}(\bar{a}\bar{b}/M; N) = \operatorname{gtp}(\bar{a}'\bar{b}/M; N)$ .
- (3) Observe that in the proof of the previous part, if the set on the right hand side has size less than  $\kappa$ , it is enough to require ( $< \kappa$ )-tameness and shortness for types of length less than ( $\alpha + \kappa$ ). Now use symmetry.

Having a nice independence relation makes the class nice. The results below are folklore:

PROPOSITION 6.4.6. Let  $\mathbf{i} = (\mathbf{K}, \downarrow)$  be a  $(< \alpha, \mathcal{F})$ -independence relation with base monotonicity. Assume **K** is an AEC in  $\mathcal{F}$  with  $LS(\mathbf{K}) = \lambda_i$ .

- (1) If  $\mathfrak{i}$  has uniqueness, and  $\kappa := \bar{\kappa}_{<\alpha}(\mathfrak{i}) < \infty$ , then **K** is  $(< \kappa)$ -tame for types of length less than  $\alpha$ .
- (2) If  $\mathfrak{i}$  has uniqueness and  $\kappa := \bar{\kappa}_{<\alpha}(\mathfrak{i}) < \infty$ , then **K** is  $(<\alpha)$ -stable in any infinite  $\mu$  such that  $\mu = \mu^{<\kappa}$ .

(3) If i has uniqueness,  $\mu > LS(\mathbf{K})$ , **K** is  $(< \alpha)$ -stable in unboundedly many  $\mu_0 < \mu$ , and cf  $\mu \ge \kappa_{<\alpha}(\mathfrak{i})$ , then **K** is  $(<\alpha)$ -stable in  $\mu$ .

Proof.

- (1) See [**GK**, p. 15], or the proof of [**Bon14a**, Theorem 3.2].
- (2) Let  $\mu = \mu^{<\kappa}$  be infinite. Let  $M \in \mathbf{K}_{\leq \mu}, \langle p_i : i < \mu^+ \rangle$  be elements in  $gS^{<\alpha}(M)$ . It is enough to show that for some  $i < j, p_i = p_j$ . For each  $i < \lambda^+$ , there exists  $M_i \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{<\kappa}$  such that  $p_i$  does not fork over  $M_i$ . Since  $\mu = \mu^{<\kappa}$ , we can assume without loss of generality that  $M_i = M_0$  for all  $i < \mu^+$ . Also,  $|gS^{<\alpha}(M_0)| \le 2^{<\kappa} \le \mu^{<\kappa} = \mu$  so there exists  $i < j < \lambda^+$ such that  $p_i \upharpoonright M_0 = p_j \upharpoonright M_0$ . By uniqueness,  $p_i = p_j$ , as needed.
- (3) As in the proof of Lemma 4.5.5.

The following technical result is also used in the last sections. Roughly, it gives conditions under which we can take the base model given by local character to be contained in both the left and right hand side.

LEMMA 6.4.7. Let  $\mathfrak{i} = (\mathbf{K}, \downarrow)$  be a  $(\langle \alpha, \mathcal{F} \rangle)$ -independence relation,  $\mathcal{F} = [\lambda, \theta)$ , with  $\alpha > \lambda$ . Assume:

- (1) **K** is an AEC with  $LS(\mathbf{K}) = \lambda$ .
- (2) i has base monotonicity and transitivity.
- (3)  $\mu$  is a cardinal,  $\lambda \leq \mu < \theta$ .
- (4) i has the left (<  $\kappa$ )-model-witness property for some regular  $\kappa \leq \mu$ .
- (5)  $\bar{\kappa}_{\mu}(i) = \mu^+$ .

Let  $M^0 \leq_{\mathbf{K}} M^\ell \leq_{\mathbf{K}} N$  be in  $\mathbf{K}$ ,  $\ell = 1, 2$  and assume  $M^1 \bigcup_{M^0}^N M^2$ . Let  $A \subseteq |M^1|$ ,

be such that  $|A| \leq \mu$ . Then there exists  $N^1 \leq_{\mathbf{K}} M^1$  and  $N^0 \leq_{\mathbf{K}} M^0$  such that:

- (1)  $A \subseteq |N^1|, A \cap |M^0| \subseteq |N^0|.$ (2)  $N^0 \leq_{\mathbf{K}} N^1$  are in  $\mathbf{K}_{\leq \mu}.$ (3)  $N^1 \bigcup_{N^0} M^2.$

**PROOF.** For  $\ell = 0, 1$ , we build  $\langle N_i^{\ell} : i \leq \kappa \rangle$  increasing continuous in  $\mathbf{K}_{<\mu}$  such that for all  $i < \kappa$  and  $\ell = 0, 1$ :

 $\begin{array}{ll} (1) & A \subseteq |N_0^1|, \ A \cap |M^0| \subseteq |N_0^0|. \\ (2) & N_i^{\ell} \leq_{\mathbf{K}} M^{\ell}. \\ (3) & N_i^0 \leq_{\mathbf{K}} N_i^1. \\ (4) & N_i^1 \bigcup_{\substack{N \\ i+1}}^N M^2. \end{array}$ 

This is possible. Pick any  $N_0^0 \leq_{\mathbf{K}} M^0$  in  $\mathbf{K}_{\leq \mu}$  containing  $A \cap |M^0|$ . Now fix  $i < \overline{\kappa}$  and assume inductively that  $\langle N_j^0 : j \leq i \rangle$ ,  $\langle N_j^1 : j < i \rangle$  have been built. If i is a limit, we take unions. Otherwise, pick any  $N_i^1 \leq_{\mathbf{K}} M^1$  in  $\mathbf{K}_{\leq \mu}$  that contains  $A, N_j^1$  for all j < i and  $N_i^0$ . Now use right transitivity and  $\overline{\kappa}_{\mu}(\mathfrak{i}) = \mu^+$  to find  $N_{i+1}^0 \leq_{\mathbf{K}} M^0$  such that  $N_i^1 \bigcup_{N_{i+1}^0}^N M^2$ . By base monotonicity, we can assume without loss of generality that  $N_i^0 \leq_{\mathbf{K}} N_{i+1}^0$ 

This is enough. We claim that  $N^{\ell} := N_{\kappa}^{\ell}$  are as required. By coherence,  $N^0 \leq_{\mathbf{K}} N^1$  and since  $\kappa \leq \mu$  they are in  $\mathbf{K}_{\leq \mu}$ . Since  $A \subseteq |N_0^1|$ ,  $A \subseteq |N^1|$ . It remains to see  $N^1 \stackrel{N}{\downarrow} M^2$ . By the left witness property<sup>20</sup>, it is enough to check it for every  $B \subseteq |N^1|$ of size less than  $\kappa$ . Fix such a *B*. Since  $\kappa$  is regular, there exists  $i < \kappa$  such that  $B \subseteq |N_i^1|$ . By assumption and monotonicity,  $B \bigcup_{\substack{N_{i+1}^0\\N_{i+1}^0}}^N M^2$ . By base monotonicity, N 

$$B \bigcup_{N_{\kappa}^{0}} M^{2}$$
, as needed.

With a similar proof, we can clarify the relationship between full model continuity and local character. Essentially, the next lemma says that local character for types up to a certain length plus full model-continuity implies local character for all lengths. It will be used in Section 6.14.

LEMMA 6.4.8. Let  $\mathfrak{i} = (\mathbf{K}, \downarrow)$  be a  $(< \theta_{\mathfrak{i}}, \mathcal{F})$ -independence relation,  $\mathcal{F} = [\lambda, \theta)$ . Assume:

- (1) **K** is an AEC with  $LS(\mathbf{K}) = \lambda$ .
- (2) i has base monotonicity, transitivity, and full model continuity.
- (3) i has the left ( $< \kappa$ )-model-witness property for some regular  $\kappa \leq \lambda$ .
- (4) For all cardinals  $\mu \leq \lambda$ ,  $\bar{\kappa}_{\mu}(\mathfrak{i}) = \lambda^+$ .

Then for all cardinals  $\mu < \theta$ ,  $\bar{\kappa}_{\mu}(\mathfrak{i}) = \lambda^{+} + \mu^{+}$ .

**PROOF.** By induction on  $\mu$ . If  $\mu \leq \lambda$ , this holds by hypothesis, so assume  $\mu > \lambda$ . Let  $\delta := \operatorname{cf} \mu$ .

$$\begin{split} \mu > \lambda. \text{ Let } \delta &:= \operatorname{ct} \mu. \\ & \text{Let } M^0 \leq_{\mathbf{K}} M^1 \text{ be in } \mathbf{K} \text{ and let } A \subseteq |M^1| \text{ have size } \mu. \text{ We want to find} \\ M &\leq_{\mathbf{K}} M^0 \text{ such that } A \underset{M}{\stackrel{N}{\downarrow}} M^0 \text{ and } ||M|| \leq \mu. \text{ Let } \langle A_i : i \leq \delta \rangle \text{ be increasing} \\ \text{continuous such that } A = A_\delta \text{ and } |A_i| < \mu \text{ for all } i < \delta. \\ & \text{For } \ell = 0, 1, \text{ we build } \langle N_i^{\ell} : i \leq \delta \rangle \text{ increasing continuous such that for all } i < \delta \end{split}$$

and  $\ell = 0, 1$ :

 $\begin{array}{ll} (1) & N_i \in \mathbf{K}_{<\mu}. \\ (2) & A_i \subseteq |N_i^1|, \, A_i \cap |M^0| \subseteq |N_i^0|. \\ (3) & N_i^\ell \leq_{\mathbf{K}} M^\ell. \\ (4) & N_i^0 \leq_{\mathbf{K}} N_i^1. \\ (5) & N_i^1 \bigcup_{\substack{M^1\\ N_{i+1}^0}}^M M^0. \end{array}$ 

<u>This is possible</u>. By (3) and (4), we have  $M^1 \stackrel{M^1}{\underset{M^0}{\downarrow}} M^0$ . Now proceed as in the proof of Lemma 6.4.7.

 $<sup>^{20}</sup>$ Note that we do *not* need to use full model continuity, as we only care about chains of cofinality  $\geq \kappa$ .

<u>This is enough</u>. As in the proof of Lemma 6.4.7, for any  $i < \delta$  of cofinality at least  $\kappa$  we have  $N_i^1 \underset{N_i^0}{\downarrow} M^0$ . Thus by full model continuity (applied to the sequences

$$\langle N_i^{\ell}: i < \delta, \mathrm{cf} \ i \ge \kappa \rangle), \ N_{\delta}^1 \bigcup_{N_{\delta}^0}^{M^1} M^0.$$
 Since  $A = A_{\delta} \subseteq |N_{\delta}^1|, \ M := N_{\delta}^0$  is as needed.  $\Box$ 

#### 6.5. Skeletons

We define what it means for an abstract class  $\mathbf{K}'$  to be a *skeleton* of an abstract class  $\mathbf{K}$ . The main examples are classes of saturated models with the usual ordering (or even universal or limit extension). Except perhaps for Lemma 6.5.7, the results of this section are either easy or well known, we simply put them in the general language of this chapter.

We will use skeletons to generalize various statements of chain local character (for example in [**GVV16**] and Chapter 4) that only ask that if  $\langle M_i : i < \delta \rangle$  is an increasing chain with respect to some restriction of the ordering of **K** (usually being universal over) and the  $M_i$ s are inside some subclass of **K** (usually some class of saturated models), then any  $p \in gS(\bigcup_{i < \delta} M_i)$  does not fork over some  $M_i$ . Lemma 6.6.8, is they key upward transfer of that property. Note that Lemma 6.6.7 shows that one can actually assume that skeletons have a particular form. However the generality is still useful when one wants to prove the local character statement.

DEFINITION 6.5.1. For  $(K, \leq_{\mathbf{K}})$  an abstract class, we say  $\mathbf{K}' = (K', \trianglelefteq)$  is a *sub-AC* of **K** if  $K' \subseteq K$ , **K**' is an AC, and  $M \trianglelefteq N$  implies  $M \leq_{\mathbf{K}} N$ . We similarly define sub-AEC, etc. When  $\trianglelefteq = \leq_{\mathbf{K}} \upharpoonright \mathbf{K}'$ , we omit it (or may abuse notation and write  $(K', \leq_{\mathbf{K}})$ ).

DEFINITION 6.5.2. For  $(K, \leq_{\mathbf{K}})$  an abstract class, we say a set  $S \subseteq K$  is *dense* in  $(K, \leq_{\mathbf{K}})$  if for any  $M \in K$  there exists  $M' \in S$  with  $M \leq_{\mathbf{K}} M'$ .

DEFINITION 6.5.3. An abstract class  $(K', \trianglelefteq)$  is a *skeleton* of  $(K, \leq_{\mathbf{K}})$  if:

- (1)  $(K', \trianglelefteq)$  is a sub-AC of  $(K, \leq_{\mathbf{K}})$ .
- (2) K' is dense in  $(K, \leq_{\mathbf{K}})$ .
- (3) If  $\langle M_i : i < \alpha \rangle$  is a  $\leq_{\mathbf{K}}$ -increasing chain in K' ( $\alpha$  not necessarily limit) and there exists  $N \in K'$  such that  $M_i <_{\mathbf{K}} N$  for all  $i < \alpha$ , then we can choose such an N with  $M_i \triangleleft N$  for all  $i < \alpha$ .

REMARK 6.5.4. The term "skeleton" is inspired from the term "skeletal" in Chapter 4, although there "skeletal" is applied to frames. The intended philosophical meaning is the same:  $\mathbf{K}'$  has enough information about  $\mathbf{K}$  so that for several purposes we can work with  $\mathbf{K}'$  rather than  $\mathbf{K}$ .

REMARK 6.5.5. Let  $(K, \leq_{\mathbf{K}})$  be an abstract class. Assume  $(K', \trianglelefteq)$  is a dense sub-AC of  $(K, \leq_{\mathbf{K}})$  with no maximal models satisfying in addition: If  $M_0 \leq_{\mathbf{K}}$  $M_1 \triangleleft M_2$  are in  $\mathbf{K}'$ , then  $M_0 \triangleleft M_2$ . Then  $(K', \trianglelefteq)$  is a skeleton of  $(K, \leq_{\mathbf{K}})$ . This property of the ordering already appears in the definition of an abstract universal ordering in Definition 4.2.11. In the terminology there, if  $(K, \leq_{\mathbf{K}})$  is an AEC and  $\triangleleft$ is an (invariant) universal ordering on  $\mathbf{K}_{\lambda}$ , then  $(K_{\lambda}, \trianglelefteq)$  is a skeleton of  $(K_{\lambda}, \leq_{\mathbf{K}})$ .

EXAMPLE 6.5.6. Let **K** be an AEC. Let  $\lambda \geq LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{\lambda}$  has amalgamation, no maximal models and is stable in  $\lambda$ . Let  $\mathbf{K}'$  be dense in  $\mathbf{K}_{\lambda}$  and

let  $\delta < \lambda^+$ . Then  $(K', \leq_{\mathbf{K}}^{\lambda, \delta})$  (recall Definition 6.2.10) is a skeleton of  $(K_{\lambda}, \leq_{\mathbf{K}})$  (use Fact 6.2.13 and Remark 6.5.5).

The next lemma is a useful tool to find extensions in the skeleton of an AEC with amalgamation:

LEMMA 6.5.7. Let  $(K', \trianglelefteq)$  be a skeleton of  $(K, \leq_{\mathbf{K}})$ . Assume **K** is an AEC in  $\mathcal{F} := [\lambda, \theta)$  with amalgamation. If  $M \leq_{\mathbf{K}} N$  are in **K**', then there exists  $N' \in \mathbf{K}'$  such that  $M \trianglelefteq N'$  and  $N \trianglelefteq N'$ .

PROOF. If N is not maximal (with respect to either of the orderings, it does not matter by definition of a skeleton), then using the definition of a skeleton with  $\alpha = 2$  and the chain  $\langle M, N \rangle$ , we can find  $N' \in \mathbf{K}'$  such that  $N \triangleleft N'$  and  $M \triangleleft N'$ , as needed.

Now assume N is maximal. We claim that  $M \leq N$ , so N' := N is as desired. Suppose not. Let  $\mu := \|N\|$ .

We build  $\langle M_i : i < \mu^+ \rangle$  and  $\langle f_i : M_i \xrightarrow{M} N : i < \mu^+ \rangle$  such that:

(1)  $\langle M_i : i < \mu^+ \rangle$  is a strictly increasing chain in  $(K', \trianglelefteq)$  with  $M_0 = M$ .

(2)  $\langle f_i : i < \mu^+ \rangle$  is a strictly increasing chain of **K**-embeddings.

This is enough. Let  $B_{\mu^+} := \bigcup_{i < \mu^+} |M_i|$  and  $f_{\mu^+} := \bigcup_{i < \mu^+} f_i$  (Note that it could be that  $\mu^+ = \theta$ , so  $B_{\mu^+}$  is just a set and we do not claim that  $f_{\mu^+}$  is a **K**-embedding). Then  $f_{\mu^+}$  is an injection from  $B_{\mu^+}$  into |N|. This is impossible because  $|B_{\mu^+}| \ge \mu^+ > \mu = ||N||$ .

This is possible. Set  $M_0 := M$ ,  $f_0 := \operatorname{id}_M$ . If  $i < \mu^+$  is limit, let  $M'_i := \bigcup_{j < i} \overline{M_j} \in \mathbf{K}$ . By density, find  $M''_i \in \mathbf{K}'$  such that  $M'_i \leq_{\mathbf{K}} M''_i$ . We have that  $M_j <_{\mathbf{K}} M''_i$  for all j < i. By definition of a skeleton, this means we can find  $M_i \in \mathbf{K}'$  with  $M_j \triangleleft M_i$  for all j < i. Let  $f'_i := \bigcup_{j < i} f_j$ . Using amalgamation and the fact that N is maximal, we can extend it to  $f_i : M_i \xrightarrow{M} N$ . If i = j + 1 is successor, we consider two cases:

- If  $M_j$  is not maximal, let  $M_i \in \mathbf{K}'$  be a  $\triangleleft$ -extension of  $M_j$ . Using amalgamation and the fact N is maximal, pick  $f_i : M_i \xrightarrow{M} N$  an extension of  $f_j$ .
- If  $M_j$  is maximal, then by amalgamation and the fact both N and  $M_j$  are maximal, we must have  $N \cong_M M_j$ . However by assumption  $M_0 \trianglelefteq M_j$  so  $M = M_0 \trianglelefteq N$ , a contradiction.

Thus we get that several properties of a class transfer to its skeletons.

PROPOSITION 6.5.8. Let  $(K, \leq_{\mathbf{K}})$  be an AEC in  $\mathcal{F}$  and let  $(K', \trianglelefteq)$  be a skeleton of  $\mathbf{K}$ .

- (1)  $(K, \leq_{\mathbf{K}})$  has no maximal models if and only if  $(K', \leq)$  has no maximal models.
- (2) If  $(K, \leq_{\mathbf{K}})$  has amalgamation, then:
  - (a)  $(K', \trianglelefteq)$  has amalgamation.
  - (b)  $(K, \leq_{\mathbf{K}})$  has joint embedding if and only if  $(K', \leq)$  has joint embedding.
  - (c) Galois types are the same in  $(K, \leq_{\mathbf{K}})$  and  $(K', \trianglelefteq)$ : For any  $N \in \mathbf{K}'$ ,  $A \subseteq |N|, \bar{b}, \bar{c} \in {}^{\alpha}|N|, \operatorname{gtp}_{\mathbf{K}}(\bar{b}/A; N) = \operatorname{gtp}_{\mathbf{K}}(\bar{c}/A; N)$  if and only if

 $\operatorname{gtp}_{\mathbf{K}'}(\overline{b}/A; N) = \operatorname{gtp}_{\mathbf{K}'}(\overline{c}/A; N)$ . Here, by  $\operatorname{gtp}_{\mathbf{K}}$  we denote the Galois type computed in  $(K, \leq_{\mathbf{K}})$  and by  $\operatorname{gtp}_{\mathbf{K}'}$  the Galois type computed in  $(K', \leq)$ .

(d)  $(K, \leq_{\mathbf{K}})$  is  $\alpha$ -stable in  $\lambda$  if and only if  $(K', \trianglelefteq)$  is  $\alpha$ -stable in  $\lambda$ .

## Proof.

(1) Directly from the definition.

- (2) (a) Let  $M_0 \leq M_\ell$  be in  $\mathbf{K}'$ ,  $\ell = 1, 2$ . By density, find  $N \in \mathbf{K}'$  and  $f_\ell : M_\ell \xrightarrow{M_0} N$  **K**-embeddings. By Lemma 6.5.7, there exists  $N_1 \in \mathbf{K}'$  such that  $N \leq N_1$ ,  $f_1[M_1] \leq N_1$ . By Lemma 6.5.7 again, there exists  $N_2 \in \mathbf{K}'$  such that  $N_1 \leq N_2$ ,  $f_2[M_2] \leq N_2$ . Thus we also have  $f_1[M_1] \leq N_2$ . It follows that  $f_\ell : M_\ell \xrightarrow{M} N_2$  is a  $\leq$ -embedding.
  - (b) If  $(K', \trianglelefteq)$  has joint embedding, then by density  $(K, \leq_{\mathbf{K}})$  has joint embedding. The converse is similar to the proof of amalgamation above.
  - (c) Note that by density any Galois type (in **K**) is realized in an element of **K'**. Since  $(K', \trianglelefteq)$  is a sub-AC of  $(K, \leq_{\mathbf{K}})$ , equality of the types in **K'** implies equality in **K** (this doesn't use amalgamation). Conversely, assume  $\operatorname{gtp}_{\mathbf{K}}(\overline{b}/A; N) = \operatorname{gtp}_{\mathbf{K}}(\overline{c}/A; N)$ . Fix  $N' \geq_{\mathbf{K}} N$  in **K** and a **K**-embedding  $f : N \xrightarrow{A} N'$  such that  $f(\overline{b}) = \overline{c}$ . By density, we can assume without loss of generality that  $N' \in \mathbf{K'}$ . By Lemma 6.5.7, find  $N'' \in \mathbf{K'}$  such that  $N \trianglelefteq N'', N' \trianglelefteq N''$ . By Lemma 6.5.7 again, find  $N''' \in \mathbf{K'}$  such that  $f[N] \trianglelefteq N''', N'' \trianglelefteq N'''$ . By transitivity,  $N \trianglelefteq N'''$  and  $f : N \xrightarrow{A} N'''$  witnesses equality of the Galois types in  $(K', \trianglelefteq)$ .
  - (d) Because Galois types are the same in  $\mathbf{K}$  and  $\mathbf{K}'$ .

We end with an observation concerning universal extensions that will be used in the proof of Lemma 6.6.7.

LEMMA 6.5.9. Let **K** be an AEC in  $\lambda := \text{LS}(\mathbf{K})$ . Assume **K** has amalgamation, no maximal models, and is stable in  $\lambda$ . Let  $(K', \trianglelefteq)$  be a skeleton of **K**. For any  $M \in \mathbf{K}'$ , there exists  $N \in \mathbf{K}'$  such that both  $M \triangleleft N$  and  $M <_{\mathbf{K}}^{\text{univ}} N$ . Thus  $(K', \trianglelefteq \cap \leq_{\mathbf{K}}^{\text{univ}})$  is a skeleton of **K**.

PROOF. For the last sentence, let  $\trianglelefteq':=\trianglelefteq \cap \leq_{\mathbf{K}}^{\mathrm{univ}}$ . Note that if  $\langle M_i : i < \alpha \rangle$  is a  $\trianglelefteq'$ -increasing chain in  $\mathbf{K}'$  and  $M \in \mathbf{K}'$  is such that  $M_i <_{\mathbf{K}} M$  for all  $i < \alpha$ , then by definition of a skeleton we can take M so that  $M_i \triangleleft M$  for all  $i < \alpha$ . If we know that there exists  $N \in \mathbf{K}'$  with  $M \triangleleft N$  and  $M <_{\mathbf{K}}^{\mathrm{univ}} N$ , then for all  $i < \alpha$ ,  $M_i \triangleleft N$ by transitivity, and  $M_i <_{\mathbf{K}}^{\mathrm{univ}} N$  by Lemma 6.2.16.

Now let  $M \in \mathbf{K}'$ . By Fact 6.2.13, there exists  $N \in \mathbf{K}$  with  $M <_{\mathbf{K}}^{\mathrm{univ}} N$ . By density (note that if  $N' \geq_{\mathbf{K}} N$  is in  $\mathbf{K}$ , then  $M <_{\mathbf{K}}^{\mathrm{univ}} N'$ ) we can take  $N \in \mathbf{K}'$ . By Lemma 6.5.7, there exists  $N' \in \mathbf{K}'$  such that  $M \leq N'$  and  $N \leq N'$ . Thus (using Fact 6.2.13 again)  $M <_{\mathbf{K}}^{\mathrm{univ}} N'$ , as desired.

## 6.6. Generating an independence relation

In [She09a, Section II.2], Shelah showed how to extend a good  $\lambda$ -frame to a pre-( $\geq \lambda$ )-frame. Later, [Bon14a] (with improvements in Chapter 5) gave conditions under which all the properties transferred. Similar ideas are used in Chapter 4 to directly build a good frame. In this section we adapt Shelah's definition to this chapter's more general setup. It is useful to think of the initial  $\lambda$ -frame as a generator<sup>21</sup> for a ( $\geq \lambda$ )-frame, since in case the frame is not good we usually can only get a nice independence relation on  $\lambda^+$ -saturated models (and thus cannot really extend the good  $\lambda$ -frame to a good ( $\geq \lambda$ )-frame). Moreover, it is often useful to work with the independence relation being only defined on a dense sub-AC of the original AEC.

DEFINITION 6.6.1. (**K**,  $\mathfrak{i}$ ) is a  $\lambda$ -generator for a (<  $\alpha$ )-independence relation if:

- (1)  $\alpha$  is a cardinal with  $2 \leq \alpha \leq \lambda^+$ .  $\lambda$  is an infinite cardinal.
- (2) **K** is an AEC in  $\lambda = LS(\mathbf{K})$
- (3) i is a  $(< \alpha, \lambda)$ -independence relation.
- (4)  $(K_i, \leq_{\mathbf{K}})$  is a dense sub-AC (recall Definitions 6.5.1, 6.5.2) of<sup>22</sup>  $(K, \leq_{\mathbf{K}})$ .
- (5)  $\mathbf{K}^{up}$  (recall Definition 6.2.3) has amalgamation.

REMARK 6.6.2. We could similarly define a  $\lambda$ -generator for a ( $< \alpha$ )-independence relation below  $\theta$ , where we require  $\theta \ge \lambda^{++}$  and only  $\mathbf{K}_{\mathcal{F}}^{\mathrm{up}}$  has amalgamation (so when  $\theta = \infty$  we recover the above definition). We will not adopt this approach as we have no use for the extra generality and do not want to complicate the notation further. We could also have required less than "**K** is an AEC in  $\lambda$ " but again we have no use for it.

DEFINITION 6.6.3. Let  $(\mathbf{K}, \mathbf{i})$  be a  $\lambda$ -generator for a  $(< \alpha)$ -independence relation. Define  $(\mathbf{K}, \mathbf{i})^{\mathrm{up}} := (\mathbf{K}^{\mathrm{up}}, {\mathbf{j}}^{\mathrm{up}})$  by  ${\mathbf{j}}^{\mathrm{up}}(M, A, B, N)$  if and only if  $M \leq_{\mathbf{K}} N$  are in  $\mathbf{K}^{\mathrm{up}}$  and there exists  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{\mathbf{i}}$  such that for all  $B_0 \subseteq B$  with  $|B_0| \leq \lambda$  and all  $N_0 \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{\mathbf{i}}$  with  $A \cup B_0 \subseteq |N_0|, M_0 \leq_{\mathbf{K}} N_0$ , we have  ${\mathbf{j}}(M_0, A, B_0, N_0)$ .

When  $\mathbf{K} = \mathbf{K}_{i}$ , we write  $i^{up}$  for  $(\mathbf{K}, i)^{up}$ .

REMARK 6.6.4. In general, we do not claim that  $(\mathbf{K}, \mathbf{i})^{\mathrm{up}}$  is even an independence relation (the problem is that given  $A \subseteq |N|$  with  $N \in \mathbf{K}^{\mathrm{up}}$  and  $|A| \leq_{\mathbf{K}} \lambda$ , there might not be any  $M \in \mathbf{K}_{\mathbf{i}}$  with  $M \leq_{\mathbf{K}} N$  and  $A \subseteq |M|$  so the monotonicity properties can fail). Nevertheless, we will abuse notation and use the restriction operations on it.

LEMMA 6.6.5. Let  $(\mathbf{K}, \mathfrak{i})$  be a  $\lambda$ -generator for a  $(< \alpha)$ -independence relation. Then:

- (1) If  $\mathbf{K} = \mathbf{K}_{i}$ , then  $i^{up} := (\mathbf{K}, i)^{up}$  is an independence relation.
- (2)  $(\mathbf{K}, \mathfrak{i})^{\mathrm{up}} \upharpoonright (\mathbf{K}^{\mathrm{up}})^{\lambda^+-\mathrm{sat}}$  is an independence relation.

PROOF. As in [She09a, Claim II.2.11], using density and homogeneity in the second case.  $\hfill \square$ 

The case (1) of Lemma 6.6.5 has been well studied (at least when  $\alpha = 2$ ): see [She09a, Section II.2] and [Bon14a], Chapter 5. We will further look at it in the last sections. We will focus on case (2) for now. It has been studied (implicitly) in

 $<sup>^{21}</sup>$ In Chapter 4, we called a generator a skeletal frame (and in earlier version a poor man's frame) but never defined it precisely.

<sup>&</sup>lt;sup>22</sup>Why not be more general and require only  $(K_i, \leq)$  to be a skeleton of **K** (for some ordering  $\leq$ )? Because some examples of skeletons do not satisfy the coherence axiom which is required by the definition of an independence relation.

Chapter 4 when i is nonsplitting and satisfies some superstability-like assumptions. We will use the same arguments as there to obtain more general results. The generality will be used, since for example we also care about what happens when i is coheir.

The following property of a generator will be very useful in the next section. The point is that  $\bigcup_{i<\lambda^+} M_i$  below is usually not a member of  $\mathbf{K}_i$  so forking is not defined on it.

DEFINITION 6.6.6. Let  $(\mathbf{K}, \mathfrak{i})$  be a  $\lambda$ -generator for a  $(< \alpha)$ -independence relation.

 $(\mathbf{K}, \mathfrak{i})$  has weak chain local character if there exists  $\trianglelefteq$  such that  $(K_{\mathfrak{i}}, \trianglelefteq)$  is a skeleton of  $\mathbf{K}$  and whenever  $\langle M_i : i < \lambda^+ \rangle$  is  $\trianglelefteq$ -increasing in  $\mathbf{K}_{\mathfrak{i}}$  and  $p \in$  $\mathrm{gS}^{<\alpha}(\bigcup_{i < \lambda^+} M_i)$ , there exists  $i < \lambda^+$  such that  $p \upharpoonright M_{i+1}$  does not fork over  $M_i$ .

The following technical lemma shows that local character in a skeleton implies local character in a bigger class with the universal ordering:

LEMMA 6.6.7. Let (**K**, i) be a  $\lambda$ -generator for a ( $< \alpha$ )-independence relation.

Assume that **K** has amalgamation, no maximal models, and is stable in  $\lambda$ . Assume  $\mathfrak{i}$  has base monotonicity. Let  $(K', \trianglelefteq)$  be a skeleton of  $(K_{\mathfrak{i}}, \leq_{\mathbf{K}})$  and let  $\mathfrak{i}' := \mathfrak{i} \upharpoonright (K', \leq_{\mathbf{K}})$ . Then:

- (1)  $\kappa_{<\alpha}(\mathfrak{i},\leq_{\mathbf{K}}^{\mathrm{univ}}) \leq \kappa_{<\alpha}(\mathfrak{i}',\trianglelefteq).$
- (2) If  $(\mathbf{K}, \mathfrak{i}')$  has weak chain local character, then  $(\mathbf{K}, \mathfrak{i})$  has it and it is witnessed by  $<^{\text{univ}}_{\mathbf{K}}$ .

Proof.

(1) By Lemma 6.5.9, we can (replacing  $\leq by \leq \cap \leq_{\mathbf{K}}^{\mathrm{univ}}$ ) assume without loss of generality that  $\leq$  is extended by  $\leq_{\mathbf{K}}^{\mathrm{univ}}$ . Let  $\langle M_i : i < \delta \rangle$  be  $\leq_{\mathbf{K}}^{\mathrm{univ}}$ -increasing in  $\mathbf{K}_i, \, \delta = \mathrm{cf} \, \delta \geq \kappa_{<\alpha}(\mathbf{i}', \leq), \, \delta < \lambda^+$ . Without loss of generality,  $\langle M_i : i < \delta \rangle$  is  $<_{\mathbf{K}}^{\mathrm{univ}}$ -increasing. Let  $M_\delta := \bigcup_{i < \delta} M_i$  and let  $p \in \mathrm{gS}^{<\alpha}(M_\delta)$ .

By density, pick  $M'_0 \in \mathbf{K}'$  such that  $M_0 <_{\mathbf{K}}^{\mathrm{univ}} M'_0$ . Now build  $\langle M'_i : i < \delta \rangle \triangleleft$ -increasing in  $\mathbf{K}'$ . Let  $M'_{\delta} := \bigcup_{i < \delta} M'_i$ . By Fact 6.2.14, there exists  $f : M'_{\delta} \cong_{M_0} M_{\delta}$  such that for every  $i < \delta$  there exists  $j < \delta$  with  $f[M'_i] \leq_{\mathbf{K}} M_j, f^{-1}[M_i] \leq_{\mathbf{K}} M'_j$ . By definition of  $\kappa_{<\alpha}(\mathbf{i}', \trianglelefteq)$ , there exists  $i < \delta$  such that  $f^{-1}(p)$  does not  $\mathbf{i}'$ -fork over  $M'_i$ . Let  $j < \delta$  be such that  $f[M'_i] \leq_{\mathbf{K}} M_j$ . By invariance, p does not  $\mathbf{i}'$ -fork over  $f[M_i]$ , so does not  $\mathbf{i}$ -fork over  $f[M_i]$ . By base monotonicity, p does not  $\mathbf{i}$ -fork over  $M_j$ , as desired.

(2) Similar.

The last lemma of this section investigates what properties directly transfer up.

LEMMA 6.6.8. Let  $(\mathbf{K}, \mathfrak{i})$  be a  $\lambda$ -generator for a  $(< \alpha)$ -independence relation. Let  $\mathfrak{i}' := (\mathbf{K}, \mathfrak{i})^{\mathrm{up}} \upharpoonright (\mathbf{K}^{\mathrm{up}})^{\lambda^+-\mathrm{sat}}$ .

- (1) If i has base monotonicity, then i' has base monotonicity.
- (2) Assume  $\mathfrak i$  has base monotonicity and  $(\mathbf K,\mathfrak i)$  has weak chain local character. Then:
  - (a)  $\bar{\kappa}_{<\alpha}(\mathfrak{i}') = \lambda^{++}$ .

(b) If  $\leq$  is an ordering such that  $(K_i, \leq)$  is a skeleton of **K**, then for any  $\alpha_0 < \alpha, \ \kappa_{\alpha_0}(\mathbf{i}') \leq \kappa_{\alpha_0}(\mathbf{i}, \leq)$ .

## Proof.

- (1) As in [She09a, Claim II.2.11.(3)]
- (2) This is a generalization of the proof of Lemma 4.4.11 (itself a variation on [She09a, Claim II.2.11.(5)]) but we have to say slightly more so we give the details. Let  $\leq^0$  be an ordering witnessing weak chain local character. We first prove (2b). Fix  $\alpha_0 < \alpha$ , and assume  $\kappa_{\alpha_0}(\mathfrak{i}, \leq) < \infty$ . Then by definition  $\kappa_{\alpha_0}(\mathfrak{i}, \leq) \leq \lambda$ . Let  $\delta = \operatorname{cf} \delta \geq \kappa_{\alpha_0}(\mathfrak{i}, \leq)$ .

Let  $\langle M_i : i < \delta \rangle$  be increasing in  $\mathbf{K}^{\lambda^+\text{-sat}}$  and write  $M_{\delta} := \bigcup_{i < \delta} M_i$ (note that we do not claim  $M_{\delta} \in \mathbf{K}^{\lambda^+\text{-sat}}$ . However,  $M_{\delta} \in \mathbf{K}_{\geq \lambda}$ ). Let  $p \in \mathrm{gS}^{\alpha_0}(M_{\delta})$ . We want to find  $i < \delta$  such that p does not fork over  $M_i$ . There are two cases:

• Case 1:  $\delta < \lambda^+$ :

We imitate the proof of [She09a, Claim II.2.11.(5)]. Assume the conclusion fails. Build  $\langle N_i : i < \delta \rangle \leq$ -increasing in  $\mathbf{K}_i, \langle N'_i : i < \delta \rangle \leq_{\mathbf{K}}$ -increasing in  $\mathbf{K}_i$  such that for all  $i < \delta$ :

- (a)  $N_i \leq_{\mathbf{K}} M_i$ .
- (b)  $N_i \leq_{\mathbf{K}} N'_i \leq_{\mathbf{K}} M_{\delta}$ .
- (c)  $p \upharpoonright N'_i$  i-forks over  $N_i$ .
- (d)  $\bigcup_{j < i} (|N'_j| \cap |M_j|) \subseteq |N_i|.$

<u>This is possible</u>. Assume  $N_j$  and  $N'_j$  have been constructed for  $j < \overline{i}$ . Choose  $N_i \leq_{\mathbf{K}} M_i$  satisfying (2d) so that  $N_j \leq N_i$  for all j < i (This is possible: use that  $M_i$  is  $\lambda^+$ -saturated and that in skeletons of AECs, chains have upper bounds). By assumption, p i'-forks over  $M_i$ , and so by definition of forking there exists  $N'_i \leq_{\mathbf{K}} M_\delta$  in  $\mathbf{K}_i$  such that  $p \upharpoonright N'_i$  forks over  $N_i$ . By monotonicity, we can of course assume  $N'_i \geq_{\mathbf{K}} N_i, N'_i \geq_{\mathbf{K}} N'_i$  for all j < i.

This is enough. Let  $N_{\delta} := \bigcup_{i < \delta} N_i$ ,  $N'_{\delta} := \bigcup_{i < \delta} N'_i$ . By local character for i, there is  $i < \delta$  such that  $p \upharpoonright N_{\delta}$  does not fork over  $N_i$ . By (2b) and (2d),  $N'_{\delta} \leq_{\mathbf{K}} N_{\delta}$ . Thus by monotonicity  $p \upharpoonright N'_i$  does not i-fork over  $N_i$ , contradicting (2c).

• <u>Case 2:</u>  $\delta \geq \lambda^+$ : Assume the conclusion fails. As in the previous case (in fact it is easier), we can build  $\langle N_i : i < \lambda^+ \rangle \leq^0$ -increasing in  $\mathbf{K}_i$  such that  $N_i \leq_{\mathbf{K}} M_\delta$  and  $p \upharpoonright N_{i+1}$  i-forks over  $N_i$ . Since i has weak chain local character, there exists  $i < \lambda^+$  such that  $p \upharpoonright N_{i+1}$  does not i-fork over  $N_i$ , contradiction.

For (2a), assume not: then there exists  $M \in \mathbf{K}^{\lambda^+\text{-sat}}$  and  $p \in \mathrm{gS}^{<\alpha}(M)$ such that for all  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}$ , p i'-forks over  $M_0$ . By stability, for any  $A \subseteq |M|$  with  $|A| \leq \lambda$ , there exists  $M_0 \leq_{\mathbf{K}} M$  containing A which is  $\lambda^+$ -saturated of size  $\lambda^+$ . As in case 2 above, we build  $\langle N_i : i < \lambda^+ \rangle$  $\leq^0$ -increasing in  $\mathbf{K}_i$  such that  $N_i \leq_{\mathbf{K}} M$  and  $p \upharpoonright N_{i+1}$  i-fork over  $N_i$ . This is possible (for the successor step, given  $N_i$ , take any  $M_0 \leq_{\mathbf{K}} M$  saturated of size  $\lambda^+$  containing  $N_i$ . By definition of i' and the fact p i'-forks over  $M_0$ , there exists  $N'_{i+1} \leq_{\mathbf{K}} M$  in  $\mathbf{K}_i$  witnessing the forking. This can further extended to  $N_{i+1}$  which is as desired). This is enough: we get a contradiction to weak chain local character.

#### 6.7. Weakly good independence relations

Interestingly, nonsplitting and (<  $\kappa$ )-coheir (for a suitable choice of  $\kappa$ ) are already well-behaved if the AEC is stable. This raises the question of whether there is an object playing the role of a good frame (see the next section) in AECs that are stable but not superstable (whatever the exact meaning of superstability should be in this context, see Section 6.10). Note that Chapter 3 proves the canonicity of independence relations that satisfy much less than the full properties of good frames, so it is reasonable to expect existence of such an object. The next definition comes from extracting all the properties we are able to prove from the construction of a good frame in Chapter 4 assuming only stability.

DEFINITION 6.7.1. Let  $\mathbf{i} = (\mathbf{K}, \downarrow)$  be a  $(< \alpha, \mathcal{F})$ -independence relation,  $\mathcal{F} = [\lambda, \theta)$ .  $\mathbf{i}$  is weakly good<sup>23</sup> if:

- (1) **K** is nonempty, is  $\lambda$ -closed (Recall Definition 6.2.5), and every chain in **K** of ordinal length less than  $\theta$  has an upper bound.
- (2) **K** is stable in  $\lambda$ .
- (3) i has base monotonicity, disjointness, existence, and transitivity.
- (4)  $\operatorname{pre}(\mathfrak{i})$  has uniqueness.
- (5) i has the left  $\lambda$ -witness property and the right  $\lambda$ -model-witness property.
- (6) Local character: For all  $\alpha_0 < \min(\lambda^+, \alpha), \ \bar{\kappa}_{\alpha_0}(\mathfrak{i}) = \lambda^+$ .
- (7) Local extension and uniqueness:  $i_{\lambda}^{<\lambda^+}$  has extension and uniqueness.

We say a pre-( $< \alpha, \mathcal{F}$ )-frame  $\mathfrak{s}$  is *weakly good* if  $cl(\mathfrak{s})$  is weakly good.  $\mathfrak{i}$  is *pre-weakly good* if  $pre(\mathfrak{i})$  is weakly good.

REMARK 6.7.2. By Propositions 6.4.3.(4), 6.4.3.(6), existence and the right  $\lambda$ -witness property follow from the others.

Our main tool to build weakly good independence relations will be to start from a  $\lambda$ -generator (see Definition 6.6.1) which satisfies some additional properties:

DEFINITION 6.7.3. (**K**,  $\mathfrak{i}$ ) is a  $\lambda$ -generator for a weakly good ( $< \alpha$ )-independence relation if:

- (1) (**K**,  $\mathfrak{i}$ ) is a  $\lambda$ -generator for a ( $< \alpha$ )-independence relation.
- (2) **K** is nonempty, has no maximal models, and is stable in  $\lambda$ .
- (3)  $(\mathbf{K}^{up})^{\lambda^+-\text{sat}}$  is  $\lambda$ -tame for types of length less than  $\alpha$ .
- (4) i has base monotonicity, existence, and is extended by  $\lambda$ -nonsplitting: whenever  $p \in \mathrm{gS}^{<\alpha}(M)$  does not i-fork over  $M_0 \leq_{\mathbf{K}} M$ , then p does not  $\mathfrak{s}_{\lambda-\mathrm{ns}}(\mathbf{K}_i)$ -fork over  $M_0$ .
- (5)  $(\mathbf{K}, \mathbf{i})$  has weak chain local character.

Both coheir and  $\lambda$ -nonsplitting induce a generator for a weakly good independence relation:

 $<sup>^{23}</sup>$ The name "weakly good" is admittedly not very inspired. A better choice may be to rename good independence relations to superstable independence relations and weakly independence relations to stable independence relations. We did not want to change Shelah's terminology here and wanted to make the relationship between "weakly good" and "good" clear.

PROPOSITION 6.7.4. Let **K** be an AEC with amalgamation and let  $\lambda \geq LS(\mathbf{K})$  be such that  $\mathbf{K}_{\lambda}$  is nonempty, has no maximal models, and **K** is stable in  $\lambda$ . Let  $2 \leq \alpha \leq \lambda^+$ .

- (1) Let  $LS(\mathbf{K}) < \kappa \leq \lambda$ . Assume that  $\mathbf{K}$  is  $(< \kappa)$ -tame and short for types of length less than  $\alpha$ . Let  $\mathfrak{i} := (\mathfrak{i}_{\kappa-ch}(\mathbf{K}))^{<\alpha}$ .
  - (a) If **K** does not have the  $(< \kappa)$ -order property of length  $\kappa$ ,  $\kappa_{<\alpha}(\mathfrak{i}) \leq \lambda^+$ , and  $\mathbf{K}_{\lambda}^{\kappa\text{-sat}}$  is dense in  $\mathbf{K}_{\lambda}$ , then  $(\mathbf{K}_{\lambda}, \mathfrak{i}_{\lambda})$  is a  $\lambda$ -generator for a weakly good  $(< \alpha)$ -independence relation.
  - (b) If  $\kappa = \beth_{\kappa}$ ,  $(\alpha_0 + 2)^{<\kappa_r} \le \lambda$  for all  $\alpha_0 < \alpha$ , then  $(\mathbf{K}_{\lambda}, \mathfrak{i}_{\lambda})$  is a  $\lambda$ -generator for a weakly good  $(<\alpha)$ -independence relation.
- (2) Assume  $\alpha \leq \omega$  and  $\mathbf{K}^{\lambda^+\text{-sat}}$  is  $\lambda\text{-tame}$  for types of length less than  $\alpha$ . Then  $(\mathbf{K}_{\lambda}, (\mathbf{i}_{\lambda\text{-ns}}(\mathbf{K}_{\lambda}))^{<\alpha})$  is a  $\lambda\text{-generator}$  for a weakly good (<  $\alpha$ )-independence relation.
- (3) Let  $\mathbf{K}'$  be a dense sub-AC of  $\mathbf{K}$  such that  $\mathbf{K}^{\lambda^+\text{-sat}} \subseteq \mathbf{K}'$  and let  $\mathbf{i}$  be a  $(< \alpha, \geq \lambda)$ -independence relation with  $\mathbf{K}_{\mathbf{i}} = \mathbf{K}'$ , such that  $\operatorname{pre}(\mathbf{i})$  has uniqueness,  $\mathbf{i}$  has base monotonicity, and  $\bar{\kappa}_{<\alpha}(\mathbf{i}) = \lambda^+$ . If  $\mathbf{K}'_{\lambda}$  is dense in  $\mathbf{K}_{\lambda}$ , then  $(\mathbf{K}_{\lambda}, \mathbf{i}_{\lambda})$  is a  $\lambda$ -generator for a weakly good  $(< \alpha)$ -independence relation.

Proof.

- (a) By Fact 6.3.19, i has base monotonicity, existence, and uniqueness. By Fact 6.3.22.(3), coheir is extended by λ-nonsplitting. The other properties are easy. For example, weak chain local character follows from κ<sub><α</sub>(i) ≤ λ<sup>+</sup> and monotonicity.
  - (b) We check that **K** and i satisfy all the conditions of the previous part. By Fact 6.2.9, **K** does not have the  $(< \kappa)$ -order property of length  $\kappa$ . By (the proof of) Proposition 6.4.3.(5) and Fact 6.3.19:

$$\kappa_{<\alpha}(\mathfrak{i}) \leq \bar{\kappa}_{<\alpha}(\mathfrak{i}) \leq \sup_{\alpha_0 < \alpha} \left( (\alpha_0 + 2)^{<\kappa_r} \right)^+ \leq \lambda^+$$

Since **K** is stable in  $\lambda$ , if  $\kappa < \lambda$  then  $\mathbf{K}_{\lambda}^{\kappa\text{-sat}}$  is dense in  $\mathbf{K}_{\lambda}$ . If  $\kappa = \lambda$ , then  $\kappa = 2^{<\kappa_r}$  so is regular, hence strongly inaccessible, so  $\kappa = \kappa^{<\kappa}$  so again it is easy to check that  $\mathbf{K}_{\lambda}^{\kappa\text{-sat}}$  is dense in  $\mathbf{K}_{\lambda}$ .

- (2) Let  $\mathbf{i} := (\mathfrak{s}_{\lambda-\mathrm{ns}}(\mathbf{K}))^{<\alpha}$ . By Fact 6.3.22.(2) and Proposition 6.4.3.(5),  $\kappa_{<\alpha}(\mathbf{i}) = \lambda^+$ . By monotonicity, weak chain local character follows. The other properties are easy to check.
- (3) By Fact 6.3.22.(3), i is extended by  $\lambda$ -nonsplitting. Weak chain character follows from  $\kappa_{<\alpha}(i) = \lambda^+$ . By (the proof of) Proposition 6.4.6,  $\mathbf{K}^{\lambda^+$ -sat} is  $\lambda$ -tame for types of length less than  $\alpha$ . The other properties are easy to check.

The next result is that a generator for a weakly good independence relation indeed induces a weakly good independence relation.

THEOREM 6.7.5. Let  $(\mathbf{K}, \mathfrak{i})$  be a  $\lambda$ -generator for a weakly good  $(< \alpha)$ -independence relation. Then  $(\mathbf{K}, \mathfrak{i})^{\mathrm{up}} \upharpoonright (\mathbf{K}^{\mathrm{up}})^{\lambda^+}$  is a pre-weakly good  $(< \alpha, \geq \lambda^+)$ -independence relation.

PROOF. This follows from the arguments of Chapter 4, but we give some details. Let  $\mathbf{i}' := (\mathbf{K}, \mathbf{i})^{\text{up}} \upharpoonright (\mathbf{K}^{\text{up}})^{\lambda^+ \text{-sat}}$ . Let  $\downarrow := \downarrow_{\mathbf{i}'}, \mathbf{s}' := \text{pre}(\mathbf{i}')$ . We check the conditions in the definition of a weakly good independence relation. Note that by Remark 6.7.2 we do not need to check existence or the right  $\lambda^+$ -witness property.

- $\mathfrak{i}'$  is a  $(< \alpha, \ge \lambda^+)$ -independence relation: By Lemma 6.6.5.
- K<sub>i'</sub> is stable in λ<sup>+</sup>: By Fact 6.2.9, K<sup>up</sup> is stable in λ<sup>+</sup>. By stability, K<sub>i'</sub> is dense in K so by Proposition 6.5.8, K<sub>i'</sub> is stable in λ<sup>+</sup>.
- $\mathbf{K}_{\mathbf{i}'} \neq \emptyset$  since it is dense in  $\mathbf{K}_{\lambda^+}^{\mathrm{up}}$  and  $\mathbf{K}_{\lambda}^{\mathrm{up}} = \mathbf{K}$  is nonempty and has no maximal models. Every chain  $\langle M_i : i < \delta \rangle$  in  $\mathbf{K}_{\mathbf{i}'}$  has an upper bound: we have  $M_{\delta} := \bigcup_{i < \delta} M_i \in \mathbf{K}$ , and by density there exists  $M \geq_{\mathbf{K}} M_{\delta}$  in  $\mathbf{K}_{\mathbf{i}'}$ .  $\mathbf{K}_{\mathbf{i}'}$  is  $\lambda^+$ -closed by an easy increasing chain argument, using stability in  $\lambda^+$ .
- Local character:  $\bar{\kappa}_{\leq \alpha}(\mathfrak{i}') = \lambda^{++}$  by Lemma 6.6.8.
- $\mathfrak{s}'$  has:
  - Base monotonicity: By Lemma 6.6.8.
  - Uniqueness: First observe that using local character, base monotonicity,  $\lambda^+$ -closure, and the fact that  $\mathbf{K}_{i'}$  is  $\lambda^+$ -tame for types of length less than  $\alpha$ , it is enough to show uniqueness for  $(\mathfrak{s}')_{\lambda^+}$ . For this imitate the proof of Lemma 4.5.3 (the key is weak uniqueness: Fact 6.3.22.(5)).
  - Local extension: Let  $p \in \mathrm{gS}^{<\alpha}(M)$ ,  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  be in  $(\mathbf{K}_{i'})_{\lambda^+}$ such that p does not fork over  $M_0$ . Let  $M'_0 \leq_{\mathbf{K}} M_0$  be in  $\mathbf{K}_i$  and witness it. By homogeneity,  $M'_0 <_{\mathbf{K}}^{\mathrm{univ}} M$  so there exists  $f : N \xrightarrow{M'_0} M$ . Let  $q := f^{-1}(p) \upharpoonright N$ . By invariance, q does not fork over  $M_0$  (as

witnessed by  $M'_0$ ). Since  $\lambda$ -nonsplitting extends nonforking,  $p \upharpoonright M'$ does not  $\mathfrak{s}_{\lambda-\mathrm{ns}}(\mathbf{K}_i)$ -fork over  $M'_0$  whenever  $M'_0 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M$  is such that  $M' \in \mathbf{K}_i$ . Let  $\mathbf{K}' := \mathbf{K}_i \cup \mathbf{K}^{\lambda^+-\mathrm{sat}}$ . By (the proof of) Fact 6.3.22.(4), p does not  $\mathfrak{s}_{\mathrm{ns}}(\mathbf{K}')$ -fork over  $M'_0$ . By weak extension (Fact 6.3.22.(5), q extends p and is algebraic if and only if q is.

- Transitivity: Imitate the proof of Lemma 4.4.10.
- Disjointness: It is enough to prove it for types of length 1 so assume  $\alpha = 2$ . Assume  $a \underset{M_0}{\downarrow} M$  (with  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  in  $\mathbf{K}^{\lambda^+ \cdot \mathrm{mh}}$ ) and  $a \in M$ . We show  $a \in M_0$ . Using local character, we can assume without loss of generality that  $||M_0|| = \lambda^+$  and (by taking a submodel of M containing a of size  $\lambda^+$ ) that also  $||M|| = \lambda^+$ . Find  $M'_0 \leq_{\mathbf{K}} M_0$  in  $\mathbf{K}_i$  witnessing the nonforking. By the proof of local extension, we can find  $p \in \mathrm{gS}(M)$  extending  $p_0 := \mathrm{gtp}(a/M_0; N)$  such that  $p_0$  is algebraic if and only if p is. Since  $a \in N$ , we must have by uniqueness that p is algebraic so  $p_0$  is algebraic, i.e.  $a \in M_0$ .
- Now by Proposition 6.4.1,  $cl(\mathfrak{s}')$  has the above properties.
- $cl(\mathfrak{s}')$  has the left  $\lambda$ -witness property: Because  $\alpha \leq \lambda^+$ .

Interestingly, the generator can always be taken to have a particular form:

LEMMA 6.7.6. Let  $(\mathbf{K}, \mathfrak{i})$  be a  $\lambda$ -generator for a weakly good  $(< \alpha)$ -independence relation. Let  $\mathfrak{i}' := \mathfrak{i}_{\lambda-ns}(\mathbf{K})^{<\alpha}$ . Then:

(1)  $(\mathbf{K}, \mathfrak{i}')$  is a  $\lambda$ -generator for a weakly good  $(< \alpha)$ -independence relation and  $<^{\text{univ}}_{\mathbf{K}}$  is the ordering witnessing weak chain local character.

(2) 
$$\operatorname{pre}((\mathbf{K},\mathfrak{i})^{\operatorname{up}}) \upharpoonright (\mathbf{K}^{\operatorname{up}})^{\lambda^{\top}-\operatorname{sat}} = \operatorname{pre}((\mathbf{K},\mathfrak{i}')^{\operatorname{up}}) \upharpoonright (\mathbf{K}^{\operatorname{up}})^{\lambda^{\top}-\operatorname{sat}}.$$

Proof.

- (1) By Lemma 6.6.7 (with  $\mathbf{K}$ ,  $\mathbf{i}'$ ,  $\mathbf{K}_i$  here standing for  $\mathbf{K}$ ,  $\mathbf{i}$ ,  $\mathbf{K}'$  there), ( $\mathbf{K}$ ,  $\mathbf{i}'$ ) has weak chain local character (witnessed by  $<^{\text{univ}}_{\mathbf{K}}$ ) and the other properties are easy to check.
- (2) Let  $\mathfrak{s} := \operatorname{pre}((\mathbf{K}, \mathfrak{i})^{\operatorname{up}}) \upharpoonright (\mathbf{K}^{\operatorname{up}})^{\lambda^+ \operatorname{-sat}}, \mathfrak{s}' := \operatorname{pre}((\mathbf{K}, \mathfrak{i}')^{\operatorname{up}}) \upharpoonright (\mathbf{K}^{\operatorname{up}})^{\lambda^+ \operatorname{-sat}}.$ We want to see that  $\underset{\mathfrak{s}}{\downarrow} = \underset{\mathfrak{s}'}{\downarrow}$ . Since  $\operatorname{pre}(\mathfrak{i})$  is extended by  $\lambda$ -nonsplitting, it is easy to check that  $\underset{\mathfrak{s}}{\downarrow} \subseteq \underset{\mathfrak{s}'}{\downarrow}$ . By the proof of Lemma 3.4.1,  $\underset{\mathfrak{s}_{\lambda^+}}{\downarrow} = \underset{\mathfrak{s}'_{\lambda^+}}{\downarrow}$ . By the right  $\lambda$ -model-witness property,  $\underset{\mathfrak{s}}{\downarrow} = \underset{\mathfrak{s}'}{\downarrow}$ .

$$\square$$

In Theorem 6.7.5,  $\mathbf{i}' := (\mathbf{K}, \mathbf{i})^{\mathrm{up}} \upharpoonright (\mathbf{K}^{\mathrm{up}})^{\lambda^+ \operatorname{-sat}}$  is only pre-weakly good, not necessarily weakly good: in general, only  $\mathbf{i}'' := \operatorname{cl}(\operatorname{pre}(\mathbf{i}'))$  will be weakly good. The following technical lemma shows that  $\mathbf{i}'$  and  $\mathbf{i}''$  agree on slightly more than  $\operatorname{pre}(\mathbf{i}')$ .

LEMMA 6.7.7. Let  $(\mathbf{K}, \mathbf{i})$  be a  $\lambda$ -generator for a weakly good  $(< \alpha)$ -independence relation. Let  $\mathbf{i}' := (\mathbf{K}, \mathbf{i})^{\mathrm{up}} \upharpoonright (\mathbf{K}^{\mathrm{up}})^{\lambda^+-\mathrm{sat}}$  and let  $\mathbf{i}'' := \mathrm{cl}(\mathrm{pre}(\mathbf{i}'))$ . Let  $M \leq_{\mathbf{K}} N$  be in  $\mathbf{K}_{\geq \lambda^+}^{\mathrm{up}}$  with  $M \in \mathbf{K}^{\lambda^+-\mathrm{sat}}$  (but maybe  $N \notin \mathbf{K}^{\lambda^+-\mathrm{sat}}$ ). Assume  $\mathbf{K}^{\mathrm{up}}$  is  $\lambda$ -tame<sup>24</sup> for types of length less than  $\alpha$ . Let  $p \in \mathrm{gS}^{<\alpha}(N)$ .

If  $||N|| = \lambda^+$  or  $\mathfrak{i}''$  has extension, then p does not  $\mathfrak{i}'$ -fork over M if and only if p does not  $\mathfrak{i}''$ -fork over M.

PROOF. Assume p does not i''-fork over M. Then by definition there exists an extension of p to a model in  $\mathbf{K}^{\lambda^+$ -sat} that does not i'-fork over M so by monotonicity p does not i'-fork over M. Assume now that p does not i'-fork over M. Note that the proof of Theorem 6.7.5 (more precisely Lemma 4.5.3 implies that p is the unique type over N that does not i'-fork over M.

Pick  $N' \geq_{\mathbf{K}} N$  in  $\mathbf{K}^{\lambda^+\text{-sat}}$  with ||N'|| = ||N||. We imitate the proof of Lemma 3.4.1. By extension (or local extension if  $||N|| = \lambda^+$ , recall that  $\mathfrak{i}''$  is weakly good, see Theorem 6.7.5), there exists  $q \in \mathrm{gS}^{<\alpha}(N')$  that does not  $\mathfrak{i}''$ -fork over M and extends  $p \upharpoonright M$ . By the above, q does not  $\mathfrak{i}'$ -fork over M. By uniqueness, q extends p, so  $q \upharpoonright N = p$  does not  $\mathfrak{i}''$ -fork over M.

Note that if the independence relation of the generator is coheir, then the weakly good independence relation obtained from it is also coheir. We first prove a slightly more abstract lemma:

LEMMA 6.7.8. Let **K** be an AEC,  $\lambda \geq LS(\mathbf{K})$ . Let **K'** be a dense sub-AC of **K** such that  $\mathbf{K}^{\lambda^+\text{-sat}} \subseteq \mathbf{K}'$  and  $\mathbf{K}'_{\lambda}$  is dense in  $\mathbf{K}_{\lambda}$ . Let **i** be a  $(< \alpha, \geq \lambda)$ -independence relation with base monotonicity and  $\mathbf{K}_i = \mathbf{K}', 2 \leq \alpha \leq \lambda^+$ . Assume that **i** has base monotonicity and the right  $\lambda$ -model-witness property.

<sup>&</sup>lt;sup>24</sup>Note that the definition of a generator for a weakly good independence relation only requires that  $(\mathbf{K}^{up})^{\lambda^+}$ -sat be  $\lambda$ -tame for types of length less than  $\alpha$ .

Assume  $\bar{\kappa}_{<\alpha}(\mathfrak{i}) = \lambda^+$  and  $(\mathbf{K}_{\lambda}, \mathfrak{i}_{\lambda})$  is a  $\lambda$ -generator for a weakly good  $(<\alpha)$ independence relation. Let  $\mathfrak{i}' := (\mathbf{K}_{\lambda}, \mathfrak{i}_{\lambda})^{\mathrm{up}} \upharpoonright \mathbf{K}^{\lambda^+ - \mathrm{sat}}$ . Then  $\mathrm{pre}(\mathfrak{i}') = \mathrm{pre}(\mathfrak{i}) \upharpoonright \mathbf{K}^{\lambda^+ - \mathrm{sat}}$ .

Moreover if  $\mathfrak{i}$  has the right  $\lambda$ -witness property, then  $\mathfrak{i}' = \mathfrak{i} \upharpoonright \mathbf{K}^{\lambda^+-\operatorname{sat}}$ .

PROOF. We prove the moreover part and it will be clear how to change the proof to prove the weaker statement (just replace the use of the witness property by the model-witness property).

Let  $M \leq_{\mathbf{K}} N$  be in  $\mathbf{K}^{\lambda^+\text{-sat}}$ ,  $p \in \mathrm{gS}^{<\alpha}(B; N)$ . We want to show that p does not i-fork over M if and only if there exists  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}'_{\lambda}$  so that for all  $B_0 \subseteq B$  of size  $\leq \lambda$ ,  $p \upharpoonright B_0$  does not i-fork over  $M_0$ . Assume first that p does not i-fork over M. Since  $\bar{\kappa}_{<\alpha}(\mathbf{i}) = \lambda^+$ , there exists  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{\lambda}$  such that p does not i-fork over  $M_0$ . By base monotonicity and homogeneity, we can assume that  $M_0 \in \mathbf{K}'_{\lambda}$ . I particular  $p \upharpoonright B_0$  does not i-fork over  $B_0$  for all  $B \subseteq B$  of size  $\leq \lambda$ .

Conversely, assume p does not i'-fork over M, and let  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}'_{\lambda}$  witness it. Then by the right  $\lambda$ -witness property, p does not i-fork over  $M_0$ , so over M, as desired.

LEMMA 6.7.9. Let **K** be an AEC,  $LS(\mathbf{K}) < \kappa \leq \kappa' \leq \lambda$ . Let  $2 \leq \alpha \leq \lambda^+$ . Let  $\mathfrak{i} := (\mathfrak{i}_{\kappa-ch}(\mathbf{K}))_{>\lambda}^{\leq \alpha} \upharpoonright \mathbf{K}^{\kappa'-sat}$ .

Assume  $\bar{\kappa}_{<\alpha}(\mathfrak{i}) = \lambda^+$  and  $(\mathbf{K}_{\lambda}, \mathfrak{i}_{\lambda})$  is a  $\lambda$ -generator for a weakly good (<  $\alpha$ )independence relation. Let  $\mathfrak{i}' := (\mathbf{K}_{\lambda}, \mathfrak{i}_{\lambda})^{\mathrm{up}} \upharpoonright \mathbf{K}^{\lambda^+}$ -sat. Then  $\mathfrak{i}' = \mathfrak{i} \upharpoonright \mathbf{K}^{\lambda^+}$ -sat.

PROOF. By Lemma 6.7.8 applied with  $\mathbf{K}' = \mathbf{K}^{\kappa'\text{-sat}}$ .

We end this section by showing how to build a weakly good independence relation in any stable fully tame and short AEC (with amalgamation and no maximal models).

THEOREM 6.7.10. Let **K** be a LS(**K**)-tame AEC with amalgamation and no maximal models. Let  $\kappa = \beth_{\kappa} > \text{LS}(\mathbf{K})$ . Assume **K** is stable and  $(< \kappa)$ -tame and short for types of length less than  $\alpha, \alpha \ge 2$ . If  $\mathbf{K}_{\kappa} \neq \emptyset$ , then  $\mathbf{i}_{\kappa\text{-ch}}(\mathbf{K})^{<\alpha} \upharpoonright \mathbf{K}^{(2^{\kappa})^{+}\text{-sat}}$  is a pre-weakly good  $(< \alpha, \ge (2^{\kappa})^{+})^{-}$ .

If  $\mathbf{K}_{\kappa} \neq \emptyset$ , then  $\mathfrak{i}_{\kappa-\mathrm{ch}}(\mathbf{K})^{<\alpha} \upharpoonright \mathbf{K}^{(2^{\kappa})^+-\mathrm{sat}}$  is a pre-weakly good  $(<\alpha, \geq (2^{\kappa})^+)$ -independence relation. Moreover if  $\alpha = \infty$ , then it is weakly good.

PROOF. Let  $\lambda := 2^{\kappa}$ . By Fact 6.3.19,  $\mathbf{i}_{\kappa-ch}(\mathbf{K})^{<\alpha} \upharpoonright \mathbf{K}^{\lambda^+\text{-sat}}$  already has several of the properties of a weakly good independence relation, and in particular has the left  $\lambda$ -witness property so it is enough to check that  $\mathbf{i} := \mathbf{i}_{\kappa-ch}(\mathbf{K})^{<(\min(\alpha,\lambda^+))} \upharpoonright$  $\mathbf{K}^{\lambda^+\text{-sat}}$  is weakly good, so assume now without loss of generality that  $\alpha \leq \lambda^+$ . Note that by Fact 2.5.15,  $\bar{\kappa}_{<\alpha}(\mathbf{i}) \leq (\lambda^{\kappa})^+ = \lambda^+$ . By Lemma 6.7.9 it is enough to check that  $(\mathbf{K}_{\lambda}, \mathbf{i}_{\lambda})$  is a  $\lambda$ -generator for a weakly good ( $< \alpha$ )-independence relation. From Fact 6.2.9, we get that  $\mathbf{K}$  is stable in  $\lambda$ . Finally, note that  $\mathbf{K}_{\lambda} \neq \emptyset$ . Now apply Proposition 6.7.4.

If  $\alpha = \infty$ , then by Fact 2.5.15, *i* has uniqueness. Since *i* is pre-weakly good,  $\operatorname{pre}(i_{\lambda})$  has extension, so by Proposition 6.4.1.(7),  $i_{\lambda}$  also has extension. The other properties of a weakly good independence relation follow from Fact 2.5.15.

### 6.8. Good independence relations

Good frames were introduced by Shelah [She09a, Definition II.2.1] as a "bare bone" definition of superstability in AECs. Here we adapt Shelah's definition to

independence relations. We also define a variation, being *fully* good. This is only relevant when the types are allowed to have length at least  $\lambda$ , and asks for more continuity (like in Chapter 5, but the continuity property asked for is different). This is used to enlarge a good frame in the last sections.

Definition 6.8.1.

- (1) A good (<  $\alpha$ ,  $\mathcal{F}$ )-independence relation  $\mathfrak{i} = (\mathbf{K}, \downarrow)$  is a (<  $\alpha$ ,  $\mathcal{F}$ )-independence relation satisfying:
  - (a) **K** is a nonempty AEC in  $\mathcal{F}$ ,  $LS(\mathbf{K}) = \lambda_i$ , **K** has no maximal models and joint embedding, **K** is stable in all cardinals in  $\mathcal{F}$ .
  - (b) i has base monotonicity, disjointness, symmetry, uniqueness, existence, extension, the left  $\lambda_i$ -witness property, and for all  $\alpha_0 < \alpha$  with  $|\alpha_0|^+ < \theta_i$ ,  $\kappa_{\alpha_0}(i) = |\alpha_0|^+ + \aleph_0$  and  $\bar{\kappa}_{\alpha_0}(i) = |\alpha_0|^+ + \lambda_i^+$ .
- (2) A type-full good (<  $\alpha$ ,  $\mathcal{F}$ )-frame  $\mathfrak{s}$  is a pre-(<  $\alpha$ ,  $\mathcal{F}$ )-frame so that cl( $\mathfrak{s}$ ) is good.
- (3) i is *pre-good* if pre(i) is good.

When we add "fully", we require in addition that the frame/independence relation satisfies full model-continuity.

REMARK 6.8.2. This chapter's definition is equivalent to that of Shelah [She09a, Definition II.2.1] if we remove the requirement there on the existence of a superlimit (as was done in almost all subsequent papers, for example in [JS13]) and assume the frame is type-full (i.e. the basic types are all the nonalgebraic types). For example, the continuity property that Shelah requires follows from  $\kappa_1(\mathfrak{s}) = \aleph_0$  ([She09a, Claim II.2.17.(3)]).

REMARK 6.8.3. If i is a good ( $< \alpha, \mathcal{F}$ )-independence relation (except perhaps for the symmetry axiom) then i is weakly good.

DEFINITION 6.8.4. An AEC **K** is [fully] (<  $\alpha$ ,  $\mathcal{F}$ )-good if there exists a [fully] (<  $\alpha$ ,  $\mathcal{F}$ )-good independence relation i with  $\mathbf{K}_i = \mathbf{K}$ . When  $\alpha = \infty$  and  $\mathcal{F} = [\mathrm{LS}(\mathbf{K}), \infty)$ , we omit them.

As in the previous section, we give conditions for a generator to induce a good independence relation:

DEFINITION 6.8.5. (**K**,  $\mathfrak{i}$ ) is a  $\lambda$ -generator for a good (<  $\alpha$ )-independence relation if:

- (1) (**K**,  $\mathfrak{i}$ ) is a  $\lambda$ -generator for a weakly good ( $< \alpha$ )-independence relation.
- (2)  $\mathbf{K}^{up}$  is  $\lambda$ -tame.
- (3) There exists  $\mu \geq \lambda$  such that  $\mathbf{K}^{up}_{\mu}$  has joint embedding.
- (4) Local character: For all  $\alpha_0 < \min(\alpha, \lambda)$ , there exists an ordering  $\leq$  such that  $(\mathbf{K}_i, \leq)$  is a skeleton of  $\mathbf{K}$  and  $\kappa_{\alpha_0}(\mathfrak{i}, \leq) = |\alpha_0|^+ + \aleph_0$ .

REMARK 6.8.6. If  $(\mathbf{K}, \mathbf{i})$  is a  $\lambda$ -generator for a good  $(< \alpha)$ -independence relation, then it is a  $\lambda$ -generator for a weakly good  $(< \alpha)$ -independence relation. Moreover if  $\alpha < \lambda^+$ , the weak chain local character axiom follows from the local character axiom.

As before, the generator can always be taken to be of a particular form:

LEMMA 6.8.7. Let  $(\mathbf{K}, \mathfrak{i})$  be a  $\lambda$ -generator for a good  $(< \alpha)$ -independence relation. Let  $\mathfrak{i}' := \mathfrak{i}_{\lambda-\mathrm{ns}}(\mathbf{K})^{<\alpha}$ . Then:

(1)  $(\mathbf{K}, \mathfrak{i}')$  is a  $\lambda$ -generator for a good  $(< \alpha)$ -independence relation and  $<_{\mathbf{K}}^{\text{univ}}$  is the ordering witnessing local character.

(2) 
$$\operatorname{pre}((\mathbf{K},\mathfrak{i})^{\operatorname{up}}) \upharpoonright (\mathbf{K}^{\operatorname{up}})^{\lambda^+-\operatorname{sat}} = \operatorname{pre}((\mathbf{K},\mathfrak{i}')^{\operatorname{up}}) \upharpoonright (\mathbf{K}^{\operatorname{up}})^{\lambda^+-\operatorname{sat}}.$$

Proof.

- (1) By Lemma 6.6.7 (with  $\mathbf{K}$ ,  $\mathbf{i}'$ ,  $\mathbf{K}_i$  here standing for  $\mathbf{K}$ ,  $\mathbf{i}$ ,  $\mathbf{K}''$  there), ( $\mathbf{K}$ ,  $\mathbf{i}'$ ) has the local character properties, witnessed by  $<^{\text{univ}}_{\mathbf{K}}$ , and the other properties are easy to check.
- (2) By Lemma 6.7.6.

Unfortunately it is not strictly true that a generator for a good ( $< \alpha$ )-independence relation induces a good independence relation. For one thing, the extension property is problematic when  $\alpha > \omega$  and this in turn creates trouble in the proof of symmetry. Also, we are unable to prove  $\mathbf{K}^{\lambda^+\text{-sat}}$  is an AEC (although we suspect it should be true, see also Fact 6.10.18). For the purpose of stating a clean result, we introduce the following definition:

DEFINITION 6.8.8. i is an almost pre-good ( $< \alpha, \mathcal{F}$ )-independence relation if:

- (1) It is a pre-weakly good ( $< \alpha, \mathcal{F}$ )-independence relation.
- (2) It satisfies all the conditions in the definition of a *pre-good* independence relation except that:
  - (a)  $\mathbf{K}_{i}$  is not required to be an AEC.
  - (b) cl(pre(i)) is not required to have extension or uniqueness, but we still ask that  $pre(i^{<\omega})$  has extension.
  - (c) cl(pre(i)) is not required to have symmetry, but we still require that pre(i<sup><ω</sup>) has full symmetry.
  - (d) We replace the condition on  $\kappa_{\alpha_0}(cl(pre(\mathfrak{i})))$  by:
    - (i)  $\kappa_{<(\min(\alpha,\omega))}(\operatorname{cl}(\operatorname{pre}(\mathfrak{i}))) = \aleph_0.$
    - (ii) For all  $\alpha_0 < \alpha$ ,  $\kappa_{\alpha_0}(\mathfrak{i}) = |\alpha_0|^+ + \aleph_0$ .

THEOREM 6.8.9. Let  $(\mathbf{K}, \mathfrak{i})$  be a  $\lambda$ -generator for a good  $(< \alpha)$ -independence relation. Then:

- (1)  $\mathbf{K}^{up}$  has joint embedding and no maximal models.
- (2)  $\mathbf{K}^{\text{up}}$  is stable in every  $\mu \geq \lambda$ .
- (3)  $\mathbf{i}' := (\mathbf{K}, \mathbf{i})^{\text{up}} \upharpoonright (\mathbf{K}^{\text{up}})^{\lambda^+\text{-sat}}$  is an almost pre-good ( $< \alpha, \ge \lambda^+$ )-independence relation.
- (4) If  $\alpha \leq \omega$  and  $\mu \geq \lambda^+$  is such that  $(\mathbf{K}^{up})^{\mu\text{-sat}}$  is an AEC with Löwenheim-Skolem-Tarski number  $\mu$ , then  $(\mathfrak{i}')^{\leq \alpha} \upharpoonright (\mathbf{K}^{up})^{\mu\text{-sat}}$  is a pre-good  $(< \alpha, \geq \mu)$ -independence relation.

PROOF. Again, this follows from the arguments in Chapter 4, but we give some details. We show by induction on  $\theta \ge \lambda^+$  that  $\mathfrak{s}' := \operatorname{pre}(\mathfrak{i}')_{[\lambda^+,\theta]}$  is a good frame, except perhaps for symmetry and the conditions in Definition 6.8.8. This gives (3) (use Proposition 6.4.3.(7) to get symmetry, the proof of Lemma 4.5.9 to get extension for types of finite length, and Lemma 6.7.7 to get (2(d)i) in Definition 6.8.8), and (4) together with (1),(2) (use Proposition 6.5.8) follow.

- $\mathfrak{s}'$  is a weakly good ( $< \alpha, [\lambda^+, \theta]$ )-frame: By Theorem 6.7.5.
- Let  $\mu \geq \lambda$  be such that  $\mathbf{K}^{up}_{\mu}$  has joint embedding. By amalgamation,  $\mathbf{K}^{up}_{\geq \mu}$  has joint embedding. Once it is shown that  $\mathbf{K}^{up}$  has no maximal

models, it will follow that  $\mathbf{K}^{up}$  has joint embedding (every model of size  $> \lambda$  extends to one of size  $\mu$ ). Note that joint embedding is never used in any of the proofs below.

- To prove that  $\mathbf{K}^{\mathrm{up}}_{[\lambda,\theta]}$  has no maximal models, we can assume without loss of generality that  $\alpha = 2$  and (by Lemma 6.8.7) that  $\mathbf{i} = \mathbf{i}_{\lambda-ns}(\mathbf{K})$ , with  $\kappa_1(\mathfrak{i}, <_{\mathbf{K}}^{\mathrm{univ}}) = \aleph_0$ . By the induction hypothesis (and the assumption that **K** has no maximal models),  $\mathbf{K}_{[\lambda,\theta)}^{\text{up}}$  has no maximal models. It remains to see that  $\mathbf{K}^{up}_{\theta}$  has no maximal models. Assume for a contradiction that  $M \in \mathbf{K}_{\theta}^{\mathrm{up}}$  is maximal. Then it is easy to check that  $M \in (\mathbf{K}^{\mathrm{up}})_{\theta}^{\theta - \mathrm{sat}}$ . Build  $\langle M_i : i < \theta \rangle$  increasing continuous and  $a \in |M|$  such that for all  $i < \theta$ : (1)  $M_i \leq_{\mathbf{K}} M.$ (2)  $M_i <_{\mathbf{K}}^{\text{univ}} M_{i+1}.$ (3)  $M_i \in \mathbf{K}_{\leq \theta}^{\text{up}}.$ 

  - (4)  $a \notin |M_i|$ .

This is enough. Let  $M_{\theta} := \bigcup_{i < \theta} M_i$ . Note that  $||M_{\theta}|| = \theta$  and  $a \in$  $|M| \setminus |M_{\theta}|$ , so  $M_{\theta} <_{\mathbf{K}} M$ . By Lemma 6.2.16,  $M_0 <_{\mathbf{K}}^{\text{univ}} M_{\theta}$ . Thus there exists  $f : M \xrightarrow{M_0} M_{\theta}$  and since M is maximal f is an isomorphism. However M is maximal whereas M witnesses that  $M_{\theta}$  is not maximal, so M cannot be isomorphic to  $M_{\theta}$ , a contradiction.

This is possible. Imitate the proof of Lemma 4.5.12 (this is where it is useful that the generator is nonsplitting and the local character is witnessed by  $<^{\text{univ}}_{\mathbf{K}}$ ).

- $\mathbf{K}^{\text{up}}$  is stable in all  $\mu \in [\lambda^+, \theta]$ : Exactly as in the proof of Theorem 4.5.6.
- $\mathfrak{s}'$  has base monotonicity, disjointness, and uniqueness because it is weakly good. For all  $\alpha_0 < \alpha$ ,  $\kappa_{\alpha_0}(\mathfrak{i}') = |\alpha_0|^+ + \aleph_0$ ,  $\bar{\kappa}_{\alpha_0}(\mathfrak{s}') = |\alpha_0|^+ + \lambda^{++} = \lambda^{++}$ by Lemma 6.6.8.

REMARK 6.8.10. Our proof of no maximal models above improves on **She09a**, Conclusion 4.13.(3)], as it does not use the symmetry property.

# 6.9. Canonicity

In Chapter 3, we gave conditions under which two independence relations are the same. There we strongly relied on the extension property, but coheir and weakly good frames only have a weak version of it. In this section, we show that if we just want to show two independence relations are the same over sufficiently saturated models, then the proofs become easier and the extension property is not needed. In addition, we obtain an explicit description of the forking relation. We conclude that coheir, weakly good frames, and good frames are (in a sense made precise below) canonical. This gives further evidence that these objects are not ad-hoc and answers several questions in Chapter 3. The results of this section are also used in Section 6.10 to show the equivalence between superstability and strong superstability.

LEMMA 6.9.1 (The canonicity lemma). Let K be an AEC with amalgamation and let  $\lambda \geq LS(\mathbf{K})$  be such that  $\mathbf{K}$  is stable in  $\lambda$ . Let  $\mathbf{K}'$  be a dense sub-AC of  $\mathbf{K}$ such that  $\mathbf{K}^{\lambda^+\text{-sat}} \subseteq \mathbf{K}'$  and  $\mathbf{K}'_{\lambda}$  is dense in  $\mathbf{K}_{\lambda}$ . Let  $\mathfrak{i}, \mathfrak{i}'$  be  $(\langle \alpha, \geq \lambda \rangle)$ -independence relation with  $\mathbf{K}_{i} = \mathbf{K}_{i'} = \mathbf{K}'$ . Let  $\alpha_0 := \min(\alpha, \lambda^+)$ .

If:

- (1)  $\operatorname{pre}(\mathfrak{i})$  and  $\operatorname{pre}(\mathfrak{i}')$  have uniqueness.
- (2) i and i' have base monotonicity, the left  $\lambda$ -witness property, and the right  $\lambda$ -model-witness property.
- (3)  $\bar{\kappa}_{<\alpha_0}(\mathfrak{i}) = \bar{\kappa}_{<\alpha_0}(\mathfrak{i}') = \lambda^+.$

Then  $\operatorname{pre}(\mathfrak{i}) \upharpoonright \mathbf{K}^{\lambda^+\operatorname{-sat}} = \operatorname{pre}(\mathfrak{i}') \upharpoonright \mathbf{K}^{\lambda^+\operatorname{-sat}}$ , and if in addition both  $\mathfrak{i}$  and  $\mathfrak{i}'$  have the right  $\lambda$ -witness property, then  $\mathfrak{i} \upharpoonright \mathbf{K}^{\lambda^+\operatorname{-sat}} = \mathfrak{i}' \upharpoonright \mathbf{K}^{\lambda^+\operatorname{-sat}}$ .

Moreover for  $M \leq_{\mathbf{K}} N$  in  $\mathbf{K}^{\lambda^+\text{-sat}}$ ,  $p \in \mathrm{gS}^{<\alpha}(N)$  does not i-fork over M if and only if for all  $I \subseteq \ell(p)$  with  $|I| \leq \lambda$ , there exists  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}'_{\lambda}$  such that  $p^I$  does not  $\mathfrak{s}_{\lambda\text{-ns}}(\mathbf{K}')$ -fork over  $M_0$ .

PROOF. By Fact 6.2.8, we can assume without loss of generality that **K** has joint embedding. If  $\mathbf{K}_{\lambda^+} = \emptyset$ , there is nothing to prove so assume  $\mathbf{K}_{\lambda^+} \neq \emptyset$ . Using joint embedding, it is easy to see that  $\mathbf{K}_{\lambda}$  is nonempty and has no maximal models. By the left  $\lambda$ -witness property, we can assume without loss of generality that  $\alpha \leq \lambda^+$ , i.e.  $\alpha = \alpha_0$ . By Proposition 6.7.4, (**K**, **i**) and (**K**, **i'**) are  $\lambda$ -generators for a weakly good ( $< \alpha$ )-independence relation. By Lemma 6.7.6, pre((**K**, **i**)<sup>up</sup>)  $\upharpoonright \mathbf{K}^{\lambda^+$ -sat.

By Lemma 6.7.8, for  $x \in \{i, i'\}$ ,  $\operatorname{pre}((\mathbf{K}, x)^{\operatorname{up}}) \upharpoonright \mathbf{K}^{\lambda^+ \operatorname{-sat}} = \operatorname{pre}(x) \upharpoonright \mathbf{K}^{\lambda^+ \operatorname{-sat}}$ , so the result follows (the definition of  $(\mathbf{K}, x)_{\geq \lambda}$  and Lemma 6.7.6 also give the moreover part). The moreover part of lemma 6.7.8 says that if  $x \in \{i, i'\}$  has the right  $\lambda$ -witness property, then  $(\mathbf{K}, x)^{\operatorname{up}} \upharpoonright \mathbf{K}^{\lambda^+ \operatorname{-sat}} = x \upharpoonright \mathbf{K}^{\lambda^+ \operatorname{-sat}}$ , so in case both i and i' have the right  $\lambda$ -witness property, we must have  $i \upharpoonright \mathbf{K}^{\lambda^+ \operatorname{-sat}} = i' \upharpoonright \mathbf{K}^{\lambda^+ \operatorname{-sat}}$ .

REMARK 6.9.2. If **K** is an AEC with amalgamation, **K'** is a dense sub-AC of **K** such that  $\mathbf{K}^{\lambda^+\text{-sat}} \subseteq \mathbf{K}'$  and  $\mathbf{K}'_{\lambda}$  is dense in  $\mathbf{K}_{\lambda}$ , and **i** is a  $(\leq 1, \geq \lambda)$ -independence relation with  $\mathbf{K}_{\mathbf{i}} = \mathbf{K}'$  and base monotonicity, uniqueness,  $\bar{\kappa}_1(\mathbf{i}) = \lambda^+$ , then by the proof of Proposition 6.4.6 and Lemma 6.5.8 **K** is stable in any  $\mu \geq \text{LS}(\mathbf{K})$  with  $\mu = \mu^{\lambda}$ .

THEOREM 6.9.3 (Canonicity of coheir). Let **K** be an AEC with amalgamation. Let  $\kappa = \beth_{\kappa} > LS(\mathbf{K})$ . Assume **K** is  $(< \kappa)$ -tame and short for types of length less than  $\alpha, \alpha \ge 2$ .

Let  $\lambda \geq \kappa$  be such that **K** is stable in  $\lambda$  and  $(\alpha_0 + 2)^{<\kappa_r} \leq \lambda$  for all  $\alpha_0 < \min(\lambda^+, \alpha)$ . Let  $\mathfrak{i}$  be a  $(<\alpha, \geq \lambda)$ -independence relation so that:

- (1)  $\mathbf{K}_i$  is a dense sub-AC of  $\mathbf{K}$  so that  $\mathbf{K}^{\lambda^+\text{-sat}} \subseteq \mathbf{K}_i$  and  $(\mathbf{K}_i)_{\lambda}$  is dense in  $\mathbf{K}_{\lambda}$ .
- (2)  $\operatorname{pre}(\mathfrak{i})$  has uniqueness.
- (3) i has base monotonicity, the left  $\lambda$ -witness property, and the right  $\lambda$ -modelwitness property.
- (4)  $\bar{\kappa}_{<\min(\lambda^+,\alpha)}(\mathfrak{i}) = \lambda^+.$

Then  $\operatorname{pre}(\mathfrak{i}) \upharpoonright \mathbf{K}^{\lambda^+\operatorname{-sat}} = \operatorname{pre}(\mathfrak{i}_{\kappa\operatorname{-ch}}(\mathbf{K})^{<\alpha}) \upharpoonright \mathbf{K}^{\lambda^+\operatorname{-sat}}$ . If in addition  $\mathfrak{i}$  has the right  $\lambda$ -witness property, then  $\mathfrak{i} \upharpoonright \mathbf{K}^{\lambda^+\operatorname{-sat}} = \mathfrak{i}_{\kappa\operatorname{-ch}}(\mathbf{K})^{<\alpha} \upharpoonright \mathbf{K}^{\lambda^+\operatorname{-sat}}$ .

PROOF. By Fact 6.2.8, we can assume without loss of generality that **K** has joint embedding. If  $\mathbf{K}_{\lambda^+} = \emptyset$ , there is nothing to prove so assume  $\mathbf{K}_{\lambda^+} \neq \emptyset$ . By Fact 6.2.7, **K** has arbitrarily large models so no maximal models. Let  $\mathbf{i}' := \mathbf{i}_{\kappa\text{-ch}}(\mathbf{K})^{<\alpha}$ . By the proof of Proposition 6.7.4,  $\mathbf{i}' \upharpoonright \mathbf{K}_{\mathbf{i}}$  satisfies the hypotheses of Lemma 6.9.1. Moreover, it has the right  $(< \kappa)$ -witness property so the result follows.  $\Box$ 

THEOREM 6.9.4 (Canonicity of weakly good independence relations). Let **K** be an AEC with amalgamation and let  $\lambda \geq \text{LS}(\mathbf{K})$ . Let **K'** be a dense sub-AC of **K** such that  $\mathbf{K}^{\lambda^+\text{-sat}} \subseteq \mathbf{K'}$  and  $\mathbf{K'}_{\lambda}$  is dense in  $\mathbf{K}_{\lambda}$ . Let **i**, **i'** be weakly good  $(\langle \alpha, \geq \lambda \rangle)$ -independence relations with  $\mathbf{K}_i = \mathbf{K}_{i'} = \mathbf{K'}$ .

Then  $\operatorname{pre}(\mathfrak{i}) \upharpoonright \mathbf{K}^{\lambda^+-\operatorname{sat}} = \operatorname{pre}(\mathfrak{i}') \upharpoonright \mathbf{K}^{\lambda^+-\operatorname{sat}}$ . If in addition both  $\mathfrak{i}$  and  $\mathfrak{i}'$  have the right  $\lambda$ -witness property, then  $\mathfrak{i} \upharpoonright \mathbf{K}^{\lambda^+-\operatorname{sat}} = \mathfrak{i}' \upharpoonright \mathbf{K}^{\lambda^+-\operatorname{sat}}$ .

PROOF. By definition of a weakly good independence relation,  $\mathbf{K}'_{\lambda}$  is stable in  $\lambda$ . Therefore by Lemma 6.5.8  $\mathbf{K}_{\lambda}$ , and hence  $\mathbf{K}$ , is stable in  $\lambda$ . Now apply Lemma 6.9.1.

THEOREM 6.9.5 (Canonicity of good independence relations). If  $\mathfrak{i}$  and  $\mathfrak{i}'$  are good (<  $\alpha$ ,  $\geq \lambda$ )-independence relations with the same underlying AEC **K**, then  $\mathfrak{i} \upharpoonright \mathbf{K}^{\lambda^+\text{-sat}} = \mathfrak{i}' \upharpoonright \mathbf{K}^{\lambda^+\text{-sat}}$ .

PROOF. By Theorem 6.9.4 (with  $\mathbf{K}' := \mathbf{K}$ ),  $\operatorname{pre}(\mathfrak{i}) \upharpoonright \mathbf{K}^{\lambda^+ \operatorname{-sat}} = \operatorname{pre}(\mathfrak{i}') \upharpoonright \mathbf{K}^{\lambda^+ \operatorname{-sat}}$ . Since good independence relations have extension, Lemma 6.4.2 implies  $\mathfrak{i} \upharpoonright \mathbf{K}^{\lambda^+ \operatorname{-sat}} = \mathfrak{i}' \upharpoonright \mathbf{K}^{\lambda^+ \operatorname{-sat}}$ .

Recall that Question 3.6.13 asked if two good  $\lambda$ -frames with the same underlying AEC should be the same. We can make progress toward this question by slightly refining our arguments. Note that the results below can be further adapted to work for not necessarily type-full frames (that is for two good frames, in Shelah's sense, with the same basic types and the same underlying AEC).

LEMMA 6.9.6. Let  $\mathfrak{s}$  and  $\mathfrak{s}'$  be good ( $\langle \alpha, \lambda \rangle$ )-frames with the same underlying AEC **K** and  $\alpha \leq \lambda$ . Let **K**' be the class of  $\lambda$ -limit models of **K** (recall Definition 6.2.10). Then  $\mathfrak{s} \upharpoonright \mathbf{K}' = \mathfrak{s}' \upharpoonright \mathbf{K}'$ .

PROOF SKETCH. By Remark 6.2.15,  $I(\mathbf{K}') = 1$ . Now refine the proof of Theorem 6.9.5 by replacing  $\lambda^+$ -saturated models by  $(\lambda, |\beta|^+ + \aleph_0)$ -limit models for each  $\beta < \alpha$ . Everything still works since one can use the weak uniqueness and extension properties of nonsplitting (Fact 6.3.22.(5)).

THEOREM 6.9.7 (Canonicity of categorical good  $\lambda$ -frames). Let  $\mathfrak{s}$  and  $\mathfrak{s}'$  be good  $(<\alpha,\lambda)$ -frames with the same underlying AEC **K** and  $\alpha \leq \lambda$ . If **K** is categorical in  $\lambda$ , then  $\mathfrak{s} = \mathfrak{s}'$ .

PROOF. By Fact 6.2.13, **K** has a limit model, and so by categoricity any model of **K** is limit. Now apply Lemma 6.9.6.  $\Box$ 

REMARK 6.9.8. The proof also gives an explicit description of forking: For  $M_0 \leq_{\mathbf{K}} M$  with  $M_0$  a limit model,  $p \in \mathrm{gS}(M)$  does not  $\mathfrak{s}$ -fork over  $M_0$  if and only if there exists  $M'_0 <_{\mathbf{K}}^{\mathrm{univ}} M_0$  such that p does not  $\mathfrak{s}_{\lambda-\mathrm{ns}}$ -fork over  $M'_0$ . Note that this is the definition of forking in Chapter 4.

Note that Shelah's construction of a good  $\lambda$ -frame in [She09a, Theorem II.3.7] relies on categoricity in  $\lambda$ , so Theorem 6.9.7 establishes that the frame there is canonical. We are still unable to show that the frame built in Theorem 6.10.16 is canonical in general, although it will be if  $\lambda$  is the categoricity cardinal or if it is weakly successful (by Theorem 3.6.12).

## 6.10. Superstability

Shelah has pointed out [She09a, p. 19] that superstability in abstract elementary classes suffers from schizophrenia, i.e. there are several different possible definitions that are equivalent in elementary classes but not necessarily in AECs. The existence of a good  $(\geq \lambda)$ -frame is a possible candidate but it is very hard to check. Instead, one would like a simple definition that implies existence of a good frame.

Shelah claims in chapter IV of his book that solvability<sup>25</sup> ([She09a, Definition IV.1.4]) is such a notion, but his justification is yet to appear (in [Sheb]). Essentially, solvability says that certain EM models are superlimits. On the other hand previous work (for example [She99, SV99, Van06, Van13, GVV16]) all rely on a local character property for nonsplitting. This is even made into a definition of superstability in [Gro02, Definition 7.12]. In Chapter 4 we gave a similar condition and used it with tameness to build a good frame. Shelah has shown [She99, Lemma I.6.3] that categoricity in a cardinal of high-enough cofinality implies the superstability condition.

We now aim to show the same conclusion under categoricity in a high-enough cardinal of arbitrary cofinality. The following definition of superstability is implicit in [SV99] and stated explicitly in [Gro02, Definition 7.12].

DEFINITION 6.10.1 (Superstability). An AEC K is  $\mu$ -superstable if:

- (1)  $LS(\mathbf{K}) \leq \mu$ .
- (2) There exists  $M \in \mathbf{K}_{\mu}$  such that for any  $M' \in \mathbf{K}_{\mu}$  there is  $f: M' \to M$ with  $f[M'] <_{\mathbf{K}}^{\operatorname{univ}} M$ . (3)  $\kappa_1(\mathfrak{s}_{\mu-\mathrm{ns}}(\mathbf{K}_{\mu}), \leq_{\mathbf{K}}^{\operatorname{univ}}) = \aleph_0$ .

We say **K** is  $\mu$ -superstable<sup>+</sup> if  $\mathbf{K}_{\geq \mu}$  is  $\mu$ -superstable, has amalgamation, and is  $\mu$ -tame. We may omit  $\mu$ , in which case we mean there exists a value such that the definition holds, e.g. **K** is *superstable* if it is  $\mu$ -superstable for some  $\mu$ .

REMARK 6.10.2. Using Fact 6.2.13, it is easy to check that Condition (2) above is equivalent to " $\mathbf{K}_{\mu}$  is nonempty, has amalgamation, joint embedding, no maximal models, and is stable in  $\mu$ ".

REMARK 6.10.3. While Definition 6.10.1 makes sense in any AEC, here we focus on tame AECs with amalgamation, and will not study what happens to Definition 6.10.1 without these assumptions (although this can be done, see [**GVV16**]). In other words, we will study "superstable<sup>+</sup>" rather than just "superstable".

For technical reasons, we will also use the following version that uses coheir rather than nonsplitting.

DEFINITION 6.10.4. An AEC K is  $\kappa$ -strongly  $\mu$ -superstable if:

- (1)  $LS(\mathbf{K}) < \kappa < \mu$ .
- (2) (2) in Definition 6.10.1 holds.
- (3) **K** does not have the  $(< \kappa)$ -order property of length  $\kappa$ .
- (4)  $\mathbf{K}_{\mu}^{\kappa\text{-sat}}$  is dense in  $\mathbf{K}_{\mu}$ . (5)  $\kappa_1(\mathfrak{i}_{\kappa\text{-ch}}(\mathbf{K})_{\mu}, \leq_{\mathbf{K}}^{\mathrm{univ}}) = \aleph_0$ .

 $<sup>^{25}</sup>$ One can ask whether there are any implications between this chapter's definition of superstability and Shelah's. We leave this to future work.

As before, we may omit some parameters and say **K** is  $\kappa$ -strongly  $\mu$ -superstable<sup>+</sup> if there exists  $\kappa_0 < \kappa$  such that  $\mathbf{K}_{\geq \kappa_0}$  is  $\kappa$ -strongly  $\mu$ -superstable, has amalgamation, and is  $(< \kappa)$ -tame.

It is not too hard to see that a  $\mu$ -superstable<sup>+</sup> AEC induces a generator for a good independence relation, but what if we have a generator of some other form (assume for example that  $<_{\mathbf{K}}^{\mathrm{univ}}$  is replaced by  $<_{\mathbf{K}}^{\mu,\delta}$  in the definition)? This is the purpose of the next definition.

DEFINITION 6.10.5. Let  $\mathbf{K}$  be an AEC.

- (1) **K** is  $(\mu, \mathfrak{i})$ -superstable<sup>+</sup> if  $LS(\mathbf{K}) \leq \mu$  and  $(\mathbf{K}_{\mu}, \mathfrak{i})$  is a  $\mu$ -generator for a good ( $\leq 1$ )-independence relation.
- (2) **K** is  $\kappa$ -strongly  $(\mu, \mathfrak{i})$ -superstable<sup>+</sup> if:
  - (a)  $LS(\mathbf{K}) < \kappa \leq \mu$ .
  - (b) There exists  $\kappa_0 < \kappa$  such that  $\mathbf{K}_{\geq \kappa_0}$  has amalgamation.
  - (c) **K** is  $(< \kappa)$ -tame.
  - (d) **K** does not have the  $(< \kappa)$ -order property of length  $\kappa$ .
  - (e) **K** is  $(\mu, \mathfrak{i})$ -superstable<sup>+</sup>.
  - (f)  $\mathbf{K}_{\mathfrak{i}} \subseteq \mathbf{K}_{\mu}^{\kappa\text{-sat}}$  and  $\mathfrak{i} = \mathfrak{i}_{\kappa\text{-ch}}(\mathbf{K})^{\leq 1} \upharpoonright \mathbf{K}_{\mathfrak{i}}$ .

The terminology is justified by the next proposition which tells us that the existence of *any* generator is equivalent to superstability. It makes checking that superstability holds easier and we will use it freely.

PROPOSITION 6.10.6. Let K be an AEC.

- (1) **K** is  $\mu$ -superstable<sup>+</sup> if and only if there exists i such that **K** is  $(\mu, i)$ -superstable<sup>+</sup>.
- (2) **K** is  $\kappa$ -strongly  $\mu$ -superstable<sup>+</sup> if and only if there exists i such that **K** is  $\kappa$ -strongly  $(\mu, i)$ -superstable<sup>+</sup>.

## Proof.

- (1) Assume first that **K** is  $\mu$ -superstable<sup>+</sup>. Then one can readily check (using Proposition 6.7.4 and Remark 6.10.2) that  $(\mathbf{K}_{\mu}, \mathbf{i}_{\mu-ns}(\mathbf{K})^{\leq 1})$  is a generator for a good independence relation, where the local character axiom is witnessed by  $\leq_{\mathbf{K}}^{\mathbf{univ}}$ . Conversely, assume that **K** is  $(\mu, \mathbf{i})$ -superstable<sup>+</sup>. By definition,  $\mathrm{LS}(\mathbf{K}) \leq \mu$  and by definition of a generator  $\mathbf{K}_{\geq \mu}$  has amalgamation and is  $\mu$ -tame. By Lemma 6.8.7,  $(\mathbf{K}_{\mu}, \mathbf{i}_{\mu-ns}(\mathbf{K})^{\leq 1})$  is a  $\mu$ -generator for a good ( $\leq$  1)-independence relation, and  $\leq_{\mathbf{K}}^{\mathbf{univ}}$  is the ordering witnessing local character. Thus **K** is  $\mu$ -superstable<sup>+</sup>.
- (2) Assume first that **K** is  $\kappa$ -strongly  $\mu$ -superstable<sup>+</sup>. Let  $\kappa_0 < \kappa$  be such that  $\mathbf{K}_{\geq \kappa_0}$  has amalgamation. Assume without loss of generality that  $\kappa_0 = \mathbf{LS}(\mathbf{K})$  and that  $\mathbf{K}_{\geq \kappa_0} = \mathbf{K}$ . By (the proof of) Proposition 6.7.4,  $(\mathbf{K}_{\mu}, \mathbf{i}_{\kappa\text{-ch}}(\mathbf{K})_{\mu}^{\leq 1})$  is a  $\mu$ -generator for a weakly good ( $\leq 1$ )-independence relation. By the other conditions, it is actually a  $\mu$ -generator for a good ( $\leq 1$ )-independence relation. Conversely, assume that **K** is  $\kappa$ -strongly ( $\mu, \mathbf{i}$ )-superstable<sup>+</sup>. We check the last two conditions in the definition of strong superstability, the others are straightforward. We know that ( $\mathbf{K}_{\mu}, \mathbf{i}$ ) is a generator and  $\mathbf{i} = \mathbf{i}_{\kappa\text{-ch}}(\mathbf{K})^{\leq 1} \upharpoonright \mathbf{K}_{\mathbf{i}}$ . Thus  $\mathbf{K}_{\mathbf{i}} \subseteq \mathbf{K}_{\mu}^{\kappa\text{-sat}}$  is dense in  $\mathbf{K}_{\mu}$ , so  $\mathbf{K}_{\mu}^{\kappa\text{-sat}}$  is dense in  $\mathbf{K}_{\mu}$ . By Lemma 6.6.7,  $\kappa_1(\mathbf{i}_{\kappa\text{-ch}}(\mathbf{K})_{\mu}, \leq_{\mathbf{M}}^{\min \vee}) \leq \kappa_1(\mathbf{i}, \trianglelefteq)$  for any  $\trianglelefteq$  such that ( $\mathbf{K}_{\mathbf{i}}, \trianglelefteq)$  is a skeleton of  $\mathbf{K}_{\mu}$  (and hence of  $\mathbf{K}_{\mu}^{\kappa\text{-sat}}$ ). By

assumption one can find such a  $\leq$  with  $\kappa_1(\mathfrak{i}, \leq) = \aleph_0$ . Thus

$$\kappa_1(\mathfrak{i}_{\kappa-\mathrm{ch}}(\mathbf{K})_{\mu},\leq^{\mathrm{univ}}_{\mathbf{K}})=leph_0$$

REMARK 6.10.7. Thus in Definitions 6.10.1 and 6.10.4, one can replace  $\leq_{\mathbf{K}}^{\text{univ}}$ by  $\leq_{\mathbf{K}}^{\mu,\delta}$  for  $1 \leq \delta < \mu^+$ .

The next result gives evidence that Definition 6.10.1 is a reasonable definition of superstability, at least in tame AECs with amalgamation. Note that most of it already appears implicitly in Chapter 4 and essentially restates Theorem 6.8.9.

THEOREM 6.10.8. Assume **K** is a  $(\mu, i)$ -superstable<sup>+</sup> AEC. Then:

- (1)  $\mathbf{K}_{\geq \mu}$  has joint embedding, no maximal models, and is stable in all  $\lambda \geq \mu$ .
- (2) Let λ ≥ μ<sup>+</sup> and let i' := (K<sub>μ</sub>, i)<sup>up</sup> ↾ K<sup>μ<sup>+</sup>-sat</sup><sub>≥λ</sub>.
  (a) i' is an almost pre-good (≤ 1, ≥ λ)-independence relation (recall Definition 6.8.8).
  - (b) If in addition **K** is  $\kappa$ -strongly  $(\mu, \mathfrak{i})$ -superstable<sup>+</sup>, then pre $(\mathfrak{i}')$  =
  - (b) If in addition  $\mathbf{L}$  is  $\lambda^{\mu^+, \text{sat}}$ . That is, the frame is  $(<\kappa)$ -coheir. (c) If  $\theta \ge \mu^+$  is such that  $\mathbf{K}' := \mathbf{K}_{\ge \lambda}^{\theta\text{-sat}}$  is an AEC with  $\text{LS}(\mathbf{K}') = \lambda$ , then  $\mathbf{i}' \upharpoonright \mathbf{K}'$  is a pre-good  $(\leq 1, \geq \lambda)$ -independence relation that will be  $(< \kappa)$ -coheir if **K** is  $\kappa$ -strongly  $(\mu, \mathfrak{i})$ -superstable<sup>+</sup>.

PROOF. Theorem 6.8.9 gives (1) and (2a), while (2c) follows from (2a) and (2b). It remains to prove (2b). Let  $\mathbf{i}'' := \mathbf{i}_{\kappa-ch}(\mathbf{K}))^{\leq 1} \upharpoonright \mathbf{K}_{\geq \lambda}^{\mu^+\text{-sat}}$ . By the proof of Lemma 6.7.8,  $\underset{\mathbf{i}'}{\downarrow} \subseteq \underset{\mathbf{i}''}{\downarrow}$ . Now by (2a), pre( $\mathbf{i}'$ ) has existence and extension and by Fact 2.5.15,  $\mathfrak{i}''$  has uniqueness. By Lemma 3.4.1,  $\operatorname{pre}(\mathfrak{i}') = \operatorname{pre}(\mathfrak{i}'')$ , as desired. 

REMARK 6.10.9. Let T be a complete first-order theory and let  $\mathbf{K} := (Mod(T), \prec$ ). Then this chapter's definitions of superstability and strong superstability coincide with the classical definition. More precisely for all  $\mu \geq |T|$ , **K** is (strongly)  $\mu$ -superstable if and only if T is stable in all  $\lambda \geq \mu$ .

Note also that [strong]  $\mu$ -superstability<sup>+</sup> is monotonic in  $\mu$ :

PROPOSITION 6.10.10. If **K** is [ $\kappa$ -strongly]  $\mu$ -superstable<sup>+</sup> and  $\mu' > \mu$ , then **K** is  $[\kappa$ -strongly]  $\mu'$ -superstable<sup>+</sup>.

**PROOF.** Say **K** is  $(\mu, \mathfrak{i})$ -superstable<sup>+</sup>. It is clearly enough to check that **K** is  $\mu'$ -superstable. Let  $\mathbf{i}' := (\mathbf{K}, \mathbf{i})^{\text{up}} \upharpoonright \mathbf{K}_{\geq \mu'}^{\mu^+\text{-sat}}$ . By Theorem 6.10.8 and Proposition 6.7.4,  $(\mathbf{K}_{\mu'}, \mathbf{i}')$  is a generator for a good  $\mu'$ -independence relation, so  $\mathbf{K}$  is  $(\mu', \mathbf{i}')$ superstable. Similarly, if **K** is  $\kappa$ -strongly  $(\mu, \mathfrak{i})$ -superstable<sup>+</sup> then **K** will be  $\kappa$ strongly  $(\mu', \mathfrak{i}')$ -superstable.  $\square$ 

Theorem 6.10.8.(2b) is the reason we introduced strong superstability. While it may seem like a detail, we are interested in extending our good frame to a frame for types longer than one element and using coheir to do so seems reasonable. Using the canonicity of coheir, we can show that superstability and strong superstability are equivalent if we do not care about the parameter  $\mu$ :

THEOREM 6.10.11. If **K** is  $\mu$ -superstable<sup>+</sup> and  $\kappa = \beth_{\kappa} > \mu$ , then **K** is  $\kappa$ -strongly  $(2^{<\kappa_r})^+$ -superstable<sup>+</sup>.

In particular a tame AEC with amalgamation is strongly superstable if and only if it is superstable.

PROOF. Let  $\mu' := (2^{<\kappa_r})^+$ . We show that **K** is  $\kappa$ -strongly  $\mu'$ -superstable<sup>+</sup>. By Theorem 6.10.8,  $\mathbf{K}_{\geq\mu}$  has joint embedding, no maximal models and is stable in all cardinals. By definition,  $\mathbf{K}_{\geq\mu}$  also has amalgamation. Also, **K** is  $\mu$ -tame, hence  $(<\kappa)$ -tame. By Fact 6.2.9, **K** does not have the  $(<\kappa)$ -order property of length  $\kappa$ . Moreover we have already observed that  $\mathbf{K}_{\mu'}$  is stable in  $\mu'$  and has joint embedding and no maximal models. Also,  $\mathbf{K}_{\mu'}^{\kappa\text{-sat}}$  is dense in  $\mathbf{K}_{\mu'}$  by stability and the fact  $\mu' > \kappa$ . It remains to check that  $\kappa_1(\mathbf{i}_{\kappa\text{-ch}}(\mathbf{K})_{\mu'}, \leq_{\mathbf{K}}^{\text{univ}}) = \aleph_0$ .

By Theorem 6.10.8, there is a  $(\leq 1, \geq \mu^+)$ -independence relation i' such that  $\mathbf{K}_{\mathbf{i}'} = \mathbf{K}^{\mu^+\text{-sat}}$  and i' is good, except that  $\mathbf{K}_{\mathbf{i}'}$  may not be an AEC. By Theorem 6.9.3 (with  $\lambda$  there standing for  $2^{<\kappa_r}$  here),  $\operatorname{pre}(\mathbf{i}') \upharpoonright \mathbf{K}^{\mu'\text{-sat}} = \operatorname{pre}(\mathbf{i}_{\kappa\text{-ch}}(\mathbf{K})^{\leq 1}) \upharpoonright \mathbf{K}^{\mu'\text{-sat}}$ . By the proof of Lemma 6.4.5, i' has the right  $(<\kappa)$ -witness property for members of  $\mathbf{K}_{\geq \mu^+}$ : If  $M \in \mathbf{K}_{\geq \mu^+}$ ,  $M_0 \leq_{\mathbf{K}} M$  is in  $\mathbf{K}^{\mu^+\text{-sat}}$ , and  $p \in \operatorname{gS}(M)$ , then p does not i-fork over  $M_0$  if and only if  $p \upharpoonright B$  does not i-fork over  $M_0$  for all  $B \subseteq |M|$  with  $|B| < \kappa$ . Therefore by the proof of Theorem 6.9.3, we actually have that for any  $M \in \mathbf{K}_{\geq \mu'}$  and  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}^{\mu'\text{-sat}}$ ,  $p \in \operatorname{gS}(M)$  does not i'-fork over  $M_0$  if and only if p hors not  $\mathbf{i}_{\kappa\text{-ch}}(\mathbf{K})$ -fork over  $M_0$ . In particular:

$$\kappa_1(\mathbf{i}_{\kappa\text{-ch}}(\mathbf{K})_{\mu'}) = \kappa_1(\mathbf{i}'_{\mu}) = \aleph_0$$
  
Therefore  $\kappa_1(\mathbf{i}_{\kappa\text{-ch}}(\mathbf{K})_{\mu'}, \leq_{\mathbf{K}}^{\mathrm{univ}}) = \aleph_0$ , as needed.

We now arrive to the main result of this section: categoricity implies strong superstability. We first recall several known consequences of categoricity.

FACT 6.10.12. Let **K** be an AEC with no maximal models, joint embedding, and amalgamation. Assume **K** is categorical in a  $\lambda > LS(\mathbf{K})$ . Then:

- (1) [She99, Claim I.1.7] K is stable in all  $\mu \in [LS(K), \lambda)$ .
- (2) [She99, Lemma 6.3] For  $LS(\mathbf{K}) \leq \mu < \operatorname{cf} \lambda, \, \kappa_1(\mathfrak{s}_{\mu-ns}(\mathbf{K}_{\mu}), \leq_{\mathbf{K}}^{\mu,\omega}) = \aleph_0.$
- (3) [**BG**, Theorem 6.8] Assume **K** does not have the weak  $\kappa$ -order property (see Definition 2.4.9) and  $LS(\mathbf{K}) < \kappa \leq \mu < \lambda$ . Then:

$$\kappa_1(\mathfrak{i}_{\kappa-\mathrm{ch}}(\mathbf{K})_{\mu},\leq^{\mathrm{univ}}_{\mathbf{K}})=\aleph_0$$

(4) [She99, Lemma II.1.5] If the model of size  $\lambda$  is  $\mu$ -saturated for  $\mu > \text{LS}(\mathbf{K})$ , then every member of  $\mathbf{K}_{\geq \chi}$  is  $\mu$ -saturated, where  $\chi := \min(\lambda, \sup_{\mu_0 < \mu} h(\mu_0))$ .

The next proposition is folklore: it derives joint embedding and no maximal models from amalgamation and categoricity. We could not find a proof in the literature, so we include one here.

PROPOSITION 6.10.13. Let **K** be an AEC with amalgamation. If there exists  $\lambda \geq LS(\mathbf{K})$  such that  $\mathbf{K}_{\lambda}$  has joint embedding, then there exists  $\chi < h(LS(\mathbf{K}))$  such that  $\mathbf{K}_{\geq \chi}$  has joint embedding and no maximal models.

PROOF. Write  $\mu := h(LS(\mathbf{K}))$ . If  $\mathbf{K}_{\mu} = \emptyset$ , then by Fact 6.2.7 there exists  $\chi < \mu$  such that  $\mathbf{K}_{\geq\chi} = \emptyset$ , so it has has joint embedding and no maximal models. Now assume  $\mathbf{K}_{\mu} \neq \emptyset$ . In particular, **K** has arbitrarily large models. By amalgamation,

 $\mathbf{K}_{\geq\lambda}$  has joint embedding, and so no maximal models. If  $\lambda < \mu$  we are done so assume  $\lambda \geq \mu$ . It is enough to show that there exists  $\chi < \mu$  such that  $\mathbf{K}_{\geq\chi}$  has no maximal model since then any model of  $\mathbf{K}_{\geq\chi}$  embeds inside a model in  $\mathbf{K}_{\geq\lambda}$  and hence  $\mathbf{K}_{>\chi}$  has joint embedding.

By Fact 6.2.8, we can write  $\mathbf{K} = \bigcup_{i \in I} \mathbf{K}^i$  where the  $\mathbf{K}^i$ 's are disjoint AECs with  $\mathrm{LS}(\mathbf{K}^i) = \mathrm{LS}(\mathbf{K})$  and each  $\mathbf{K}^i$  has joint embedding and amalgamation. Note that  $|I| \leq I(\mathbf{K}, \mathrm{LS}(\mathbf{K})) \leq 2^{\mathrm{LS}(\mathbf{K})}$ . For  $i \in I$ , let  $\chi_i$  be the least  $\chi < \mu$  such that  $\mathbf{K}^i_{\geq \chi} = \emptyset$ , or  $\mathrm{LS}(\mathbf{K})$  if  $\mathbf{K}^i_{\mu} \neq \emptyset$ . Let  $\chi := \sup_{i \in I} \chi_i$ . Note that cf  $\mu = (2^{\mathrm{LS}(\mathbf{K})})^+ > 2^{\mathrm{LS}(\mathbf{K})} \geq |I|$ , so  $\chi < \mu$ .

Now let  $M \in \mathbf{K}_{\geq \chi}$ . Let  $i \in I$  be such that  $M \in \mathbf{K}^i$ . M witnesses that  $\mathbf{K}^i_{\chi} \neq \emptyset$ so by definition of  $\chi$ ,  $\mathbf{K}^i$  has arbitrarily large models. Since  $\mathbf{K}^i$  has joint embedding, this implies that  $\mathbf{K}^i$  has no maximal models. Therefore there exists  $N \in \mathbf{K}^i \subseteq \mathbf{K}$ with  $M <_{\mathbf{K}} N$ , as desired.

The next two results are simple consequences of Fact 6.10.12.(2).

PROPOSITION 6.10.14. Let **K** be an LS(**K**)-tame AEC with amalgamation and no maximal models. If **K** is categorical in a  $\lambda$  with cf  $\lambda >$  LS(**K**), then **K** is LS(**K**)-superstable<sup>+</sup>.

PROOF. By amalgamation, categoricity, and no maximal models, **K** has joint embedding. By Fact 6.10.12.(1), **K** is stable in LS(**K**). Now apply Fact 6.10.12.(2) and Proposition 6.10.6 (with Remark 6.10.7).  $\Box$ 

PROPOSITION 6.10.15. Let **K** be an LS(**K**)-tame AEC with amalgamation. If **K** is categorical in a  $\lambda$  with cf  $\lambda \ge h(\text{LS}(\mathbf{K}))$ , then there exists  $\mu < h(\text{LS}(\mathbf{K}))$  such that **K** is  $\mu$ -superstable<sup>+</sup>.

PROOF. By Proposition 6.10.13, there exists  $\mu < h(\text{LS}(\mathbf{K}))$  such that  $\mathbf{K}_{\geq \mu}$  has joint embedding and no maximal models. Now apply Proposition 6.10.14 to  $\mathbf{K}_{\geq \mu}$ .

We now remove the restriction on the cofinality and get strong superstability. The downside is that  $h(LS(\mathbf{K}))$  is replaced by a fixed point of the beth function above  $LS(\mathbf{K})$ .

THEOREM 6.10.16. Let **K** be an AEC with amalgamation. Let  $\kappa = \beth_{\kappa} >$  LS(**K**) and assume **K** is  $(< \kappa)$ -tame. If **K** is categorical in a  $\lambda > \kappa$ , then:

- (1) **K** is  $\kappa$ -strongly  $\kappa$ -superstable<sup>+</sup>.
- (2) **K** is stable in all cardinals above or equal to  $h(LS(\mathbf{K}))$ .
- (3) The model of size  $\lambda$  is saturated.
- (4) **K** is categorical in  $\kappa$ .
- (5) For  $\chi := \min(\lambda, h(\kappa))$ , pre  $\left(\mathbf{i}_{\kappa-ch}(\mathbf{K})^{\leq 1}_{\geq \chi}\right)$  is a good  $(\leq 1, \geq \chi)$ -frame with underlying AEC  $\mathbf{K}_{\geq \chi}$ .

PROOF. Note that  $\mathbf{K}_{\lambda}$  has joint embedding so by Proposition 6.10.13, there exists  $\chi_0 < h(\mathrm{LS}(\mathbf{K}))$  such that  $\mathbf{K}_{\geq\chi_0}$  (and thus  $\mathbf{K}_{\geq\kappa}$ ) has joint embedding and no maximal models. By Fact 6.10.12.(1),  $\mathbf{K}_{\geq\chi_0}$  is stable everywhere below  $\lambda$ . Since  $\kappa = \beth_{\kappa}$ , Fact 6.2.9 implies that  $\mathbf{K}$  does not have the (<  $\kappa$ )-order property of length  $\kappa$ .

Let  $\kappa \leq \mu < \lambda$ . By Fact 6.10.12.(3),  $\kappa_1(i_{\kappa-ch}(\mathbf{K})_{\mu}, \leq_{\mathbf{K}}^{\text{univ}}) = \aleph_0$ . Now using Proposition 6.10.6, **K** is  $\kappa$ -strongly  $\mu$ -superstable if and only if  $\mathbf{K}_{\mu}^{\kappa-\text{sat}}$  is dense in

 $\mathbf{K}_{\mu}$ . If  $\kappa < \mu$ , then  $\mathbf{K}_{\mu}^{\kappa\text{-sat}}$  is dense in  $\mathbf{K}_{\mu}$  (by stability), so  $\mathbf{K}$  is  $\kappa\text{-strongly }\mu\text{-superstable}$ . However we want  $\kappa\text{-strong }\kappa\text{-superstability}$ . We proceed in several steps.

First, we show **K** is  $\mu$ -superstable for some  $\mu < \lambda$ . If  $\lambda = \kappa^+$ , then this follows directly from Proposition 6.10.14 with  $\mu = \kappa$ , so assume  $\lambda > \kappa^+$ . Then by the previous paragraph **K** is  $\kappa$ -strongly  $\mu$ -superstable for  $\mu := \kappa^+$ .

Second, we prove (2). We have already observed  $\mathbf{K}_{\geq\chi_0}$  is stable everywhere below  $\lambda$ . By Theorem 6.10.8, **K** is stable in every  $\mu' \geq \mu$ . In particular, it is stable in and above  $\lambda$ , so (2) follows.

Third, we show (3). Since **K** is stable in  $\lambda$ , we can build a  $\lambda_0^+$ -saturated model of size  $\lambda$  for all  $\lambda_0 < \lambda$ . Thus the model of size  $\lambda$  is  $\lambda_0^+$ -saturated for all  $\lambda_0 < \lambda$ , and hence  $\lambda$ -saturated.

Fourth, we prove (4). Since the model of size  $\lambda$  is saturated, it is  $\kappa$ -saturated. By Fact 6.10.12.(4), every model of size  $\sup_{\kappa_0 < \kappa} h(\kappa_0) = \kappa$  is  $\kappa$ -saturated. By uniqueness of saturated models, **K** is categorical in  $\kappa$ .

Fifth, observe that since every model of size  $\kappa$  is saturated,  $\mathbf{K}_{\kappa}^{\kappa\text{-sat}} = \mathbf{K}_{\kappa}$  is dense in  $\mathbf{K}_{\kappa}$ . By the second paragraph above,  $\mathbf{K}$  is  $\kappa$ -strongly  $\kappa$ -superstable so (1) holds.

Finally, we prove (5). We have seen that the model of size  $\lambda$  is saturated, thus  $\kappa^+$ -saturated. By Fact 6.10.12.(4), every model of size  $\geq \chi$  is  $\kappa^+$ -saturated. Now use (1) with Theorem 6.10.8.

REMARK 6.10.17. If one just wants to get strong superstability from categoricity, we suspect it should be possible to replace the  $\beth_{\kappa} = \kappa$  hypothesis by something more reasonable (maybe just asking for the categoricity cardinal to be above  $2^{\kappa}$ ). Since we are only interested in eventual behavior here, we leave this to future work.

As a final remark, we point out that it is always possible to get a good independence relation from superstability (i.e. even without categoricity) if one is willing to restrict the class to sufficiently saturated models:

FACT 6.10.18 (Corollary 7.3.5). Let **K** be an AEC. If **K** is  $\kappa$ -strongly  $\mu$ -superstable<sup>+</sup>, then whenever  $\lambda > (\mu^{<\kappa_r})^+$ ,  $\mathbf{K}^{\lambda\text{-sat}}$  is an AEC with  $\mathrm{LS}(\mathbf{K}^{\lambda\text{-sat}}) = \lambda$ .

COROLLARY 6.10.19. Let **K** be an AEC. If **K** is  $\kappa$ -strongly  $\mu$ -superstable<sup>+</sup>, then  $\mathbf{K}^{(\mu^{<\kappa_r})^{+2}\text{-sat}}$  is  $(\leq 1)$ -good. Moreover the good frame is induced by  $(<\kappa)$ -coheir.

PROOF. Combine Theorem 6.10.8.(2c) and Fact 6.10.18.

REMARK 6.10.20. Let **K** be an AEC in  $\lambda := LS(\mathbf{K})$  with amalgamation, joint embedding, and no maximal models. If  $\mathbf{K}^{\lambda\text{-sat}}$  is a nonempty AEC in  $\lambda$ , then the saturated model is superlimit (see [She09a, Definition 1.13]). Thus we even obtain a good frame in the sense of [She09a, Chapter II].

#### 6.11. Domination

Our next aim is to take a sufficiently nice good  $\lambda$ -frame (for types of length 1) and show that it can be extended to types of any length at most  $\lambda$ . To do this, we will give conditions under which a good  $\lambda$ -frame is *weakly successful* (a key technical property of [She09a, Chapter II], see Definition 6.11.4), and even  $\omega$ -successful (Definition 6.11.20).

The hypotheses we will work with are:

Hypothesis 6.11.1.

- (1)  $i = (\mathbf{K}, \downarrow)$  is a  $(< \infty, \ge \mu)$ -independence relation.
- (2)  $\mathfrak{s} := \operatorname{pre}(\mathfrak{i}^{\leq 1})$  is a type-full good  $(\geq \mu)$ -frame.
- (3)  $\lambda > \mu$  is a cardinal.
- (4) For all  $n < \omega$ :
  - (a)  $\mathbf{K}^{\lambda^{+n}\text{-sat}}$  is an AEC<sup>26</sup> with Löwenheim-Skolem-Tarski number  $\lambda^{+n}$ . (b)  $\kappa_{\lambda^{+n}}(\mathfrak{i}) = \lambda^{+n+1}$ .
- (5) i has base monotonicity, pre(i) has uniqueness.
- (6) i has the left and right ( $\leq \mu$ )-model-witness properties.

REMARK 6.11.2. We could have given more local hypotheses (e.g. by replacing  $\infty$  by  $\theta$  or only assuming (4) for *n* below some fixed  $m < \omega$ ) and made some of the required properties more precise (this is part of what should be done to improve "short" to "diagonally tame" in the main theorem, see the discussion in Section 6.15).

The key is that we assume there is *already* an independence notion for longer types. However, it is potentially weak compared to what we want. The next fact shows that the hypotheses above are reasonable.

FACT 6.11.3. Assume  $\mathbf{K}^0$  is a fully  $(<\kappa)$ -tame and short  $\kappa$ -strongly  $\mu_0$ -superstable AEC with amalgamation. Then for any  $\mu \geq (\mu_0^{<\kappa_r})^{+2}$  and any  $\lambda > \mu$  with  $\lambda = \lambda^{<\kappa_r}$ , Hypothesis 6.11.1 holds for  $\mathbf{K} := (\mathbf{K}^0)^{\mu\text{-sat}}$  and  $\mathfrak{i} := \mathfrak{i}_{\kappa\text{-ch}}(\mathbf{K}^0) \upharpoonright \mathbf{K}$ .

PROOF. By Fact 6.10.18, for any  $\mu' \ge \mu$ ,  $\mathbf{K}^{\mu'\text{-sat}}$  is an AEC with  $\mathrm{LS}(\mathbf{K}^{\mu'\text{-sat}}) = \mu'$ . By Theorem 6.10.8.(2c),  $(< \kappa)$ -coheir induces a good  $(\ge \mu)$ -frame for  $\mu$ -saturated models. The other conditions follow directly from the definition of strong superstability and the properties of coheir (Fact 2.5.15). For example, the local character condition holds because  $\lambda^{<\kappa_r} = \lambda$  implies  $(\lambda^{+n})^{<\kappa_r} = \lambda^{+n}$  for any  $n < \omega$ .

The next technical property is of great importance in Chapter II and III of [She09a]. The definition below follows [JS13, Definition 4.1.5] (but as usual, we work only with type-full frames).

DEFINITION 6.11.4. Let  $\mathfrak{t}$  be a type-full good  $\lambda_{\mathfrak{t}}$ -frame.

- (1) For  $M_0 \leq_{\mathbf{K}} M_\ell$  in  $\mathbf{K}$ ,  $\ell = 1, 2$ , an amalgam of  $M_1$  and  $M_2$  over  $M_0$  is a triple  $(f_1, f_2, N)$  such that  $N \in \mathbf{K}_t$  and  $f_\ell : M_\ell \xrightarrow[M_0]{} N$ .
- (2) Let  $(f_1^x, f_2^x, N^x)$ , x = a, b be amalgams of  $M_1$  and  $M_2$  over  $M_0$ . We say  $(f_1^a, f_2^a, N^a)$  and  $(f_1^b, f_2^b, N^b)$  are equivalent over  $M_0$  if there exists  $N_* \in \mathbf{K}_t$  and  $f^x : N^x \to N_*$  such that  $f^b \circ f_1^b = f^a \circ f_1^a$  and  $f^b \circ f_2^b = f^a \circ f_2^a$ , namely, the following commutes:

<sup>&</sup>lt;sup>26</sup>Thus we have a superlimit of size  $\lambda^{+n}$ , see Remark 6.10.20.


Note that being "equivalent over  $M_0$ " is an equivalence relation ([**JS13**, Proposition 4.3]).

- (3) Let  $\mathbf{K}_{t}^{3,\mathrm{uq}}$  be the set of triples (a, M, N) such that  $M \leq_{\mathbf{K}} N$  are in  $\mathbf{K}$ ,  $a \in |N| \setminus |M|$  and for any  $M_1 \geq_{\mathbf{K}} M$  in **K**, there exists a unique (up to equivalence over M) amalgam  $(f_1, f_2, N_1)$  of N and  $M_1$  over M such that  $gtp(f_1(a)/f_2[M_1]; N_1)$  does not fork over M. We call the elements of  $\mathbf{K}^{3,\mathrm{uq}}_{\mathfrak{t}}$  uniqueness triples.
- (4)  $\mathbf{K}_{\mathfrak{t}}^{3,\mathrm{uq}}$  has the *existence property* if for any  $M \in \mathbf{K}_{\mathfrak{t}}$  and any nonalgebraic  $p \in gS(M)$ , one can write p = gtp(a/M; N) with  $(a, M, N) \in \mathbf{K}_t^{3, uq}$ . We also talk about the existence property for uniqueness triples. (5)  $\mathfrak{s}$  is weakly successful if  $\mathbf{K}_{t}^{3,\mathrm{uq}}$  has the existence property.

The uniqueness triples can be seen as describing a version of domination. They were introduced by Shelah for the purpose of starting with a good  $\lambda$ -frame and extending it to a good  $\lambda^+$ -frame. The idea is to first extend the good  $\lambda$ -frame to a forking notion for types of models of size  $\lambda$  (and really this is what interests us here, since tameness already gives us a good  $\lambda^+$ -frame). Now, since we already have an independence notion for longer types, we can follow [MS90, Definition 4.21] and give a more explicit version of domination that is exactly as in the first-order case.

Definition 6.11.5 (Domination). Fix  $N \in \mathbf{K}$ . For  $M \leq_{\mathbf{K}} N, B, C \subseteq |N|, B$ dominates C over M in N if for any  $N' \geq_{\mathbf{K}} N$  and any  $D \subseteq |N'|, B \bigcup_{M}^{N'} D$  implies M'

$$B \cup C \bigcup_{M}^{N} D.$$

We say that B model-dominates C over M in N if for any  $N' \geq_{\mathbf{K}} N$  and any  $M \leq_{\mathbf{K}} N'_0 \leq_{\mathbf{K}} N', \ B \downarrow_M^{N'} N'_0 \text{ implies } B \cup C \downarrow_M^{N'} N'_0.$ 

Model-domination turns out to be the technical variation we need, but of course if i has extension, then it is equivalent to domination. We start with two easy ambient monotonicity properties:

LEMMA 6.11.6. Let  $M \leq_{\mathbf{K}} N$ . Let  $B, C \subseteq |N|$  and assume B [model-]dominates C over M in N. Then:

- (1) If  $N' \geq_{\mathbf{K}} N$ , then B [model-]dominates C over M in N'.
- (2) If  $M \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$  contains  $B \cup C$ , then B [model-]dominates C over M in  $N_0$ .

PROOF. We only do the proofs for the non-model variation but of course the model variation is completely similar.

- (1) By definition of domination.
- (2) Let  $N' \geq_{\mathbf{K}} N_0$  and  $D \subseteq |N'|$  be given such that  $B \stackrel{N'}{\downarrow} D$ . By amalgamation, there exists  $N'' \geq_{\mathbf{K}} N$  and  $f : N' \xrightarrow[N_0]{} N''$ . By invariance,  $B \stackrel{N''}{\downarrow} f[D]$ . By definition of domination,  $B \cup C \stackrel{f[N']}{\downarrow} f[D]$ . By invariance again,  $B \cup C \stackrel{N'}{\downarrow} D$ , as desired.

The next result is key for us: it ties domination with the notion of uniqueness triples:

LEMMA 6.11.7. Assume  $M_0 \leq_{\mathbf{K}} M_1$  are in  $\mathbf{K}_{\lambda}$ , and  $a \in M_1$  model-dominates  $M_1$  over  $M_0$  (in  $M_1$ ). Then  $(a, M_0, M_1) \in \mathbf{K}^{3, uq}_{\mathfrak{s}_{\lambda}}$ .

PROOF. Let  $M_2 \geq_{\mathbf{K}} M_0$  be in  $\mathbf{K}_{\lambda}$ . First, we need to show that there exists  $(b, M_2, N)$  such that  $\operatorname{gtp}(b/M_2; N)$  extends  $\operatorname{gtp}(a/M_0; M_1)$  and  $\operatorname{gtp}(b/M_2; N)$  does not fork over  $M_0$ . This holds by the extension property of good frames.

Second, we need to show that any such amalgam is unique: Let  $(f_1^x, f_2^x, N^x)$ ,  $x \in \{a, b\}$  be amalgams of  $M_1$  and  $M_2$  over  $M_0$  such that  $f_1^x(a) \bigcup_{M_0}^{N^x} f_2^x[M_2]$ . We want to show that the two amalgams are equivalent: we want  $N_* \in \mathbf{K}_{\lambda}$  and  $f^x : N^x \to N_*$  such that  $f^b \circ f_1^b = f^a \circ f_1^a$  and  $f^b \circ f_2^b = f^a \circ f_2^a$ , namely, the following commutes:



For x = a, b, rename  $f_2^x$  to the identity to get amalgams  $((f_1^x)', \mathrm{id}_{M_2}, (N^x)')$ of  $M_1$  and  $M_2$  over  $M_0$ . For x = a, b, the amalgams  $((f_1^x)', \mathrm{id}_{M_2}, (N^x)')$  and  $(f_1^x, f_2^x, N^x)$  are equivalent over  $M_0$ , hence we can assume without loss of generality that the renaming has already been done and  $f_2^x = \mathrm{id}_{M_2}$ 

Thus we know that  $f_1^x(a) \underset{M_0}{\overset{N^x}{\downarrow}} M_2$  for x = a, b. By domination,  $f_1^x[M_1] \underset{M_0}{\overset{N^x}{\downarrow}} M_2$ . Let  $\overline{M}_1$  be an enumeration of  $M_1$ . Using amalgamation, we can obtain the following diagram:



#### 6.11. DOMINATION

This shows  $\operatorname{gtp}(f_1^a(\bar{M}_1)/M_0; N^a) = \operatorname{gtp}(f_1^b(\bar{M}_1)/M_0; N^b)$ . By uniqueness,  $\operatorname{gtp}(f_1^a(\bar{M}_1)/M_2; N^a) = \operatorname{gtp}(f_1^b(\bar{M}_1)/M_2; N^b)$ . Let  $N_*$  and  $f^x : N^x \xrightarrow{M_2} N_*$  witness the equality. Since  $f_2^x = \operatorname{id}_{M_2}, f^b \circ f_2^b = f^b \upharpoonright M_2 = \operatorname{id}_{M_2} = f^a \circ f_2^a$ . Moreover,  $(f^b \circ f_1^b)(\bar{M}_1) = f^b(f_1^b(\bar{M}_1)) = f^a(f_2^a(\bar{M}_1))$  by definition, so  $f^b \circ f_1^b = f^a \circ f_1^a$ . This completes the proof.

REMARK 6.11.8. The converse holds if i has left extension.

REMARK 6.11.9. The relationship of uniqueness triples with domination is already mentioned in [**JS13**, Proposition 4.1.7], although the definition of domination there is different.

Thus to prove the existence property for uniqueness triples, it will be enough to imitate the proof of [**MS90**, Proposition 4.22], which gives conditions under which the hypothesis of Lemma 6.11.7 holds. We first show that we can work inside a local monster model.

LEMMA 6.11.10. Let  $M \leq_{\mathbf{K}} N$  and  $B \subseteq |N|$ . Let  $\mathfrak{C} \geq N$  be  $||N||^+$ -saturated. Then B model-dominates N over M in  $\mathfrak{C}$  if and only if for any  $M' \leq_{\mathbf{K}} \mathfrak{C}$  with  $\overset{\mathfrak{C}}{\underset{M}{\longrightarrow}} M'$ ,  $B \underset{M}{\overset{\mathfrak{C}}{\longrightarrow}} M'$  implies  $N \underset{M}{\overset{\mathfrak{C}}{\longrightarrow}} M'$ . Moreover if  $\mathfrak{i}$  has the right  $(\leq \mu)$ -witness property, we get an analogous result for domination instead of model-domination.

PROOF. We prove the non-trivial direction for model-domination. The proof of the moreover part for domination is similar. Assume  $\mathfrak{C}' \geq \mathfrak{C}$  and  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} \mathfrak{C}'$ is such that  $B \underset{M}{\cup} M'$ . We want to show that  $N \underset{M}{\cup} M'$ . Suppose not. Then we can use the  $(\leq \mu)$ -model-witness property to assume without loss of generality that  $||M'|| \leq \mu + ||M||$ , and so we can find  $N \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} \mathfrak{C}'$  containing M'with ||N'|| = ||N|| and  $B \underset{M}{\cup} M'$ ,  $N \underset{M}{\not{}} M'$ . By homogeneity, find  $f : N' \underset{N}{\to} \mathfrak{C}$ . By invariance,  $B \underset{M}{\overset{f[N']}{\cup}} f[M']$  but  $N \underset{M}{\overset{f[N']}{\not{}}} f[M']$ . By monotonicity,  $B \underset{M}{\overset{\mathfrak{C}}{\to}} f[M']$  but  $N \underset{M}{\not{}} f[M']$ , a contradiction.  $\Box$ 

LEMMA 6.11.11 (Lemma 4.20 in [**MS90**]). Let  $\langle M_i : i < \lambda^+ \rangle$ ,  $\langle N_i : i < \lambda^+ \rangle$ be increasing continuous in  $\mathbf{K}_{\lambda}$  such that  $M_i \leq_{\mathbf{K}} N_i$  for all  $i < \lambda^+$ . Let  $M_{\lambda^+} := \bigcup_{i < \lambda^+} M_i$ ,  $N_{\lambda^+} := \bigcup_{i < \lambda^+} N_i$ .

Then there exists  $i < \lambda^+$  such that  $N_i \bigcup_{M_i}^{N_{\lambda^+}} M_{\lambda^+}$ .

PROOF. For each  $i < \lambda^+$ , let  $j_i < \lambda^+$  be least such that  $N_i \underset{M_{j_i}}{\bigcup} M_{\lambda^+} (\text{exists}$ since  $\kappa_{\lambda}(\mathfrak{i}) = \lambda^+$ ). Let  $i^*$  be such that  $j_i < i^*$  for all  $i < i^*$  and  $\operatorname{cf} i^* \ge \mu^+$ . By definition of  $j_i$  and base monotonicity we have that for all  $i < i^*$ ,  $N_i \underset{M_{i^*}}{\bigcup} M_{\lambda^+}$ . By the left  $(\leq \mu)$ -model-witness property,  $N_{i^*} \underset{M_{i^*}}{\bigcup} M_{\lambda^+}$ .

LEMMA 6.11.12 (Proposition 4.22 in [MS90]). Let  $M \in \mathbf{K}_{\lambda}$  be saturated. Let

 $\mathfrak{C} \geq_{\mathbf{K}} M$  be saturated of size  $\lambda^+$ . Work inside  $\mathfrak{C}$ . Write  $A \underset{M}{\downarrow} B$  for  $A \underset{M}{\downarrow} \mathfrak{C} B$ .

- There exists a saturated  $N \leq_{\mathbf{K}} \mathfrak{C}$  in  $\mathbf{K}_{\lambda}$  such that  $M \leq_{\mathbf{K}} N$ , N contains a, and a model-dominates N over M (in  $\mathfrak{C}$ ).
- In fact, if  $M^* \leq_{\mathbf{K}} M$  is in  $\mathbf{K}_{<\lambda}$ ,  $a \downarrow_{M^*} M$ , and  $r \in gS^{\leq \lambda}(M^*a)$ , then N can be chosen so that it realizes r

be chosen so that it realizes r.

PROOF. Since  $\bar{\kappa}_1(\mathfrak{s}) = \mu^+ \leq \lambda$ , it suffices to prove the second part. Assume it fails.

<u>Claim</u>: For any saturated  $M' \geq_{\mathbf{K}} M$  in  $\mathbf{K}_{\lambda}$ , if  $a \underset{M}{\downarrow} M'$ , then the second part fails with M' replacing M.

<u>Proof of claim</u>: By transitivity,  $a \underset{M^*}{\downarrow} M'$ . By uniqueness of saturated models, there exists  $f: M' \cong_{M^*} M$ , which we can extend to an automorphism of  $\mathfrak{C}$ . Thus we also have  $f(a) \underset{M^*}{\downarrow} M$ . By uniqueness, we can assume without loss of generality that f fixes a as well. Since the second part above is invariant under applying  $f^{-1}$ , the result follows.

We now construct increasing continuous chains  $\langle M_i : i \leq \lambda^+ \rangle$ ,  $\langle N_i : i \leq \lambda^+ \rangle$ such that for all  $i < \lambda^+$ :

(1)  $M_0 = M$ . (2)  $M_i \leq_{\mathbf{K}} N_i$ . (3)  $M_i \in \mathbf{K}_{\lambda}$  is saturated. (4)  $a \downarrow M_i$ . (5)  $N_i \not\downarrow M_{i+1}$ .

This is enough: the sequences contradict Lemma 6.11.11. This is possible: take  $M_0 = M$ , and  $N_0$  any saturated model of size  $\lambda$  containing  $M_0$  and a and realizing r. At limits, take unions (we are using that  $\mathbf{K}^{\lambda\text{-sat}}$  is an AEC). Now assume everything up to i has been constructed. By the claim, the second part above fails for  $M_i$ , so in particular  $N_i$  cannot be model-dominated by a over  $M_i$ . Thus (implicitly using Lemma 6.11.10) there exists  $M'_i \geq_{\mathbf{K}} M_i$  with  $a \underset{M_i}{\downarrow} M'_i$  and  $N_i \underset{M_i}{\not{\downarrow}} M'_i$ . By the model-witness property, we can assume without loss of generality that  $||M'_i|| \leq \lambda$ , so using extension and transitivity, we can find  $M_{i+1} \in \mathbf{K}_{\lambda}$  saturated containing  $M'_i$  so that  $a \underset{M_i}{\downarrow} M_{i+1}$ . By monotonicity we still have  $N_i \underset{M_i}{\not{\downarrow}} M_{i+1}$ . Let  $N_{i+1} \in \mathbf{K}_{\lambda}$  be any saturated model containing  $N_i$  and  $M_{i+1}$ .

THEOREM 6.11.13.  $\mathfrak{s}_{\lambda} \upharpoonright \mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  is a *weakly successful* type-full good  $\lambda$ -frame.

PROOF. Since  $\mathfrak{s}_{\lambda}$  is a type-full good frame,  $\mathfrak{s}_{\lambda} \upharpoonright \mathbf{K}_{\lambda}^{\lambda-\operatorname{sat}}$  also is. To show it is weakly successful, we want to prove the existence property for uniqueness triples. So let  $M \in \mathbf{K}_{\lambda}^{\lambda-\operatorname{sat}}$  and  $p \in \operatorname{gS}(M)$  be nonalgebraic. Say  $p = \operatorname{gtp}(a/M; N')$ . Let  $\mathfrak{C}$ be a monster model with  $N' \leq_{\mathbf{K}} \mathfrak{C}$ . By Lemma 6.11.12, there exists  $N \leq_{\mathbf{K}} \mathfrak{C}$  in  $\mathbf{K}_{\lambda}^{\lambda-\operatorname{sat}}$  such that  $M \leq_{\mathbf{K}} N$ ,  $a \in |N|$ , and a dominates N over M in  $\mathfrak{C}$ . By Lemma 6.11.6, a dominates N over M in N. By Lemma 6.11.7,  $(a, M, N) \in \mathbf{K}_{\mathfrak{s}_{\lambda} \upharpoonright \mathbf{K}_{\lambda}^{\lambda-\operatorname{sat}}}^{3,\operatorname{uq}}$ . Now,  $p = \operatorname{gtp}(a/M; N') = \operatorname{gtp}(a/M; \mathfrak{C}) = \operatorname{gtp}(a/M; N)$ , as desired.  $\Box$  The term "weakly successful" suggests that there must exist a definition of "successful". This is indeed the case:

DEFINITION 6.11.14 (Definition 10.1.1 in [**JS13**]). A type-full good  $\lambda_t$ -frame t is successful if it is weakly successful and  $\leq_{\lambda_t^+}$  has smoothness: whenever  $\langle N_i : i \leq \delta \rangle$  is a  $\leq_{\lambda_t^+}$ -increasing continuous chain of saturated models in  $(\mathbf{K}_t^{up})_{\lambda_t^+}, N \in (\mathbf{K}_t^{up})_{\lambda_t^+}$  is saturated and  $i < \delta$  implies  $N_i \leq_{\lambda_t^+} N$ , then  $N_\delta \leq_{\lambda_t^+} N$ .

We will not define  $\leq_{\lambda_{\iota}^+}$  (the interested reader can consult e.g. [JS13, Definition 6.14]). The only fact about it we will need is:

FACT 6.11.15 (Theorem 7.8 in [**Jar16**]). If t is a weakly successful type-full good  $\lambda_t$ -frame,  $(\mathbf{K}_t^{up})_{[\lambda_t, \lambda_t^+]}$  has amalgamation and is  $\lambda_t$ -tame, then  $\leq_{\mathbf{K}} \upharpoonright (\mathbf{K}_t^{up})_{\lambda_t^+}^{\lambda_t^+$ -sat  $=\leq_{\lambda_t^+}$ .

COROLLARY 6.11.16.  $\mathfrak{s}_{\lambda} \upharpoonright \mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  is a *successful* type-full good  $\lambda$ -frame.

PROOF. By Theorem 6.11.13,  $\mathfrak{s}_{\lambda} \upharpoonright \mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  is weakly successful. To show it is successful, it is enough (by Fact 6.11.15), to see that  $\leq_{\mathbf{K}}$  has smoothness. But this holds since **K** is an AEC.

For a good  $\lambda_t$ -frame  $\mathfrak{t}$ , Shelah also defines a  $\lambda_t^+$ -frame  $\mathfrak{t}^+$  ([She09a, Definition III.1.7]). He then goes on to show:

FACT 6.11.17 (Claim III.1.9 in [She09a]). If t is a successful good  $\lambda_t$ -frame, then t<sup>+</sup> is a good<sup>27</sup>  $\lambda_t^+$ -frame.

REMARK 6.11.18. This does not use the weak continuum hypothesis.

Note that in our case, it is easy to check that:

FACT 6.11.19.  $(\mathfrak{s}_{\lambda})^+ = \mathfrak{s}_{\lambda^+} \upharpoonright \mathbf{K}_{\lambda^+}^{\lambda^+-\mathrm{sat}}.$ 

DEFINITION 6.11.20 (Definition III.1.12 in [She09a]). Let  $\mathfrak{t}$  be a pre- $\lambda_{\mathfrak{t}}$ -frame.

- (1) By induction on  $n < \omega$ , define  $\mathfrak{t}^{+n}$  as follows:
  - (a)  $\mathfrak{t}^{+0} = \mathfrak{t}$ .
  - (b)  $\mathfrak{t}^{+(n+1)} = (\mathfrak{t}^{+n})^+.$
- (2) By induction on  $n < \omega$ , define "t is *n*-successful" as follows:
  - (a)  $\mathfrak{t}$  is 0-successful if and only if it is a good  $\lambda$ -frame.
  - (b)  $\mathfrak{t}$  is (n + 1)-successful if and only if it is a successful good  $\lambda$ -frame and  $\mathfrak{t}^+$  is *n*-successful.
- (3)  $\mathfrak{t}$  is  $\omega$ -successful if it is n-successful for all  $n < \omega$ .

Thus by Fact 6.11.17, t is 1-successful if and only if it is a successful good  $\lambda_t$ -frame. More generally a good  $\lambda_t$ -frame t is *n*-successful if and only if  $t^{+m}$  is a successful good  $\lambda_t^{+m}$ -frame for all m < n.

THEOREM 6.11.21.  $\mathfrak{s}_{\lambda} \upharpoonright \mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  is an  $\omega$ -successful type-full good  $\lambda$ -frame.

PROOF. By induction on  $n < \omega$ , simply observing that we can replace  $\lambda$  by  $\lambda^{+n}$  in Corollary 6.11.16.

 $<sup>^{27}\</sup>mathrm{Shelah}$  proves that  $\mathfrak{t}^+$  is actually good<sup>+</sup>. There is no reason to define what this means here.

We emphasize again that we did *not* use the weak continuum hypothesis (as Shelah does in [She09a, Chapter II]). We pay for this by using tameness (in Fact 6.11.3). Note that all the results of [She09a, Chapter III] apply to our  $\omega$ -successful good frame.

Recall that part of Shelah's point is that  $\omega$ -successful good  $\lambda$ -frames extend to  $(\geq \lambda)$ -frames. However this is secondary for us (since tameness already implies that a frame extends to larger models, see [Bon14a] and Chapter 5). Really, we want to extend the good frame to longer *types*. We show that it is possible in the next section.

# 6.12. A fully good long frame

HYPOTHESIS 6.12.1.  $\mathfrak{s} = (\mathbf{K}, \downarrow)$  is a weakly successful type-full good  $\lambda$ -frame.

This is reasonable since the previous section showed us how to build such a frame. Our goal is to extend  $\mathfrak{s}$  to obtain a fully good ( $\leq \lambda, \lambda$ )-independence relation. Most of the work has already been done by Shelah:

FACT 6.12.2 (Conclusion II.6.34 in [She09a]). There exists a relation NF  $\subseteq$  <sup>4</sup>K satisfying:

- (1) NF $(M_0, M_1, M_2, M_3)$  implies  $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$  are in **K** for  $\ell = 1, 2$ .
- (2) NF $(M_0, M_1, M_2, M_3)$  and  $a \in |M_1| \setminus |M_2|$  implies  $gtp(a/M_2; M_3)$  does not  $\mathfrak{s}$ -fork over  $M_0$ .
- (3) Invariance: NF is preserved under isomorphisms.
- (4) Monotonicity: If  $NF(M_0, M_1, M_2, M_3)$ :

  - (a) If  $M_0 \leq_{\mathbf{K}} M'_{\ell} \leq_{\mathbf{K}} M_{\ell}$  for  $\ell = 1, 2$ , then NF $(M_0, M'_1, M'_2, M'_3)$ . (b) If  $M'_3 \leq_{\mathbf{K}} M_3$  contains  $|M_1| \cup |M_2|$ , then NF $(M_0, M_1, M_2, M'_3)$ .
  - (c) If  $M'_3 \geq_{\mathbf{K}} M_3$ , then NF $(M_0, M_1, M_2, M'_3)$ .
- (5) Symmetry:  $NF(M_0, M_1, M_2, M_3)$  if and only if  $NF(M_0, M_2, M_1, M_3)$ .
- (6) Long transitivity: If  $\langle M_i : i \leq \alpha \rangle$ ,  $\langle N_i : i \leq \alpha \rangle$  are increasing continuous and NF $(M_i, N_i, M_{i+1}, N_{i+1})$  for all  $i < \alpha$ , then NF $(M_0, N_0, M_\alpha, N_\alpha)$ .
- (7) Independent amalgamation: If  $M_0 \leq_{\mathbf{K}} M_\ell$ ,  $\ell = 1, 2$ , then for some  $M_3 \in \mathbf{K}$ ,  $f_\ell : M_\ell \xrightarrow{M_0} M_3$ , we have  $\operatorname{NF}(M_0, f_1[M_1], f_2[M_2], M_3)$ .
- (8) Uniqueness: If NF $(M_0^{\ell}, M_1^{\ell}, M_2^{\ell}, M_3^{\ell}), \ \ell = 1, 2, \ f_i : M_i^1 \cong M_i^2$  for  $i = 1, 2, \ f_i : M_i^1 \cong M_i^2$ 0, 1, 2, and  $f_0 \subseteq f_1$ ,  $f_0 \subseteq f_2$ , then  $f_1 \cup f_2$  can be extended to  $f_3 : M_3^1 \to M_4^2$ , for some  $M_4^2$  with  $M_3^2 \leq_{\mathbf{K}} M_4^2$ .

NOTATION 6.12.3. We write  $M_1 \underset{M_0}{\overset{M_3}{\downarrow}} M_2$  instead of NF $(M_0, M_1, M_2, M_3)$ . If  $\bar{a}$  is a sequence, we write  $\bar{a} \underset{M_0}{\overset{M_3}{\downarrow}} M_2$  for ran $(\bar{a}) \underset{M_0}{\overset{M_3}{\downarrow}} M_2$ , and similarly if sequences appear

at other places.

REMARK 6.12.4. Shelah's definition of NF ([She09a, Definition II.6.12]) is very complicated. It is somewhat simplified in [JS13].

REMARK 6.12.5. Shelah calls such an NF a nonforking relation which respects  $\mathfrak{s}$  ([She09a, Definition II.6.1]). While there are similarities with this chapter's definition of a good ( $\leq \lambda$ )-frame, note that NF is only defined for types of models while we would like to make it into a relation for arbitrary types of length at most λ.

We start by showing that uniqueness is really the same as the uniqueness property stated for frames. We drop Hypothesis 6.12.1 for the next lemma.

LEMMA 6.12.6. Let **K** be an AEC in  $\lambda$  and assume **K** has amalgamation. The following are equivalent for a relation NF  $\subseteq$  <sup>4</sup>**K** satisfying (1), (3), (4) of Fact 6.12.2:

- (1) Uniqueness in the sense of Fact 6.12.2.(8).
- (2) Uniqueness in the sense of frames: If  $M \bigcup_{M_0}^N M_1$  and  $M' \bigcup_{M_0}^{N'} M_1$  for models  $M, M' \in$

 $K, \bar{a} \text{ and } \bar{a}' \text{ are enumerations of } M \text{ and } M' \text{ respectively, } p := \operatorname{gtp}(\bar{a}/M_1; N), q := \operatorname{gtp}(\bar{a}'/M_1; N'), \text{ and } p \upharpoonright M_0 = q \upharpoonright M_0, \text{ then } p = q.$ 

Proof.

• (1) implies (2): Since  $p \upharpoonright M_0 = q \upharpoonright M_0$ , there exists  $N'' \geq_{\mathbf{K}} N'$  and  $f: N \xrightarrow[M_0]{} N''$  such that  $f(\bar{a}) = \bar{a}'$ . Therefore by invariance,  $\bar{a}' \bigcup_{M_0}^{N''} f[M_1]$ . Let  $f_0 := \operatorname{id}_{M_0}, f_1 := f^{-1} \upharpoonright f[M_1], f_2 := \operatorname{id}_{M'}$ . By uniqueness, there exists  $N''' \geq_{\mathbf{K}} N'', g \supseteq f_1 \cup f_2, g : N'' \to N'''$ . Consider the map  $h := g \circ f : N \to N'''$ . Then  $g \upharpoonright M_1 = \operatorname{id}_{M_1}$  and  $h(\bar{a}) = g(\bar{a}') = \bar{a}'$ , so hwitnesses p = q.

• (2) implies (1): By some renaming, it is enough to prove that whenever  $M_2 \underset{M_0}{\downarrow} M_1$  and  $M_2 \underset{M_0}{\downarrow} M_1$ , there exists  $N'' \ge_{\mathbf{K}} N'$  and  $f: N' \xrightarrow[|M_1|\cup|M_2|]{} N''$ . Let  $\bar{a}$  be an enumeration of  $M_2$ . Let  $p := \operatorname{gtp}(\bar{a}/M_1; N), q := \operatorname{gtp}(\bar{a}/M_1; N')$ . We have that  $p \upharpoonright M_0 = \operatorname{gtp}(\bar{a}/M_1; M_2) = q \upharpoonright M_0$ . Thus p = q, so there exists  $N'' \ge_{\mathbf{K}} N'$  and  $f: N \xrightarrow[M_1]{} N''$  such that  $f(\bar{a}) = \bar{a}$ . In other words, f fixes  $M_2$ , so is the desired map.

We now extend NF to take sets on the left hand side. This step is already made by Shelah in [She09a, Claim III.9.6], for singletons rather than arbitrary sets. We check that Shelah's proofs still work.

DEFINITION 6.12.7. Define NF'( $M_0, A, M, N$ ) to hold if and only if  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  are in  $\mathbf{K}, A \subseteq |N|$ , and there exists  $N' \geq_{\mathbf{K}} N, N_A \geq_{\mathbf{K}} M$  with  $N_A \leq_{\mathbf{K}} N'$  and  $N_A \downarrow_{M_0}^{N'} M$ . We abuse notation and also write  $A \downarrow_{M_0}^{N} M$  instead of NF'( $M_0, A, M, N$ ). We let  $\mathfrak{t} := (\mathbf{K}, \downarrow)$ .

REMARK 6.12.8. Compare with the definition of cl (Definition 6.3.9).

PROPOSITION 6.12.9.

- (1) If  $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$ ,  $\ell = 1, 2$ , then NF $(M_0, M_1, M_2, M_3)$  if and only if NF' $(M_0, M_1, M_2, M_3)$ .
- (2)  $\mathfrak{t}$  is a (type-full) pre-( $\leq \lambda, \lambda$ )-frame.
- (3) t has base monotonicity, full symmetry, uniqueness, existence, and extension.

PROOF. Exactly as in [She09a, Claim III.9.6]. Shelah omits the proof of uniqueness, so we give it here. For notational simplicity, let us work in a local monster model  $\mathfrak{C} \in \mathbf{K}_{\lambda^+}^{\lambda^+}$ , and write  $A \underset{M_0}{\downarrow} M_1$  instead of  $A \underset{M_0}{\downarrow} M_1$ . Let  $\alpha \leq \lambda$  and assume that  $p, q \in \mathrm{gS}^{\alpha}(M)$  are given such that  $p = \mathrm{gtp}(\bar{a}/M), q = \mathrm{gtp}(\bar{a}'/M)$ . Assume that  $M_0 \leq_{\mathbf{K}} M$  is such that both p and q do not fork over  $M_0$  (in the sense of NF'). We want to see that p = q.

By definition, there exists  $M_{\bar{a}} \in \mathbf{K}_{\lambda}$  such that  $M_0 \leq_{\mathbf{K}} M_{\bar{a}}, \bar{a} \in {}^{\alpha}|M_{\bar{a}}|$ , and  $M_{\bar{a}} \downarrow M$ . By symmetry for NF,  $M \downarrow M_{\bar{a}}$ . Similarly, there exists a model  $M_{\bar{a}'} \in \mathbf{K}_{\lambda}$  containing  $\bar{a}'$  such that  $M_0 \leq_{\mathbf{K}} M_{\bar{a}'}$  and  $M \downarrow M_{\bar{a}'}$ . Since  $p \upharpoonright M_0 = q \upharpoonright M_0$ , there exists an automorphism f of  $\mathfrak{C}$  fixing  $M_0$  such that  $M_0 \leq_{\mathbf{K}} M_{\bar{a}'}$  and  $M \downarrow M_{\bar{a}'}$ .

Since  $p \upharpoonright M_0 = q \upharpoonright M_0$ , there exists an automorphism f of  $\mathfrak{C}$  fixing  $M_0$  such that  $f(\bar{a}) = \bar{a}'$ . By invariance,  $M \underset{M_0}{\cup} M_{\bar{a}'}$  and  $f[M] \underset{M_0}{\cup} f[M_{\bar{a}}]$ , and both  $M_{\bar{a}'}$  and  $f[M_{\bar{a}}]$  contain  $\bar{a}'$ . By Lemma 6.12.6 and the proof of Lemma 3.5.4(3), we have that (for some enumeration  $\bar{c}$  of M) gtp $(\bar{c}/M_0\bar{a}') = \text{gtp}(f(\bar{c})/M_0\bar{a}')$ . Thus we can pick an automorphism g of  $\mathfrak{C}$  fixing  $M_0\bar{a}'$  and sending  $f(\bar{c})$  back to  $\bar{c}$ . Now  $f \circ g^{-1}$  shows that gtp $(\bar{a}/M) = \text{gtp}(\bar{a}'/M)$ , i.e. p = q as needed.

We now turn to local character. The key is:

FACT 6.12.10 (Claim III.1.17 in [She09a]). Let  $\delta \leq \lambda^+$  be a limit ordinal. Given  $\langle M_i : i \leq \delta \rangle$  increasing continuous, we can build  $\langle N_i : i \leq \delta \rangle$  increasing continuous such that for all  $i \leq j \leq \delta$  with  $j < \lambda^+$ ,  $N_i \downarrow M_j$  and  $M_{\delta} <_{\mathbf{K}}^{\mathrm{univ}} N_{\delta}$ .

LEMMA 6.12.11. For all  $\alpha < \lambda$ ,  $\kappa_{\alpha}(\mathfrak{t}) = |\alpha|^{+} + \aleph_{0}$ . Moreover if  $\langle M_{i} : i < \lambda^{+} \rangle$  is increasing in  $\mathbf{K}_{\lambda}$  and  $p \in \mathrm{gS}^{\lambda}(\bigcup_{i < \lambda^{+}} M_{i})$ , there exists  $i < \lambda^{+}$  such that  $p \upharpoonright M_{j}$  does not fork over  $M_{i}$  for all  $j \geq i$ .

PROOF. Let  $\alpha < \lambda$ . Let  $\langle M_i : i \leq \delta + 1 \rangle$  be increasing continuous with  $\delta = \operatorname{cf} \delta > |\alpha|$ . Let  $A \subseteq |M_{\delta+1}|$  have size  $\leq \alpha$ . Let  $\langle N_i : i \leq \delta \rangle$  be as given by Fact 6.12.10. By universality, we can assume without loss of generality that  $M_{\delta+1} \leq_{\mathbf{K}} N_{\delta}$ . Thus  $A \subseteq |N_{\delta}|$  and by the cofinality hypothesis, there exists  $i < \delta$  such that  $A \subseteq |N_i|$ . In particular,  $A \underset{M_i}{\downarrow} M_{\delta}$ , so  $A \underset{M_i}{\downarrow} M_{\delta}$ , as needed. The proof of the moreover part is completely similar.

REMARK 6.12.12. In **[JS13**] (and later in **[JS12, Jar, Jar16**]), the authors have considered *semi-good*  $\lambda$ -frames, where the stability condition is replaced by almost stability  $(|gS(M)| \leq \lambda^+ \text{ for all } M \in \mathbf{K}_{\lambda})$ , and an hypothesis called the conjugation property is often added. Several of the above results carry through in that setup but we do not know if Lemma 6.12.11 would also hold.

We come to the last property: disjointness. The situation is a bit murky: At first glance, Fact 6.12.2.(2) seems to give it to us for free (since we are assuming  $\mathfrak{s}$  has disjointness), but unfortunately we are assuming  $a \notin |M_2|$  there. We will obtain it with the additional hypothesis of categoricity in  $\lambda$  (this is reasonable since if the frame has a superlimit, see Remark 6.10.20, one can always restrict oneself to the class generated by the superlimit). Note that disjointness is never used in a crucial way in this chapter (but it is always nice to have, as it implies for example disjoint amalgamation when combined with independent amalgamation).

LEMMA 6.12.13. If **K** is categorical in  $\lambda$ , then  $\mathfrak{t}$  has disjointness and  $\mathfrak{t}^{\leq 1} = \mathfrak{s}$ .

**PROOF.** We have shown that  $\mathfrak{t}^{\leq 1}$  has all the properties of a good frame except perhaps disjointness so by the proof of Theorem 6.9.7 (which never relied on disjointness),  $\mathfrak{s} = \mathfrak{t}^{\leq 1}$ . Since  $\mathfrak{s}$  has disjointness,  $\mathfrak{t}^{\leq 1}$  also does, and therefore  $\mathfrak{t}$  has disjointness.  $\square$ 

What about continuity for chains? The long transitivity property seems to suggest we can say something, and indeed we can:

FACT 6.12.14. Assume  $\lambda = \lambda_0^{+3}$  and there exists an  $\omega$ -successful good  $\lambda_0$ -frame  $\mathfrak{s}'$  such that  $\mathfrak{s} = (\mathfrak{s}')^{+3}$ .

Assume  $\delta$  is a limit ordinal and  $\langle M_i^{\ell} : i \leq \delta \rangle$  is increasing continuous in  $\mathbf{K}_{\lambda}$ ,  $\ell \leq 3. \text{ If } M_i^1 \underset{M_i^0}{\overset{M_i^3}{\downarrow}} M_i^2 \text{ for each } i < \delta, \text{ then } M_{\delta}^1 \underset{M_{\delta}^0}{\overset{M_{\delta}^3}{\downarrow}} M_{\delta}^2.$ 

PROOF. By [She09a, Claim III.12.2], all the hypotheses at the beginning of each section of Chapter III in the book hold for  $\mathfrak{s}$ . Now apply Claim III.8.19 in the book.  $\square$ 

REMARK 6.12.15. Where does the hypothesis  $\lambda = \lambda_0^{+3}$  come from? Shelah's analysis in chapter III of his book proceeds on the following lines: starting with an  $\omega$ -successful frames  $\mathfrak{s}$ , we want to show  $\mathfrak{s}$  has nice properties like existence of prime triples, weak orthogonality being orthogonality, etc. They are hard to show in general, however it turns out  $\mathfrak{s}^+$  has some nicer properties than  $\mathfrak{s}$  (for example,  $\mathbf{K}_{\mathfrak{s}^+}$  is always categorical)... In general,  $\mathfrak{s}^{+(n+1)}$  has even nicer properties than  $\mathfrak{s}^{+n}$ ; and Shelah shows that the frame has all the nice properties he wants after going up three successors.

We obtain:

THEOREM 6.12.16.

- (1) If **K** is categorical in  $\lambda$ , then t is a good ( $\leq \lambda, \lambda$ )-frame.
- (2) If  $\lambda = \lambda_0^{+3}$  and there exists an  $\omega$ -successful good  $\lambda_0$ -frame  $\mathfrak{s}'$  such that  $\mathfrak{s} = (\mathfrak{s}')^{+3}$ , then  $\mathfrak{t}$  is a fully good  $(\leq \lambda, \lambda)$ -frame.

PROOF. t is good by Proposition 6.12.9, Lemma 6.12.11, and Lemma 6.12.13. The second part follows from Fact 6.12.14 (note that by definition of the successor frame, **K** will be categorical in  $\lambda$  in that case).  $\square$ 

REMARK 6.12.17. In Corollary 5.6.10, it is shown that  $\lambda$ -tameness and amalgamation imply that a good  $\lambda$ -frame extends to a good ( $< \infty, \lambda$ )-frame. However, the definition of a good frame there is not the same as it does not assume that the frame is type-full (the types on which forking is defined are only the types of independent sequences). Thus the conclusion of Theorem 6.12.16 is much stronger (but uses more hypotheses).

# 6.13. Extending the base and right hand side

Hypothesis 6.13.1.

- (1)  $i = (\mathbf{K}, \downarrow)$  is a fully good  $(\leq \lambda, \lambda)$ -independence relation.
- (2)  $\mathbf{K}' := \mathbf{K}^{up}$  has amalgamation and is  $\lambda$ -tame for types of length less than  $\lambda^+$ .

In this section, we give conditions under which i becomes a fully good ( $\leq \lambda, \geq \lambda$ )-independence relation. In the next section, we will make the left hand side bigger and get a fully good ( $< \infty, \geq \lambda$ )-independence relation.

Recall that extending a  $(\leq 1, \lambda)$ -frame to bigger models was investigated in [She09a, Chapter II] and [Bon14a], Chapter 5. Here, most of the arguments are similar but the longer types cause some additional difficulties (e.g. in the proof of local character).

NOTATION 6.13.2. Let  $\mathfrak{i}' := \mathfrak{i}^{up}$  (recall Definition 6.6.3). Write  $\mathfrak{s} := \operatorname{pre}(\mathfrak{i})$ ,  $\mathfrak{s}' := \operatorname{pre}(\mathfrak{i}')$ ,  $\mathbf{K}' := \mathbf{K}_{\mathfrak{i}'}$ . We abuse notation and also denote  $\downarrow$  by  $\downarrow$ .

We want to investigate when the properties of i carry over to i'.

## LEMMA 6.13.3.

- (1)  $\mathfrak{i}'$  is a  $(\leq \lambda, \geq \lambda)$ -independence relation.
- (2)  $\mathbf{K}'$  has joint embedding, no maximal models, and is stable in all cardinals.
- (3) i' has base monotonicity, transitivity, uniqueness, and disjointness.
- (4) i' has full model continuity.

# Proof.

- (1) By Proposition 6.6.5.
- (2) By Corollary 5.6.9,  $(\mathfrak{s}')^{\leq 1}$  is a good  $(\geq \lambda)$ -frame, so in particular **K**' has joint embedding, no maximal models, and is stable in all cardinals.
- (3) See [She09a, Claim II.2.11] for base monotonicity and transitivity. Disjointness is straightforward from the definition of i', and uniqueness follows from the tameness hypothesis and the definition of i'.
- (4) Assume  $\langle M_i^{\ell} : i \leq \delta \rangle$  is increasing continuous in  $\mathbf{K}', \ell \leq 3, \delta$  is regular,  $M_i^0 \leq_{\mathbf{K}} M_i^{\ell} \leq_{\mathbf{K}} M_i^3$  for  $\ell = 1, 2, ||M_{\delta}^1|| < \lambda^+$  (recall the definition of full model continuity),  $i < \delta$ , and  $M_i^1 \underset{M_i^0}{\cup} M_i^2$  for all  $i < \delta$ . Let  $N := M_{\delta}^3$ .

By ambient monotonicity,  $M_i^1 \stackrel{N}{\underset{M_i^0}{\downarrow}} M_i^2$  for all  $i < \delta$ . We want to see that

 $M^1_{\delta} \bigcup_{M^0_{\delta}}^{N} M^2_{\delta}$ . Since  $||M^1_{\delta}|| < \lambda^+$ ,  $M^1_{\delta}$  and  $M^0_{\delta}$  are in **K**. Thus it is enough

to show that for all  $M' \leq_{\mathbf{K}} M_{\delta}^2$  in  $\mathbf{K}$  with  $M_{\delta}^0 \leq_{\mathbf{K}} M', M_{\delta}^1 \bigcup_{M_{\delta}^0}^N M'$ . Fix

such an M'. We consider two cases:

• <u>Case 1:</u>  $\delta < \lambda^+$ : Then we can find  $\langle M'_i : i \leq \delta \rangle$  increasing continuous in **K** (as opposed to just in **K**') such that  $M'_{\delta} = M'$  and for all  $i < \delta$ ,  $M^0_i \leq_{\mathbf{K}} M'_i \leq_{\mathbf{K}} M^2_i$ . By monotonicity, for all  $i < \delta$ ,  $M^1_i \stackrel{N}{\underset{M^0_i}{\longrightarrow}} M'_i$ . By full model continuity in **K**  $M^1 \stackrel{N}{\underset{M^0_i}{\longrightarrow}} M'_i$  as desired

full model continuity in  $\mathbf{K}$ ,  $M_{\delta}^1 \stackrel{N}{\underset{M_{\delta}^0}{\downarrow}} M'$ , as desired.

• Case 2:  $\delta \ge \lambda^+$ : Since  $M^0_{\delta}, M^1_{\delta} \in \mathbf{K}$ , the chains  $\langle M^{\ell}_i : i \le \delta \rangle$  for  $\ell = 0, 1$  must be eventually constant, so we can assume without loss of generality that  $M^0_{\delta} = M^0_0, M^1_{\delta} = M^1_0$ . Since  $\delta$  is regular, there

exists 
$$i < \delta$$
 such that  $M' \leq_{\mathbf{K}} M_i^2$ . By assumption,  $M_0^1 \bigcup_{M_0^0}^N M_i^2$ , so by  
monotonicity,  $M_0^1 \bigcup_{M_0^0}^N M'$ , as needed.

We now turn to local character.

LEMMA 6.13.4. Assume  $\langle M_i : i \leq \delta \rangle$  is increasing continuous,  $p \in gS^{\alpha}(M_{\delta})$ ,  $\alpha < \lambda^+$  a cardinal and  $\delta = \operatorname{cf} \delta > \alpha$ .

- (1) If  $\alpha < \lambda$ , then there exists  $i < \delta$  such that p does not fork over  $M_i$ .
- (2) If  $\alpha = \lambda$  and i has the left ( $\langle \operatorname{cf} \lambda \rangle$ )-witness property, then there exists  $i < \delta$  such that p does not fork over  $M_i$ .

Proof.

- (1) As in the proof of Lemma 6.6.8.(2b) Note that weak chain local character holds for free because  $\alpha < \lambda$  and  $\kappa_{\alpha}(\mathfrak{i}) = \alpha^{+} + \aleph_{0}$  by assumption.
- (2) By the proof of Lemma 6.6.8.(2b) again, it is enough to see that i has weak chain local character: Let  $\langle M_i : i < \lambda^+ \rangle$  be increasing in **K** and let  $M_{\lambda^+} := \bigcup_{i < \lambda^+} M_i$ . Let  $p \in \mathrm{gS}^{\lambda}(M_{\lambda^+})$ . We will show that there exists  $i < \lambda^+$  such that p does not fork over  $M_i$ . Say  $p = \mathrm{gtp}(\bar{a}/M_{\lambda^+}; N)$  and let  $A := \mathrm{ran}(\bar{a})$ . Write  $A = \bigcup_{j < \mathrm{cf} \lambda} A_j$  with  $\langle A_j : i < \mathrm{cf} \lambda \rangle$  increasing continuous and  $|A_j| < \lambda$ . By the first part, for each  $j < \mathrm{cf} \lambda$  there exists  $i_j < \lambda^+$  such that  $A_j \underset{M_{i_j}}{\overset{N}{\longrightarrow}} M_{\lambda^+}$ . Let  $i := \mathrm{sup}_{j < \mathrm{cf} \lambda} i_j$ . We claim that

 $A \stackrel{N}{\underset{M_i}{\downarrow}} M_{\lambda^+}$ . By the (< cf  $\lambda$ )-witness property and the definition of i' (here

we use that  $M_i \in \mathbf{K}$ ), it is enough to show this for all  $B \subseteq A$  of size less than cf  $\lambda$ . But any such B is contained in an  $A_j$ , and so the result follows from base monotonicity.

LEMMA 6.13.5. Assume  $\mathfrak{i}'$  has existence. Then  $\mathfrak{i}'$  has independent amalgamation.

PROOF. As in, for example, the proof of [Bon14a, Theorem 5.3], using full model continuity.  $\hfill\square$ 

Putting everything together, we obtain:

THEOREM 6.13.6. If **K** is  $(\langle \text{cf } \lambda \rangle)$ -tame and short for types of length less than  $\lambda^+$ , then  $\mathfrak{i}'$  is a fully pre-good  $(\leq \lambda, \geq \lambda)$ -independence relation.

PROOF. We want to show that  $\mathfrak{s}'$  is fully good. The basic properties are proven in Lemma 6.13.3. By Lemma 6.4.5,  $\mathfrak{i}$  has the left ( $\langle \operatorname{cf} \lambda \rangle$ )-witness property. Thus by Lemma 6.13.4, for any  $\alpha < \lambda^+$ ,  $\kappa_{\alpha}(\mathfrak{i}') = |\alpha|^+ + \aleph_0$ . In particular,  $\mathfrak{i}'$  has existence, and thus by the definition of  $\mathfrak{i}'$  and transitivity in  $\mathfrak{i}$ ,  $\bar{\kappa}_{\alpha}(\mathfrak{i}') = \lambda^+ = |\alpha|^+ + \lambda^+$ . Finally by Lemma 6.13.5,  $\mathfrak{i}'$  has independent amalgamation and so by Proposition 6.4.3.(3),  $\mathfrak{i}'$  has extension.

#### 6.14. Extending the left hand side

We now enlarge the left hand side of the independence relation built in the previous section.

Hypothesis 6.14.1.

(1)  $i = (\mathbf{K}, \bot)$  is a fully good  $(\leq \lambda, \geq \lambda)$ -independence relation.

(2) **K** is fully  $\lambda$ -tame and short.

DEFINITION 6.14.2. Define  $\mathbf{i}^{\text{long}} = (\mathbf{K}, \bigcup^{\text{long}})$  by setting  $\bigcup^{\text{long}}(M_0, A, B, N)$  if and only if for all  $A_0 \subseteq A$  of size less than  $\lambda^+$ ,  $A_0 \bigcup_{M_0}^N B$ .

REMARK 6.14.3. The idea is the same as for Definition 5.4.3: we extend the frame to have longer types. The difference is that  $i^{\text{long}}$  is type-full.

REMARK 6.14.4. We could also have defined extension to types of length less than  $\theta$  for  $\theta$  a cardinal or  $\infty$  but this complicates the notation and we have no use for it here.

NOTATION 6.14.5. Write  $\mathfrak{i}' := \mathfrak{i}^{\text{long}}$ . We abuse notation and also write  $\downarrow$  for  $\downarrow^{\text{long}}$ .

Lemma 6.14.6.

- (1) i' is a  $(<\infty, \geq \lambda)$ -independence relation.
- (2) K has joint embedding, no maximal models, and is stable in all cardinals.
- (3) i' has base monotonicity, transitivity, disjointness, existence, symmetry, the left and right  $\lambda$ -witness properties, and uniqueness.

# Proof.

- (1) Straightforward.
- (2) Because i is good.
- (3) Base monotonicity, transitivity, disjointness, existence, and the left and right λ-witness property are straightforward (recall that i has the right λ-witness property by Lemma 6.4.5). Uniqueness is by the shortness hypothesis. Symmetry follows easily from the witness properties.

LEMMA 6.14.7. Assume there exists a regular  $\kappa \leq \lambda$  such that i has the left  $(\langle \kappa \rangle)$ -model-witness property. Then i' has full model continuity.

PROOF. Let  $\langle M_i^{\ell} : i \leq \delta \rangle$ ,  $\ell \leq 3$  be increasing continuous in **K** such that  $M_i^0 \leq_{\mathbf{K}} M_i^{\ell} \leq_{\mathbf{K}} M_i^3$ ,  $\ell = 1, 2$ , and  $M_i^1 \bigcup_{M_i^0} M_i^2$ . Without loss of generality,  $\delta$  is regular. Let  $N := M_{\delta}^3$ . We want to show that  $M_{\delta}^1 \bigcup_{M_{\delta}^0} M_{\delta}^2$ . Let  $A \subseteq |M_{\delta}^1|$  have size less than  $\lambda^+$ . Write  $\mu := |A|$ . By monotonicity, assume without loss of generality that  $\lambda + \kappa \leq \mu$ . We show that  $A \bigcup_{M_{\delta}^0} M_{\delta}^2$ , which is enough by definition of i'. We consider two cases.

- Case 1:  $\delta > \mu$ : By local character in i there exists  $i < \delta$  such that  $A \stackrel{\sim}{\downarrow} M_{\delta}^2$ . By right transitivity,  $A \underset{M_{\delta}^0}{\downarrow} M_{\delta}^2$ , so by base monotonicity,  $A \underset{M_{\delta}^0}{\downarrow} M_{\delta}^2$ .
- <u>Case 2</u>:  $\delta \leq \mu$ : For  $i \leq \overset{i}{\delta}$ , let  $A_i := A \cap |M_i^1|$ . Build  $\langle \overset{o}{N_i} : i \leq \delta \rangle$ ,  $\overline{\langle N_i^0 : i \leq \delta \rangle}$  increasing continuous in  $\mathbf{K}_{\leq \mu}$  such that for all  $i < \delta$ : (1)  $A_i \subseteq |N_i|$ .

  - (1)  $N_i \cong |I^{*i}|$ . (2)  $N_i \leq_{\mathbf{K}} M_i^1, A \subseteq |N_i|$ . (3)  $N_i^0 \leq_{\mathbf{K}} M_i^0, N_i^0 \leq_{\mathbf{K}} N_i$ . (4)  $N_i \underset{N_i^0}{\downarrow} M_i^2$ .

This is possible. Fix  $i \leq \delta$  and assume  $N_j, N_j^0$  have already been constructed for j < i. If i is limit, take unions. Otherwise, recall that we are assuming  $M_i^1 \bigcup_{M_i^0}^N M_i^2$ . By Lemma 6.4.7 (with  $A_i \cup \bigcup_{j < i} |N_j|$  standing for A there, this is where we use the  $(< \kappa)$ -model-witness property), we can find  $N_i^0 \leq_{\mathbf{K}} M_i^0$  and  $N_i \leq_{\mathbf{K}} M_i^1$  in  $\mathbf{K}_{\leq \mu}$  such that  $N_i^0 \leq_{\mathbf{K}} N_i, N_i \bigcup_{N_i^0}^N M_i^2$ ,  $A_i \subseteq |N_i|, N_j \leq_{\mathbf{K}} N_i$  for all j < i, and  $N_j^0 \leq_{\mathbf{K}} N_i^0$  for all j < i. Thus they are as desired. <u>This is enough</u>. Note that  $A_{\delta} = A$ , so  $A \subseteq |N_{\delta}|$ . By full model continuity in i,  $N_{\delta} \stackrel{N}{\underset{N_{\delta}^{0}}{\downarrow}} M_{\delta}^{2}$ . By monotonicity,  $A \stackrel{N}{\underset{M_{\delta}^{0}}{\downarrow}} M_{\delta}^{2}$ , as desired.

LEMMA 6.14.8. Assume there exists a regular  $\kappa \leq \lambda$  such that i has the left  $(<\kappa)$ -model-witness property. Then for all cardinals  $\mu$ :

- (1)  $\bar{\kappa}_{\mu}(\mathfrak{i}') = \lambda^{+} + \mu^{+}.$ (2)  $\kappa_{\mu}(\mathfrak{i}') = \aleph_{0} + \mu^{+}.$

PROOF. By Lemma 6.14.7, i' has full model continuity. By Lemma 6.4.8, (1) holds. For (2), if  $\mu \leq \lambda$ , this holds because i is good and if  $\mu > \lambda$ , this follows from Proposition 6.4.3.(5) and (1). 

We now turn to proving extension. The proof is significantly more complicated than in the previous section. We attempt to explain why and how our proof goes. Of course, it suffices to show independent amalgamation (Proposition 6.4.3.(3)). We work by induction on the size of the models but land in trouble when all models have the same size. Suppose for example that we want to amalgamate  $M^0 \leq_{\mathbf{K}} M^\ell$ ,  $\ell = 1, 2$  that are all in  $\mathbf{K}_{\lambda^+}$ . If  $M^1$  (or, by symmetry,  $M^2$ ) had smaller size, we could use local character to assume without loss of generality that  $M^0$  is in  $\mathbf{K}_{\lambda}$ and then imitate the usual directed system argument (as in for example the proof of [Bon14a, Theorem 5.3]).

Here however it seems we have to take at least two resolutions at once so we fix  $\langle M_i^{\ell} : i < \lambda^+ \rangle$ ,  $\ell = 0, 1$ , satisfying the usual conditions. Letting p := $gtp(M^1/M^0; M^1)$  and its resolution  $p_i := gtp(M_i^1/M_i^0; M^1)$ , it is natural to build  $\langle q_i : i < \lambda^+ \rangle$  such that  $q_i$  is the nonforking extension of  $p_i$  to  $M^2$ . If everything

works, we can take the direct limit of the  $q_i$ s and get the desired nonforking extension of p. However with what we have said so far it is not clear that  $q_{i+1}$  is even an extension of  $q_i$ ! In the usual argument, this is the case since both  $p_i$  and  $p_{i+1}$  do not fork over the same domain but we cannot expect it here. Thus we require in addition that  $M_i^1 \underset{M_i^0}{\downarrow} M^0$  and this turns out to be enough for successor steps. To

achieve this extra requirement, we use Lemma 6.4.7. Unfortunately, we also do not know how to go through limit steps without making one extra locality hypothesis:

DEFINITION 6.14.9 (Type-locality).

- (1) Let  $\delta$  be a limit ordinal, and let  $\bar{p} := \langle p_i : i < \delta \rangle$  be an increasing chain of Galois types, where for  $i < \delta$ ,  $p_i \in gS^{\alpha_i}(M)$  and  $\langle \alpha_i : i \leq \delta \rangle$  are increasing continuous. We say  $\bar{p}$  is type-local if whenever  $p, q \in gS^{\alpha_\delta}(M)$  are such that  $p^{\alpha_i} = q^{\alpha_i} = p_i$  for all  $i < \delta$ , then p = q.
- (2) We say **K** is *type-local* if every  $\bar{p}$  as above is type-local.
- (3) We say **K** is densely type-local above  $\lambda$  if for every  $\lambda_0 > \lambda$ ,  $M \in \mathbf{K}_{\lambda_0}$ ,  $p \in \mathrm{gS}^{\lambda_0}(M)$ , there exists  $\langle N_i : i \leq \delta \rangle$  such that:
  - (a)  $\delta = \operatorname{cf} \lambda_0$ .
  - (b) For all  $i < \delta$ ,  $N_i \in \mathbf{K}_{<\lambda_0}$ .
  - (c)  $\langle N_i : i \leq \delta \rangle$  is increasing continuous.
  - (d)  $N_{\delta} \geq_{\mathbf{K}} M$  is in  $\mathbf{K}_{\lambda_0}$ .
  - (e) Letting  $q_i := \operatorname{gtp}(N_i/M; N_{\delta})$  (seen as a member of  $\operatorname{gS}^{\alpha_i}(M)$ , where of course  $\langle \alpha_i : i \leq \delta \rangle$  are increasing continuous), we have that  $q_{\delta}$  extends p and  $\langle q_j : j < i \rangle$  is type-local for all limit  $i \leq \delta$ .

We say **K** is *densely type-local* if it is densely type-local above  $\lambda$  for some  $\lambda$ .

Intuitively, the relationship between type-locality and locality (see [**Bal09**, Definition 11.4]) is the same as the relationship between type-shortness and tameness (in the later, we look at *domain* of types, in the former we look at *length* of types). We suspect that dense type-locality should hold in our context, see the discussion in Section 6.15 for more. The following lemma says that increasing the elements in the resolution of the type preserves type-locality.

LEMMA 6.14.10. Let  $\delta$  be a limit ordinal. Assume  $\bar{p} := \langle p_i : i < \delta \rangle$  is an increasing chain of Galois types,  $p_i \in gS^{\alpha_i}(M)$  and  $\langle \alpha_i : i \leq \delta \rangle$  are increasing continuous. Assume  $\bar{p}$  is type-local and assume  $p_{\delta} \in gS^{\alpha_{\delta}}(M)$  is such that  $p^{\alpha_i} = p_i$  for all  $i < \delta$ . Say  $p = gtp(\bar{a}_{\delta}/M; N)$  and let  $\bar{a}_i := \bar{a}_{\delta} \upharpoonright \alpha_i$  (so  $p_i = gtp(\bar{a}_i/M; N)$ ).

Assume  $\langle \bar{b}_i : i \leq \delta \rangle$  are increasing continuous sequences such that  $\bar{a}_{\delta} = \bar{b}_{\delta}$ and  $\bar{a}_i$  is an initial segment of  $\bar{b}_i$  for all  $i < \delta$ . Let  $q_i := \text{gtp}(\bar{b}_i/M; N)$ . Then  $\bar{q} := \langle q_i : i < \delta \rangle$  is type-local.

PROOF. Say  $\bar{b}_i$  is of type  $\beta_i$ . So  $\langle \beta_i : i \leq \delta \rangle$  is increasing continuous and  $\alpha_{\delta} = \beta_{\delta}$ .

If  $q \in gS^{\beta_{\delta}}(M)$  is such that  $q^{\beta_i} = q_i$  for all  $i < \delta$ , then  $q^{\alpha_i} = (q_i)^{\alpha_i} = p_i$  for all  $i < \delta$  so by type-locality of  $\bar{p}$ , p = q, as desired.

Before proving Lemma 6.14.13, let us make precise what was meant above by "direct limit" of a chain of types. It is known that (under some set-theoretic hypotheses) there exists AECs where some chains of Galois types do not have an upper bound, see [**BS08**, Theorem 3.3]. However a *coherent* chain of types (see

below) always has an upper bound. We adapt Definition 5.1 in [Bon14a] (which is implicit already in [She01a, Claim 0.32.2] or [GV06a, Lemma 2.12]) to our purpose.

DEFINITION 6.14.11. Let  $\delta$  be an ordinal. An increasing chain of types  $\langle p_i : i < \delta \rangle$  is said to be *coherent* if there exists a sequence  $\langle (\bar{a}_i, M_i, N_i) : i < \delta \rangle$  and maps  $f_{i,j} : N_i \to N_j, i \leq j < \delta$ , so that for all  $i \leq j \leq k < \delta$ :

- (1)  $f_{j,k} \circ f_{i,j} = f_{i,k}.$
- (2)  $gtp(\bar{a}_i/M_i; N_i) = p_i.$
- (3)  $\langle M_i : i < \delta \rangle$  and  $\langle N_i : i < \delta \rangle$  are increasing.
- (4)  $M_i \leq_{\mathbf{K}} N_i, \, \bar{a}_i \in {}^{<\infty}N_i.$
- (5)  $f_{i,j}$  fixes  $M_i$ .
- (6)  $f_{i,j}(\bar{a}_i)$  is an initial segment of  $\bar{a}_j$ .

We call the sequence and maps above a *witnessing sequence* to the coherence of the  $p_i$ 's.

Given a witnessing sequence  $\langle (\bar{a}_i, M_i, N_i) : i < \delta \rangle$  with maps  $f_{i,j} : N_i \to N_j$ , we can let  $N_{\delta}$  be the direct limit of the system  $\langle N_i, f_{i,j} : i \leq j < \delta \rangle$ ,  $M_{\delta} := \bigcup_{i < \delta} M_i$ , and  $\bar{a}_{\delta} := \bigcup_{i < \delta} f_{i,\delta}(\bar{a}_i)$  (where  $f_{i,\delta} : N_i \to N_{\delta}$  is the canonical embedding). Then  $p := \operatorname{gtp}(\bar{a}/M_{\delta}; N_{\delta})$  extends each  $p_i$ . Note that p depends on the witness but we sometimes abuse language and talk about "the" direct limit (where really some witnessing sequence is fixed in the background).

Finally, note that full model continuity also applies to coherent sequences. More precisely:

PROPOSITION 6.14.12. Assume i has full model continuity. Let  $\langle (\bar{a}_i, M_i, N_i) : i < \delta \rangle$ ,  $\langle f_{i,j} : N_i \to N_j, i \leq j < \delta \rangle$  be witnesses to the coherence of  $p_i := \operatorname{gtp}(\bar{a}_i/M_i; N_i)$ . Assume that for each  $i < \delta$ ,  $\bar{a}_i$  enumerates a model  $M'_i$  and that  $\langle M^0_i : i < \delta \rangle$  are increasing such that  $M^0_i \leq_{\mathbf{K}} M_i, M^0_i \leq_{\mathbf{K}} M'_i$ , and  $p_i$  does not fork over  $M^0_i$ . Let p be the direct limit of the  $p_i$ s (according to the witnessing sequence). Then p does not fork over  $M^0_\delta := \bigcup_{i < \delta} M^0_i$ .

PROOF. Use full model continuity inside the direct limit.

LEMMA 6.14.13. Assume **K** is densely type-local above  $\lambda$ , and assume there exists a regular  $\kappa \leq \lambda$  such that **K** is fully  $(\langle \kappa \rangle)$ -tame. Then i' has extension.

PROOF. By Lemma 6.4.5 and symmetry, i has the left ( $< \kappa$ )-model-witness property. By Lemmas 6.14.7 and 6.14.8, i' has full model continuity and the local character properties. Let  $\lambda_0 \geq \lambda$  be a cardinal. We prove by induction on  $\lambda_0$  that i' has extension for base models in  $\mathbf{K}_{\lambda_0}$ . By Proposition 6.4.3.(3), it is enough to prove independent amalgamation.

Let  $M^0 \leq_{\mathbf{K}} M^{\ell}$ ,  $\ell = 1, 2$  be in  $\mathbf{K}$  with  $||M^0|| = \lambda_0$ . We want to find  $q \in gS^{\lambda_0}(M^2)$  a nonforking extension of  $p := gtp(M^1/M^0; M^1)$ . Let  $\lambda_{\ell} := ||M^{\ell}||$  for  $\ell = 1, 2$ .

Assume we know the result when  $\lambda_0 = \lambda_1 = \lambda_2$ . Then we can work by induction on  $(\lambda_1, \lambda_2)$ : if they are both  $\lambda_0$ , the result holds by assumption. If not, we can assume by symmetry that  $\lambda_1 \leq \lambda_2$ , find an increasing continuous resolution of  $M^2$ ,  $\langle M_i^2 \in \mathbf{K}_{<\lambda_2} : i < \lambda_2 \rangle$  and do a directed system argument as in [**Bon14a**, Theorem 5.3] (using full model continuity and the induction hypothesis).

Now assume that  $\lambda_0 = \lambda_1 = \lambda_2$ . If  $\lambda_0 = \lambda$ , we get the result by extension in i, so assume  $\lambda_0 > \lambda$ . Let  $\delta := \operatorname{cf} \lambda_0$ . By dense type-locality, we can assume (extending

 $M^1$  if necessary) that there exists  $\langle N_i : i \leq \delta \rangle$  an increasing continuous resolution of  $M^1$  with  $N_i \in \mathbf{K}_{<\lambda_0}$  for  $i < \delta$  so that  $\langle \operatorname{gtp}(N_j/M^0; M^1) : j < i \rangle$  is type-local for all limit  $i \leq \delta$ .

**Step 1**. Fix increasing continuous  $\langle M_i^{\ell} : i \leq \delta \rangle$  for  $\ell < 2$  such that for all  $i < \delta$ ,  $\ell < 2$ :

(1)  $M^{\ell} = M^{\ell}_{\delta}$ . (2)  $M^{\ell}_i \in \mathbf{K}_{<\lambda_0}$ . (3)  $N_i \leq_{\mathbf{K}} M^1_i$ . (4)  $M^0_i \leq_{\mathbf{K}} M^1_i$ . (5)  $M^1_i \stackrel{M^1}{\downarrow} M^0$ .

This is possible by repeated applications of Lemma 6.4.7 (as in the proof of Lemma 6.14.7), starting with  $M^1 \downarrow_{M^0}^{M^1} M^0$  which holds by existence.

Step 2. Fix enumerations of  $M_i^1$  of order type  $\alpha_i$  such that  $\langle \alpha_i : i \leq \delta \rangle$  is increasing continuous,  $\alpha_{\delta} = \lambda_0$  and i < j implies that  $M'_i$  appears as the initial segment up to  $\alpha_i$  of the enumeration of  $M'_j$ . For  $i \leq \delta$ , let  $p_i := \operatorname{gtp}(M_i^1/M_i^0; M^1)$ (seen as an element of  $\operatorname{gS}^{\alpha_i}(M_i^0)$ ). We want to find  $q \in \operatorname{gS}^{\lambda_0}(M^2)$  extending  $p = p_{\delta}$ and not forking over  $M^0$ . Note that since for all  $j < \delta$ ,  $N_j \leq_{\mathbf{K}} M_j^1$ , we have by Lemma 6.14.10 that  $\langle \operatorname{gtp}(M_j^1/M^0; M^1) : j < i \rangle$  is type-local for all limit  $i \leq \delta$ .

Build an increasing, coherent  $\langle q_i : i \leq \delta \rangle$  such that for all  $i \leq \delta$ ,

- (1)  $q_i \in gS^{\alpha_i}(M^2).$
- (1)  $q_i \in \mathcal{S}_i^{\circ}$  (1) (2)  $q_i \upharpoonright M_i^0 = p_i$ .
- (3)  $q_i$  does not fork over  $M_i^0$ .

This is enough: then  $q_{\delta}$  is an extension of  $p = p_{\delta}$  that does not fork over  $M_{\delta}^0 = M^0$ .

This is possible: We work by induction on  $i \leq \delta$ . While we do not make it explicit, the sequence witnessing the coherence is also built inductively in the natural way (see also [**Bon14a**, Proposition 5.2]): at base and successor steps, we use the definition of Galois types. At limit steps, we take direct limits.

Now fix  $i \leq \delta$  and assume everything has been defined for j < i.

- **Base step:** When i = 0, let  $q_0 \in gS^{\alpha_0}(M^2)$  be the nonforking extension of  $p_0$  to  $M_0^2$  (exists by extension below  $\lambda_0$ ).
- Successor step: When i = j+1,  $j < \delta$ , let  $q_i$  be the nonforking extension (of length  $\alpha_i$ ) of  $p_i$  to  $M^2$ . We have to check that  $q_i$  indeed extends  $q_j$ (i.e.  $q_i^{\alpha_j} = q_j$ ). Note that  $q_j \upharpoonright M^0$  does not fork over  $M_j^0$  so by step 1 and uniqueness,  $q_j \upharpoonright M^0 = \operatorname{gtp}(M_j^1/M^0; M^1)$ . In particular,  $q_j \upharpoonright M_i^0 = \operatorname{gtp}(M_j^1/M_i^0; M^1)$ . Since  $q_i$  extends  $p_i$ ,  $q_i \upharpoonright M_i^0 = \operatorname{gtp}(M_i^1/M_i^0; M^1)$  so  $q_i^{\alpha_j} \upharpoonright M_i^0 = \operatorname{gtp}(M_j^1/M_i^0; M^1) = q_j \upharpoonright M_i^0$ . By base monotonicity,  $q_j$  does not fork over  $M_i^0$  so by uniqueness  $q_i^{\alpha_j} = q_j$ . A picture is below.



• Limit step: Assume *i* is limit. Let  $q_i$  be the direct limit of the coherent sequence  $\langle q_j : j < i \rangle$ . Note that  $q_i \in gS^{\alpha_i}(M^2)$  and by Proposition 6.14.12,  $q_i$  does not fork over  $M_i^0$ . It remains to see  $q_i \upharpoonright M_i^0 = p_i$ . For j < i, let  $p'_j \in gS^{\alpha_j}(M_i^0)$  be the nonforking extension of  $p_j$  to  $M_i^0$ .

For j < i, let  $p'_j \in gS^{\alpha_j}(M_i^0)$  be the nonforking extension of  $p_j$  to  $M_i^0$ . By step 1,  $p'_j = gtp(M_j^1/M_i^0; M^1)$ . Thus  $\langle p'_j : j < i \rangle$  is type-local. By an argument similar to the successor step above, we have that for all j < i,  $p_i^{\alpha_j} = p'_j$ . Moreover, for all j < i,  $q_i^{\alpha_j} \upharpoonright M_j^0 = p_j$  and  $q_j$  does not fork over  $M_j^0$  so by uniqueness,  $q_i^{\alpha_j} \upharpoonright M_i^0 = (q_i \upharpoonright M_i^0)^{\alpha_j} = p'_j$ . By type-locality, it follows that  $q_i \upharpoonright M_i^0 = p_i$ , as desired.

Putting everything together, we get:

# THEOREM 6.14.14. If:

- (1) For some regular  $\kappa \leq \lambda$ , **K** is fully  $(< \kappa)$ -tame.
- (2) **K** is densely type-local above  $\lambda$ .

Then i' is a fully good  $(<\infty, \geq \lambda)$ -independence relation.

PROOF. Lemma 6.14.6 gives most of the properties of a good independence relation. By Lemma 6.4.5 and symmetry, i has the left ( $< \kappa$ )-model-witness property. By Lemma 6.14.7, i' has full model continuity. By Lemma 6.14.8, it has the local character properties. By Lemma 6.14.13, i' has extension.

We suspect that dense type-locality is not necessary, at least when i comes from our construction (see the proof of Theorem 6.15.1). For example, by the proof below, it would be enough to see that  $\operatorname{pre}(i_{\mu}^{\leq 1})$  is weakly successful for all  $\mu \geq \lambda$ . We delay a full investigation to a future work. For now, here is what we can say without dense type-locality:

THEOREM 6.14.15. Assume that for some regular  $\kappa \leq \lambda$ , **K** is fully  $(< \kappa)$ -tame. Then:

- (1) i' is a fully good independence relation, except perhaps for the extension property. Moreover, it has the right  $\lambda$ -witness property.
- (2) Assume that<sup>28</sup> for all  $\mu \geq \lambda$ ,  $(\mathfrak{i}')_{\geq \mu}$  satisfies Hypothesis 6.11.1. Then  $\mathfrak{i}'$  has the extension property when the base is saturated.

PROOF. The first part has been observed in the proof of Theorem 6.14.14 (see also Lemma 6.14.6). To see the second part, let  $\mu \geq \lambda$ . By Theorem 6.11.13 and Theorem 6.12.16, there exists a good ( $\leq \mu, \mu$ )-independence relation i" with underlying class  $\mathbf{K}^{\mu\text{-sat}}_{\mu}$ . Using the witness properties and the arguments of Chapter 3 we have that (i')<sup> $\leq \mu$ </sup> |  $\mathbf{K}^{\mu\text{-sat}}_{\mu} = i$ ". By the proof of Lemma 6.14.13, i' has extension when the base model is in  $\mathbf{K}^{\mu\text{-sat}}_{\mu}$ .

# 6.15. The main theorems

Recall (Definition 6.8.4) that an AEC  $\mathbf{K}$  is fully good if there is a fully good independence relation with underlying class  $\mathbf{K}$ . Intuitively, a fully good independence relation is one that satisfies all the basic properties of forking in a superstable firstorder theory. We are finally ready to show that densely type-local fully tame and

 $<sup>^{28}\</sup>mathrm{If}$  for example  $\mathfrak{i}'$  is constructed as in the proof of Theorem 6.15.1, this will be the case.

short superstable classes are fully good, at least on a class of sufficiently saturated models<sup>29</sup>.

THEOREM 6.15.1. Let **K** be a fully ( $< \kappa$ )-tame and short AEC with amalgamation. Assume that **K** is densely type-local above  $\kappa$ .

- (1) If **K** is  $\mu$ -superstable,  $\kappa = \beth_{\kappa} > \mu$ , and  $\lambda := (\mu^{<\kappa_r})^{+7}$ , then  $\mathbf{K}^{\lambda\text{-sat}}$  is fully good.
- (2) If **K** is  $\kappa$ -strongly  $\mu$ -superstable and  $\lambda := (\mu^{<\kappa_r})^{+6}$ , then  $\mathbf{K}^{\lambda\text{-sat}}$  is fully good.
- (3) If  $\kappa = \beth_{\kappa} > \mathrm{LS}(\mathbf{K})$ , and **K** is categorical in a  $\mu > \lambda_0 := (\kappa^{<\kappa_r})^{+5}$ , then  $\mathbf{K}_{\geq \lambda}$  is fully good, where  $\lambda := \min(\mu, h(\lambda_0))$ .

PROOF. Given what has been proven already, the proofs are short. However to help the reader reflect on all the ground that was covered, we start by giving a summary in plain language of what the main steps in the construction are. Assume for example that **K** is categorical in a high-enough cardinal  $\mu > \kappa = \beth_{\kappa} > \text{LS}(\mathbf{K})$ . By the results of Section 6.10 (using results in [**BG**], which ultimately rely on [**SV99**]), we get that **K** is  $\kappa$ -strongly  $\kappa$ -superstable (note that, as opposed to [**She99**], nothing is assumed about the cofinality of  $\mu$ ). Thus coheir induces a good ( $\leq 1, \lambda$ )-frame  $\mathfrak{s}$  with underlying class  $\mathbf{K}_{\lambda}$ , for  $\lambda$  a high-enough cardinal. Moreover, coheir (seen as a global independence relation) has the properties in Hypothesis 6.11.1. Thus from the material of Section 6.11, we conclude that the good frame is well-behaved: it is  $\omega$ -successful.

By Section 6.12, this means that  $\mathfrak{s}$  can be extended to a good  $(\leq \lambda, \lambda)$ -frame  $\mathfrak{s}'$  (so forking is defined not only for types of length one but for all types of length at most  $\lambda$ ). With slightly more hypotheses on  $\lambda$ , we even can even make  $\mathfrak{s}'$  a *fully* good  $(\leq \lambda, \lambda)$ -frame, and by the "minimal closure" trick, into a fully good  $(\leq \lambda, \lambda)$  independence relation i. By Section 6.13, i can be extended further to a fully good  $(\leq \lambda, \geq \lambda)$ -independence relation i' (that is, forking is not only defined over models of size  $\lambda$ , but over models of all sizes at least  $\lambda$ ). Finally, by Section 6.14, we can extend i' to types of any length (not just length at most  $\lambda$ ), hence getting the desired global independence relation (a fully good  $(< \infty, \geq \lambda)$ -independence relation).

Now on to the actual proofs:

- (1) By Theorem 6.10.11 and Proposition 6.10.10, **K** is  $\kappa$ -strongly  $(2^{<\kappa_r})^+$ -superstable. Now apply (2).
- (2) By Fact 6.11.3, Hypothesis 6.11.1 holds for  $\mu' := (\mu^{<\kappa_r})^{+2}$ ,  $\lambda$  there standing for  $(\mu')^+$  here, and  $\mathbf{K}' := \mathbf{K}^{\mu'\text{-sat}}$ . By Theorem 6.11.21, there is an  $\omega$ -successful type-full good  $(\mu')^+$ -frame  $\mathfrak{s}$  on  $\mathbf{K}^{(\mu')^+\text{-sat}}$ . By Theorem 6.12.16,  $\mathfrak{s}^{+3}$  induces a fully good  $(\leq \lambda, \lambda)$ -independence relation i on  $\mathbf{K}^{(\mu')^{+4}\text{-sat}} = \mathbf{K}^{\lambda\text{-sat}}$ . By Theorem 6.13.6,  $\mathfrak{i}' := \operatorname{cl}(\operatorname{pre}(\mathfrak{i}_{\geq \lambda}))$  is a fully good  $(\leq \lambda, \geq \lambda)$ -independence relation on  $\mathbf{K}^{\lambda\text{-sat}}$ . By Theorem 6.14.14,  $(\mathfrak{i}')^{\operatorname{long}}$  is a fully good  $(<\infty, \geq \lambda)$ -independence relation on  $\mathbf{K}^{\lambda\text{-sat}}$ . Thus  $\mathbf{K}^{\lambda\text{-sat}}$  is fully good.

 $<sup>^{29}</sup>$ The number 7 in (1) is possibly the largest natural number ever used in a statement about abstract elementary classes!

(3) By Theorem 6.10.16, **K** is  $\kappa$ -strongly  $\kappa$ -superstable. By (2),  $\mathbf{K}^{\lambda_0^+\text{-sat}}$  is fully good. By Fact 6.10.12.(4), all the models in  $\mathbf{K}_{\geq\lambda}$  are  $\lambda_0^+$ -saturated, hence  $\mathbf{K}_{\geq\lambda}^{\lambda_0^+\text{-sat}} = \mathbf{K}_{\geq\lambda}$  is fully good.

We now discuss the necessity of the hypotheses of the above theorem. It is easy to see that a fully good AEC is superstable<sup>+</sup>. Moreover, the existence of a relation  $\downarrow$  with disjointness and independent amalgamation directly implies disjoint amalgamation. An interesting question is whether there is a general framework in which to study independence without assuming amalgamation, but this is out of the scope of this chapter. To justify full tameness and shortness, one can ask:

QUESTION 6.15.2. Let **K** be a fully good AEC. Is **K** fully tame and short?

If the answer is positive, we believe the proof to be nontrivial. We suspect however that the shortness hypothesis of our main theorem can be weakened to a condition that easily holds in all fully good classes. In fact, we propose the following:

DEFINITION 6.15.3. An AEC **K** is diagonally  $(< \kappa)$ -tame if for any  $\kappa' \ge \kappa$ , **K** is  $(< \kappa')$ -tame for types of length less than  $\kappa'$ . **K** is diagonally  $\kappa$ -tame if it is diagonally  $(< \kappa^+)$ -tame. **K** is diagonally tame if it is diagonally  $(< \kappa)$ -tame for some  $\kappa$ .

It is easy to check that if i is a good ( $< \infty, \geq \lambda$ )-independence relation, then  $\mathbf{K}_i$  is diagonally  $\lambda$ -tame. Thus we suspect the answer to the following should be positive:

QUESTION 6.15.4. In Theorem 6.15.1, can "fully (<  $\kappa$ )-tame and short" be replaced by "diagonally (<  $\kappa$ )-tame?

Finally, we believe the dense type-locality hypothesis can be removed<sup>30</sup>. Indeed, chapter III of [**She09a**] has several results on getting models "generated" by independent sequences. Since independent sequences exhibit a lot of finite character (see also Chapter 5), we suspect the answer to the following should be positive.

QUESTION 6.15.5. Is dense type-locality needed in Theorem 6.15.1?

The construction also gives a more localized independence relation if we do not assume dense type-locality. Note that we can replace categoricity by superstability or strong superstability as in the proof of Theorem 6.15.1.

THEOREM 6.15.6. Let **K** be a fully (<  $\kappa$ )-tame and short AEC with amalgamation. Let  $\lambda$ ,  $\mu$  be cardinals such that:

$$LS(\mathbf{K}) < \kappa = \beth_{\kappa} < \lambda = \beth_{\lambda} \le \mu$$

Assume further that  $\operatorname{cf} \lambda \geq \kappa$ . If **K** is categorical in  $\mu$ , then:

- (1) There exists an  $\omega$ -successful type-full good  $\lambda$ -frame  $\mathfrak{s}$  with  $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\lambda}$ . Furthermore, the frame is induced by  $(<\kappa)$ -coheir:  $\mathfrak{s} = \operatorname{pre}(\mathfrak{i}_{\kappa-\operatorname{ch}}(\mathbf{K})_{\lambda}^{\leq 1})$ .
- (2)  $\mathbf{K}_{\lambda}$  is  $(\leq \lambda, \lambda)$ -good.

 $<sup>^{30}</sup>$ In fact, the result was initially announced without this hypothesis but Will Boney found a mistake in our proof of Lemma 6.14.13. This is the only place where type-locality is used

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- (3)  $\mathbf{K}^{\lambda^{+3}\text{-sat}}$  is fully  $(\leq \lambda^{+3})$ -good.
- (4)  $\mathbf{K}^{\lambda^{+3}\text{-sat}}$  is fully good, except it may not have extension. Moreover it has extension over saturated models.
- (5) Let  $\mathbf{i} := \mathbf{i}_{\kappa-ch}(\mathbf{K}^{\lambda^{+4}-\text{sat}})$ . Then  $\mathbf{i}$  is fully good, except it may not have extension. Moreover it has extension over saturated models.

PROOF. By cardinal arithmetic,  $\lambda = \lambda^{<\kappa_r}$ . By Fact 6.11.3 and Theorem 6.11.21, there is an  $\omega$ -successful type-full good  $\lambda$ -frame  $\mathfrak{s}$  with  $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\lambda}^{\lambda-\mathrm{sat}}$ . Now (by Theorem 6.10.16 if  $\mu > \lambda$ ),  $\mathbf{K}$  is categorical in  $\lambda$ . Thus  $\mathbf{K}^{\lambda-\mathrm{sat}} = \mathbf{K}_{\geq \lambda}$ . Theorem 6.12.16 and Theorem 6.13.6 give the next two parts. Theorem 6.14.15 gives the fourth part. For the fifth part, let i' witness the fourth part. We use Theorem 6.9.3 with  $\alpha$ ,  $\lambda$ , i there standing for  $\lambda^{+4}$ ,  $\lambda^{+3}$ ,  $(\mathbf{i'})^{<\lambda^{+4}}$  here. We obtain that  $(\mathbf{i'})^{<\lambda^{+4}} \upharpoonright \mathbf{K}^{\lambda^{+4}-\mathrm{sat}} = \mathbf{i}^{<\lambda^{+4}}$ . Since both i and i' have the left  $\lambda^{+3}$ -witness property,  $\mathbf{i'} \upharpoonright \mathbf{K}^{\lambda^{+4}-\mathrm{sat}} = \mathbf{i}$ , as desired.

# 6.16. Applications

We give three contexts in which the construction of a global independence relation can be carried out. To simplify the statement of the results, we adopt the following convention:

NOTATION 6.16.1. When we say "For any high-enough cardinal  $\lambda$ ", this should be replaced by "There exists an infinite cardinal  $\lambda_0$  such that for all  $\lambda \geq \lambda_0$ ".

# **6.16.1.** Fully $(\langle \aleph_0 \rangle)$ -tame and short AECs.

LEMMA 6.16.2. Let **K** be a fully  $(<\aleph_0)$ -tame and short AEC with amalgamation. Then **K** is type-local.

PROOF. Straightforward since types are determined by finite restrictions of their length.  $\hfill \Box$ 

Note that the framework of fully  $(<\aleph_0)$ -tame and short AEC with amalgamation generalizes homogeneous model theory. It is more general since we are not assuming that all sets are amalgamation bases, nor that we are working in a class of models of a first-order theory omitting a set of types. As a result, we do not have the weak compactness of homogeneous model theory, so Corollary 6.16.3 is (to the best of our knowledge) new.

COROLLARY 6.16.3. Let **K** be a fully  $(\langle \aleph_0 \rangle)$ -tame and short AEC with amalgamation. Assume that **K** is LS(**K**)-superstable. For any high-enough cardinal  $\lambda$ ,  $\mathbf{K}^{\lambda-\text{sat}}$  is fully good.

PROOF. By Lemma 6.16.2, K is type-local. Now apply Theorem 6.15.1.  $\Box$ 

**6.16.2.** Fully tame and short eventually categorical AECs. An AEC is eventually categorical (or categorical on a tail) if it is categorical in any high-enough cardinal. Note that Theorem 6.1.5 gives conditions under which this follows from categoricity in a single cardinal. In this context, we can also construct a global independence relation. To the best of our knowledge, this is new.

COROLLARY 6.16.4. Let **K** be a fully tame and short AEC with amalgamation. If **K** is eventually categorical, then for any high-enough cardinal  $\lambda$ ,  $\mathbf{K}_{\geq\lambda}$  is fully good.

PROOF. Let  $\kappa$  be such that **K** is fully  $(< \kappa)$ -tame and short and let  $\lambda_0$  be such that **K** is categorical in any  $\lambda \geq \lambda_0$ . We can make  $\kappa$  bigger if necessary and replace **K** by  $\mathbf{K}_{\geq \mu}$  for an appropriate  $\mu$  to assume without loss of generality that  $\mathrm{LS}(\mathbf{K}) < \kappa = \beth_{\kappa}$ , and **K** is categorical in all  $\lambda \geq \kappa$ . We now apply Theorem 6.15.6 to obtain that for any high-enough  $\lambda$ ,  $\mathbf{K}_{\geq \lambda}$  is fully good, except it may only have extension over saturated models. However any model is saturated by categoricity, so  $\mathbf{K}_{\geq \lambda}$  is fully good.

**6.16.3.** Large cardinals. Categoricity together with a large cardinal axiom implies that coheir is a well-behaved global independence relation. This was observed in [**MS90**] when the AEC is a class of models of an  $\mathbb{L}_{\kappa,\omega}$  theory ( $\kappa$  a strongly compact cardinal), and in [**BG**] for any AEC. Here we can improve on these results by proving that in this framework coheir is *fully good*. In particular, it has full model continuity and  $\kappa_{\alpha}(\mathfrak{i}) = \alpha^{+} + \aleph_{0}$ . Full model continuity is not discussed in [**MS90**, **BG**], and  $\kappa_{\alpha}(\mathfrak{i}) = \alpha^{+} + \aleph_{0}$  is only proven when  $\alpha < \kappa$  or  $\alpha = \alpha^{<\kappa}$  (see [**BG**, Theorem 8.2(3)]). Further, the proof uses the large cardinal axiom whereas we use it only to prove that coheir has the extension property.

COROLLARY 6.16.5. Let **K** be an AEC and let  $\kappa > \text{LS}(\mathbf{K})$  be a strongly compact cardinal. For any high-enough cardinal  $\lambda > \kappa$ , if **K** is categorical in  $\lambda$  then  $\mathbf{K}_{\geq \lambda}$  is fully good as witnessed by coheir (that is,  $\mathbf{i}_{\kappa-\text{ch}}(\mathbf{K}_{\geq \lambda})$  is fully good).

PROOF. By Fact 6.1.1, **K** is fully  $(< \kappa)$ -tame and short and  $\mathbf{K}_{\geq \kappa}$  has amalgamation. By the last part of Theorem 6.15.6, there exists  $\mu < \lambda$  such that  $\mathbf{i} := \mathbf{i}_{\kappa-ch}(\mathbf{K}^{\mu^{+4}\text{-sat}})$  is fully good, except perhaps for the extension property. By [**BG**, Theorem 8.2(1)] (using that  $\kappa$  is strongly compact)  $\mathbf{i}$  also has extension, hence it is fully good. Now the model of size  $\lambda$  is saturated (Theorem 6.10.16), so  $\mathbf{K}_{\geq \lambda} \subseteq \mathbf{K}^{\mu^{+4}\text{-sat}}$ . Hence  $\mathbf{i}_{\kappa-ch}(\mathbf{K}_{\geq \lambda})$  is also fully good.  $\Box$ 

REMARK 6.16.6. We can replace the categoricity hypothesis by amalgamation and  $\kappa$ -superstability. Moreover instead of asking for  $\kappa$  to be a large cardinal, it is enough to assume that **K** has amalgamation, is fully ( $< \kappa$ )-tame and short,  $LS(\mathbf{K}) < \kappa = \beth_{\kappa}$ , and coheir has the extension property (as in hypothesis (3) of [**BG**, Theorem 5.1]).

# CHAPTER 7

# Chains of saturated models in AECs

This chapter is based [BV17] and is joint work with Will Boney.

# Abstract

We study when a union of saturated models is saturated in the framework of tame abstract elementary classes (AECs) with amalgamation. We prove:

THEOREM 7.0.7. If K is a tame AEC with amalgamation satisfying a natural definition of superstability (which follows from categoricity in a high-enough cardinal), then for all high-enough  $\lambda$ :

- (1) The union of an increasing chain of  $\lambda$ -saturated models is  $\lambda$ -saturated.
- (2) There exists a type-full good  $\lambda$ -frame with underlying class the saturated models of size  $\lambda$ .
- (3) There exists a unique limit model of size  $\lambda$ .

Our proofs use independence calculus and a generalization of averages to this non first-order context.

# 7.1. Introduction

Determining when a union of  $\lambda$ -saturated models is  $\lambda$ -saturated is an important dividing line in first-order model theory. Recall that Harnik and Shelah have shown:

FACT 7.1.1 ([Har75], III.3.11 in [She90] for the case  $\lambda \leq |T|$ ). Let T be a first-order theory.

- If T is superstable, then any increasing union of  $\lambda$ -saturated models is  $\lambda$ -saturated.
- If T is stable, then any increasing union of  $\lambda$ -saturated models of cofinality at least  $|T|^+$  is  $\lambda$ -saturated.

A converse was later proven by Albert and Grossberg [AG90, Theorem 13]. Fact 7.1.1 can be used to prove:

FACT 7.1.2 (The saturation spectrum theorem, VIII.4.7 in [She90]). Let T be a stable first-order theory. Then T has a saturated model of size  $\lambda$  if and only if [Tis stable in  $\lambda$  or  $\lambda = \lambda^{<\lambda} + |D(T)|$ ].

Although not immediately evident from the statement, the proof of Fact 7.1.1 relies on the heavy machinery of forking and averages.

While the saturation spectrum theorem has been generalized to homogeneous model theory (see [She75c, 1.13] or [GL02, 5.9]), to the best of our knowledge no explicit generalization of Fact 7.1.1 has been published in this context (Shelah asserts it without proof in [She75c, 1.15]). Grossberg [Gro91b] has proven a

version of Fact 7.1.1 in the framework of stability theory inside a model. The proof uses averages but relies on a strong negation of the order property. Makkai and Shelah [**MS90**, 4.18] have given a generalization in the class of models of an  $L_{\kappa,\omega}$ sentence where  $\kappa$  is a strongly compact cardinal. The proof uses independence calculus.

One can ask whether Fact 7.1.1 can also be generalized to abstract elementary classes (AECs), a general framework for classification theory introduced in [She87a] (see [Gro02] for an introduction to AECs). In [She09a, I.5.39], Shelah proves a generalization of the superstable case of Fact 7.1.1 to "definable-enough" AECs with countable Löwenheim-Skolem number, using the weak continuum hypothesis.

In chapter II of [She09a], Shelah starts with a (weakly successful) good  $\lambda$ -frame (a local notion of superstability) on an abstract elementary class (AEC) **K** and wants to show that a union of saturated models is saturated in  $\mathbf{K}_{\lambda^+}$ . For this purpose, he introduces a restriction  $\leq^*$  of the ordering that allows him to prove the result for  $\leq^*$ -increasing chains (II.7.7 there). Restricting the ordering of the AEC is somewhat artificial and one can ask what happens in the general case, and also if  $\lambda^+$  is replaced by an arbitrary cardinal. Moreover, Shelah's methods to obtain a weakly successful good  $\lambda$ -frame typically use categoricity in two successive cardinals and the weak continuum hypothesis<sup>1</sup>.

In [She99], Shelah had previously proven that a union of  $\lambda$ -saturated models is  $\lambda$ -saturated, for **K** an AEC with amalgamation, joint embedding, and no maximal models categorical in a successor  $\lambda' > \lambda$  (see [Bal09, Chapter 15] for a writeup), but left the case  $\lambda \geq \lambda'$  (or  $\lambda'$  not a successor) unexamined.

In this chapter, we replace the local model-theoretic assumptions of Shelah with *global* ones, including *tameness*, a locality notion for types introduced by Grossberg and VanDieren [**GV06b**]. We take advantage of recent developments in the study of forking in tame AECs (especially by Boney and Grossberg [**BG**] and the author (see Chapters 2 and 6) to generalize Fact 7.1.1 to tame abstract elementary classes with amalgamation. Our main result is:

COROLLARY 7.3.5. Assume **K** is a  $(< \kappa)$ -tame AEC with amalgamation. If  $\kappa = \beth_{\kappa} > \text{LS}(\mathbf{K})$  and **K** is categorical in some cardinal strictly above  $\kappa$ , then for all  $\lambda > 2^{\kappa}$ ,  $\mathbf{K}^{\lambda-\text{sat}}$  (the class of  $\lambda$ -saturated models of **K**) is an AEC with  $\text{LS}(\mathbf{K}^{\lambda-\text{sat}}) = \lambda$ .

Notice that if  $\mathbf{K}^{\lambda\text{-sat}}$  is an AEC, then any increasing union of  $\lambda$ -saturated models is  $\lambda$ -saturated. Thus, in contrast to Shelah's [She99] result, we obtain a *global* theorem that holds for all high-enough  $\lambda$  and not just those under the categoricity cardinal. Furthermore categoricity at a successor is not assumed. We can also replace the categoricity by various notions of superstability defined in terms of the local character for independence notions such as coheir or splitting. In fact, we can combine this result with the construction of a good frame in Chapter 6 to obtain the theorem in the abstract:

<sup>&</sup>lt;sup>1</sup>See for example [She09a, II.3.7]. Shelah also shows how to build a good frame in ZFC from more model-theoretic hypotheses in [She09a, IV.4.10], but he has to change the class and it is not clear his frame is weakly successful.

#### 7.1. INTRODUCTION

THEOREM 7.6.1. If **K** is a tame AEC with amalgamation satisfying a natural definition of superstability (see Definition 7.5.12), then for all high-enough  $\lambda$ , there exists a *unique* limit model of size  $\lambda$ .

This proves an eventual version of a statement appearing in early versions of [**GVV16**] (see the discussion in Section 7.6).

It is very convenient to have  $\mathbf{K}^{\lambda\text{-sat}}$  an AEC, as saturated models are typically better behaved than arbitrary ones. This is crucial for example in Shelah's upward transfer of frames in [She09a, Chapter II], and is also used in Chapter 6 to build an  $\omega$ -successful good frame (and later a global independence notion). We also prove a result for the strictly stable case:

THEOREM 7.5.16. Let **K** be a  $\kappa$ -tame AEC with amalgamation,  $\kappa \geq \text{LS}(\mathbf{K})$ , stable in some cardinal above  $\kappa$ . Then there exists  $\chi_0 \leq \lambda_0 < \beth_{(2^{\kappa})^+}$  such that whenever  $\lambda \geq \lambda_0$  is such that  $\mu^{<\chi_0} < \lambda$  for all  $\mu < \lambda$ , the union of an increasing chain of  $\lambda$ -saturated models of cofinality at least  $\chi_0$  is  $\lambda$ -saturated.

One caveat here (compared to Fact 7.1.1, where there are no restrictions on  $\lambda$ ) is the introduction of cardinal arithmetic. When dealing with compact classes (or even just ( $< \omega$ )-tame classes), the map  $\lambda \mapsto \lambda^{<\omega}$  can be used freely. Even in the work of Makkai and Shelah [**MS90**], where  $\kappa$  is strongly compact and the class is ( $< \kappa$ )-tame, the map  $\lambda \mapsto \lambda^{<\kappa}$  is constant on most cardinals (those with cofinality at least  $\kappa$ ) by a result of Solovay. However, in our context of ( $< \kappa$ )-tameness for  $\kappa > \omega$  but not strongly compact, this function can be much wilder. Thus, we need to introduce assumptions that this map is well-behaved. Using various tricks, we can bypass these assumptions in the superstable case but are unable to do so in the stable case. For example in Theorem 7.5.16, the cardinal arithmetic assumption can be replaced by "**K** is stable in  $\mu$  for unboundedly many  $\mu < \lambda$ ", which is always true in case **K** is superstable.

We use two main methods: The first method is pure independence calculus, relying on a well-behaved independence relation (coheir), whose existence in our context is proven in [**BG**] and Chapter 2. This works well in the superstable case if we define superstability in terms of coheir (called strong superstability in Chapter 6) but we do not know how to make it work for weaker definitions of superstability (such as superstability defined in terms of splitting, a more classical definition implicit for example in [**GVV16**]). The second method is the use of syntactic averages, developed by Shelah in [**She09b**, Chapter V]. We end up proving a result on chains of saturated models in the framework of stability theory inside a model and then translate to AECs using *Galois Morleyization*, introduced in Chapter 2. This method allows us to use superstability defined in terms of splitting. The two methods give incomparable results: in case we know that **K** is  $(< \kappa)$ -tame, with  $\kappa = \beth_{\kappa} > LS(\mathbf{K})$ , the first gives better Hanf numbers than the second. However if we know that **K** is  $LS(\mathbf{K})$ -tame, then we get better bounds using the second method, since we do not need to work above a fixed point of the beth function.

The chapter is organized as follows. Section 7.2 gives the argument using independence calculus culminating in Theorems 7.2.14 and 7.3.4. Both of these arguments work just using forking relations, drawing inspiration from Makkai and Shelah, rather than the classical first-order argument using averages. Section 7.4

develops averages in our context based on earlier work of Shelah and culminates in the more local Theorem 7.4.27. Section 7.5 translates the local result to AECs and Section 7.6 proves consequences such as the uniqueness of limit models from superstability.

#### 7.2. Using independence calculus: the stable case

We assume that the reader is familiar with the basics of AECs, as presented in [**Bal09**] or [**Gro**]. We will use the notation from Chapter 2. In particular,  $gtp(\bar{b}/A; N)$  denotes the Galois type of  $\bar{b}$  over A as computed in N.

All throughout this section, we assume:

# Hypothesis 7.2.1.

- (1) **K** is an AEC with amalgamation, joint embedding, and arbitrarily large models. We work inside a monster model  $\mathfrak{C}$ .
- (2)  $LS(\mathbf{K}) < \kappa = \beth_{\kappa}$ .
- (3) **K** is  $(< \kappa)$ -tame.
- (4) **K** is stable (in some cardinal above  $\kappa$ ).

We will use the independence notion of coheir for AECs, introduced in [BG].

DEFINITION 7.2.2 (Coheir). Define a tertiary relation  $\downarrow$  by  $\downarrow(M, A, B)$  if and only if:

- (1)  $M \leq_{\mathbf{K}} \mathfrak{C}$ , M is  $\kappa$ -saturated, and  $A, B \subseteq |\mathfrak{C}|$ .
- (2) For any  $\bar{a} \in {}^{<\kappa}A$  and  $B_0 \subseteq |M| \cup B$  of size less than  $\kappa$ , there exists  $\bar{a}' \in {}^{<\kappa}|M|$  such that  $\operatorname{gtp}(\bar{a}/B_0) = \operatorname{gtp}(\bar{a}'/B_0)$  (here, the Galois types are computed inside  $\mathfrak{C}$ ).

We write  $A \downarrow B$  instead of  $\downarrow (M, A, B)$ . We will also say that  $gtp(\bar{a}/B)$  is a

 $(<\kappa)$ -coheir over M when  $\operatorname{ran}(\bar{a}) \underset{M}{\bigcup} B$  (it is straightforward to check that this does not depend on the choice of  $\bar{a}$ ).

The following locality cardinals will play an important role:

Definition 7.2.3. Let  $\alpha$  be a cardinal.

- (1) Let  $\bar{\kappa}_{\alpha}(\downarrow)$  be the minimal cardinal  $\mu \geq |\alpha|^{+} + \kappa^{+}$  such that for any  $M \leq_{\mathbf{K}} \mathfrak{C}$  that is  $\kappa$ -saturated, any  $A \subseteq |\mathfrak{C}|$  with  $|A| \leq \alpha$ , there exists  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{<\mu}$  with  $A \downarrow M$ . When  $\mu$  does not exist, we set  $\bar{\kappa}_{\alpha_0}(\mathfrak{i}) = \infty$ .
- (2) Let  $\kappa_{\alpha}(\downarrow)$  be the minimal cardinal  $\mu \ge |\alpha|^+ + \aleph_0$  such that for any regular  $\delta \ge \mu$ , any increasing chain  $\langle M_i : i < \delta \rangle$  in **K** and any A of size at most  $\alpha$ , there exists  $i < \delta$  such that  $A \downarrow \bigcup_{i < \delta} M_i$ . When  $\mu$  does not exist, we

set  $\kappa_{\alpha_0}(\mathfrak{i}) = \infty$ . For  $\mathbf{K}^*$  a subclass of  $\mathbf{K}$ , we similarly define  $\kappa_{\alpha}(\downarrow \upharpoonright \mathbf{K}^*)$ , where in addition we require that  $M_i \in \mathbf{K}^*$  for all  $i < \delta$  (we will use this when  $\mathbf{K}^*$  is a class of saturated models).

REMARK 7.2.4. For any cardinal  $\alpha$ , we always have that  $\kappa_{\alpha}(\downarrow) \leq \bar{\kappa}_{\alpha}(\downarrow)$ .

FACT 7.2.5. Under Hypothesis 7.2.1,  $\downarrow$  satisfies the following properties:

- (1) <u>Invariance</u>: If f is an automorphism of  $\mathfrak{C}$  and  $A \underset{M}{\downarrow} B$ , then  $f[A] \underset{f[M]}{\downarrow} B$ .
- (2) <u>Monotonicity</u>: Assume  $A \underset{M}{\cup} B$ , then:

- (a) Left and right monotonicity: If  $A_0 \subseteq A$ ,  $B_0 \subseteq B$ , then  $A_0 \downarrow B_0$ .
- (b) Base monotonicity: If  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} \mathfrak{C}$ ,  $|M'| \subseteq B$ , and M' is  $\kappa$ saturated, then  $A \perp B$ .
- (3) Left and right normality: If  $A \underset{M}{\downarrow} B$ , then  $AM \underset{M}{\downarrow} BM$ .
- (4) <u>Symmetry</u>:  $A \downarrow_{M} B$  if and only if  $B \downarrow_{M} A$ . (5) <u>Strong transitivity</u>: If  $M_0 \leq_{\mathbf{K}} \mathfrak{C}$ ,  $M_1 \leq_{\mathbf{K}} \mathfrak{C}$ ,  $A \downarrow_{M_0} M_1$ , and  $A \downarrow_{M_1} B$ , then  $A \downarrow_{M_0} B$  (note that we do *not* assume that  $M_0 \leq_{\mathbf{K}} M_1$ ).

- (6) Uniqueness for types of length one: If  $M \leq_{\mathbf{K}} M'$ ,  $p, q \in \mathrm{gS}(M')$  are both  $(<\kappa)$ -coheir over M and  $p \upharpoonright M = q \upharpoonright M$ , then p = q
- (7) <u>Set local character</u>: For any  $\alpha$ ,  $\bar{\kappa}_{\alpha}(\downarrow) \leq ((\alpha+2)^{<\kappa_r})^+$ .

Moreover **K** is stable in all  $\mu \geq \kappa$  such that  $\mu = \mu^{2^{<\kappa_r}}$ .

REMARK 7.2.6. We will not use the exact definition of coheir, just that it satisfies the conclusion of Fact 7.2.5.

REMARK 7.2.7. Strong transitivity will be used in the proof that the relation  $\downarrow$ (Definition 7.2.8) is transitive, see Proposition 7.2.11. We do not know if transitivity would suffice.

For what comes next, it will be convenient if we could say that  $A \, {\downarrow} \, B$  and  $M \leq_{\mathbf{K}} N$  implies  $A \underset{N}{\cup} B$ . By base monotonicity, this holds if  $|N| \subseteq B$  but in general this is not part of our assumptions (and in practice this need not hold). Thus we will close  $\downarrow$  under this property. This is where we depart from [**MS90**]; there the authors used that the singular cardinal hypothesis holds above a strongly compact to prove the result corresponding to our Lemma 7.3.2. Here we need to be more clever.

DEFINITION 7.2.8.  $A \overline{\downarrow}_C B$  means that there exists  $M_0 \leq_{\mathbf{K}} \mathfrak{C}, |M_0| \subseteq C$  such that  $A \downarrow B$ .

REMARK 7.2.9.  $\overline{\downarrow}$  need not satisfy the normality property from Fact 7.2.5.

In what follows, we will apply the definition of  $\kappa_{\alpha}$  and  $\bar{\kappa}_{\alpha}$  (Definition 7.2.3) to other independence relations than coheir.

DEFINITION 7.2.10. We write  $A \overline{\downarrow}_C[B]^1$  to mean that  $A \overline{\downarrow}_C b$  for all  $b \in B$ . Similarly define  $[A]^1 \overline{\downarrow}_C B$ . Let  $(\overline{\downarrow})^1$  denote the relation defined by  $A(\overline{\downarrow}_C)^1 B$  if and only if  $A\overline{\bigcup}_C[B]^1$ . For  $\alpha$  a cardinal, let  $\bar{\kappa}^1_{\alpha} = \bar{\kappa}^1_{\alpha}(\overline{\bigcup}) := \bar{\kappa}_{\alpha}((\overline{\bigcup})^1)$ .

Note that  $A \overline{\bigcup}_C B$  implies  $[A]^1 \overline{\bigcup}_C B$  by monotonicity.

Proposition 7.2.11.

- (1)  $\downarrow$  satisfies invariance, monotonicity, symmetry, and strong right transitivity (see Fact 7.2.5).
- (2) For all  $\alpha$ ,  $\bar{\kappa}_{\alpha}(\overline{\downarrow}) = \bar{\kappa}_{\alpha}(\downarrow)$ ,  $\kappa_{\alpha}(\overline{\downarrow}) = \kappa_{\alpha}(\downarrow)$ .
- (3)  $\overline{\downarrow}$  has strong base monotonicity: If  $A \overline{\downarrow}_C B$  and  $C \subseteq C'$ , then  $A \overline{\downarrow}_{C'} B$ .
- (4) If  $A \underset{M}{\downarrow} B$ , then  $A \overline{\downarrow}_M B$ .

- (5) If  $A \overline{\downarrow}_M B$  and M is  $\kappa$ -saturated such  $|M| \subseteq B$ , then  $A \downarrow B$ .
- (6) For all  $\alpha$ ,  $\bar{\kappa}^1_{\alpha}(\overline{\downarrow}) \leq \bar{\kappa}_{\alpha}(\downarrow)$ .

PROOF. All quickly follow from the definition. As an example, we prove that  $\overline{\downarrow}$  has strong right transitivity. Assume  $A \overline{\downarrow}_{M_0} M_1$  and  $A \overline{\downarrow}_{M_1} B$ . Then there exists  $M'_0 \leq_{\mathbf{K}} M_0$  and  $M'_1 \leq_{\mathbf{K}} M_1$  such that  $A \downarrow M_1$  and  $A \downarrow B$ . By monotonicity for  $\downarrow$ ,  $A \downarrow M'_1$ . By strong right transitivity for  $\downarrow$ ,  $A \downarrow B$ . Thus  $M'_0$  witnesses  $A \overline{\downarrow}_{M'_0} B$ .

PROPOSITION 7.2.12. Assume  $\langle M_i : i < \delta \rangle$ ,  $\langle N_i : i < \delta \rangle$  are increasing chains of  $\kappa$ -saturated models, A is a set. If  $A \downarrow_{M_i} N_i$  for all  $i < \delta$  and  $\kappa_{|A|}(\downarrow) \leq \operatorname{cf} \delta$ , then  $A \downarrow_{M_\delta} N_\delta$ , where<sup>2</sup>  $M_\delta := \bigcup_{i < \delta} M_i$  and  $N_\delta := \bigcup_{i < \delta} N_i$ .

PROOF. Without loss of generality,  $\delta = \operatorname{cf} \delta$ . By definition of  $\kappa_{|A|}(\downarrow)$ , there exists  $i < \delta$  such that  $A \underset{N_i}{\downarrow} N_{\delta}$ , so  $A \underset{N_i}{\downarrow} N_{\delta}$ . By strong right transitivity for  $\overline{\downarrow}$ ,  $A \underset{M_i}{\downarrow} N_{\delta}$ . By strong base monotonicity,  $A \underset{M_{\delta}}{\downarrow} N_{\delta}$ .

As already discussed, the reason we use  $\overline{\downarrow}$  is that we want to generalize [**MS90**, 4.17] to our context. In their proof, Makkai and Shelah use that cardinal arithmetic behaves nicely above a strongly compact, and we cannot make use of this fact here. Thus we are only able to prove this lemma for  $\overline{\downarrow}$  instead of  $\downarrow$  (see Lemma 7.3.2). Fortunately, this turns out to be enough. The reader can also think of  $\overline{\downarrow}$  as a trick to absorb some quantifiers.

The next lemma imitates [MS90, 4.18].

LEMMA 7.2.13. Let  $\lambda_0 \geq \kappa_r$  be regular, let  $\lambda > \lambda_0$  be regular such that **K** is stable in unboundedly-many cardinals below  $\lambda$  and let  $\langle M_i : i < \delta \rangle$  be an increasing chain with  $M_i \lambda$ -saturated for all  $i < \delta$ . Assume that  $\kappa_1(\downarrow \upharpoonright \mathbf{K}^{\lambda_0\text{-sat}}) \leq \operatorname{cf} \delta$ .

If  $\bar{\kappa}^1_{<\lambda}(\downarrow) \leq \lambda$ , then  $M_{\delta} := \bigcup_{i < \delta} M_i$  is  $\lambda$ -saturated.

PROOF. Without loss of generality,  $\delta = \operatorname{cf} \delta$ . Let  $A \subseteq |M_{\delta}|$  have size less than  $\lambda$ . If  $\lambda \leq \delta$ , then  $A \subseteq |M_i|$  for some  $i < \delta$  and so any type over A is realized in  $M_i \subseteq |M_{\delta}|$ . Now assume without loss of generality that  $\lambda > \delta$ . We need to show every Galois type over A is realized in  $M_{\delta}$ . Let  $\mu := \lambda_0 + \delta$ . Note that  $\mu = \operatorname{cf} \mu < \lambda$ . First, we build an array of  $\lambda_0$ -saturated models  $\langle N_i^{\alpha} \in \mathbf{K}_{<\lambda} : i < \delta, \alpha < \mu \rangle$  such that:

- (1) For all  $i < \delta$ ,  $\langle N_i^{\alpha} : \alpha < \mu \rangle$  is increasing.
- (2) For all  $\alpha < \mu$ ,  $\langle N_i^{\alpha} : i < \delta \rangle$  is increasing.
- (3) For all  $i < \delta$  and all  $\alpha < \mu$ ,  $N_i^{\alpha} \leq M_i$ .
- (4)  $A \subseteq \bigcup_{i < \delta} |N_i^0|$ .
- (5) For all  $\alpha < \mu$  and all  $i < \delta$ ,  $\bigcup_{i < \delta} N_i^{\alpha} \overline{\bigcup}_{N_i^{\alpha+1}} [M_i]^1$ .

For  $\alpha < \mu$ , write  $N_{\delta}^{\alpha} := \bigcup_{i < \delta} N_i^{\alpha}$  and for  $i \leq \delta$ , write  $N_i^{\mu} := \bigcup_{\alpha < \mu} N_i^{\alpha}$ . The following is a picture of the array constructed.

<sup>&</sup>lt;sup>2</sup>Note that  $M_{\delta}$  and  $N_{\delta}$  need not be  $\kappa$ -saturated.



This is enough: Note that for  $i < \delta$ ,  $N_i^{\mu}$  is  $\lambda_0$ -saturated and has size less than  $\lambda$  (since  $\lambda > \mu$  and  $\lambda$  is regular). Note also that since  $\delta \le \mu < \lambda$ ,  $N_{\delta}^{\mu}$  has size less than  $\lambda$  (but we do *not* claim that it is  $\lambda_0$ -saturated).

**Claim:** For all  $i < \delta$ ,  $N^{\mu}_{\delta} \downarrow_{N^{\mu}_{i}} [M_{i}]^{1}$ .

**Proof of claim:** Fix  $i < \delta$  and let  $a \in M_i$ . Fix  $j < \delta$ . By (5), monotonicity, and symmetry,  $a \downarrow_{N_i^{\alpha+1}} N_j^{\alpha}$  for all  $\alpha < \mu$ . By Proposition 7.2.12 applied to  $\langle N_i^{\alpha+1} : \alpha < \mu \rangle$  and  $\langle N_j^{\alpha} : \alpha < \mu \rangle$ ,  $a \downarrow_{N_i^{\mu}} N_j^{\mu}$  (note that  $\mu = \operatorname{cf} \mu \ge \delta \ge \kappa_1(\downarrow)$ ). Since j was arbitrary, we can apply Proposition 7.2.12 again with the constantly  $N_i^{\mu}$  sequence and  $\langle N_j^{\mu} : j < \delta \rangle$  (note that  $\delta = \operatorname{cf} \delta \ge \kappa_1(\downarrow)$ ) to get that  $a \downarrow_{N_i^{\mu}} N_{\delta}^{\mu}$ . By symmetry,  $N_i^{\mu} \downarrow_{N_i^{\mu}} a$ , as desired.

Now let  $p \in \mathrm{gS}(A)$ . By (4),  $A \subseteq N_{\delta}^{\mu}$  so we can extend p to some  $q \in \mathrm{gS}(N_{\delta}^{\mu})$ . Since  $\delta \geq \kappa_1(\downarrow)$ , we can find  $i < \delta$  such that q does not fork over  $N_i^{\mu}$ . Since  $N_i^{\mu} \leq M_i$ ,  $M_i$  is  $\lambda$ -saturated, and  $\|N_i^{\mu}\| < \lambda$ , we can find  $a \in M_i$  realizing  $q \upharpoonright N_i^{\mu}$ . Since by the claim  $N_{\delta}^{\mu} \overline{\bigcup}_{N_i^{\mu}} [M_i]^1$ , we can use symmetry to conclude  $a \overline{\bigcup}_{N_i^{\mu}} N_{\delta}^{\mu}$ , and hence (Proposition 7.2.11(5))  $a \underset{N_i^{\mu}}{\bigcup} N_{\delta}^{\mu}$ . By uniqueness for types of length one, a must realize q, so in particular a realizes p. This concludes the proof that  $M_{\delta}$  is  $\lambda$ -saturated.

This is possible: We define  $\langle N_i^{\alpha} : i < \delta \rangle$  by induction on  $\alpha$ . For a fixed  $i < \delta$ , choose any  $N_i^0 \leq_{\mathbf{K}} M_i$  in  $\mathbf{K}_{<\lambda}$  that contains  $A \cap |M_i|$  and is  $\lambda_0$ -saturated (this is possible since  $\mathbf{K}$  is stable in unboundedly-many cardinals below  $\lambda$ ). For  $\alpha < \mu$  limit and  $i < \delta$ , pick any  $N_i^{\alpha} \leq_{\mathbf{K}} M_i$  containing  $\bigcup_{\beta < \alpha} N_i^{\beta}$  which is in  $\mathbf{K}_{<\lambda}$  and  $\lambda_0$ -saturated (this is possible for the same reason as in the base case). Now assume  $\alpha = \beta + 1 < \mu$ , and  $N_i^{\beta}$  has been defined for  $i < \delta$ . Define  $N_i^{\alpha}$  by induction on i. Assume  $N_j^{\alpha}$  has been defined for all j < i. Pick  $N_i^{\alpha}$  containing  $\bigcup_{j < i} N_j^{\alpha}$  which is in  $\mathbf{K}_{<\lambda}$ , is  $\lambda_0$ -saturated, and satisfies  $N_i^{\alpha} \leq_{\mathbf{K}} M_i$  and (5). This is possible by strong base monotonicity and definition of  $\bar{\kappa}_{<\lambda}^1$ .

Below, we give a more natural formulation of the hypotheses.

THEOREM 7.2.14. Let  $\lambda > \kappa$ . Let  $\langle M_i : i < \delta \rangle$  be an increasing chain with  $M_i$  $\lambda$ -saturated for all  $i < \delta$ . If:

(1) cf  $\delta \ge \kappa_1(\downarrow)$ ; and (2)  $\chi^{2^{<\kappa_r}} < \lambda$  for all  $\chi < \lambda$ ,

then  $\bigcup_{i < \delta} M_i$  is  $\lambda$ -saturated.

PROOF. Let  $M_{\delta} := \bigcup_{i < \delta} M_i$ . Note that  $\lambda > \kappa_r$ : since  $\lambda > \kappa$ ,  $\lambda \ge \kappa^+$  and if  $\lambda = \kappa^+$  then  $\kappa^{<\kappa} < \lambda$  so  $\kappa = \kappa^{<\kappa}$  hence  $\kappa$  is regular:  $\kappa_r = \kappa$ .

Let  $\chi < \lambda$  be such that  $\chi^+ > \kappa_r$ . We show that  $M_{\delta}$  is  $\chi^+$ -saturated. By hypothesis,  $\chi^{2^{\leq \kappa_r}} < \lambda$ , so replacing  $\chi$  by  $\chi^{<\kappa_r}$  if necessary, we might as well assume that  $\chi = \chi^{2^{\leq \kappa_r}}$ . We check that  $\chi^+$  satisfies the conditions of Lemma 7.2.13 (with  $\lambda_0$  there standing for  $\kappa_r$  here) as  $\lambda$  there. By assumption,  $\chi^+$  is regular and  $\chi^+ > \kappa_r$ . Also, **K** is stable in unboundedly-many cardinals below  $\chi^+$  because by the moreover part of Fact 7.2.5, **K** is stable in  $\chi$ .

Now by Proposition 7.2.11(6),  $\bar{\kappa}^1_{\chi}(\overline{\downarrow}) \leq \bar{\kappa}_{\chi}(\downarrow)$ . By Fact 7.2.5,  $\bar{\kappa}_{\chi}(\downarrow) \leq (\chi^{<\kappa_r})^+ = \chi^+$ . Thus  $\bar{\kappa}^1_{\chi}(\overline{\downarrow}) \leq \chi^+$ , as needed.

Thus Lemma 7.2.13 applies and so  $M_{\delta}$  is  $\chi^+$ -saturated. Since  $\chi < \lambda$  was arbitrary,  $M_{\delta}$  is  $\lambda$ -saturated.

For the next corollaries to AECs, we repeat our hypotheses.

COROLLARY 7.2.15. Let **K** be an AEC with amalgamation. Let  $\kappa = \beth_{\kappa} >$ LS(**K**) be such that **K** is  $(< \kappa)$ -tame. Assume that **K** is stable in some cardinal greater than or equal to  $\kappa$  and let  $\langle M_i : i < \delta \rangle$  be an increasing chain of  $\lambda$ -saturated models. If:

(1) cf  $\delta > 2^{<\kappa_r}$ . (2)  $\chi^{2^{<\kappa_r}} < \lambda$  for all  $\chi < \lambda$ . Then  $\bigcup_{i < \delta} M_i$  is  $\lambda$ -saturated.

PROOF. Without loss of generality,  $\delta = \operatorname{cf} \delta < \lambda$ . Also without loss of generality, **K** has joint embedding (otherwise, partition it into disjoint classes, each of which has joint embedding), and arbitrarily large models (since **K** has a model of cardinality  $\kappa = \beth_{\kappa} > \operatorname{LS}(\mathbf{K})$ ). Therefore Hypothesis 7.2.1, and hence the conclusion of Fact 7.2.5, hold.

Note (Remark 7.2.4) that  $\kappa_1(\downarrow) \leq \bar{\kappa}_1(\downarrow) \leq (2^{<\kappa_r})^+$ . Now use Theorem 7.2.14.

# 7.3. Using independence calculus: the superstable case

Next we show that in the *superstable* case we can remove the cardinal arithmetic condition (2) in Corollary 7.2.15.

HYPOTHESIS 7.3.1. Same as in the previous section: Hypothesis 7.2.1.

In the proof of Theorem 7.2.14, we estimated  $\bar{\kappa}^1_{\alpha}(\downarrow)$  using  $\bar{\kappa}_{\alpha}(\downarrow)$ . Using superstability, we can prove a better bound. This is adapted from [MS90, 4.17].

LEMMA 7.3.2. Assume that  $\kappa_1(\downarrow) = \aleph_0$  and **K** is stable in all  $\lambda \ge \kappa_r$ . Then for any cardinal  $\alpha$ ,  $\bar{\kappa}^1_{\alpha}(\overline{\downarrow}) \le \bar{\kappa}_{\kappa_r}(\downarrow) + \alpha^+$ .

PROOF. Let A have size  $\alpha$  and N be a  $\kappa$ -saturated model. We show by induction on  $\alpha$  that there exists an  $M \leq_{\mathbf{K}} N$  with  $||M|| < \mu := \bar{\kappa}_{\kappa_r}(\bot) + \alpha^+$  and  $A \overline{\bigcup}_M [N]^1$ . Note that  $\mu > \kappa_r$ .

If  $\alpha \leq \kappa_r$ , then apply the definition of  $\bar{\kappa}_{\kappa_r}(\downarrow)$  to get a  $M \leq N$  with  $||M|| < \bar{\kappa}_{\kappa_r}(\downarrow)$ ,  $A \downarrow_M N$ , which is more than what we need.

Now, assume  $\alpha > \kappa_r$ , and that the result has been proven for all  $\alpha_0 < \alpha$ . Closing A to a  $\kappa$ -saturated model (using the stability assumptions) if necessary, we can assume without loss of generality that A is a  $\kappa$ -saturated model. Let  $\langle A_i : i < \alpha \rangle$  be an increasing resolution of A such that  $A_i$  is  $\kappa$ -saturated in  $\mathbf{K}_{<\alpha}$  for all  $i < \alpha$ . Now define an increasing chain  $\langle M_i : i < \alpha \rangle$  such that for all  $i < \alpha$ :

- (1)  $M_i \in \mathbf{K}_{<\mu}$  and  $M_i$  is  $\kappa$ -saturated.
- (2)  $M_i \leq_{\mathbf{K}} N$ .
- (3)  $A_i {ot}_{M_i} [N]^1$ .

<u>This is possible</u>: For  $i < \alpha$ , use the induction hypothesis to find  $M_i \leq_{\mathbf{K}} N$ such that  $A_i \overline{\bigcup}_{M_i} [N]^1$  and  $||M_i|| < \mu$ . By strong base monotonicity of  $\overline{\bigcup}$  and the closure assumption, we can assume that  $M_i$  contains  $\bigcup_{i < i} M_i$ .

This is enough: Let  $M \in \mathbf{K}_{<\mu}$  be  $\kappa$ -saturated and contain  $\bigcup_{i < \alpha} M_i$ . We claim that  $\overline{A \cup_M}[N]^1$ . Let  $a \in N$ . By symmetry, it is enough to see  $a \cup_M A$ . This follows from strong base monotonicity and Proposition 7.2.12 applied to  $\langle M_i : i < \alpha \rangle$  and  $\langle A_i : i < \alpha \rangle$  since  $\kappa_1(\overline{\cup}) = \aleph_0 \leq \mathrm{cf}\,\alpha$  by Proposition 7.2.11(2) and the hypothesis.

REMARK 7.3.3. The heavy use of strong base monotonicity in the above proof was the reason for introducing  $\overline{\bigcup}$ .

THEOREM 7.3.4. Let  $\lambda_0 \geq \kappa_r$  be regular. Assume that  $\kappa_1(\downarrow \upharpoonright \mathbf{K}^{\lambda_0 \text{-sat}}) = \aleph_0$ and **K** is stable in all  $\lambda \geq \lambda_0$ . Let  $\lambda \geq \bar{\kappa}_{\kappa_r}(\downarrow) + \lambda_0^+$ .

Let  $\langle M_i : i < \delta \rangle$  be an increasing chain with  $M_i \lambda$ -saturated for all  $i < \delta$ . Then  $M_{\delta} := \bigcup_{i < \delta} M_i$  is  $\lambda$ -saturated.

PROOF. Let  $\chi < \lambda$  be such that  $\chi^+ \geq \bar{\kappa}_{\kappa_r}(\downarrow) + \lambda_0^+$ . We claim that  $\chi^+$  satisfies the hypotheses of Lemma 7.2.13 (as  $\lambda$  there). Indeed by Lemma 7.3.2,  $\bar{\kappa}_{\chi}^1(\overline{\downarrow}) \leq \bar{\kappa}_{\kappa_r}(\downarrow) + \chi^+ = \chi^+$ .

Thus Lemma 7.2.13 applies:  $M_{\delta}$  is  $\chi^+$ -saturated. Since  $\chi < \lambda$  was arbitrary,  $M_{\delta}$  is  $\lambda$ -saturated.

For the next corollary to AECs, we drop our hypotheses.

COROLLARY 7.3.5. Let **K** be an AEC with amalgamation and no maximal models. Let  $\kappa = \beth_{\kappa} > \text{LS}(\mathbf{K})$  be such that **K** is  $(< \kappa)$ -tame. If **K** is categorical in some cardinal strictly above  $\kappa$ , then for all  $\lambda > 2^{\kappa}$ ,  $\mathbf{K}^{\lambda\text{-sat}}$  is an AEC with Löwenheim-Skolem number  $\lambda$ .

PROOF. Using categoricity and amalgamation, it is easy to check that **K** has joint embedding. Let  $\lambda_0 := \kappa^+$ . By Theorem 6.10.8 and Proposition 6.10.10, **K** is stable in all  $\mu \geq \kappa$  and  $\kappa_1(\downarrow \upharpoonright \mathbf{K}^{\lambda_0\text{-sat}}) = \aleph_0$ . In particular, Hypothesis 7.2.1 holds. Remembering (Fact 7.2.5) that  $\bar{\kappa}_{\kappa_r}(\downarrow) \leq \kappa_r^{<\kappa_r} \leq 2^{\kappa}$ , we obtain the result from Theorem 7.3.4 (to show that  $\mathrm{LS}(\mathbf{K}^{\lambda\text{-sat}}) = \lambda$ , imitate the proof of [**She90**, III.3.12]).

## 7.4. Averages

In this section, we write in the framework of stability theory inside a model:

Hypothesis 7.4.1.

- (1)  $\kappa$  is an infinite cardinal.
- (2) L is a  $(< \kappa)$ -ary language.
- (3)  $\mathcal{N}$  is a fixed *L*-structure.
- (4) We work inside  $\mathcal{N}$ .
- (5) Hypotheses 7.4.2 and 7.4.4, see the discussion below.

Midway through, we will also assume Hypothesis 7.4.22.

We use the same notation and convention as the preliminaries of Chapter 2: although we may forget to say it, we always work with quantifier-free  $L_{\kappa,\kappa}$  formulas and types (so the arity of all the variables inside a given formula is less than  $\kappa$ ). Also, since we work inside  $\mathcal{N}$ , everything is defined relative to  $\mathcal{N}$ . For example  $\operatorname{tp}(\bar{c}/A)$ means  $\operatorname{tp}_{qL_{\kappa,\kappa}}(\bar{c}/A;\mathcal{N})$ , the quantifier-free  $L_{\kappa,\kappa}$ -type of  $\bar{c}$  over A, and saturated means saturated in  $\mathcal{N}$ . Similarly, we write  $\models \phi[\bar{b}]$  instead of  $\mathcal{N} \models \phi[\bar{b}]$ . By "type", we mean a member of  $\mathrm{S}^{<\infty}(A)$  for some set A. Whenever we mention a set of formulas (meaning a possibly incomplete type), we mean a  $(L_{\kappa,\kappa}$ -quantifier free) set of formulas that is satisfiable by an element in  $\mathcal{N}$ .

Unless otherwise noted, the letters  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{c}$  denote tuples of elements of length less than  $\kappa$ . The letters A, B, C, will denote subsets of  $\mathcal{N}$ . We say  $\langle A_i : i < \delta \rangle$  is increasing if  $A_i \subseteq A_j$  for all  $i < j < \delta$ .

We say  $\mathcal{N}$  is  $\alpha$ -stable in  $\lambda$  if  $|S^{\alpha}(A)| \leq \lambda$  for all A with  $|A| \leq \lambda$  (the default value is for  $\alpha$  is 1). We say  $\mathcal{N}$  has the order property of length  $\chi$  if there exists a (quantifier-free) formula  $\phi(\bar{x}, \bar{y})$  and elements  $\langle \bar{a}_i : i < \chi \rangle$  of the same arity (less than  $\kappa$ ) such that for  $i, j < \chi, \models \phi[\bar{a}_i, \bar{a}_i]$  if and only if i < j.

Boldface letters like  $\mathbf{I}$ ,  $\mathbf{J}$  will always denote sequences of tuples of the same arity (less than  $\kappa$ ). We will use the corresponding non-boldface letter to denote the linear ordering indexing the sequence (writing for example  $\mathbf{I} = \langle \bar{a}_i : i \in I \rangle$ , where I is a linear order). We sometimes treat such sequences as sets of tuples, writing statements like  $\bar{a} \in \mathbf{I}$ , but then we are really looking at the range of the sequence. To avoid potential mistakes, we do not necessarily assume that the elements of  $\mathbf{I}$  are all distinct although it should always hold in cases of interest. We write  $|\mathbf{I}|$  for the cardinality of the range, i.e. the number of distinct elements in  $\mathbf{I}$ . We will sometimes use the interval notation on linear order. For example, if I is a linear order and  $i \in I$ ,  $[i, \infty)_I := \{j \in I \mid j \geq i\}$ .

As the reader will see, this section builds on earlier work of Shelah from [She09b, Chapter V.A]. Note that Shelah works in an arbitrary logic. We work only with quantifier-free  $L_{\kappa,\kappa}$ -formulas in order to be concrete and because this is the case we are interested in to translate the syntactic results to AECs.

The reader may wonder what the right notion of submodel is in this context. We could simply say that it is "subset" but this does not quite work when translating to AECs. Thus we fix a set of subsets of  $\mathcal{N}$  that by definition will be the substructures of  $\mathcal{N}$ . We require that this set satisfies some axioms akin to those of AECs. This can be taken to be the full powerset if one is not interested in doing an AEC translation.

HYPOTHESIS 7.4.2.  $\mathcal{S} \subseteq \mathcal{P}(|\mathcal{N}|)$  is a fixed set of subsets of  $\mathcal{N}$  satisfying:

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- (1) Closure under chains: If  $\langle A_i : i < \delta \rangle$  is an increasing sequence of members of S, then  $\bigcup_{i < \delta} A_i$  is in S.
- (2) Löwenheim-Skolem axiom: If  $A \subseteq B$  are sets and  $B \in S$ , there exists  $A' \in S$  such that  $A \subseteq A' \subseteq B$  and  $|A'| \leq (|L|+2)^{<\kappa} + |A|$ .

We exclusively use the letters M and N to denote elements of S and call such elements *models*. We pretend they are *L*-structures and write |M| and |N| for their universe and ||M|| and ||N|| for their cardinalities.

REMARK 7.4.3. An element M of S is *not* required to be an L-structure. Note however that if it is  $\kappa$ -saturated for types of length less than  $\kappa$  (see below), then it will be one.

We also need to discuss the definition of saturated: define M to be  $\lambda$ -saturated for types of length  $\alpha$  if for any  $A \subseteq |M|$  of size less than  $\lambda$ , any  $p \in S^{\alpha}(A)$  is realized in M. Similarly define  $\lambda$ -saturated for types of length less than  $\alpha$ . Now in the framework we are working in,  $\mu$ -saturated for types of length less than  $\kappa$ seems to be the right notion, so we say that M is  $\mu$ -saturated if it is  $\mu$ -saturated for types of length less than  $\kappa$ . Unfortunately it is not clear that it is equivalent to  $\mu$ -saturated for types of length one (or length less than  $\omega$ ), even when  $\mu > \kappa$ . However [**She09a**, II.1.14] (the "model-homogeneity = saturativity" lemma) tells us that in case  $\mathcal{N}$  comes from an AEC, then this is the case. Thus we will make the following additional assumption. Note that it is possible to work without it, but then everywhere below "stability" must be replaced by "( $\langle \kappa \rangle$ -stability".

HYPOTHESIS 7.4.4. If  $\mu > (|L| + 2)^{<\kappa}$ , then whenever M is  $\mu$ -saturated for types of length one, it is  $\mu$ -saturated (for types of length less than  $\kappa$ ).

Our goal in this section is to use Shelah's notion of average in this framework to prove a result about chains of saturated models. Recall:

DEFINITION 7.4.5 (Definition V.A.2.6 in [She09b]). For I a sequence,  $\chi$  an infinite cardinal such that  $|\mathbf{I}| \geq \chi$ , and A a set, define  $\operatorname{Av}_{\chi}(\mathbf{I}/A)$  to be the set of formulas  $\phi(\bar{x})$  over A so that the set  $\{\bar{b} \in \mathbf{I} \mid \models \neg \phi[\bar{b}]\}$  has size less than  $\chi$ .

Note that if  $|\mathbf{I}| \geq \chi$  (say all the elements of  $\mathbf{I}$  have the same arity  $\alpha$ ) and  $\phi(\bar{x})$ is a formula with  $\ell(\bar{x}) = \alpha$ , then *at most one* of  $\phi$ ,  $\neg \phi$  is in  $\operatorname{Av}_{\chi}(\mathbf{I}/A)$ . Thus the average is not obviously contradictory, but we do *not* claim that there is an element in  $\mathcal{N}$  realizing it. Also,  $\operatorname{Av}_{\chi}(\mathbf{I}/A)$  might be empty. However, we give conditions below (see Fact 7.4.13 and Theorem 7.4.21) where it is in fact complete (i.e. exactly one of  $\phi$  and  $\neg \phi$  is in the average).

The next lemma is a simple counting argument allowing us to find such an element:

LEMMA 7.4.6. Let **I** be a sequence with  $|\mathbf{I}| \geq \chi$  and let A be a set. Let  $p := \operatorname{Av}_{\chi}(\mathbf{I}/A)$ . Assume that

$$|\mathbf{I}| > \chi + \min((|A| + |L| + 2)^{<\kappa}, |\mathbf{S}^{\ell(p)}(A)|)$$

Then there exists  $\bar{b} \in \mathbf{I}$  realizing p.

PROOF. Assume first that the minimum is realized by  $(|A| + |L| + 2)^{<\kappa}$ . By definition of the average, for every every formula  $\phi(\bar{x}) \in p$ ,  $\mathbf{J}_{\phi} := \{\bar{b} \in \mathbf{I} \mid \models \neg \phi[\bar{b}]\}$  has size less than  $\chi$ . Let  $\mathbf{J} := \bigcup_{\phi \in p} \mathbf{J}_{\phi}$ . Note that  $|\mathbf{J}| \leq \chi + (|A| + |L| + 2)^{<\kappa}$  and by definition any  $\bar{b} \in \mathbf{I} \setminus \mathbf{J}$  realizes p.

Now assume that the minimum is realized by  $|S^{\ell(p)}(A)|$ . Let  $\mu := \chi + |S^{\ell(p)}(A)|$ . By the pigeonhole principle, there exists  $\mathbf{I}_0 \subseteq \mathbf{I}$  of size  $\mu^+$  such that  $\bar{c}, \bar{c}' \in \mathbf{I}_0$  implies  $q := \operatorname{tp}(\bar{c}/A) = \operatorname{tp}(\bar{c}'/A)$ . We claim that  $p \subseteq q$ , which is enough: any  $\bar{b} \in \mathbf{I}_0$  realizes p. If not, there exists  $\phi(\bar{x}) \in p$  such that  $\neg \phi(\bar{x}) \in q$ . By definition of the average, fewer than  $\chi$ -many elements of  $\mathbf{I}$  satisfy  $\neg \phi(\bar{x})$ . However,  $\neg \phi(\bar{x})$  is in q which means that it is realized by all the elements of  $\mathbf{I}_0$  and  $|\mathbf{I}_0| = \mu^+ > \chi$ , a contradiction.  $\Box$ 

We now recall the definition of splitting and study how it interacts with averages.

DEFINITION 7.4.7. A set of formulas p splits over A if there exists  $\phi(\bar{x}, \bar{b}) \in p$ and  $\bar{b}'$  with  $\operatorname{tp}(\bar{b}'/A) = \operatorname{tp}(\bar{b}/A)$  and  $\neg \phi(\bar{x}, \bar{b}') \in p$ .

The following result is classical:

LEMMA 7.4.8 (Uniqueness for nonsplitting). Let  $A \subseteq |M| \subseteq B$ . Assume p, q are complete sets of formulas (say in the variable  $\bar{x}$ , with  $\ell(\bar{x}) < \kappa$ ) over B that do not split over A and M is  $|A|^+$ -saturated. If  $p \upharpoonright M = q \upharpoonright M$ , then p = q.

PROOF. Let  $\phi(\bar{x}, \bar{b}) \in p$  with  $\bar{b} \in B$ . We show  $\phi(\bar{x}, \bar{b}) \in q$  and the converse is symmetric. By saturation<sup>3</sup>, find  $\bar{b}' \in M$  such that  $\operatorname{tp}(\bar{b}'/A) = \operatorname{tp}(\bar{b}/A)$ . Since p does not split over A,  $\phi(\bar{x}, \bar{b}') \in p$ . Since  $p \upharpoonright M = q \upharpoonright M$ ,  $\phi(\bar{x}, \bar{b}') \in q$ . Since again q does not split,  $\phi(\bar{x}, \bar{b}) \in q$ .

We would like to study when the average is a nonsplitting extension. This is the purpose of the next definition.

DEFINITION 7.4.9. I is  $\chi$ -based on A if for any B,  $\operatorname{Av}_{\chi}(\mathbf{I}/B)$  does not split over A.

The next lemma tells us that any sequence is based on a set of small size.

LEMMA 7.4.10 (IV.1.23(2) in [She09a]). If I is a sequence and  $\mathbf{J} \subseteq \mathbf{I}$  has size at least  $\chi$ , then I is  $\chi$ -based on J.

PROOF. Let *B* be a set. Let  $p := \operatorname{Av}_{\chi}(\mathbf{I}/B)$ . Note that  $p \subseteq \operatorname{Av}_{\chi}(\mathbf{J}/B)$ . Let  $\bar{b}, \bar{b}' \in B$  be such that  $\operatorname{tp}(\bar{b}/\mathbf{J}) = \operatorname{tp}(\bar{b}'/\mathbf{J})$ . Assume  $\phi(\bar{x}, \bar{b}) \in p$ . Then since  $p \subseteq \operatorname{Av}_{\chi}(\mathbf{J}/B)$ , let  $\bar{a} \in \mathbf{J}$  be such that  $\models \phi[\bar{a}, \bar{b}]$ . Since  $\bar{a} \in \mathbf{J}$ ,  $\models \phi[\bar{a}, \bar{b}']$ . Since there are at least  $\chi$ -many such  $\bar{a}$ 's,  $\neg \phi(\bar{x}, \bar{b}') \notin p$ .

We know that at most one of  $\phi$ ,  $\neg \phi$  is in the average. It is very desirable to have that *exactly* one is in, i.e. that the average is a *complete* type. This is the purpose of the next definition. Recall from the beginning of this section that **I** always denotes a sequence of elements of the same arity less than  $\kappa$ .

DEFINITION 7.4.11 (V.A.2.1 in [She09b]). A sequence I is said to be  $\chi$ convergent if  $|\mathbf{I}| \geq \chi$  and for any set A,  $\operatorname{Av}_{\chi}(\mathbf{I}/A)$  is a complete type over A. That is, whenever  $\phi(\bar{x})$  is a formula with  $\ell(\bar{x})$  equal to the arity of all the elements of I, then we have that exactly one of  $\phi$  or  $\neg \phi$  is in  $\operatorname{Av}_{\chi}(\mathbf{I}/A)$ .

REMARK 7.4.12 (Monotonicity). If  $\mathbf{I}$  is  $\chi$ -convergent,  $\mathbf{J} \subseteq \mathbf{I}$ , and  $|\mathbf{J}| \geq \chi' \geq \chi$ , then for any set A,  $\operatorname{Av}_{\chi}(\mathbf{I}/A) = \operatorname{Av}_{\chi'}(\mathbf{J}/A)$ . In particular,  $\mathbf{J}$  is  $\chi'$ -convergent.

<sup>&</sup>lt;sup>3</sup>Note that we are really using saturation for types of length less than  $\kappa$  here.

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Recall [She90, III.1.7(1)] that if T is a first-order stable theory and  $\mathbf{I}$  is an infinite sequence of indiscernibles (in its monster model), then  $\mathbf{I}$  is  $\aleph_0$ -convergent. The proof relies heavily on the compactness theorem. We would like a replacement of the form "if  $\mathcal{N}$  has some stability and  $\mathbf{I}$  is nice, then it is convergent. The next result is key. It plays the same role as the ability to extract indiscernible subsequences in first-order stable theories.

FACT 7.4.13 (The convergent set existence theorem: V.A.2.8 in [She09b]). Let  $\chi_0 \geq (|L|+2)^{<\kappa}$  be such that  $\mathcal{N}$  does not have the order property of length  $\chi_0^+$ . Let  $\mu$  be an infinite cardinal such that  $\mu = \mu^{\chi_0} + 2^{2^{\chi_0}}$ .

Let **I** be a sequence with  $|\mathbf{I}| = \mu^+$ . Then there is  $\mathbf{J} \subseteq \mathbf{I}$  of size  $\mu^+$  which is  $\chi_0$ -convergent.

However having to extract a subsequence every time is too much for us. One issue is with the cardinal arithmetic condition on  $\mu$ : what if we have a sequence of length  $\mu^+$  when  $\mu$  is a singular cardinal of low cofinality? We work toward proving a more constructive result: *Morley* sequences (defined below) are always convergent. The parameters represent respectively a bound on the size of A, the degree of saturation of the models, and the length of the sequence. They will be assigned default values in Hypothesis 7.4.22.

DEFINITION 7.4.14. We say  $\langle \bar{a}_i : i \in I \rangle \land \langle N_i : i \in I \rangle$  is a  $(\chi_0, \chi_1, \chi_2)$ -Morley sequence for p over A if:

- (1)  $\chi_0 \leq \chi_1 \leq \chi_2$  are infinite cardinals, I is a linear order, A is a set,  $p(\bar{x})$  is a set of formulas with parameters and  $\ell(\bar{x}) < \kappa$ , and there is  $\alpha < \kappa$  such that for all  $i \in I$ ,  $\bar{a}_i \in {}^{\alpha}\mathcal{N}$ .
- (2) For all  $i \in I$ ,  $A \subseteq |N_i|$  and  $|A| < \chi_0$ .
- (3)  $\langle N_i : i \in I \rangle$  is increasing, and each  $N_i$  is  $\chi_1$ -saturated.
- (4) For all  $i \in I$ ,  $\bar{a}_i$  realizes<sup>4</sup>  $p \upharpoonright N_i$  and for all j > i in I,  $\bar{a}_i \in {}^{\alpha}N_j$ .
- (5) i < j in I implies  $\bar{a}_i \neq \bar{a}_j$ .
- (6)  $|I| \ge \chi_2$ .
- (7) For all i < j in I,  $\operatorname{tp}(\bar{a}_i/N_i) = \operatorname{tp}(\bar{a}_j/N_i)$ .
- (8) For all  $i \in I$ ,  $tp(\bar{a}_i/N_i)$  does not split over A.

When p or A is omitted, we mean "for some p or A". We call  $\langle N_i : i \in I \rangle$  the witnesses to  $\mathbf{I} := \langle \bar{a}_i : i \in I \rangle$  being Morley, and when we omit them we simply mean that  $\mathbf{I} \cap \langle N_i : i \in I \rangle$  is Morley for some witnesses  $\langle N_i : i \in I \rangle$ .

REMARK 7.4.15 (Monotonicity). Let  $\langle \bar{a}_i : i \in I \rangle \frown \langle N_i : i \in I \rangle$  be  $(\chi_0, \chi_1, \chi_2)$ -Morley for p over A. Let  $\chi'_0 \ge \chi_0, \chi'_1 \le \chi_1$ , and  $\chi'_2 \le \chi_2$ . Let  $I' \subseteq I$  be such that  $|I'| \ge \chi'_2$ , then  $\langle \bar{a}_i : i \in I' \rangle \frown \langle N_i : i \in I' \rangle$  is  $(\chi'_0, \chi'_1, \chi'_2)$ -Morley for p over A.

REMARK 7.4.16. By the proof of [She90, I.2.5], a Morley sequence is indiscernible (this will not be used).

The next result tells us how to build Morley sequences inside a given model:

LEMMA 7.4.17. Let  $A \subseteq |M|$  and let  $\chi \ge (|L|+2)^{<\kappa}$  be such that  $|A| \le \chi$ . Let  $p \in S^{\alpha}(M)$  be nonalgebraic (that is,  $a_i \notin |M|$  for all  $i < \alpha$  for any  $\bar{a}$  realizing p) such that p does not split over A, and let  $\mu > \chi$ . If:

(1) M is  $\mu^+$ -saturated.

<sup>&</sup>lt;sup>4</sup>Note that dom p might be smaller than  $N_i$ .

(2)  $\mathcal{N}$  is stable in  $\mu$ .

Then there exists  $\langle \bar{a}_i : i < \mu^+ \rangle \land \langle N_i : i < \mu^+ \rangle$  inside M which is  $(\chi^+, \chi^+, \mu^+)$ -Morley for p over A.

PROOF. We build  $\langle \bar{a}_i : i < \mu^+ \rangle$  and  $\langle N_i : i < \mu^+ \rangle$  increasing such that for all  $i < \mu^+$ :

- (1)  $A \subseteq |N_0|$ .
- (2)  $|N_i| \subseteq |M|$ .
- (3)  $||N_i|| \le \mu$ .
- (4)  $N_i$  is  $\chi^+$ -saturated.
- (5)  $\bar{a}_i \in {}^{\alpha}N_{i+1}$ .
- (6)  $\bar{a}_i$  realizes  $p \upharpoonright N_i$ .

This is enough by definition of a Morley sequence (note that for all  $i < \mu^+$ ,  $\bar{a}_i \notin {}^{\alpha}N_i$  by nonalgebraicity of p, so  $\bar{a}_i \neq \bar{a}_j$  for all j < i).

This is possible: assume inductively that  $\langle \bar{a}_j : j < i \rangle \land \langle N_j : j < i \rangle$  has been defined. Pick  $N_i \subseteq M$  which is  $\chi^+$ -saturated, has size  $\leq \mu$ , and contains  $A \cup \bigcup_{j < i} N_j$ . Such an  $N_i$  exists: simply build an increasing chain  $\langle M_k : k < \chi^+ \rangle$ with  $M_0 := A \cup \bigcup_{j < i} N_j$ ,  $||M_k|| \leq \mu$ , and  $M_k$  realizing all elements of  $S(\bigcup_{k' < k} M_{k'})$ (this is where we use stability in  $\mu$ ). Then  $N_i := \bigcup_{k < \chi^+} M_k$  is as desired (we are using Hypothesis 7.4.4 to deduce that it is  $\chi^+$ -saturated for types of length less than  $\kappa$ ). Now pick  $\bar{a}_i \in {}^{\alpha}M$  realizing  $p \upharpoonright N_i$  (exists by saturation of M).  $\Box$ 

Before proving that Morley sequences are convergent (Theorem 7.4.21), we prove several useful lemmas:

LEMMA 7.4.18. Let  $\mathbf{I} := \langle \bar{a}_i : i \in I \rangle$  be  $(\chi_0, \chi_1, \chi)$ -Morley, as witnessed by  $\langle N_i : i \in I \rangle$ . Let  $i \in I$  be such that  $[j, \infty)_I$  has size at least  $\chi$ . Then  $\operatorname{Av}_{\chi}(\mathbf{I}/N_i) \subseteq \operatorname{tp}(\bar{a}_i/N_i)$ .

PROOF. Let  $\phi(\bar{x})$  be a formula over  $N_i$  with  $\ell(\bar{x}) = \ell(\bar{a}_i)$ . Assume  $\phi(\bar{x}) \in$ Av<sub> $\chi$ </sub>(I/ $N_i$ ). By definition of average and assumption there exists  $j \in [i, \infty)$  such that  $\models \phi[\bar{a}_j]$ . By (7) in Definition 7.4.14,  $\models \phi[\bar{a}_i]$  so  $\phi(\bar{x}) \in \text{tp}(\bar{a}_i/N_i)$ .

LEMMA 7.4.19. Let I be a linear order and let  $\chi < |I|$  be infinite. Then there exists  $i \in I$  such that both  $(-\infty, i]_I$  and  $[i, \infty)_I$  have size at least  $\chi$ .

PROOF. Without loss of generality,  $|I| = \chi^+$ . Let  $I_0 := \{i \in I \mid |(-\infty, i]_I| < \chi\}$ and let  $I_1 := \{i \in I \mid |[i, \infty)_I| < \chi\}$ . Assume the conclusion of the lemma fails. Then  $I_0 \cup I_1 = I$ . Thus either  $|I_0| = \chi^+$  or  $|I_1| = \chi^+$ . Assume that  $|I_0| = \chi^+$ , the proof in case  $|I_1| = \chi^+$  is symmetric. Let  $\delta := \operatorname{cf} I_0$  and let  $\langle a_\alpha \in I_0 : \alpha < \delta \rangle$  be a cofinal sequence. If  $\delta < \chi^+$ , then, since  $I_0 = \bigcup_{\alpha < \delta} (-\infty, a_\alpha]_I$  has size  $\chi^+$ , there is  $\alpha < \delta$  such that  $|(-\infty, a_\alpha)_I| = \chi^+$ . If  $\delta \ge \chi^+$ , then  $|(-\infty, a_\chi)_I| \ge \chi$ . Either of these contradict the definition of  $I_0$ .

LEMMA 7.4.20. Let **I** be  $(\chi^+, \chi^+, \chi^+)$ -Morley over A (for some type). If **I** is  $\chi$ -convergent, then **I** is  $\chi$ -based on A.

PROOF. Let  $\mathbf{I} := \langle \bar{a}_i : i \in I \rangle$  and let  $\langle N_i : i \in I \rangle$  witness that  $\mathbf{I}$  is  $\chi$ -Morley over A. By assumption,  $|I| \ge \chi^+$ , so let  $i \in I$  be as given by Lemma 7.4.19: both  $(-\infty, i]_I$  and  $[i, \infty)_I$  have size at least  $\chi$ . By Lemma 7.4.10 and the definition of i, we can find  $A' \subseteq |N_i|$  containing A of size at most  $\chi$  such that  $\mathbf{I}$  is  $\chi$ -based on A'.
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Let  $p := \operatorname{Av}_{\chi}(\mathbf{I}/\mathcal{N})$ . Assume for a contradiction that p splits over A and pick witnesses such that  $\phi(\bar{x}, \bar{b}), \neg \phi(\bar{x}, \bar{b}') \in p$  and  $\operatorname{tp}(\bar{b}/A') = \operatorname{tp}(\bar{b}'/A')$ . Note that  $p \upharpoonright N_i = \operatorname{tp}(\bar{a}_i/N_i)$  by convergence and Lemma 7.4.18. Since  $N_i$  is  $\chi^+$ -saturated, we can find  $\bar{b}'' \in {}^{<\kappa}|N_i|$  such that  $\operatorname{tp}(\bar{b}''/A') = \operatorname{tp}(\bar{b}/A')$ . Now either  $\phi(\bar{x}, \bar{b}'') \in p$  or  $\neg \phi(\bar{x}, \bar{b}'') \in p$ . If  $\phi(\bar{x}, \bar{b}'') \in p$ , then  $\phi(\bar{x}, \bar{b}''), \neg \phi(\bar{x}, \bar{b}')$  witness that p splits over Aand if  $\neg \phi(\bar{x}, \bar{b}'') \in q$ , then  $\phi(\bar{x}, \bar{b}), \neg \phi(\bar{x}, \bar{b}'')$  witness the splitting. Either way, we can replace  $\bar{b}$  or  $\bar{b}'$  by  $\bar{b}''$ . So (swapping the role of  $\bar{b}$  and  $\bar{b}'$  if necessary), assume without loss of generality that  $\bar{b}'' = \bar{b}$  (so  $\bar{b} \in {}^{<\kappa}|N_i|$ ).

By definition of a Morley sequence,  $p \upharpoonright N_i$  does not split over A, so  $\bar{b}' \notin \langle \kappa | N_i |$ . Let  $p'_i := p \upharpoonright N_i \cup \{\phi(\bar{x}, \bar{b}), \phi(\bar{x}, \bar{b}')\}$ . We claim that  $p'_i$  does not split over A: if it does, since  $\phi(\bar{x}, \bar{b}')$  is the only formula of  $p'_i$  with parameters outside of  $N_i$ , the splitting must be witnessed by  $\phi(\bar{x}, \bar{c}), \neg \phi(\bar{x}, \bar{c}')$ , and one of them must be outside  $N_i$ , so  $\bar{c} = \bar{b}'$ . Now  $\operatorname{tp}(\bar{b}/A) = \operatorname{tp}(\bar{b}'/A) = \operatorname{tp}(\bar{c}'/A)$ , and we have  $\bar{c}' \in \langle \kappa | N_i |$  so by nonsplitting of  $p \upharpoonright N_i$ , also  $\neg \phi(\bar{x}, \bar{b}) \in p \upharpoonright N_i$ . This is a contradiction since we know  $\phi(\bar{x}, \bar{b}) \in p \upharpoonright N_i$ .

Now, since **I** is  $\chi$ -based on A', p does not split over A' and by monotonicity  $p'_i$  also does not split over A'. Now use the proof of Lemma 7.4.8 (with  $M = N_i$ ) to get a contradiction.

We are now ready to prove the relationship between Morley and convergent:

THEOREM 7.4.21. Let  $\chi_0 \ge (|L|+2)^{<\kappa}$  be such that  $\mathcal{N}$  does not have the order property of length  $\chi_0^+$ . Let  $\chi := (2^{2^{\chi_0}})^+$ .

If **I** is a  $(\chi_0^+, \chi_0^+, \chi)$ -Morley sequence, then **I** is  $\chi$ -convergent.

PROOF. Write  $\mathbf{I} = \langle \bar{a}_i : i \in I \rangle$  and let  $\langle N_i : i \in I \rangle$  witness that it is Morley for p over A.

Assume for a contradiction that **I** is *not*  $\chi$ -convergent. Then there exists a formula  $\phi(\bar{x})$  (over  $\mathcal{N}$ ) and linear orders  $I_{\ell} \subseteq I$ ,  $\ell = 0, 1$  such that  $|I_{\ell}| = \chi$  and  $i \in I_{\ell}$  implies<sup>5</sup>  $\models \phi^{\ell}[\bar{a}_i]$ . By Fact 7.4.13, we can assume without loss of generality that  $\mathbf{I}_{\ell} := \langle \bar{a}_i : i \in I_{\ell} \rangle$  is  $\chi_0$ -convergent. By Lemma 7.4.20 (with  $\chi_0$  here standing for  $\chi$  there),  $\mathbf{I}_{\ell}$  is  $\chi_0$ -based on A for  $\ell = 0, 1$ . Let  $p_{\ell} := \operatorname{Av}_{\chi_0}(\mathbf{I}_{\ell}/\mathcal{N})$ . Since  $\mathbf{I}_{\ell}$  is  $\chi_0$ -based on A,  $p_{\ell}$  does not split over A. By Lemma 7.4.19, pick  $i_{\ell} \in I$  so that  $|(i, \infty)_{I_{\ell}}| \geq \chi_0$  for  $\ell = 0, 1$ . let  $i := \min(i_0, i_1)$ . By Lemma 7.4.18 and convergence,  $p_{\ell} \upharpoonright N_{i_{\ell}} = \operatorname{tp}(a_{i_{\ell}}/N_{i_{\ell}})$  so  $p_{\ell} \upharpoonright N_i = \operatorname{tp}(a_{i_{\ell}}/N_i)$  so  $p_0 \upharpoonright N_i = p_1 \upharpoonright N_i$ . By assumption,  $N_i$  is  $\chi_0^+$ -saturated. By uniqueness for nonsplitting (Lemma 7.4.8),  $p_0 = p_1$ . However  $\phi(\bar{x}) \in p_0$  while  $\neg \phi(\bar{x}) \in p_1$ , contradiction.

From now on we assume:

Hypothesis 7.4.22.

- (1)  $\chi_0 \ge (|L|+2)^{<\kappa}$  is an infinite cardinal.
- (2)  $\mathcal{N}$  does not have the order property of length  $\chi_0^+$ .
- (3)  $\chi := (2^{2^{\chi_0}})^+$ .
- (4) The default parameters for Morley sequences are (χ<sup>+</sup><sub>0</sub>, χ<sup>+</sup>, χ<sup>+</sup>), and the default parameter for averages and convergence is χ. That is, Morley means (χ<sup>+</sup><sub>0</sub>, χ<sup>+</sup>, χ<sup>+</sup>)-Morley, convergent means χ-convergent, Av(I/A) means Av<sub>χ</sub>(I/A), and based means χ-based.

<sup>&</sup>lt;sup>5</sup>Where  $\phi^0$  stands for  $\phi$ ,  $\phi^1$  for  $\neg \phi$ .

Note that Theorem 7.4.21 and Hypothesis 7.4.22 imply that any Morley sequence is convergent. Moreover by Lemma 7.4.20, any Morley sequence over A is based on A. We will use this freely.

Before studying chains of saturated models, we generalize Lemma 7.4.20 to independence notions that are very close to splitting (the reason has to do with the translation to AECs (Section 7.5)):

DEFINITION 7.4.23. A splitting-like notion is a binary relation R(p, A), where  $p \in S^{<\infty}(B)$  for some set B and  $A \subseteq B$ , satisfying the following properties:

- (1) Monotonicity: If  $A \subseteq A' \subseteq B_0 \subseteq B$ ,  $p \in S^{<\infty}(B)$ , and R(p, A), then  $R(p \upharpoonright B_0, A')$ .
- (2) Weak uniqueness: If  $A \subseteq |M| \subseteq B$ , M is  $(|A| + (|L| + 2)^{<\kappa})^+$ -saturated, and for  $\ell = 1, 2, q_{\ell} \in S^{<\infty}(B)$ ,  $R(q_{\ell}, A)$ , and  $q_1 \upharpoonright M = q_2 \upharpoonright M$ , then  $q_1 = q_2$ .
- (3) R extends nonsplitting: If  $p \in S^{<\infty}(B)$  does not split over  $A \subseteq B$ , then R(p, A).

We also say "p does not R-split over A" instead of R(p, A).

REMARK 7.4.24. If R(p, A) holds if and only if p does not split over A, then R is a splitting-like notion: monotonicity is easy to check and R is nonsplitting. Weak uniqueness is Lemma 7.4.8.

LEMMA 7.4.25. Let R be a splitting-like notion. Let  $p \in S^{<\kappa}(B)$  be such that p does not R-split over  $A \subseteq B$  with  $|A| \leq \chi_0$ .

Let  $\mathbf{I} := \langle \bar{a}_i : i \in I \rangle \land \langle N_i : i \in I \rangle$  be Morley for p over A. If  $|\bigcup_{i \in I} N_i| \subseteq B$ , then  $\operatorname{Av}(\mathbf{I}/B) = p$ .

PROOF. Since **I** is Morley, **I** is convergent. By Lemma 7.4.20, **I** is based on A. Thus we have that  $\operatorname{Av}(\mathbf{I}/B)$  does not split over A, so it does not R-split over A. Let  $i \in I$  be such that  $|(i, \infty)_I| \ge \chi$  (use Lemma 7.4.19). Then  $\operatorname{Av}(\mathbf{I}/N_i) = \operatorname{tp}(\bar{a}_i/N_i) = p \upharpoonright N_i$  by Lemma 7.4.18. By the weak uniqueness axiom of splitting-like relations (with  $N_i$  here standing for M there),  $\operatorname{Av}(\mathbf{I}/B) = p$ .

To construct Morley sequences, we will also use:

Fact 7.4.26.

- (1) If  $\mu = \mu^{\chi_0} + 2^{2^{\chi_0}}$ , then  $\mathcal{N}$  is  $(< \kappa)$ -stable in  $\mu$ .
- (2) Let M be  $\chi_0^+$ -saturated. Then for any  $p \in S^{<\kappa}(M)$ , there exists  $A \subseteq |M|$  of size at most  $\chi_0$  such that p does not split over A.

PROOF. The first result is [She09b, V.A.1.19]. The second follows from [She09b, V.A.1.12]: one only has to observe that the condition between M and  $\mathcal{N}$  there holds when M is  $\chi_0^+$ -saturated.

We can now get a (completely local) result on unions of saturated models.

THEOREM 7.4.27. Assume:

- (1)  $\lambda > \chi^+$  is such that  $\mathcal{N}$  is stable in  $\mu$  for unboundedly many  $\mu < \lambda$ .
- (2)  $\langle M_i : i < \delta \rangle$  is increasing and for all  $i < \delta$ ,  $M_i$  is  $\lambda$ -saturated. Write  $M_{\delta} := \bigcup_{i < \delta} M_i$ .
- (3) For any  $q \in S(M_{\delta})$ , there exists a splitting-like notion R,  $i < \delta$  and  $A \subseteq |M_i|$  of size at most  $\chi_0$  such that q does not R-split over A.

Then  $M_{\delta}$  is  $\lambda$ -saturated.

PROOF. By Hypothesis 7.4.4, it is enough to check that  $M_{\delta}$  is  $\lambda$ -saturated for types of length one. Let  $p \in S(B)$ ,  $B \subseteq |M_{\delta}|$  have size less than  $\lambda$ . Let q be an extension of p to  $S(M_{\delta})$ . If q is algebraic, then p is realized inside  $M_{\delta}$  so assume without loss of generality that q is not algebraic. By assumption, there exists a splitting-like notion R,  $i < \delta$  and  $A \subseteq |M_i|$  such that p does not R-split over A and  $|A| \leq \chi_0$ . Without loss of generality, i = 0. Now  $M_0$  is  $\chi_0^+$ -saturated so (by Fact 7.4.26) there exists  $A' \subseteq |M_0|$  of size at most  $\chi_0$  such that  $q \upharpoonright M_0$  does not split over A'. By making A larger if necessary, we can assume A = A'.

Pick  $\mu < \lambda$  such that  $\mu \ge \chi^+ + |B|$  and  $\mathcal{N}$  is stable in  $\mu$ . Such a  $\mu$  exists by the hypothesis on  $\lambda$ . By Lemma 7.4.17, there exists a sequence **I** of length  $\mu^+$  which is Morley for  $q \upharpoonright M_0$  over A, with the witnesses living inside  $M_0$ . Thus **I** is also Morley for q over A.

By Lemma 7.4.25,  $\operatorname{Av}(\mathbf{I}/M_{\delta}) = q$ , and so in particular  $\operatorname{Av}(\mathbf{I}/B) = q \upharpoonright B = p$ . By Lemma 7.4.6, p is realized by an element of  $\mathbf{I} \subseteq |M_0| \subseteq |M_{\delta}|$ , as needed.

The condition (3) in Theorem 7.4.27 is useful in case we know that the local character cardinal for *chains*  $\kappa_{\alpha}$  is significantly lower than the local character cardinal for *sets*  $\bar{\kappa}_{\alpha}$ . This is the case when a superstability-like condition holds. If we do not care about the local character cardinal for chains, we can state a version of Theorem 7.4.27 without condition (3).

COROLLARY 7.4.28. Assume:

(1)  $\lambda > \chi^+$  is such that  $\mu^{\chi_0} < \lambda$  for all  $\mu < \lambda$ .

(2)  $\langle M_i : i < \delta \rangle$  is increasing and for all  $i < \delta$ ,  $M_i$  is  $\lambda$ -saturated.

If cf  $\delta \geq \chi_0^+$ , then  $\bigcup_{i < \delta} M_i$  is  $\lambda$ -saturated.

PROOF. Fix  $\alpha < \kappa$ . By Fact 7.4.26(1),  $\mathcal{N}$  is  $\alpha$ -stable in  $\mu$  for any  $\mu < \lambda$  with  $\mu^{\chi_0} = \mu$  and  $\mu \geq \chi$ . By hypothesis, there are unboundedly many such  $\mu$ 's.

Let  $M_{\delta} := \bigcup_{i < \delta} M_i$ . By an easy argument using the cofinality condition on  $\delta$ ,  $M_{\delta}$  is  $\chi_0^+$ -saturated. By Fact 7.4.26(2), for any  $p \in S^{<\kappa}(M_{\delta})$ , there exists  $A \subseteq |M_{\delta}|$  of size  $\leq \chi_0$  such that p does not split over A. By the cofinality assumption on  $\delta$ , we can find  $i < \delta$  such that  $A \subseteq |M_i|$ . Now apply Theorem 7.4.27 and get the result.

REMARK 7.4.29. The proof shows that we can still replace (1) with " $\lambda > \chi^+$  is such that  $\mathcal{N}$  is stable in  $\mu$  for unboundedly many  $\mu < \lambda$ ".

We end this section with the following interesting variation: the cardinal arithmetic condition on  $\lambda$  is improved, and we do not even need that the  $M_i$ 's be  $\lambda$ -saturated, only that they realize enough types from the previous  $M_i$ 's.

THEOREM 7.4.30. Assume:

- (1)  $\lambda > \chi$  is such that  $\mu^{<\kappa} < \lambda$  for all  $\mu < \lambda$  (or such that  $\mathcal{N}$  is stable in  $\mu$  for unboundedly many  $\mu < \lambda$ ).
- (2) M is such that for any  $q \in S(M)$  there exists  $\langle M_i : i < \delta \rangle$  strictly increasing so that:
  - (a)  $\delta \geq \lambda$  is a limit ordinal.
  - (b)  $M = \bigcup_{i < \delta} M_i$
  - (c) For all  $i < \delta$ ,  $M_i$  is  $\chi^+$ -saturated and  $M_{i+1}$  realizes  $q \upharpoonright M_i$ .

(d) There exists a splitting-like notion R,  $i < \delta$  and  $A \subseteq |M_i|$  of size at most  $\chi_0$  such that q does not R-split over A.

Then M is  $\lambda$ -saturated.

PROOF. By Hypothesis 7.4.4, it is enough to check that M is  $\lambda$ -saturated for types of length one. Let  $p \in S(B)$ ,  $B \subseteq |M|$  have size less than  $\lambda$ . Let q be an extension of p to S(M). If q is algebraic, then p is realized inside M, so assume qis not algebraic. Let  $\langle M_i : i < \delta \rangle$  be as given by (2) for q. Let R be a splitting-like notion for which there is  $i < \delta$  and  $A \subseteq |M_i|$  such that q does not R-split over Aand  $|A| \leq \chi_0$ . Without loss of generality, i = 0.

Let  $\mu := (\chi + |B|)^{<\kappa}$  (or take  $\mu < \lambda$  such that  $\mu \ge \chi + |B|$  and  $\mathcal{N}$  is stable in  $\mu$ ). Note that  $\mu < \lambda$ . For  $i < \mu^+$ , let  $a_i \in |M_{i+1}|$  realize  $q \upharpoonright M_i$ . By cofinality considerations,  $\bigcup_{i < \mu^+} M_i$  is  $\chi^+$ -saturated. By Fact 7.4.26, there exists  $i < \mu^+$  and  $A' \subseteq |M_i|$  such that  $q \upharpoonright \bigcup_{i < \mu^+} M_i$  does not split over A'. By some renaming we can assume without loss of generality that A' = A. It is now easy to check that  $\mathbf{I} := \langle a_i : i < \mu^+ \rangle$  is Morley for q over A, as witnessed by  $\langle M_i : i < \mu^+ \rangle$ .

By Lemma 7.4.25,  $\operatorname{Av}(\mathbf{I}/M) = q$ , and so in particular  $\operatorname{Av}(\mathbf{I}/B) = q \upharpoonright B = p$ . By Lemma 7.4.6, p is realized by an element of  $\mathbf{I} \subseteq |M_0| \subseteq |M|$ , as needed.  $\Box$ 

# 7.5. Translating to AECs

To translate the result of the previous section to AECs, we will use the *Galois Morleyization* of an AEC, a tool introduced in Chapter 2: Essentially, we expand the language of the AEC with a symbol for each Galois type. With enough tameness, Galois types then become syntactic.

DEFINITION 7.5.1 (Definition 2.3.3). Let **K** be an AEC and let  $\kappa$  be an infinite cardinal. Define an (infinitary) expansion  $\hat{L}$  of  $L(\mathbf{K})$  by adding a relation symbol  $R_p$  of arity  $\ell(p)$  for each  $p \in \mathrm{gS}^{<\kappa}(\emptyset)$ . Expand each  $N \in \mathbf{K}$  to a  $\hat{L}$ -structure  $\hat{N}$  by specifying that for each  $\bar{a} \in \hat{N}$ ,  $R_p^{\hat{N}}(\bar{a})$  holds exactly when  $\mathrm{gtp}(\bar{a}/\emptyset; N) = p$ . We write  $\hat{K}^{<\kappa}$  for  $\hat{K}$ . We call  $\hat{K}^{<\kappa}$  the  $(<\kappa)$ -Galois Morleyization of **K**.

REMARK 7.5.2. Let **K** be an AEC and  $\kappa$  be an infinite cardinal. Then  $|L(\widehat{K}^{<\kappa})| \leq |gS^{<\kappa}(\emptyset)| + |L| \leq 2^{<(\kappa + LS(\mathbf{K})^+)}$ .

FACT 7.5.3 (Theorem 2.3.15). Let **K** be a  $(< \kappa)$ -tame AEC, and let  $M \leq_{\mathbf{K}} N_{\ell}$ ,  $a_{\ell} \in |N_{\ell}|, \ell = 1, 2$ . Then  $\operatorname{gtp}(a_1/M; N_1) = \operatorname{gtp}(a_2/M; N_2)$  if and only if<sup>6</sup>  $\operatorname{tp}_{\widehat{qL}_{\kappa,\kappa}}(a_1/M; \widehat{N_1}) = \operatorname{tp}_{\widehat{qL}_{\kappa,\kappa}}(a_2/M; \widehat{N_2})$ .

Moreover the left to right direction does not need tameness: if  $M \leq_{\mathbf{K}} N_{\ell}, \bar{a}_{\ell} \in \langle \infty | N_{\ell} |, \ell = 1, 2, \text{ and } \operatorname{gtp}(\bar{a}_1/M; N_1) = \operatorname{gtp}(\bar{a}_2/M; N_2), \text{ then } \operatorname{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{a}_1/M; \widehat{N}_1) = \operatorname{tp}_{q\widehat{L}_{\kappa,\kappa}}(\bar{a}_2/M; \widehat{N}_2).$ 

Note that this implies in particular that (if **K** is  $(< \kappa)$ -tame and has amalgamation) the Galois version of saturation and stability coincide with their syntactic analog in  $\hat{K}^{<\kappa}$ . There is also a nice correspondence between the syntactic version of the order property defined at the beginning of Section 7.4 and Shelah's semantic version [**She99**, 4.3]:

<sup>&</sup>lt;sup>6</sup>Recall that  $tp_{q\hat{L}_{\kappa,\kappa}}$  stands for quantifier-free  $L_{\kappa,\kappa}$ -type.

DEFINITION 7.5.4. Let  $\alpha$  and  $\mu$  be cardinals and let **K** be an AEC. A model  $M \in \mathbf{K}$  has the  $\alpha$ -order property of length  $\mu$  if there exists  $\langle \bar{a}_i : i < \mu \rangle$  inside M with  $\ell(\bar{a}_i) = \alpha$  for all  $i < \mu$ , such that for any  $i_0 < j_0 < \mu$  and  $i_1 < j_1 < \mu$ ,  $\operatorname{gtp}(\bar{a}_{i_0}\bar{a}_{j_0}/\emptyset; N) \neq \operatorname{gtp}(\bar{a}_{j_1}\bar{a}_{i_1}/\emptyset; N).$ 

*M* has the  $(< \alpha)$ -order property of length  $\mu$  if it has the  $\beta$ -order property of length  $\mu$  for some  $\beta < \alpha$ . *M* has the order property of length  $\mu$  if it has the  $\alpha$ -order property of length  $\mu$  for some  $\alpha$ .

**K** has the  $\alpha$ -order of length  $\mu$  if some  $M \in \mathbf{K}$  has it. **K** has the order property if it has the order property for every length.

FACT 7.5.5 (Proposition 2.4.6). Let **K** be an AEC. Let  $\widehat{K} := \widehat{K}^{<\kappa}$ . If  $\widehat{N} \in \widehat{K}$  has the (syntactic) order property of length  $\chi$ , then N has the (Galois) ( $< \kappa$ )-order property of length  $\chi$ . Conversely, if  $\chi \geq 2^{<(\kappa + \mathrm{LS}(\mathbf{K})^+)}$  and N has the (Galois) ( $< \kappa$ )-order property of length  $(2^{\chi})^+$ , then  $\widehat{N}$  has the (syntactic) order property of length  $\chi$ .

We will use Facts 7.5.3 and 7.5.5 freely in this section. We will also use the following results about stability and the order property:

FACT 7.5.6 (Fact 2.4.7 and Theorem 2.4.15). Let **K** be an  $(< \kappa)$ -tame AEC with amalgamation. The following are equivalent:

- (1) **K** is stable in some  $\lambda \ge \kappa + \text{LS}(\mathbf{K})$ .
- (2) There exists  $\mu \leq \lambda_0 < h^*(\kappa + \mathrm{LS}(\mathbf{K})^+)$  (see Definition 2.2.2) such that **K** is stable in any  $\lambda \geq \lambda_0$  with  $\lambda = \lambda^{<\mu}$ .
- (3) **K** does not have the order property.
- (4) There exists  $\chi < h^*(\kappa + \mathrm{LS}(\mathbf{K})^+)$  such that **K** does not have the  $(< \kappa)$ -order property of length  $\chi$ .

It remains to find an independence notion to satisfy condition (3) in Theorem 7.4.27. The splitting-like notion R there will be given by the following:

DEFINITION 7.5.7. Let **K** be an AEC and let  $\kappa$  be an infinite cardinal. For  $p \in gS^{<\infty}(B; N)$  and  $A \subseteq B$ , say  $p \kappa$ -explicitly does not split over A if whenever  $p = gtp(\bar{c}/B; N)$ , for any  $\bar{b}, \bar{b}' \in {}^{<\kappa}B$ , if  $gtp(\bar{b}/A; N) = gtp(\bar{b}'/A; N)$ , then  $tp_{q\widehat{L}_{\kappa,\kappa}}(\bar{c}\bar{b}/A; N) = tp_{q\widehat{L}_{\kappa,\kappa}}(\bar{c}\bar{b}'/A; N)$ , where  $\widehat{L} = L(\widehat{K}^{<\kappa})$ .

REMARK 7.5.8. This is closely related to explicit nonsplitting defined in Definition 3.3.13. The definition there is that p explicitly does not split if and only if it  $\kappa$ -explicitly does not split for all  $\kappa$ . When **K** is fully ( $< \kappa$ )-tame and short (see **[Bon14b**, 3.3]), this is equivalent to just asking for p to  $\kappa$ -explicitly not split.

REMARK 7.5.9 (Syntactic invariance). Let  $\widehat{K} := \widehat{K}^{<\kappa}$ . Assume  $\operatorname{tp}_{q\widehat{L}_{\kappa,\kappa}}(\overline{c}/B; N) = \operatorname{tp}_{q\widehat{L}_{\kappa,\kappa}}(\overline{c}'/B; N)$  and  $\operatorname{gtp}(\overline{c}/B; N)$   $\kappa$ -explicitly does not split over  $A \subseteq B$ . Then  $\operatorname{gtp}(\overline{c}'/B; N)$   $\kappa$ -explicitly does not split over A.

We will use the following definition of an independence relation, which appears implicitly in Lemma 4.4.8.

DEFINITION 7.5.10. Let **K** be an AEC with amalgamation and let  $\lambda \geq \text{LS}(\mathbf{K})$ be such that **K** is  $\lambda$ -tame and stable in  $\lambda$ . For  $M \leq_{\mathbf{K}} N$  with  $M \lambda^+$ -saturated, we say that  $p \in \text{gS}(N)$  does not  $\lambda$ -fork over M if there exists  $M_0 \in \mathbf{K}_{\lambda}$  such that  $M_0 \leq_{\mathbf{K}} M$  and p does not  $\lambda$ -split over  $M_0$  (that is [She99, I.3.2], whenever  $N_{\ell}$ ,  $\ell = 1, 2$ , are of size  $\lambda$  such that  $M_0 \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} N$  and  $f : N_1 \cong_{M_0} N_2$ , we have that  $f(p \upharpoonright N_1) = p \upharpoonright N_2$ ). We write  $a \downarrow_M^{\lambda-\mathrm{nf}} N$  to say that  $\operatorname{gtp}(a/N)$  does not  $\lambda$ -fork over M and will apply the definition of the properties Definition 7.2.3 and Fact 7.2.5 to it.

FACT 7.5.11 (§4.4 and §4.5 in Chapter 4). Let **K** be an AEC with amalgamation and let  $\lambda \geq \text{LS}(\mathbf{K})$  be such that **K** is  $\lambda$ -tame, stable in  $\lambda$ , and has no maximal models in  $\lambda$ .

Then  $\lambda$ -nonforking satisfies invariance, monotonicity, transitivity (i.e. if  $M_1 \leq_{\mathbf{K}} M_2 \leq_{\mathbf{K}} M_3$  are such that  $M_1$  and  $M_2$  are  $\lambda^+$ -saturated,  $p \in \mathrm{gS}(M_3)$ , p does not  $\lambda$ -fork over  $M_2$ ,  $p \upharpoonright M_2$  does not  $\lambda$ -fork over  $M_1$ , then p does not  $\lambda$ -fork over  $M_1$ ), and uniqueness. Moreover  $\bar{\kappa}_1(\bigcup \upharpoonright \mathbf{K}^{\lambda^+}$ -sat) =  $\lambda^{++}$ .

We recall the definition of superstability from Definition 6.10.1 using local character of nonsplitting. Note that it coincides with the first-order definition and is equivalent to the definition implicit in [**GVV16**] and Chapter 4 and explicit in [**Gr002**, 7.12].

DEFINITION 7.5.12 (Superstability). Let  $\mathbf{K}$  be an AEC.

- (1) For  $M, N \in \mathbf{K}$ , say N is universal over M if and only if  $M \leq_{\mathbf{K}} N$  and whenever we have  $M' \geq_{\mathbf{K}} M$  such that ||M'|| = ||M||, then there exists  $f: M' \xrightarrow{M} N$ .
- (2) **K** is  $\lambda$ -superstable if:
  - (a)  $LS(\mathbf{K}) \leq \lambda$  and  $\mathbf{K}_{\lambda} \neq \emptyset$ .
  - (b)  $\mathbf{K}_{\lambda}$  has amalgamation, joint embedding, and no maximal models.
  - (c) **K** is stable in  $\lambda$ .
  - (d) **K** has no long splitting chains in  $\lambda$ : for any limit  $\delta < \lambda^+$  and increasing continuous  $\langle M_i : i \leq \delta \rangle$  in  $\mathbf{K}_{\lambda}$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \delta$ , and any  $p \in \mathrm{gS}(M_{\delta})$ , there exists  $i < \delta$  such that p does not  $\lambda$ -split over  $M_i$ .

Note that superstability implies local character of  $\lambda$ -forking, and superstability transfers up assuming tameness:

FACT 7.5.13. Let **K** be an AEC with amalgamation that is  $\lambda$ -tame and  $\lambda$ -superstable.

- (1) (Lemma 4.4.11)  $\kappa_1(\stackrel{\lambda-\mathrm{nf}}{\bigcup} \upharpoonright \mathbf{K}^{\lambda^+\mathrm{-sat}}) = \aleph_0.$
- (2) **K** is  $\lambda'$ -superstable for all  $\lambda' \geq \lambda$ .

The next result imitates Lemma 3.5.6:

LEMMA 7.5.14. Let **K** be an AEC with amalgamation and let  $\lambda \geq \text{LS}(\mathbf{K})$  be such that **K** is  $\lambda$ -tame, stable in  $\lambda$ , and has no maximal models in  $\lambda$ . Let  $\kappa \leq \lambda^+$ .

Let  $M \leq_{\mathbf{K}} N$  be given with  $M \lambda^+$ -saturated. Let  $p \in gS(N)$ . If p does not  $\lambda$ -fork over M, then  $p \kappa$ -explicitly does not split over A.

PROOF. By definition of  $\lambda$ -nonforking, there exists  $M_0 \leq_{\mathbf{K}} M$  of size  $\lambda$  such that p does not  $\lambda$ -split over  $M_0$ . We will show that p explicitly does not  $\kappa$ -split over  $M_0$  which is enough by base monotonicity of explicit  $\kappa$ -nonsplitting.

Work inside a monster model  $\mathfrak{C}$  and write  $p = \operatorname{gtp}(c/N)$ . Let  $\overline{b}, \overline{b'} \in \langle \kappa | N |$  be such that  $\operatorname{gtp}(b/M_0) = \operatorname{gtp}(\overline{b'}/M_0)$ . Let f be an automorphism of  $\mathfrak{C}$  fixing  $M_0$  such that  $f(\overline{b}) = \overline{b'}$ . By invariance, f(p) does not  $\lambda$ -split over  $M_0$ . Now using uniqueness of  $\lambda$ -splitting (see [Van06, I.4.12]),  $f(p \upharpoonright M_0 \overline{b}) = p \upharpoonright M_0 \overline{b'}$ . The result follows.  $\Box$ 

The next technical lemma captures the essence of our translation:

LEMMA 7.5.15. Let **K** be a ( $< \kappa$ )-tame AEC with amalgamation and no maximal models. Let  $\chi_0$  be such that:

- (1)  $\chi_0 \ge 2^{<(\kappa + \mathrm{LS}(\mathbf{K})^+)}$ .
- (2) **K** does not have the  $(< \kappa)$ -order property of length  $\chi_0^+$ .

Set  $\chi := (2^{2^{\chi_0}})^+$ . Let  $\lambda$  be such that:

- (1)  $\lambda > \chi^+$ .
- (2) **K** is stable in  $\mu$  for unboundedly many  $\mu < \lambda$ .

Let  $\theta := \kappa_1 ( \stackrel{\chi\text{-nf}}{\cup} \upharpoonright \mathbf{K}^{\chi^+\text{-sat}} )$ . Then:

- (1) If  $\langle M_i : i < \delta \rangle$  is an increasing chain of  $\lambda$ -saturated models and cf  $\delta \ge \theta$ , then  $\bigcup_{i < \delta} M_i$  is  $\lambda$ -saturated.
- (2) If  $M \in \mathbf{K}$  is such that for any  $q \in gS(M)$  there exists  $\langle M_i : i < \delta \rangle$  strictly increasing so that:
  - (a)  $\delta \ge \lambda$  and cf  $\delta \ge \theta$ .
  - (b)  $M = \bigcup_{i < \delta} M_i$ .
  - (c) For all  $i < \delta$ ,  $M_i$  is  $\chi^+$ -saturated and  $M_{i+1}$  realizes  $q \upharpoonright M_i$ . Then M is  $\lambda$ -saturated.

PROOF. We prove the first statement. The proof of the second is analogous but uses Theorem 7.4.30 instead of Theorem 7.4.27. Set  $M_{\delta} := \bigcup_{i < \delta} M_i$ . Let  $N \geq_{\mathbf{K}} M_{\delta}$ be such that N realizes all types in  $\mathrm{gS}^{<\kappa}(M_{\delta})$ . We check that  $M_{\delta}$  is  $\lambda$ -saturated in N. Let  $\widehat{K} := \widehat{K}^{<\kappa}$  be the  $(<\kappa)$ -Galois Morleyization of  $\mathbf{K}$ . Let  $\mathcal{N} := \widehat{N}$ . By  $(<\kappa)$ -tameness, it is enough to show that  $\widehat{M}_{\delta}$  is (syntactically)  $\lambda$ -saturated in  $\mathcal{N}$ . Work inside  $\mathcal{N}$  in the language of  $\widehat{K}$ . We also let  $\mathcal{S} := \{|M| \mid M \leq_{\mathbf{K}} N\}$ . Note that  $\mathcal{S}$  satisfies Hypothesis 7.4.2.

First observe that Hypothesis 7.4.22 holds as (Remark 7.5.2)  $|L(\widehat{K})| \leq 2^{\langle (\kappa + \text{LS}(\mathbf{K})^+)}$ , so  $\chi_0$  has all the required properties. Also, Hypothesis 7.4.4 holds by [**She09a**, II.1.14]. Note that **K** is stable in  $\chi$  by Fact 7.4.26(1). By hypothesis,  $\lambda > \chi^+$ . We want to use Theorem 7.4.27, and it remains to check that (3) there holds.

For  $A \subseteq B$  and  $p \in S^{<\infty}(B)$ , define the relation R(p, A) to hold if and only if  $p = \operatorname{tp}(\bar{c}/B)$  and  $\operatorname{gtp}(\bar{c}/B; N)$   $\kappa$ -explicitly does not split over A. Note that this is well-defined by Remark 7.5.9. We want to check that this is a splitting-like notion (Definition 7.4.23). By definition of  $\kappa$ -explicit nonsplitting, if  $p \in S^{<\infty}(B)$ does not split over  $A \subseteq B$ , then R(p, A). Also, it is easy to check that R satisfies the monotonicity axiom. It remains to check the weak uniqueness axiom. So let M be  $\mu := \left(|A| + (|L(\hat{K})| + 2)^{<\kappa}\right)^+$ -saturated,  $A \subseteq |M| \subseteq B$ , and for  $\ell = 1, 2,$  $q_{\ell} \in S^{<\infty}(B), R(q_{\ell}, A)$  and  $q_1 \upharpoonright M = q_2 \upharpoonright M$ . Note that M is also  $\mu$ -saturated in the Galois sense (by tameness and Remark 3). Thus we can imitate the proof of Lemma 7.4.8, using Galois saturation instead of syntactic saturation to get  $\bar{b}'$ satisfying  $\operatorname{gtp}(\bar{b}'/A) = \operatorname{gtp}(\bar{b}/A)$  (instead of just  $\operatorname{tp}(\bar{b}'/A) = \operatorname{tp}(\bar{b}'/A)$  as there). The definition of  $\kappa$ -explicit nonsplitting then makes the proof go through.

Now let  $q \in gS(M_{\delta})$ . By definition of  $\theta$ , there exists  $i < \delta$  such that q does not  $\chi$ -fork over  $M_i$ . Now by set local character, there exists  $M \leq_{\mathbf{K}} M_i$  of size  $\chi^+$  such that  $q \upharpoonright M_i$  does not  $\chi$ -fork over M. By transitivity, q does not  $\chi$ -fork over M. By Lemma 7.5.14, working syntactically inside  $\mathcal{N}$ , q does not R-split over M. Thus (3) holds. Therefore  $M_{\delta}$  is  $\lambda$ -saturated, as desired.  $\square$ 

We obtain the following result on chains of saturated models in stable AECs:

THEOREM 7.5.16. Let **K** be a ( $< \kappa$ )-tame AEC with amalgamation,  $\kappa \geq$  $LS(\mathbf{K})^+$ . If **K** is stable, then there exists  $\chi_0 \leq \lambda_0 < h^*(\kappa)$  (see Definition 2.2.2) satisfying the following property:

If  $\lambda \geq \lambda_0$  is such that  $\mu^{\chi_0} < \lambda$  for all  $\mu < \lambda$  (or just that **K** is stable in  $\mu$  for unboundedly many  $\mu < \lambda$ ), then whenever  $\langle M_i : i < \delta \rangle$  is an increasing chain of  $\lambda$ -saturated models with cf  $\delta \geq \lambda_0$ , we have that  $\bigcup_{i < \delta} M_i$  is  $\lambda$ -saturated.

PROOF. Using Fact 7.5.6, pick  $\chi_0 \leq \mu_0 < h^*(\kappa)$  such that:

- (1)  $\chi_0^+ \ge 2^{<(\kappa + \mathrm{LS}(\mathbf{K})^+)} + \kappa^+.$
- (2) K is stable in any μ ≥ μ<sub>0</sub> with μ = μ<sup>χ<sub>0</sub></sup>.
  (3) K does not have the (< κ)-order property of length χ<sub>0</sub><sup>+</sup>.

Now set  $\lambda_0 := (2^{2^{\chi_0}})^{+3}$  and apply Lemma 7.5.15.

The statement becomes much nicer in superstable AECs:

THEOREM 7.5.17. Let **K** be a ( $< \kappa$ )-tame AEC with amalgamation, LS(**K**)<sup>+</sup>  $\leq$  $\kappa$ . Let  $\mu \geq LS(\mathbf{K})$  be a cardinal with  $\mu^+ \geq \kappa$  and assume that  $\mathbf{K}$  is  $\mu$ -superstable. Then there exists  $\lambda_0 < h^*(\kappa) + \mu^{++}$  with  $\lambda_0 > \mu$  such that for any  $\lambda \ge \lambda_0$ :

- (1)  $\mathbf{K}^{\lambda-\text{sat}}$  is an AEC with  $\text{LS}(\mathbf{K}^{\lambda-\text{sat}}) = \lambda$ .
- (2) If  $M \in \mathbf{K}^{\lambda_0 \text{-sat}}$  is such that for any  $q \in \mathrm{gS}(M)$  there exists  $\langle M_i : i < \lambda \rangle$ a resolution of M in  $\mathbf{K}^{\lambda_0$ -sat such that  $q \upharpoonright M_i$  is realized in  $M_{i+1}$  for all  $i < \lambda$ , then  $M \in \mathbf{K}^{\lambda-\text{sat}}$ .

# Proof.

- (1) We first show that any increasing union of  $\lambda$ -saturated models is saturated. Let  $\lambda_{00} < h^*(\kappa)$  be as given by the proof of Theorem 7.5.16 and let  $\lambda_0 := \lambda_{00} + \mu^+$ . By Fact 7.5.13,  $\kappa_1(\downarrow \models \mathbf{K}^{\lambda_0 \text{-sat}}) = \aleph_0$ . Now apply Lemma 7.5.15 (note that by Fact 7.5.13, **K** is stable in any  $\mu' \ge \mu$ ). To see that  $LS(\mathbf{K}^{\lambda-\text{sat}}) = \lambda$ , imitate the proof of [She90, Theorem III.3.12].
- (2) Similar: use the second conclusion of Lemma 7.5.15.

## 7.6. On superstability in AECs

In the introduction to [She09a], Shelah points out the importance of finding a definition of superstability for AECs. He also remarks (p. 19) that superstability in AECs suffers from "schizophrenia": definitions that are equivalent in the first-order case might not be equivalent in AECs. In this section, we point out that Definition 7.5.12 implies several other candidate definitions of superstability. Recall from Fact 7.5.13 that Definition 7.5.12 implies that the class is stable on a tail of cardinals. We will focus on five other definitions:

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- (1) For every high-enough  $\lambda$ , the union of any increasing chain of  $\lambda$ -saturated models is  $\lambda$ -saturated. This is the focus of this chapter and is equivalent to first-order superstability by [**AG90**, Theorem 13].
- (2) The existence of a saturated model of size  $\lambda$  for every high-enough  $\lambda$ . In first-order, this is an equivalent definition of superstability by the saturation spectrum theorem (Fact 7.1.2).
- (3) The existence of a superlimit model of size  $\lambda$  for every high-enough  $\lambda$ . This is the definition of superstability listed by Shelah in [She09a, N.2.4]. Recall that a model  $M \in \mathbf{K}_{\lambda}$  is *superlimit* if it is universal, has an isomorphic proper extension in  $\mathbf{K}_{\lambda}$ , and whenever  $\langle M_i : i < \delta \rangle$  is increasing in  $\mathbf{K}_{\lambda}$ ,  $\delta < \lambda^+$ , and  $M_i \cong M$  for all  $i < \delta$ , then  $\bigcup_{i < \delta} M_i \cong M$ .
- (4) The existence of a good λ-frame on a subclass of saturated models (e.g. for every high-enough λ). Recall that a good frame is essentially a forking-like notion for types of length one (see [She09a, II.2.1] for the formal definition). Good frames are *the* central notion in [She09a] and are described by Shelah as a "bare bone" definition of superstability.
- (5) The uniqueness of limit models of size λ for every high-enough λ: Recall that a model M is (λ, δ)-limit over M<sub>0</sub> if M<sub>0</sub> ≤<sub>K</sub> M are in K<sub>λ</sub>, δ < λ<sup>+</sup> is a limit ordinal and there exists ⟨M<sub>i</sub> : i ≤ δ⟩ increasing continuous such that M<sub>δ</sub> = M and i < δ implies M<sub>i</sub> <<sup>univ</sup> M<sub>i+1</sub> (recall Definition 7.5.12). We say K<sub>λ</sub> has uniqueness of limit models if for any M<sub>0</sub> ∈ K<sub>λ</sub>, any limit δ<sub>1</sub>, δ<sub>2</sub> < λ<sup>+</sup>, any M<sub>ℓ</sub> which are (λ, δ<sub>ℓ</sub>)-limit over M<sub>0</sub> are isomorphic over M<sub>0</sub>. Uniqueness of limit models is central in [She99, SV99, Van06, Van13] and is further examined in [GVV16] (Theorem 6.1 there proves that the condition is equivalent to first-order superstability). These papers all prove the uniqueness under a categoricity (or no Vaughtian pair) assumption. In [She09a, II.4.8], uniqueness of limit models is proven from a good frame (see also [Bon14a, 9.2] for a detailed writeup). This is used in [BG] to get eventual uniqueness of limit models from categoricity, but the authors have to make an extra assumption (the extension property for coheir).

Note that some easy implications between these definitions are already known (see for example [**Dru13**, 2.3.12]). We now show that assuming amalgamation and tameness, if **K** is superstable, then all five of these conditions hold. This gives an eventual version of [**Dru13**, Conjecture 4.2.5]. It also shows how to build a good frame without relying on categoricity (as opposed to all previous constructions, see [**She09a**, II.3.7], Theorem 4.7.3, or Theorem 6.10.16).

THEOREM 7.6.1. If **K** is a  $\mu$ -tame,  $\mu$ -superstable AEC with amalgamation, then there exists  $\lambda_0 < h(\mu)$  such that for all  $\lambda \ge \lambda_0$ :

- (1) The union of any increasing chain of  $\lambda$ -saturated models in **K** is  $\lambda$ -saturated.
- (2) **K** has a saturated model of size  $\lambda$ .
- (3) **K** has a superlimit model of size  $\lambda$ .
- (4) There exists a type-full good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$ .
- (5)  $\mathbf{K}_{\lambda}$  has uniqueness of limit models.

PROOF. Note that by Fact 7.5.13,  $\mathbf{K}_{\geq\mu}$  has no maximal models, joint embedding, and is stable in every cardinal. Let  $\lambda_0 < h(\mu)$  be as given by Theorem 7.5.17 and let  $\lambda \geq \lambda_0$ . Then  $\mathbf{K}^{\lambda\text{-sat}}$  is an AEC with  $\mathrm{LS}(\mathbf{K}^{\lambda\text{-sat}}) = \lambda$ . Thus (1) and (2)

hold. If M is the saturated model of size  $\lambda$ , then it is easy to check that M is superlimit: it is universal as  $\mathbf{K}_{\geq\mu}$  has joint embedding, it has a saturated proper extension of size  $\lambda$  since  $\mathrm{LS}(\mathbf{K}^{\lambda-\mathrm{sat}}) = \lambda$ , and any increasing chain of saturated models in  $\mathbf{K}_{\lambda}$  of length less than  $\lambda^+$  has a saturated union. Thus (3) holds. To see (4), use Theorem 6.10.8.(2c).

We are now ready to prove (5). As observed above, a good frame implies uniqueness of limit models. Thus  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  has uniqueness of limit models. It follows that  $\mathbf{K}_{\lambda}$  has uniqueness of limit models: Let  $M_{\ell}$  be  $(\lambda, \delta_{\ell})$ -limit over  $M_0$ ,  $\ell = 1, 2$ . Pick  $M'_0 \geq_{\mathbf{K}} M_0$  in  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$ . By universality,  $M_{\ell}$  is also  $(\lambda, \delta_{\ell})$ -limit over some copy of  $M'_0$ , so after some renaming we can assume without loss of generality that  $M_0 = M'_0$ . For  $\ell = 1, 2$ , build  $\langle M'_i : i \leq \delta_{\ell} \rangle$  increasing continuous such that for all  $i < \delta_{\ell}, M'_i \in \mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  and  $M'_i <_{\mathbf{K}}^{\text{univ}} M'_{i+1}$ . This is easy to do and by a back and forth argument,  $M_{\ell} \cong_{M_0} M'_{\delta_{\ell}}$ . By uniqueness of limit models in  $\mathbf{K}^{\lambda-\text{sat}}, M'_{\delta_1} \cong_{M_0} M'_{\delta_2}$ .

# CHAPTER 8

# Shelah's eventual categoricity conjecture in universal classes: part I

This chapter is based on [**Vasg**]. I thank Will Boney for pointing me to AECs which admit intersections, for helpful conversations, and for good feedback. I thank John Baldwin, Adi Jarden, and the referee for useful feedback that greatly helped me improve the presentation of this chapter.

#### Abstract

# We prove:

THEOREM 8.0.2. Let  $\mathbf{K}$  be a universal class. If  $\mathbf{K}$  is categorical in cardinals of arbitrarily high cofinality, then  $\mathbf{K}$  is categorical on a tail of cardinals.

The proof stems from ideas of Adi Jarden and Will Boney, and also relies on a deep result of Shelah. As opposed to previous works, the argument is in ZFC and does not use the assumption of categoricity in a successor cardinal. The argument generalizes to abstract elementary classes (AECs) that satisfy a locality property and where certain prime models exist. Moreover assuming amalgamation we can give an explicit bound on the Hanf number and get rid of the cofinality restrictions:

THEOREM 8.0.3. Let **K** be an AEC with amalgamation. Assume that **K** is fully LS(**K**)-tame and short and has primes over sets of the form  $M \cup \{a\}$ . Write  $H_2 := \beth_{\left(2^{2}(2^{\text{LS}(\mathbf{K})})^+\right)^+}$ . If **K** is categorical in a  $\lambda > H_2$ , then **K** is categorical in all  $\lambda' \ge H_2$ .

# 8.1. Introduction

Morley's categoricity theorem [Mor65] states that a first-order countable theory that is categorical in some uncountable cardinal must be categorical in all uncountable cardinals. The result motivated much of the development of first-order classification theory (it was later generalized by Shelah [She74] to uncountable theories).

Toward developing a classification theory for non-elementary classes, one can ask whether there is such a result for infinitary logics, e.g. for an  $L_{\omega_1,\omega}$  sentence. In 1971, Keisler proved [**Kei71**, Section 23] a generalization of Morley's theorem to this framework assuming in addition that the model in the categoricity cardinal is sequentially homogeneous. Unfortunately Shelah later observed using an example of Marcus [**Mar72**] that Keisler's assumption does not follow from categoricity. Still in the summer of 1976, Shelah proposed the following far-reaching conjecture (this is listed as Open problem D.(3a) in [**She90**]): CONJECTURE 8.1.1 (see p. 218 of [She83a]). If L is a countable language and  $\psi \in L_{\omega_1,\omega}$  is categorical in one  $\lambda \geq \beth_{\omega_1}$ , then it is categorical in all  $\lambda' \geq \beth_{\omega_1}$ .

This has now become the central test problem in classification theory for nonelementary classes. Shelah alone has more than 2000 pages of approximations (for example [She75a, She83a, She83b, MS90, She99, She01a, She09a, She09b]). Shelah's results led him to introduce a semantic framework encompassing several different infinitary logics and algebraic classes [She87a]: abstract elementary classes (AECs). In this framework, we can state an eventual version of the conjecture<sup>1</sup>:

CONJECTURE 8.1.2 (Shelah's eventual categoricity conjecture for AECs). An AEC that is categorical in a high-enough cardinal is categorical on a tail of cardinals.

REMARK 8.1.3. A more precise statement is that there should be a function  $\mu \mapsto \lambda_{\mu}$  such that every AEC **K** categorical in some  $\lambda \geq \lambda_{\text{LS}(\mathbf{K})}$  is categorical in all  $\lambda' \geq \lambda_{\text{LS}(\mathbf{K})}$ . By a similar argument as for the existence of Hanf numbers [Han60] (see [Bal09, Conclusion 15.13]), Shelah's eventual categoricity conjecture for AECs is equivalent to the statement that an AEC categorical in *unboundedly* many cardinals is categorical on a tail of cardinals. We will use this equivalence freely. Note that Theorem 8.0.3 gives an *explicit*<sup>2</sup> bound for  $\lambda_{\mu}$ , so proves a stronger statement than just Shelah's eventual categoricity conjecture for universal classes with amalgamation<sup>3</sup>.

Positive results are known in less general frameworks: For homogeneous model theory by Lessmann [Les00] and more generally for  $\aleph_0$ -tame<sup>4</sup> simple finitary AECs by Hyttinen and Kesälä [HK11] (note that these results apply only to countable languages). In uncountable languages, Grossberg and VanDieren proved the conjecture in tame AECs categorical in a successor cardinal [GV06c, GV06a]. Later Will Boney pointed out that tameness follows<sup>5</sup> from large cardinals [Bon14b], a result that (as pointed out in [LR16]) can also be derived from a 25 year old theorem of Makkai and Paré ([MP89, Theorem 5.5.1]). A combination of this gives that statements much stronger than Shelah's categoricity conjecture for a successor hold if there exists a proper class of strongly compact cardinals.

The question of whether categoricity in a sufficiently high *limit* cardinal implies categoricity on a tail remains open (even in tame AECs). The central tool there is the notion of a *good*  $\lambda$ -*frame*, a local axiomatization of forking which is the main concept in [**She09a**]. After developing the theory of good  $\lambda$ -frames over several hundreds of pages, Shelah claims to be able to prove the following (see [**She09a**, Discussion III.12.40], a proof should appear in [**Sheb**]):

<sup>&</sup>lt;sup>1</sup>The statement here appears in [She09a, Conjecture N.4.2].

 $<sup>^2\</sup>mathrm{We}$  thank John Baldwin for helpful conversation on the topic.

<sup>&</sup>lt;sup>3</sup>We are not sure how to make the distinction precise. Maybe one can call the *computable* eventual categoricity conjecture the statement that has the additional requirement that  $\mu \mapsto \lambda_{\mu}$  be computable, where computable can be defined as in [**BS14**]. Note that in Shelah's original categoricity conjecture,  $\lambda_{\mu}$  is  $\beth_{(2^{\mu})^+}$ , see [**She00**, 6.14.(3)].

 $<sup>^4{\</sup>rm Tameness}$  is a locality property for orbital types introduced by Grossberg and VanDieren in  $[{\bf GV06b}].$ 

 $<sup>{}^{5}</sup>$ Recently Boney and Unger [**BU**] established that the statement "all AECs are tame" is in fact *equivalent* to a large cardinal axioms (the existence of a proper class of almost strongly compact cardinals). This result does not however say anything on the consistency strength of Shelah's eventual categoricity conjecture.

CLAIM 8.1.4. Assume that  $2^{\theta} < 2^{\theta^+}$  for all cardinals  $\theta$ . Let **K** be an AEC such that there is an  $\omega$ -successful<sup>6</sup> good<sup>+</sup>  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}$ . If **K** is categorical in  $\lambda$  and in some  $\mu > \lambda^{+\omega}$ , then K is categorical in all  $\mu > \lambda^{+\omega}$ .

Assuming amalgamation and Claim 8.1.4, Shelah obtains the eventual categoricity conjecture [She09a, Theorem IV.7.12] (or see Section 15.5 for an exposition):

FACT 8.1.5. Assume Claim 8.1.4 and  $2^{\theta} < 2^{\theta^+}$  for all cardinals  $\theta$ . Then an AEC with amalgamation categorical in some  $\lambda \ge h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$  is categorical in all  $\lambda' \ge h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$ .

Note that Fact 8.1.5 applies in particular to homogeneous model theory and finitary AECs with uncountable language (the latter case could not previously be dealt with).

Now a conjecture of Grossberg made in 1986 (see Grossberg **Gro02**. Conjecture 2.3) is that categoricity of an AEC in a high-enough cardinal should imply amalgamation (above a certain Hanf number). This is especially relevant considering that all the positive results above assume amalgamation. In the presence of large cardinals, Grossberg's conjecture is known to be true (This was first pointed out by Will Boney for general AECs, see [Bon14b, Theorem 4.3] and the discussion around Theorem 7.6 there. The key is that the proofs in [MS90, Proposition 1.13]or the stronger  $[{\bf SK96}]$  which are for classes of models of an  $L_{\kappa,\omega}$  sentence,  $\kappa$  a large cardinal, carry over to AECs **K** with  $LS(\mathbf{K}) < \kappa$ ). In recent years it has been shown that several results that could be proven using large cardinals can be proven using just the model-theoretic assumption of tameness or shortness (see all of the above works on tameness and for example Chapters 2 and 7). Thus one can ask whether tameness suffices to get amalgamation from categoricity. In general, this is not known. The only approximation is a result of Adi Jarden [Jar16] discussed more at length in Section 8.4. Our contribution is a weak version of amalgamation which one can assume alongside tameness to prove Grossberg's conjecture:

COROLLARY 8.4.17. Let **K** be tame AEC categorical in unboundedly many cardinals. If **K** is eventually syntactically characterizable<sup>7</sup> and has weak amalgamation (see Definition 8.4.11), then there exists  $\lambda$  such that  $\mathbf{K}_{\geq \lambda}$  has amalgamation.

The proof uses a deep result of Shelah showing that a categorical AEC is well-behaved in a specific cardinal, then uses tameness and weak amalgamation to transfer the good behavior up.

We apply our result to *universal classes*. Universal classes were introduced by Shelah in [She87b] as an important framework where he thought finding dividing lines should be possible<sup>8</sup>. For many years, Shelah has claimed a main gap theorem for these classes but the full proof has not appeared in print. The most recent version is Chapter V of [She09b] which contains hundreds of pages of approximations. The methods used are stability theory inside a model (averages) as well as

<sup>&</sup>lt;sup>6</sup>See Section 8.6 for a definition of good frames and the related technical terms.

<sup>&</sup>lt;sup>7</sup>A technical condition discussed more at length in Section 8.4.

 $<sup>^{8}\</sup>mathrm{We}$  were told by Rami Grossberg that another motivation was to study certain non first-order classes of modules.

combinatorial tools to build many models. Here we show that universal classes are tame<sup>9</sup> (in fact fully ( $< \aleph_0$ )-tame and short) and have weak amalgamation. Moreover Shelah has shown<sup>10</sup> that categoricity in cardinals of arbitrarily high cofinality implies that the class is eventually syntactically characterizable. Thus combining Corollary 8.4.17 and Fact 8.1.5 we can already prove Theorem 8.0.2 assuming the weak generalized continuum hypothesis and Claim 8.1.4. If the universal class is categorical in unboundedly many successor cardinals, we can use [**GV06a**] instead to get a categoricity transfer in ZFC.

By relying on Shelah's analysis of frames in [She09a, Chapter III] as well as the frame transfer theorems in [Bon14a], Chapter 5, we can also prove that Claim 8.1.4 holds in ZFC for universal classes (this uses the proof of Corollary 8.4.17). We deduce Theorem 8.0.2 in the abstract (see Corollary 8.5.28). Note that the result also holds in uncountable languages.

Our results apply to a more general context than universal classes: fully tame and short AECs with amalgamation which have a prime model over every set of the form  $M \cup \{a\}$  for M a model (this is Theorem 8.0.3 in the abstract, see Theorem 8.5.18 for a proof). Note that existence of prime models over sets of the form  $M \cup \{a\}$ already played a crucial role in the proof of the categoricity transfer theorem for excellent classes of models of an  $L_{\omega_1,\omega}$  sentences [**She83b**, Theorem 5.9] (in fact, our proof works also in this setting). Since in that case we are assuming amalgamation, there is no cofinality restrictions and the Hanf number can be explicitly computed.

Theorem 8.0.3 shows that, at least assuming amalgamation, tameness and shortness, the existence of primes is the *only* obstacle. Since amalgamation and full tameness and shortness follow from large cardinals [Bon14b], we obtain:

THEOREM 8.1.6. Let **K** be an AEC and let  $\kappa > \text{LS}(\mathbf{K})$  be strongly compact. Assume that in  $\mathbf{K}_{\geq\kappa}$  there are prime models over sets of the form  $M \cup \{a\}$ . If **K** is categorical in a  $\lambda > h(h(\kappa))$ , then **K** is categorical in all  $\lambda' \geq h(h(\kappa))$ .

When  $\mathbf{K}$  is a universal class, we can replace the strongly compact with a measurable (Theorem 8.5.27).

There remains one question: can the conclusion of Theorem 8.0.2 be obtained from only categoricity in a single cardinal (without cofinality restriction)? We answer positively in a sequel<sup>11</sup> (Chapter 16). Here, let us note that Theorem 8.0.2 generalizes to fully tame and short AECs with primes, but universal classes have better properties (as demonstrated by Shelah in [**She09b**, Chapter V]), so there is still room for improvement<sup>12</sup>.

The chapter is organized as follows. In Section 8.2, we recall the definition of universal classes and more generally of AECs which admit intersections (a notion introduced by Baldwin and Shelah in [**BS08**]), give examples, and prove some basic

<sup>&</sup>lt;sup>9</sup>This uses an argument of Will Boney.

 $<sup>^{10}</sup>$ In fact, Shelah asserts that the cofinality restriction is not necessary but Will Boney and the author have found a gap in Shelah's argument, and Shelah's fix has not yet been published. See the beginning of Section 8.4.

<sup>&</sup>lt;sup>11</sup>Since the initial circulation of this chapter (in June 2015), there have been several other improvements: Question 8.5.19 has been answered positively (see Chapter 11) and the categoricity threshold of Theorem 8.0.3 has been improved from  $H_2$  to  $H_1$ , see Chapter 14. All these improvements rely on the results of this chapter.

 $<sup>^{12}</sup>$ In fact, we had claimed in an earlier version of this work to be able to prove the full categoricity conjecture for universal classes but our argument contained an error.

properties. In Section 8.3, we prove that universal classes are fully  $(<\aleph_0)$ -tame and short. In Section 8.4 we give conditions under which amalgamation follows from categoricity (in more general classes than universal classes). In Section 8.5, we prove a categoricity transfer in universal classes that have amalgamation and more generally in fully tame and short AECs with primes and amalgamation.

To avoid cluttering the chapter, we have written the technical definitions and results on independence needed for the paper (but not crucial to a conceptual understanding) in Section 8.6. In Section 8.7, we prove Fact 8.5.10, a result of Shelah which is crucial to our argument but whose proof is only implicit in Shelah's book. In Section 8.8, we give some properties of independence in AECs which admit intersections.

A word on the background needed to read this chapter: we assume familiarity with a basic text on AECs such as [**Bal09**] or [**Gro**] and refer the reader to the preliminaries of Chapter 2 for more details and motivations on the notation and concepts used here. Familiarity with good frames [**She09a**, Chapter II] would be helpful, although the basics are reviewed in Section 8.6. The proof of the two theorems in the abstract relies on the construction of a good frame in Chapter 4, and more generally<sup>13</sup> on the study of global independence relations in Chapter 6. Some material from Chapter III of [**She09a**] is implicitly used there. To get amalgamation and prove Theorem 8.0.2, the hard arguments of [**She09a**, Chapter IV] are used. However we do not rely on them once amalgamation has been obtained (so for example Theorem 8.0.3 does *not* rely on [**She09a**, Chapter IV]). Finally, let us note that a more leisurely overview of the proof of Theorem 8.0.2 will appear in [**BVd**]. We have also written a short outline of the proof in [**Vasb**].

# 8.2. AECs which admit intersections

Recall:

DEFINITION 8.2.1 ([She87b]). A class of structures K is universal if:

- (1) It is a class of L-structures for a fixed language L = L(K), closed under isomorphisms.
- (2) If  $\langle M_i : i < \delta \rangle$  is  $\subseteq$ -increasing in K, then  $\bigcup_{i < \delta} M_i \in K$ .
- (3) If  $M \in K$  and  $M_0 \subseteq M$ , then  $M_0 \in K$ .

Example 8.2.2.

- (1) The class of models of a universal  $L_{\lambda,\omega}$  theory (that is, a set of sentences of the form  $\forall x_0 \dots \forall x_{n-1}\psi$ , with  $\psi$  a quantifier-free  $L_{\lambda,\omega}$ -formula) is universal.
- (2) Not all elementary classes are universal but some universal classes are not elementary (locally finite groups are one example).
- (3) Coloring classes [**KLH16**] are universal classes. This shows that the behavior of amalgamation is non-trivial even in universal classes: some coloring classes can have amalgamation up to  $\beth_{\alpha}$  for some  $\alpha < LS(\mathbf{K})^+$  and fail to have it above  $\beth_{LS(\mathbf{K})^+}$ . Other universal classes with non-trivial amalgamation spectrum appear in [**BKL**].
- (4) If **K** is a universal class in a countable vocabulary with:

 $<sup>^{13}</sup>$ Although since this chapter has first been made public, an improvement has been published that avoids dealing with global independence relation, see Chapter 11 (we have kept the original proof to avoid changing history and also because Section 8.6 is useful in other contexts).

- (a) Arbitrarily large models.
- (b) Joint embedding.
- (c) Disjoint amalgamation (see Definition 8.4.1).

Then  $(K, \subseteq)$  is a finitary abstract elementary class in the sense of Hyttinen and Kesälä [**HK06**, Definition 2.9]. We do not know whether  $(K, \subseteq)$  is always simple (in the sense of [**HK11**, Definition 6.1]. In any case, the arguments of Hyttinen and Kesälä only deal with countable languages.

Universal classes are abstract elementary classes:

REMARK 8.2.3. If K is a universal class, then  $\mathbf{K} := (K, \subseteq)$  is an AEC with  $LS(\mathbf{K}) = |L(K)| + \aleph_0$ . We will use this fact freely. Note that K may have finite models, and it is the case in several examples, see [**BKL**].

We now recall the definition of AECs that admit intersections, a notion introduced by Baldwin and Shelah. It is interesting to note that Baldwin and Shelah thought of admitting intersections as a weak version of amalgamation (see the conclusion of [**BS08**]).

DEFINITION 8.2.4 (Definition 1.2 in [BS08]). Let K be an AEC.

- (1) Let  $N \in \mathbf{K}$  and let  $A \subseteq |N|$ . N admits intersections over A if there is  $M_0 \leq_{\mathbf{K}} N$  such that  $|M_0| = \bigcap \{M \leq_{\mathbf{K}} N \mid A \subseteq |M|\}$ . N admits intersections if it admits intersections over all  $A \subseteq |N|$ .
- (2) **K** admits intersections if N admits intersections for all  $N \in \mathbf{K}$ .

Example 8.2.5.

- (1) If **K** is a universal class, then **K** admits intersections (and see Remark 8.2.12).
- (2) If **K** is a class of models of a first-order theory, then when  $(K, \subseteq)$  admits intersections has been characterized by Rabin [**Rab62**].
- (3) Several classes appearing in algebra admit intersections. For example [**Gro02**, Example 1.15], let **K** be the class of algebraically closed valued fields (we code the value group with an additional predicate), ordered by  $F_1 \leq_{\mathbf{K}} F_2$  if and only if  $F_1$  is a subfield of  $F_2$ , the value groups are the same, and the valuations coincide on  $F_1$ . Then **K** admits intersections. Again, **K** is not universal (as it is not closed under substructure).
- (4) The examples in [**BS08**] admit intersections. Since they are not  $(<\aleph_0)$ -tame, they cannot be universal classes (see Theorem 8.3.6).
- (5) The Hart-Shelah example [HS90, BK09] admits intersections but is also not (< ℵ<sub>0</sub>)-tame.
- (6) If C is a quasiminimal excellent pregeometry class (see [Zil05a, Kir10]) then the AEC K that it induces admits intersections and is categorical in every uncountable cardinal. Moreover it will be fully tame and short (at least assuming the existence of large cardinals). However it need not be finitary (take C to be the class of pseudo-exponential fields [Kir13, Theorem 2]).

In the rest of this section, we give several equivalent definitions of admitting intersections and deduce some properties of these classes. All throughout this chapter, we assume:

Hypothesis 8.2.6. K is an AEC.

DEFINITION 8.2.7. Let  $M \in \mathbf{K}$  and let  $A \subseteq |M|$  be a set. M is minimal over A if whenever  $M' \leq_{\mathbf{K}} M$  contains A, then M' = M. M is minimal over A in N if in addition  $M \leq_{\mathbf{K}} N$ .

DEFINITION 8.2.8. Let  $N \in \mathbf{K}$ . We say  $\mathcal{F}$  is a set of Skolem functions for N if:

- (1)  $\mathcal{F}$  is a non-empty set, and each element f of  $\mathcal{F}$  is a function from  $N^n$  to N, for some  $n < \omega$ .
- (2) For all  $A \subseteq |N|$ ,  $M := \mathcal{F}[A] := \bigcup \{ f[A] \mid f \in \mathcal{F} \}$  is such that  $M \leq_{\mathbf{K}} N$  and contains A.

REMARK 8.2.9. The proof of Shelah's presentation theorem (see [She09a, Lemma I.1.7]) gives that for each  $N \in \mathbf{K}$ , there is  $\mathcal{F}$  a set of Skolem functions for N with  $|\mathcal{F}| \leq \mathrm{LS}(\mathbf{K})$ .

THEOREM 8.2.10. Let **K** be an AEC and let  $N \in \mathbf{K}$ . The following are equivalent:

- (1) N admits intersections.
- (2) There is an operator  $cl := cl^N : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$  such that for all  $A, B \subseteq |N|$  and all  $M \leq_{\mathbf{K}} N$ :
  - (a)  $\operatorname{cl}(A) \leq_{\mathbf{K}} N$ .
  - (b)  $A \subseteq \operatorname{cl}(A)$ .
  - (c)  $A \subseteq B$  implies  $cl(A) \subseteq cl(B)$ .
  - (d)  $\operatorname{cl}(M) = M$ .
- (3) For each  $A \subseteq |N|$ , there is a unique minimal model over A in N.
- (4) There is a set  $\mathcal{F}$  of Skolem functions for N such that:
  - (a)  $|\mathcal{F}| \leq \mathrm{LS}(\mathbf{K}).$
  - (b) For all  $M \leq_{\mathbf{K}} N$ , we have  $\mathcal{F}[M] = M$ .

Moreover the operator  $cl^N : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$  with the properties in (2) is unique and if it exists then it has the following characterizations:

- $\operatorname{cl}^{N}(A) = \bigcap \{ M \leq_{\mathbf{K}} N \mid A \subseteq |M| \}.$
- $\operatorname{cl}^{N}(A) = \mathcal{F}[A]$ , for any set of Skolem functions  $\mathcal{F}$  for N such that  $\mathcal{F}[M] = M$  for all  $M \leq_{\mathbf{K}} N$ .
- $\operatorname{cl}^{N}(A)$  is the unique minimal model over A in N.

Proof.

- (1) implies (2): Let  $cl^N(A) := \bigcap \{M \leq_{\mathbf{K}} N \mid A \subseteq |M|\}$ . Even without hypotheses on N, (2b), (2c), and (2d) are satisfied. Since N admits intersections, (2a) is also satisfied.
- (2) implies (3): Let  $A \subseteq |N|$ . Let cl be as given by (2). Let M := cl(A). By (2a),  $M \leq_{\mathbf{K}} N$ . By (2b),  $A \subseteq |M|$ . Moreover if  $M' \leq_{\mathbf{K}} N$  contains A, then by (2c),  $|M| \subseteq |cl(M')|$  but by (2d), cl(M') = M'. Thus by coherence and (2a)  $M \leq_{\mathbf{K}} M'$ . This shows both that M is minimal over A and that it is unique.
- (3) implies (4): We slightly change the proof of [She09a, Lemma I.1.7] as follows: Let  $\chi := \mathrm{LS}(\mathbf{K})$ . For each  $\bar{a} \in {}^{<\omega}|N|$ , let  $\langle b_i^{\bar{a}} : i < \chi \rangle$  be an enumeration (possibly with repetitions) of the unique minimal model over ran $(\bar{a})$  in N. For each  $n < \omega$  and  $i < \chi$ , we let  $f_i^n : N^n \to N$  be  $f_i^n(\bar{a}) := b_i^{\bar{a}}$ . Let  $\mathcal{F} := \{f_i^n \mid i < \chi, n < \omega\}$ . Then  $|\mathcal{F}| \leq \mathrm{LS}(\mathbf{K})$  and

if  $A \subseteq |N|$ , we claim that  $\mathcal{F}[A]$  is minimal over A in N. This shows in particular that  $\mathcal{F}$  is as required.

Let  $M := \mathcal{F}[A]$ . By definition,  $M = \bigcup_{\bar{a} \in {}^{<\omega}|A|} \mathcal{F}[\operatorname{ran}(\bar{a})]$ . Now if  $\bar{a} \in {}^{<\omega}A, M_{\bar{a}} := \mathcal{F}[\operatorname{ran}(\bar{a})] = \{b_i^{\bar{a}} : i < \chi\}$  is the unique minimal model over  $\operatorname{ran}(\bar{a})$  in N. Thus if  $\operatorname{ran}(\bar{a}) \subseteq \operatorname{ran}(\bar{b})$ , we must have (by coherence)  $M_{\bar{a}} \leq_{\mathbf{K}} M_{\bar{b}}$ . It follows that  $M \in \mathbf{K}$  and by the axioms of AECs also  $M \leq_{\mathbf{K}} N$ . Of course, M contains A. Now if  $M' \leq_{\mathbf{K}} M$  contains A, then for all  $\bar{a} \in {}^{<\omega}A, \bar{a} \in {}^{<\omega}|M'|$ , so as  $M_{\bar{a}}$  is minimal over  $\operatorname{ran}(\bar{a}), M_{\bar{a}} \leq_{\mathbf{K}} M'$ . It follows that  $M \leq_{\mathbf{K}} M$  so M = M'.

• (4) implies (1): Let  $\mathcal{F}$  be as given by (4). Let  $A \subseteq |N|$ . Let  $M := \mathcal{F}[A]$ . By definition of Skolem functions, M contains A and  $M \leq_{\mathbf{K}} N$ . We claim that  $M = \bigcap \{M' \leq_{\mathbf{K}} N \mid A \subseteq |M'|\}$ . Indeed, if  $M' \leq_{\mathbf{K}} N$  contains A, then by the hypothesis on  $\mathcal{F}, M = \mathcal{F}[A] \subseteq \mathcal{F}[M'] = M'$ .

The moreover part follows from the arguments above.

DEFINITION 8.2.11. For  $N \in \mathbf{K}$  let  $\mathrm{cl}^N : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$  be defined by  $\mathrm{cl}^N(A) := \bigcap \{M \leq_{\mathbf{K}} N \mid A \subseteq |M|\}.$ 

REMARK 8.2.12. If **K** is a universal class, then one can take the set  $\mathcal{F}$  of Skolem functions in (4) to consist of all the functions of N. Thus  $cl^{N}(A)$  is just the closure of A under the functions of N.

Theorem 8.2.10 allows us to deduce several properties of the operator  $cl^N$ .

PROPOSITION 8.2.13. Let  $M \leq_{\mathbf{K}} N \in \mathbf{K}$  and let  $A, B \subseteq |N|$ .

- (1) Invariance: If  $f: N \cong N'$ , then  $f[cl^N(A)] = cl^{N'}(f[A])$ .
- (2) Monotonicity 1:  $A \subseteq cl^N(A)$ .
- (3) Monotonicity 2:  $A \subseteq B$  implies  $\operatorname{cl}^N(A) \subseteq \operatorname{cl}^N(B)$ .
- (4) Monotonicity 3: If  $A \subseteq |M|$ , then  $\operatorname{cl}^N(A) \subseteq \operatorname{cl}^M(A)$ . Moreover if N admits intersections over A, then M admits intersections over A and  $\operatorname{cl}^N(A) = \operatorname{cl}^M(A)$ .
- (5) Idempotence:  $\operatorname{cl}^N(M) = M$ .
- (6) Finite character: If N admits intersections, then if  $B \subseteq cl^N(A)$  is finite, there exists a finite  $A_0 \subseteq A$  such that  $B \subseteq cl^N(A_0)$ .

PROOF. Straightforward given Theorem 8.2.10: For finite character, use the characterization in terms of Skolem functions. For monotonicity 3, let  $M_0 := \operatorname{cl}^N(A)$ . We have  $M_0 \leq_{\mathbf{K}} N$  since N admits intersections over A. Since  $M \leq_{\mathbf{K}} N$  contains A, we must have  $|M_0| \subseteq |M|$ . By coherence,  $M_0 \leq_{\mathbf{K}} M$ , and by minimality,  $M_0 = \operatorname{cl}^M(A)$ .

Note in particular the following:

Corollary 8.2.14.

- (1) Assume that for every  $M \in \mathbf{K}$  and every  $A \subseteq |M|$ , there is  $N \geq_{\mathbf{K}} M$  such that N admits intersections over A. Then **K** admits intersections.
- (2)  $N \in \mathbf{K}$  admits intersections if and only if it admits intersections over every finite  $A \subseteq |N|$ .

# Proof.

(1) By Monotonicity 3.

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(2) By the proof of Theorem 8.2.10.

REMARK 8.2.15. The second result is implicit in the discussion after Remark 4.3 in [BS08].

The next result says that in AECs admitting intersections, equality of Galois types is witnessed by an isomorphism. This can be seen as a weak version of amalgamation (see Section 8.4).

PROPOSITION 8.2.16. Assume **K** admits intersections. Then  $gtp(\bar{a}_1/A; N_1) = gtp(\bar{a}_2/A; N_2)$  if and only if there exists  $f : cl^{N_1}(\bar{a}_1) \cong_A cl^{N_2}(\bar{a}_2)$  such that  $f(\bar{a}_1) = \bar{a}_2$ .

PROOF. Let  $M_1 := \operatorname{cl}^{N_1}(\bar{a}_1), M_2 := \operatorname{cl}^{N_2}(\bar{a}_2)$ . Since  $N_\ell$  admits intersections, we have  $M_\ell \leq_{\mathbf{K}} N_\ell, \ell = 1, 2$  so the right to left direction follows. Now assume  $\operatorname{gtp}(\bar{a}_1/A; N_1) = \operatorname{gtp}(\bar{a}_2/A; N_2)$ . It suffices to prove the result when the equality is atomic (then we can compose the isomorphisms in the general case). So let  $N \in \mathbf{K}$ and  $f_\ell : N_\ell \xrightarrow{A} N$  witness atomic equality, i.e.  $f_1(\bar{a}_1) = f_2(\bar{a}_2)$ . By invariance and monotonicity 3,  $f_\ell[M_\ell] = \operatorname{cl}^{f[N_\ell]}(f_\ell(\bar{a}_\ell)) = \operatorname{cl}^N(f_\ell(\bar{a}_\ell))$ . Since  $f_1(\bar{a}_1) = f_2(\bar{a}_2)$ , we must have that  $f_1[M_1] = f_2[M_2]$ . Thus  $f := (f_2 \upharpoonright M_2)^{-1} \circ (f_1 \upharpoonright M_1)$  is as desired.  $\Box$ 

REMARK 8.2.17. Proposition 8.2.16 was already observed (without proof) in [**BS08**, Lemma 1.3]. Baldwin and Shelah also assert that  $E = E_{\text{at}}$  (see Definition 2.2.17), but this does not seem to follow.

Before ending this section, we point out a technical disadvantage of the definition of admitting intersection. The notion is not closed under the tail of the AEC: If **K** admits intersections and  $\lambda$  is a cardinal, then it is not clear that  $\mathbf{K}_{\geq\lambda}$  admits intersections. Thus we will work with a slightly weaker definition:

DEFINITION 8.2.18. For **K** an AEC and  $M \in \mathbf{K}$ , let  $\mathbf{K}_M$  be the AEC defined by adding constant symbols for the elements of M and requiring that M embeds inside every model of  $\mathbf{K}_M$ . That is,  $L(\mathbf{K}_M) = L(\mathbf{K}) \cup \{c_a \mid a \in |M|\}$ , where the  $c_a$ 's are new constant symbols, and

 $\mathbf{K}_M := \{ (N, c_a^N)_{a \in |M|} \mid N \in \mathbf{K} \text{ and } a \mapsto c_a^N \text{ is a } \mathbf{K} \text{-embedding from } M \text{ into } N \}$ 

We order  $\mathbf{K}_M$  by  $(N_1, c_a^N)_{a \in |M|} \leq_{\mathbf{K}} (N_2, c_a^{N_2})$  if and only if  $N_1 \leq_{\mathbf{K}} N_2$  and  $c_a^{N_1} = c_a^{N_2}$  for all  $a \in |M|$ .

DEFINITION 8.2.19. **K** has *P* above *M* For *P* a property of AECs and  $M \in \mathbf{K}$ , **K** has *P* above *M* if  $\mathbf{K}_M$  has *P*. **K** locally has *P* if it has *P* above every  $M \in \mathbf{K}$ .

REMARK 8.2.20. **K** locally admits intersections if and only if for every  $M \leq_{\mathbf{K}} N$ in **K** and every  $A \subseteq |N|$  which contains M,  $\operatorname{cl}^{N}(A) \leq_{\mathbf{K}} N$ .

REMARK 8.2.21. If **K** locally has P, then for every cardinal  $\lambda$ ,  $\mathbf{K}_{\geq\lambda}$  locally has P.

#### 8.3. Universal classes are fully tame and short

In this section, we show that universal classes are fully  $(<\aleph_0)$ -tame and short. The basic argument for Theorem 8.3.6 is due to Will Boney and will also appear in [**Bonc**].

Note that it is impossible to extend this result to AECs which admits intersections: **[BS08]** gives several counterexamples. One could hope that showing that categoricity in a high-enough cardinal implies tameness (a conjecture of Grossberg and VanDieren, see **[GV06a**, Conjecture 1.5]) is easier in AECs which admits intersections, but we have been unable to make progress in that direction and leave it to future work.

The key of the argument for tameness of universal classes is that the isomorphism characterizing the equality of Galois type is unique. We abstract this feature into a definition:

DEFINITION 8.3.1. **K** is *pseudo-universal* if it admits intersections and for any  $N_1, N_2, \bar{a}_1, \bar{a}_2$ , if  $gtp(\bar{a}_1/\emptyset; N_1) = gtp(\bar{a}_2/\emptyset; N_2)$  and  $f, g: cl^{N_1}(\bar{a}_1) \cong cl^{N_2}(\bar{a}_2)$  are such that  $f(\bar{a}_1) = g(\bar{a}_1) = \bar{a}_2$ , then f = g.

Example 8.3.2.

- (1) In universal classes,  $cl^N(A)$  is just the substructure of N generated by A (see Remark 8.2.12). Thus universal classes are pseudo-universal.
- (2) Let  $\mathbf{K}$  be the class of groups in the language containing only the multiplication symbol. Then  $\mathbf{K}$  is not a universal class but it is pseudo-universal (the inverse function and the unit constant are first-order definable).
- (3) We show below that pseudo-universal classes are  $(<\aleph_0)$ -tame, hence the AECs in [**BS08**] admit intersections but are not pseudo-universal.
- (4) More simply, the class **K** of algebraically closed fields is elementary (hence  $(<\aleph_0)$ -tame), admits intersections, but is not pseudo-universal. Indeed, let M be the algebraic closure of  $\mathbb{Q}$  and let x be transcendental. Let N be the algebraic closure of  $M \cup \{x\}$ . Then there exists two different automorphisms of N that fix  $M \cup \{x\}$ : the identity and one that sends  $\sqrt{x}$  to  $-\sqrt{x}$ .

DEFINITION 8.3.3. Let  $\bar{a}_{\ell} \in {}^{\alpha}|N_{\ell}|$  and let  $\kappa$  be an infinite cardinal. We write  $(\bar{a}_1, N_1) \equiv_{<\kappa} (\bar{a}_2, N_2)$  if for every  $I \subseteq \alpha$  of size less than  $\kappa$ ,  $\operatorname{gtp}(\bar{a}_1 \upharpoonright I/\emptyset; N_1) = \operatorname{gtp}(\bar{a}_2 \upharpoonright I/\emptyset; N_2)$ .

The next proposition says roughly that it is enough to show shortness for types over the empty set. This appears already as [Bon14b, Theorem 3.5]. We repeat the argument here for convenience.

PROPOSITION 8.3.4. Let  $\kappa$  be an infinite cardinal. Assume that for every  $\alpha$ ,  $N_{\ell} \in \mathbf{K}$ ,  $\bar{a}_{\ell} \in {}^{\alpha}|N_{\ell}|$ ,  $\ell = 1, 2$ , we have that  $(\bar{a}_1, N_1) \equiv_{<\kappa} (\bar{a}_2, N_2)$  implies  $\operatorname{gtp}(\bar{a}_1/\emptyset; N_1) = \operatorname{gtp}(\bar{a}_2/\emptyset; N_2)$ . Then **K** is fully  $(<\kappa)$ -tame and short.

PROOF. Let  $\beta$  be an ordinal,  $M \in \mathbf{K}$ ,  $p, q \in \mathrm{gS}^{\beta}(M)$ . Assume that  $p^{I} \upharpoonright A = q^{I} \upharpoonright A$  for all  $I \subseteq \beta$  and  $A \subseteq |M|$  of size less than  $\kappa$ . Say  $p = \mathrm{gtp}(\bar{a}_{1}/M; N_{1})$ ,  $q = \mathrm{gtp}(\bar{a}_{2}/M; N_{2})$ . Let  $\bar{b}$  be an enumeration of |M| and let  $p' := \mathrm{gtp}(\bar{a}_{1}\bar{b}/\emptyset; N_{1})$ ,  $q' := \mathrm{gtp}(\bar{a}_{2}\bar{b}/\emptyset; N_{2})$ . By assumption,  $(p')^{I'} = (q')^{I'}$  for all I' of size less than  $\kappa$ . In other words,  $(\bar{a}_{1}\bar{b}, N_{1}) \equiv_{<\kappa} (\bar{a}_{2}\bar{b}, N_{2})$ . Therefore by our locality assumption p' = q', and from the definition of Galois types this implies that p = q.

REMARK 8.3.5. By a similar argument, we can show that pseudo-universal classes are locally pseudo-universal (recall Definition 8.2.19).

THEOREM 8.3.6. If **K** is pseudo-universal, then **K** is fully  $(\langle \aleph_0 \rangle)$ -tame and short.

PROOF. Let  $N_{\ell} \in \mathbf{K}$ ,  $\bar{a}_{\ell} \in {}^{\alpha}|N_{\ell}|$ ,  $\ell = 1, 2$ . Assume that  $(\bar{a}_1, N_1) \equiv_{<\aleph_0} (\bar{a}_2, N_2)$ . We show that  $\operatorname{gtp}(\bar{a}_1/\emptyset; N_1) = \operatorname{gtp}(\bar{a}_2/\emptyset; N_1)$ , which is enough by Proposition 8.3.4. Let  $M_{\ell} := \operatorname{cl}^{N_{\ell}}(\operatorname{ran} \bar{a}_{\ell})$ .

For each finite  $I \subseteq \alpha$ , let  $M_{\ell,I} := \operatorname{cl}^{N_{\ell}}(\operatorname{ran} \bar{a}_{\ell} \upharpoonright I)$ . By definition of  $\equiv_{<\aleph_0}$ , for each finite  $I \subseteq \alpha$ ,  $\operatorname{gtp}(\bar{a}_1 \upharpoonright I/\emptyset; N_1) = \operatorname{gtp}(\bar{a}_2 \upharpoonright I/\emptyset; N_2)$ . Therefore (because **K** admits intersections) there exists  $f_I : M_{1,I} \cong M_{2,I}$  such that  $f_I(\bar{a}_1 \upharpoonright I) = \bar{a}_2 \upharpoonright I$ . Moreover by definition of pseudo-universal,  $f_I$  is unique with that property. This means in particular that if  $I \subseteq J \subseteq \alpha$  are both finite, we must have  $f_I \subseteq f_J$ . By finite character of the closure operator,  $M_{\ell} = \bigcup_{I \in [\alpha] < \aleph_0} M_{\ell,I}$  and so letting  $f := \bigcup_{I \in [\alpha] < \aleph_0} f_I$ , we have that  $f : M_1 \cong M_2$  and  $f(\bar{a}_1) = \bar{a}_2$ . This witnesses that  $\operatorname{gtp}(\bar{a}_1/\emptyset; M_1) = \operatorname{gtp}(\bar{a}_2/\emptyset; M_2)$  and so (since  $M_{\ell} \leq_{\mathbf{K}} N_{\ell}$ ),  $\operatorname{gtp}(\bar{a}_1/\emptyset; N_1) =$  $\operatorname{gtp}(\bar{a}_2/\emptyset; N_2)$ .

REMARK 8.3.7. If **K** is a universal class (i.e. not "pseudo"), then the proof of Theorem 8.3.6 (together with Remark 8.2.12) shows that Galois types are the same as quantifier-free types. That is,  $gtp(\bar{a}_1/A; N_1) = gtp(\bar{a}_2/A; N_2)$  if and only if  $tp_{qf}(\bar{a}_1/A; N_1) = tp_{qf}(\bar{a}_2/A; N_2)$ , where  $tp_{qf}(\bar{a}/A; N)$  denotes the quantifier-free types of  $\bar{a}$  over A computed in N.

We can localize Theorem 8.3.6 to obtain more generally (Remark 8.3.5):

COROLLARY 8.3.8. If  $\mathbf{K}$  is locally pseudo-universal, then  $\mathbf{K}$  is fully  $LS(\mathbf{K})$ -tame and short.

PROOF. Let  $M \in \mathbf{K}$  and let  $p, q \in \mathrm{gS}^{\alpha}(M)$ . Assume that  $p^{I} \upharpoonright A = q^{I} \upharpoonright A$  for all  $A \subseteq |M|$  of at most size  $\mathrm{LS}(\mathbf{K})$  and all  $I \subseteq \alpha$  of size at most  $\mathrm{LS}(\mathbf{K})$ . We want to see that p = q. Without loss of generality  $||M|| \ge \mathrm{LS}(\mathbf{K})$ . Let  $M_0 \le_{\mathbf{K}} M$  have size  $\mathrm{LS}(\mathbf{K})$ . We know that  $p^{I} \upharpoonright M_0 = q^{I} \upharpoonright M_0$  for all  $I \subseteq \alpha$  of size at most  $\mathrm{LS}(\mathbf{K})$ . Since  $\mathbf{K}$  is locally pseudo-universal,  $\mathbf{K}_{M_0}$  (see Definition 8.2.18) is pseudo-universal. By Theorem 8.3.6,  $\mathbf{K}_{M_0}$  is fully  $(<\aleph_0)$ -tame and short. Translating to  $\mathbf{K}$ , this means that for any  $N \ge_{\mathbf{K}} M_0$ , any  $p', q' \in \mathrm{gS}^{\beta}(N)$ , if  $(p')^{I} \upharpoonright (|M_0| \cup A) = (q')^{I} \upharpoonright (|M_0| \cup A)$ for all finite I and A, then p' = q'. Setting N, p', q' to stand for M, p, q, we get that p = q, as desired.  $\Box$ 

## 8.4. Amalgamation from categoricity

We investigate how to get amalgamation from categoricity in tame AECs admitting intersections. In what follows, we will often use Remark 8.1.3 without comments. Recall:

DEFINITION 8.4.1. An AEC **K** has amalgamation if for any  $M_0 \leq_{\mathbf{K}} M_\ell$ ,  $\ell = 1, 2$ , there exists  $N \in \mathbf{K}$  and embeddings  $f_\ell : M_\ell \xrightarrow{M_0} N$ . We say that **K** has  $\lambda$ -amalgamation if this holds for the models in  $\mathbf{K}_{\lambda}$ . We define similarly *disjoint amalgamation*, where we require in addition that  $f_1[M_1] \cap f_2[M_2] = M_0$ .

We will use the concept of a good  $\lambda$ -frame, a notion of forking for types of length one over models of size  $\lambda$ , see [She09a, Definition II.2.1] or Section 8.6. The

following claim is a deep result of Shelah which says that good  $\lambda$ -frames exist in categorical classes.

CLAIM 8.4.2. If **K** is categorical in unboundedly many cardinals, then there exists a categoricity cardinal  $\lambda \geq LS(\mathbf{K})$  such that **K** has a good  $\lambda$ -frame (i.e. there exists a good  $\lambda$ -frame  $\mathfrak{s}$  such that  $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\lambda}$ ). In particular, **K** has  $\lambda$ -amalgamation.

The statement is implicit in Chapter IV of [She09a], but in June 2015 Will Boney and the author identified a gap in a key part of Shelah's proof [BVb]. In September 2016, Shelah communicated a fix to the author, which should appear as an online revision of Sh:734. As of December 2016, Shelah's fix has not yet been made public.

The key notion in the proof of Claim 8.4.2 is:

DEFINITION 8.4.3 (Definition 2.1 in [**BVb**]). An AEC **K** is  $L_{\infty,\theta}$ -syntactically characterizable if whenever  $M, N \in \mathbf{K}$ , if  $M \leq_{\mathbf{K}} N$  then  $M \preceq_{L_{\infty,\theta}} N$ . We say that **K** is eventually syntactically characterizable if for every infinite cardinal  $\theta$ , there exists  $\lambda$  such that  $\mathbf{K}_{>\lambda}$  is  $L_{\infty,\theta}$ -syntactically characterizable.

REMARK 8.4.4. Using that saturated models are model-homogeneous, it is easy to see that any AEC *with amalgamation* categorical in a proper class of cardinals is eventually syntactically characterizable [**BVb**, Proposition 1.3].

The problematic part of Shelah's proof is a claim that an AEC categorical in unboundedly many cardinals is eventually syntactically characterizable (see [She09a, Conclusion IV.2.14]). However the following weakening is true:

FACT 8.4.5 (Conclusion IV.2.12.(1) in [She09a]). If **K** is categorical in cardinals of arbitrarily high cofinality (that is, for every  $\theta$  there exists  $\lambda$  such that **K** is categorical in  $\lambda$  and cf  $\lambda \geq \theta$ ), then **K** is eventually syntactically characterizable.

From an eventually syntactically characterizable AEC that is categorical in unboundedly many cardinals, Shelah's proof of Claim 8.4.2 goes through:

FACT 8.4.6 (Theorem 2.12 in [**BVb**]). If **K** is eventually syntactically characterizable and categorical in unboundedly many cardinals, then there exists a categoricity cardinal  $\lambda \geq LS(\mathbf{K})$  such that **K** has a good  $\lambda$ -frame.

Thus it is reasonable to assume that we have a good  $\lambda$ -frame, and we want to transfer amalgamation above it. Our inspiration is a recent result of Adi Jarden, presented at a talk in South Korea in the Summer of 2014.

FACT 8.4.7 (Corollary 7.16 in [Jar16]). Assume **K** has a good  $\lambda$ -frame where the class of uniqueness triples satisfies the existence property and **K** is strongly  $\lambda$ -tame, then **K** has  $\lambda^+$ -amalgamation.

We will not give the definition of the class of uniqueness triples here (but see Definition 8.6.17 and Fact 8.6.18). It suffices to say that they are a version of domination for good frames. As for strong tameness, it is a variation of tameness relevant when amalgamation fails to hold. Recall that  $\lambda$ -tameness asks for two types that are equal on all their restrictions of size  $\lambda$  to be equal. The strong version asks them to be *atomically equal*, i.e. there is a map witnessing it that amalgamates the two models in which the types are computed, see Definition 2.2.17. Jarden's result is interesting, since it shows that tameness, a locality property that we see as quite mild compared to assuming amalgamation, can be of some use to proving amalgamation. The downside is that we have to ask for a strengthened version.

While Jarden proved much more than  $\lambda^+$ -amalgamation, it has been pointed out by Will Boney (in a private communication) that if one only wants amalgamation, the hypothesis that uniqueness triples satisfy the existence property is not necessary. The reason is that the argument of [**Bon14a**] can be used to transfer enough of the good frame to  $\lambda^+$  so that the extension property holds, and the extension property implies amalgamation.

We make the argument precise here and also show that less than strong tameness is needed (in particular, it suffices to assume tameness and that the AEC admits intersections). We first fix some notation.

DEFINITION 8.4.8. Let  $\lambda \geq LS(\mathbf{K})$ .

- (1)  $K^{3,1}$  is the set of triples (a, M, N) such that  $M \leq_{\mathbf{K}} N$  and  $a \in N$ .  $K^{3,1}_{\lambda}$  is the set of such triples where the models are in  $\mathbf{K}_{\lambda}$  (the difference with Definition 2.2.17 is that we require the base to be a model and the sequence  $\bar{b}$  to have length one).
- (2) We say  $(a_1, M_1, N_1) \in K^{3,1}$  atomically extends  $(a_0, M_0, N_0) \in K^{3,1}$  if  $M_1 \geq_{\mathbf{K}} M_0$  and  $(a_1, M_0, N_1) E_{\mathrm{at}}(a_0, M_0, N_0)$  (recall Definition 2.2.17)
- (3) We say  $M \in \mathbf{K}_{\lambda}$  has the type extension property if for any  $N \geq_{\mathbf{K}} M$  in  $\mathbf{K}_{\lambda}$  and any  $p \in gS(M)$ , there exists  $q \in gS(N)$  extending p.
- (4) We say M has the strong type extension property if for any  $N \geq_{\mathbf{K}} M$ , whenever  $(a, M, M') \in K_{\lambda}^{3,1}$ , there exists  $(b, N, N') \in K_{\lambda}^{3,1}$  atomically extending (a, M, M').

We say  $\mathbf{K}_{\lambda}$  has the [strong] type extension property (or **K** has the [strong] type extension property in  $\lambda$ ) if every  $M \in \mathbf{K}_{\lambda}$  has it.

REMARK 8.4.9. It is well-known (see for example [**Gro**]) that if **K** has amalgamation, then  $E = E_{\text{at}}$ . Similarly if  $\lambda \geq \text{LS}(\mathbf{K})$  and **K** has  $\lambda$ -amalgamation, then  $E \upharpoonright K_{\lambda}^{3,1} = E_{\text{at}} \upharpoonright K_{\lambda}^{3,1}$ . Moreover, **K** has  $\lambda$ -amalgamation if and only if  $\mathbf{K}_{\lambda}$  has the strong type extension property.

We think of the type extension property as saying that amalgamation cannot fail because there are "fundamentally incompatible" elements in the two models we want to amalgamate. Rather, the reason amalgamation fails is because we simply "do not have enough models" to witness that two types are equal in one step. It would be useful to formalize this intuition but so far we have failed to do so.

We are interested in conditions implying that the type extension property (not the strong one) is enough to get amalgamation. For this, it turns out that it is enough to require that the AEC admits intersections. However we can even require a weaker condition:

DEFINITION 8.4.10 (Weak atomic equivalence). Let  $(a_{\ell}, M, N_{\ell}) \in K_{\lambda}^{3,1}$ ,  $\ell = 1, 2$ . We say  $(a_1, M, N_1) E_{\mathrm{at}}^-(a_2, M, N_2)$  (in words,  $(a_1, M, N_1)$  and  $(a_2, M, N_2)$  are weakly atomically equivalent) if for  $\ell = 1, 2$ , there exists  $N'_{\ell} \leq_{\mathbf{K}} N_{\ell}$  containing  $a_{\ell}$  and M such that  $(a_{\ell}, M, N'_{\ell}) E_{\mathrm{at}}(a_{3-\ell}, M, N_{3-\ell})$ .

DEFINITION 8.4.11. **K** has weak amalgamation if  $E \upharpoonright K^{3,1} = E_{\text{at}}^- \upharpoonright K^{3,1}$ , i.e. equivalence of triples is the same as weak atomic equivalence of triples. Similarly define what it means for **K** to have weak  $\lambda$ -amalgamation.

REMARK 8.4.12. **K** has weak amalgamation if and only if whenever  $gtp(a_1/M; N_1) = gtp(a_2/M; N_2)$ , there exists  $N'_1 \leq_{\mathbf{K}} N_1$  containing  $a_1$  and M and there exists  $N \geq_{\mathbf{K}} N_2$  and  $f: N'_1 \xrightarrow{\longrightarrow} N$  so that  $f(a_1) = a_2$ .

REMARK 8.4.13. If **K** locally admits intersections,  $(a_{\ell}, M, N_{\ell}) \in K_{\lambda}^{3,1}$ ,  $\ell = 1, 2$ and  $(a_1, M, N_1)E(a_2, M, N_2)$ , then by Proposition 8.2.13,  $N'_{\ell} := \operatorname{cl}^{N_{\ell}}(|M| \cup \{a_{\ell}\})$ witnesses that  $(a_1, M, N_1)E_{\operatorname{at}}^-(a_2, M, N_2)$ . Thus in that case,  $E \upharpoonright K_{\lambda}^{3,1} = E_{\operatorname{at}}^- \upharpoonright K_{\lambda}^{3,1}$ , so **K** has weak amalgamation.

Intuitively, weak amalgamation requires only that points that have the same Galois types can be amalgamated. The key result is:

THEOREM 8.4.14. Let **K** be an AEC and  $\lambda \geq LS(\mathbf{K})$ . Assume  $\mathbf{K}_{\lambda}$  has the type extension property. The following are equivalent:

- (1) **K** has  $\lambda$ -amalgamation.
- (2)  $E \upharpoonright K_{\lambda}^{3,1} = E_{\text{at}} \upharpoonright K_{\lambda}^{3,1}$  (i.e. equivalence of triples is the same as atomic equivalence of triples).
- (3) **K** has weak  $\lambda$ -amalgamation (i.e. equivalence of triples is the same as *weak* atomic equivalence of triples).

In particular, if  $\mathbf{K}$  admits intersections and has the type extension property, then it has amalgamation.

**PROOF.** (1) implies (2) implies (3) is easy. We prove (3) implies (1).

Assume  $E \upharpoonright K_{\lambda}^{3,1} = E_{\mathrm{at}}^{-} \upharpoonright K_{\lambda}^{3,1}$ . The idea of the proof is as follows: we want to amalgamate a triple  $(M_0, M, N)$ ,  $M_0 \leq_{\mathbf{K}} M$ ,  $M_0 \leq_{\mathbf{K}} N$ . We use weak amalgamation first to amalgamate some smaller triple  $(M_0, M', N')$  with  $M_0 <_{\mathbf{K}} M' \leq_{\mathbf{K}} M$ ,  $M_0 <_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ , then proceed inductively to amalgamate the entire triple. Claim 1 below shows that there exists a smaller triple which can be amalgamated and Claim 2 is a renaming of Claim 1. We then use Claim 2 repeatedly to amalgamate the full triple.

<u>Claim 1</u>. For every triple  $(M_0, M_1, M_2)$  of models in  $\mathbf{K}_{\lambda}$  so that  $M_0 <_{\mathbf{K}} M_1$  and  $M_0 \leq_{\mathbf{K}} M_2$ , there exists  $M'_1 \leq_{\mathbf{K}} M_1$  and  $M'_2 \geq_{\mathbf{K}} M_2$  in  $\mathbf{K}_{\lambda}$  such that  $M_0 <_{\mathbf{K}} M'_1$ , and there exists  $g: M'_1 \xrightarrow{M_2} M'_2$ .



<u>Proof of claim 1</u>. Let  $M_0 <_{\mathbf{K}} M_\ell$  be models in  $\mathbf{K}_\lambda$ ,  $\ell = 1, 2$ . Pick any  $a_1 \in |M_1| \setminus |M_0|$ . Let  $p := \operatorname{gtp}(a_1/M_0; M_1)$ . By the type extension property, there exists  $q \in \operatorname{gS}(M_2)$  extending p. Pick  $M_2^* \geq_{\mathbf{K}} M_2$  and  $a_2 \in |M_2^*|$  such that  $q = \operatorname{gtp}(a_2/M_2; M_2^*)$ . Since E is  $E_{\operatorname{at}}^-$  over the domain of interest, we have  $(a_1, M_0, M_1)E_{\operatorname{at}}^-(a_2, M_0, M_2^*)$ . Let  $M_1' \leq_{\mathbf{K}} M_1$  contain  $a_1$  and  $M_0$  such that  $(a_1, M_0, M_1')E_{\operatorname{at}}(a_2, M_0, M_2^*)$ . By definition, we have that there exists  $M_2' \geq_{\mathbf{K}} M_2^*$  such that  $M_1'$  embeds into  $M_2'$  over  $M_0$ , as needed.  $\dagger_{\operatorname{Claim} 1}$ 

Now we obtain amalgamation by repeatedly applying Claim 1. Since the result is key to subsequent arguments, we give full details below.

<u>Claim 2</u>. For every triple  $(M_0, M_1, M_2)$  of models in  $\mathbf{K}_{\lambda}$  so that  $M_0 <_{\mathbf{K}} M_1$ and  $f: M_0 \to M_2$ , there exists  $M'_1 \leq_{\mathbf{K}} M_1, M'_2 \geq_{\mathbf{K}} M_2$  in  $\mathbf{K}_{\lambda}$  and  $g: M'_1 \to M'_2$ such that  $M_0 <_{\mathbf{K}} M'_1$  and  $f \subseteq g$ .



<u>Proof of claim 2</u>. Let  $M_0, M_1, M_2$  and f be as given by the hypothesis. Let  $\widehat{M_2} \text{ and } \widehat{f} \text{ be such that } f \subseteq \widehat{f}, M_0 \leq_{\mathbf{K}} \widehat{M_2} \text{ and } \widehat{f} : \widehat{M_2} \cong M_2. \text{ Now apply} \\ \text{Claim 1 to } (M_0, M_1, \widehat{M_2}) \text{ to obtain } M_1' \leq_{\mathbf{K}} M_1 \text{ with } M_0 <_{\mathbf{K}} M_1', \widehat{M_2}' \geq_{\mathbf{K}} \widehat{M_2} \text{ and } \\ \widehat{g} : M_1' \xrightarrow{M_0} \widehat{M_2}'. \text{ Now let } \widehat{f}', M_2' \text{ be such that } M_2' \geq_{\mathbf{K}} M_2 \text{ and } \widehat{f}' : \widehat{M_2}' \cong M_2' \\ \end{array}$ extends  $\widehat{f}$ . Let  $g := \widehat{f'} \circ \widehat{g}$ . Since  $\widehat{g}$  fixes  $M_0$  and  $\widehat{f'}$  extends f, g extends f, as desired.  $\dagger_{\text{Claim 2}}$ 

Now let  $M_0 \leq_{\mathbf{K}} M$  and  $M_0 \leq_{\mathbf{K}} N$  be in  $\mathbf{K}_{\lambda}$ . We want to amalgamate M and N over  $M_0$ . We try to build  $\langle M_i : i < \lambda^+ \rangle$ ,  $\langle N_i : i < \lambda^+ \rangle$  increasing continuous in  $\mathbf{K}_{\lambda}$  and  $\langle f_i : i < \lambda^+ \rangle$  an increasing continuous sequence of embeddings such that for all  $i < \lambda^+$ :

- (1)  $M_i \leq_{\mathbf{K}} M.$ (2)  $f_i : M_i \xrightarrow[M_0]{} N_i.$
- (3)  $N_0 = N$ .
- (4)  $M_i <_{\mathbf{K}} M_{i+1}$ .

This is impossible since then  $\bigcup_{i < \lambda^+} M_i$  has cardinality  $\lambda^+$  but is a **K**-substructure of M which has cardinality  $\lambda$ . Now for i = 0, we can take  $N_0 = N$  and  $f_0 = \mathrm{id}_{M_0}$ and for i limit we can take unions. Therefore there must be some  $\alpha < \lambda^+$  such that  $f_{\alpha}, M_{\alpha}, N_{\alpha}$  are defined but we cannot define  $f_{\alpha+1}, M_{\alpha+1}, N_{\alpha+1}$ . If  $M_{\alpha} <_{\mathbf{K}} M$ , we can use Claim 2 (with  $M_0$ ,  $M_1$ ,  $M_2$ , f there standing for  $M_\alpha$ , M,  $N_\alpha$ ,  $f_\alpha$  here) to get  $M_{\alpha+1} \leq_{\mathbf{K}} M$  with  $M_{\alpha} <_{\mathbf{K}} M_{\alpha+1}$  and  $N_{\alpha+1} \geq_{\mathbf{K}} N_{\alpha}$  with  $f_{\alpha+1} : M_{\alpha+1} \to N_{\alpha+1}$ extending  $f_{\alpha}$  (so  $M'_1, M'_2, g$  in Claim 2 stand for  $M_{\alpha+1}, N_{\alpha+1}, f_{\alpha+1}$  here). Thus we can continue the induction, which we assumed was impossible. Therefore  $M_{\alpha} = M$ , so  $f_{\alpha}: M \xrightarrow{M_0} N_{\alpha}$  amalgamates M and N over  $M_0$ , as desired.  $\square$ 

REMARK 8.4.15. It is enough to assume that the type extension property holds on a set of types satisfying what Shelah calls the *density* property of basic types (see axiom (D)(c) in [She09a, Definition II.2.1]): for any  $M <_{\mathbf{K}} N$  in  $\mathbf{K}_{\lambda}$ , there exists  $b \in |N| \setminus |M|$  such that gtp(b/M; N) can be extended to any  $M' \geq_{\mathbf{K}} M$  with  $M' \in \mathbf{K}_{\lambda}$ . This generalization is the reason the last part of the proof is done non-constructively rather than first enumerating M and amalgamating it element by element. This is used to prove Theorem 8.4.16 in full generality (i.e. without assuming that the good frame is type-full).

We are now ready to formally state the amalgamation transfer:

THEOREM 8.4.16. Let **K** be an AEC. Let  $\lambda \geq \text{LS}(\mathbf{K})$  and assume  $\mathfrak{s}$  is a good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}$ . If:

(1) **K** is  $\lambda$ -tame.

(2)  $\mathbf{K}_{>\lambda}$  has weak amalgamation.

Then  $\mathbf{K}_{>\lambda}$  has amalgamation.

PROOF. We extend  $\mathfrak{s}$  to models of size greater than  $\lambda$  by defining  $\geq \mathfrak{s}$  as in [She09a, Section II.2] (or see [Bon14a, Definition 2.7]). Even without assuming tameness or weak amalgamation, Shelah has shown that  $\geq \mathfrak{s}$  has local character, density of basic types, and transitivity. Moreover, tameness implies that it has uniqueness. Now work by induction on  $\mu \geq \lambda$  to show that **K** has  $\mu$ -amalgamation. When  $\mu = \lambda$  this follows from the definition of a good frame so assume  $\mu > \lambda$ . As in [Bon14a, Theorem 5.13], we can prove that  $\geq \mathfrak{s}$  has the extension property for models of size  $\mu$  (the key is that the directed system argument only uses amalgamation below  $\mu$ ). In particular,  $\mathbf{K}_{\mu}$  has the type extension property for basic types. The proof of Theorem 8.4.14 together with the density of basic types (see Remark 8.4.15) shows that this suffices to get  $\mu$ -amalgamation.

COROLLARY 8.4.17. Let **K** be a tame AEC that is eventually syntactically characterizable and categorical in unboundedly many cardinals. If **K** has weak amalgamation, then there exists  $\lambda$  such that  $\mathbf{K}_{>\lambda}$  has amalgamation.

PROOF. By Fact 8.4.6, we can find  $\lambda \geq LS(\mathbf{K})$  such that  $\mathbf{K}_{\lambda}$  has a good frame and  $\mathbf{K}$  is  $\lambda$ -tame. By Theorem 8.4.16,  $\mathbf{K}_{>\lambda}$  has amalgamation.

COROLLARY 8.4.18. Let **K** be an eventually syntactically characterizable AEC categorical in unboundedly many cardinals. If **K** is tame and locally admits intersections, then there exists  $\lambda$  such that  $\mathbf{K}_{\geq \lambda}$  has amalgamation.

PROOF. By Remark 8.4.13, **K** has weak amalgamation. Now apply Corollary 8.4.17.  $\hfill \square$ 

COROLLARY 8.4.19. Let **K** be locally pseudo-universal AEC. If **K** is eventually syntactically characterizable and categorical in unboundedly many cardinals, then there exists  $\lambda$  such that  $\mathbf{K}_{>\lambda}$  has amalgamation.

PROOF. By Corollary 8.3.8, **K** is tame. Now apply Corollary 8.4.18.  $\Box$ 

We can apply these results to Shelah's categoricity conjecture and improve Fact 8.1.5. When **K** has primes, this will be further improved in Section 8.5.

COROLLARY 8.4.20. Let K be a tame AEC with weak amalgamation.

- (1) If  $\mathbf{K}$  is categorical in a high-enough successor cardinal, then  $\mathbf{K}$  is categorical on a tail of cardinals.
- (2) Assume  $2^{\theta} < 2^{\theta^+}$  for every cardinal  $\theta$  and an unpublished claim of Shelah (Claim 8.1.4). If **K** is eventually syntactically characterizable and categorical in unboundedly many cardinals, then **K** is categorical on a tail of cardinals.

PROOF. By Corollary 8.4.17 (using Fact 8.4.5 to see that  $\mathbf{K}$  is eventually syntactically characterizable in (1)), we can assume without loss of generality that  $\mathbf{K}$  has amalgamation. Now:

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- (1) Apply [GV06a] (and [She99] can also give a downward transfer).
- (2) Apply Fact 8.1.5.

Note that even if  $\mathbf{K}$  is a universal class which *already* has amalgamation, Theorem 8.4.16 is still key to transfer categoricity (see Theorem 8.5.16).

# 8.5. Categoricity transfer in AECs with primes

In this section, we prove a categoricity transfer for AECs that have amalgamation and primes. Prime triples were introduced in [She09a, Section III.3], see also [Jar].

DEFINITION 8.5.1.

- (1) Let  $M \in \mathbf{K}$  and let  $A \subseteq |M|$ . M is prime over A if for any enumeration  $\bar{a}$  of A and any  $N \in \mathbf{K}$ , whenever  $\operatorname{gtp}(\bar{a}/\emptyset; M) = \operatorname{gtp}(\bar{b}/\emptyset; N)$ , there exists  $f: M \to N$  such that  $f(\bar{a}) = \bar{b}$ .
- (2) (a, M, N) is a prime triple if  $M \leq_{\mathbf{K}} N$ ,  $a \in |N|$ , and N is prime over  $|M| \cup \{a\}$ .
- (3) **K** has primes if for any  $p \in gS(M)$  there exists a prime triple (a, M, N) such that p = gtp(a/M; N).
- (4) **K** weakly has primes if whenever  $gtp(a_1/M; N_1) = gtp(a_2/M; N_2)$ , there exists  $M_1 \leq_{\mathbf{K}} M$  containing  $a_1$  and  $N_1$  and  $f : M_1 \xrightarrow{}{M} N_2$  such that  $f(a_1) = a_2$ . Similarly define what it means for  $\mathbf{K}_{\lambda}$  to have or weakly have primes.

REMARK 8.5.2. For  $M \leq_{\mathbf{K}} N$  and  $a \in |N|$ , (a, M, N) is a prime triple if and only if whenever gtp(b/M; N') = gtp(a/M; N), there exists  $f : N \xrightarrow{M} N'$  such that f(a) = b. Thus if **K** has primes, then **K** weakly has primes.

REMARK 8.5.3. If **K** admits intersections,  $M \leq_{\mathbf{K}} N$ , and  $a \in |N|$ ,  $(a, M, cl^{N}(|M| \cup \{a\}))$  is a prime triple. Thus **K** has primes.

Assume **K** is an AEC categorical in  $\lambda := \text{LS}(\mathbf{K})$  (this is a reasonable assumption as we can always restrict ourselves to the class of  $\lambda$ -saturated models of **K**). Our goal is to prove (with more hypotheses) that if **K** is categorical in a  $\theta > \lambda$  then it is categorical in all  $\theta' \geq \lambda$ . To accomplish this, we will show that  $\mathbf{K}_{\lambda}$  is *unidimensional*. In [She09a, Section III.2], Shelah gives several possible generalization of the first-order definition in [She90, Definition V.2.2]. We have picked what seems to be the most convenient to work with:

DEFINITION 8.5.4 (Definition III.2.2.6 in [She09a]). Let  $\lambda \geq LS(\mathbf{K})$ .  $\mathbf{K}_{\lambda}$  is weakly uni-dimensional if for every  $M <_{\mathbf{K}} M_{\ell}$ ,  $\ell = 1, 2$  all in  $\mathbf{K}_{\lambda}$ , there is  $c \in |M_2| \setminus |M|$  such that  $gtp(c/M; M_2)$  has more than one extension in  $gS(M_1)$ .

To understand this definition, it might be helpful to look at the negation: there exists  $M <_{\mathbf{K}} M_{\ell}$ ,  $\ell = 1, 2$  all in  $\mathbf{K}_{\lambda}$  such that for all  $c \in |M_2| \setminus |M|$ ,  $\operatorname{gtp}(c/M; M_2)$  has exactly one extension in  $\operatorname{gS}(M_1)$ . Working in a good frame, this one extension must be the nonforking extension (so in particular  $\operatorname{gtp}(c/M; M_2)$  is omitted in  $M_1$ ). It turns out that for any  $c \in |M_2| \setminus |M|$  and  $d \in |M_1| \setminus |M|$ ,  $\operatorname{gtp}(c/M; M_2)$  and  $\operatorname{gtp}(d/M; M_1)$  are orthogonal (in a suitable sense, see Section 8.7), so they will generate two different dimensions.

FACT 8.5.5 (Claim III.2.3.(4) in [She09a]). Let  $\lambda \geq LS(\mathbf{K})$ . If  $\mathbf{K}_{\lambda}$  is weakly uni-dimensional, is categorical in  $\lambda$ , is stable in  $\lambda$ , and has  $\lambda$ -amalgamation, then<sup>14</sup> **K** is categorical in  $\lambda^+$ .

If **K** is  $\lambda$ -tame and has amalgamation, then categoricity in  $\lambda^+$  is enough by the categoricity transfer of Grossberg and VanDieren:

FACT 8.5.6 (Theorem 6.3 in  $[\mathbf{GV06a}]$ ). Assume **K** is an LS(**K**)-tame AEC with amalgamation and no maximal models. If **K** is categorical in LS(**K**) and LS(**K**)<sup>+</sup>, then **K** is categorical in all  $\mu \geq \text{LS}(\mathbf{K})$ .

Thus the hard part is showing that  $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  is weakly uni-dimensional. We proceed by contradiction.

DEFINITION 8.5.7 (III.12.39.(d) in [She09a]). Let  $M \in \mathbf{K}$  and let  $p \in \mathrm{gS}(M)$ . We define<sup>15</sup>  $\mathbf{K}_{\neg^* p}$  to be the class of  $N \in \mathbf{K}_M$  (recall Definition 8.2.18) such that f(p) has a unique extension to  $\mathrm{gS}(N \upharpoonright L(\mathbf{K}))$ . Here  $f : M \to N$  is given by  $f(a) := c_a^N$ . We order  $\mathbf{K}_{\neg^* p}$  with the strong substructure relation induced from  $\mathbf{K}_M$ .

REMARK 8.5.8. Let  $p \in gS(M)$  be nonalgebraic and let  $M \leq_{\mathbf{K}} N$ . If we are working in a good frame and p has a unique extension to gS(N), then it must be the nonforking extension. Thus p is omitted in N. However even if p is omitted in N, p could have two nonalgebraic extensions to gS(N), so  $\mathbf{K}_{\neg^*p}$  need *not* be the same as the class  $\mathbf{K}_{\neg p}$  of models omitting p.

In general, we do *not* claim that  $\mathbf{K}_{\neg^* p}$  is an AEC. Nevertheless it is an abstract class in the sense introduced by Grossberg in [**Gro**], see Definition 2.2.7. Thus we can define notions such as amalgamation, Galois types, and tameness there just as in AECs. The following gives an easy criterion for when  $\mathbf{K}_{\neg^* p}$  is an AEC:

PROPOSITION 8.5.9. Let  $\mathfrak{s} = (\mathbf{K}, \downarrow)$  be a type-full good  $(\geq \lambda)$ -frame (so  $\lambda = \mathrm{LS}(\mathbf{K})$  and  $\mathbf{K}_{<\lambda} = \emptyset$ ). Let  $M \in \mathbf{K}$  and let  $p \in \mathrm{gS}(M)$ . Then  $\mathbf{K}_{\neg^* p}$  is an AEC.

PROOF. All the axioms are easy except closure under chains. So let  $\delta$  be a limit ordinal and let  $\langle N_i : i < \delta \rangle$  be increasing continuous in  $\mathbf{K}_{\neg^* p}$ . Identify models in  $\mathbf{K}$ with their expansions in  $\mathbf{K}_M$ , assuming without loss of generality that  $M \leq_{\mathbf{K}} N_0$ , i.e. the map  $a \mapsto c_a^{N_0}$  for  $a \in M$  is the identity. Let  $N_{\delta} := \bigcup_{i < \delta} M_i$ . We have that  $N_{\delta} \upharpoonright L(\mathbf{K}) \in \mathbf{K}$ . Now if  $p_1, p_2 \in \mathrm{gS}(N_{\delta} \upharpoonright L(\mathbf{K}))$  are two extensions of p, by local character there exists  $i < \delta$  such that  $p_1$  and  $p_2$  do not fork over  $N_i$ . Since p has a unique extension to  $N_i, p_1 \upharpoonright N_i = p_2 \upharpoonright N_i$ . By uniqueness,  $p_1 \upharpoonright N_{\delta} = p_2 \upharpoonright N_{\delta}$ .  $\Box$ 

In fact, Shelah gave a criterion for when  $\mathbf{K}_{\neg^* p}$  has a good  $\lambda$ -frame:

FACT 8.5.10 (Claim III.12.39 in [She09a]). Let  $\mathfrak{s}$  be a good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}$ . Assume  $\mathfrak{s}$  is type-full, good<sup>+</sup>, successful (see Section 8.6 for the definitions of these terms), and  $\mathbf{K}_{\lambda}$  has primes. Assume further that  $\mathbf{K}$  is categorical in  $\lambda$ .

If  $\mathbf{K}_{\lambda}$  is not weakly uni-dimensional, then there exists  $M \in \mathbf{K}_{\lambda}$  and  $p \in \mathrm{gS}(M)$  such that  $\mathfrak{s} \upharpoonright \mathbf{K}_{\neg^* p}$  (the restriction of  $\mathfrak{s}$  to models in  $\mathbf{K}_{\neg^* p}$ ) is a type-full good  $\lambda$ -frame.

<sup>&</sup>lt;sup>14</sup>In [She09a, Claim III.2.3.(4)], Shelah assumes more generally the existence of a good  $\lambda$ -frame, but the proof shows that the hypotheses mentioned here suffice. In any case, we will only use Fact 8.5.5 inside a good frame.

<sup>&</sup>lt;sup>15</sup>Shelah calls the class  $\mathbf{K}^*$ .

Since this result is crucial to our argument and Shelah's proof is only implicit, we have included a proof in Section 8.7.

Note that the hypotheses of Fact 8.5.10 are reasonable. In fact, it is known that they follow from categoricity in fully tame and short AECs with amalgamation:

FACT 8.5.11 (Theorem 6.15.6). Let **K** be a fully ( $< \kappa$ )-tame and short AEC with amalgamation. Let  $\lambda$ ,  $\mu$  be cardinals such that:

$$LS(\mathbf{K}) < \kappa = \beth_{\kappa} < \lambda = \beth_{\lambda} \le \mu$$

Assume further that  $\operatorname{cf} \lambda \geq \kappa$ . If **K** is categorical in  $\mu$ , then **K** is categorical in  $\lambda$  and there exists a type-full successful good  $\lambda$ -frame  $\mathfrak{s}$  with underlying class  $\mathbf{K}_{\lambda}$ .

From Proposition 8.6.20, it will follow that the frame given by Fact 8.5.11 is also  $\text{good}^+$ . If in addition the AEC has primes (e.g. if it is universal), then the hypotheses are satisfied. Of course, the Hanf numbers in Fact 8.5.11 are not optimal. We give the following improvement in Section 8.6:

THEOREM 8.5.12. Let **K** be a fully  $LS(\mathbf{K})$ -tame and short AEC with amalgamation and no maximal models. If **K** is categorical in a  $\mu > LS(\mathbf{K})$ , then there exists  $\lambda_0 < h(LS(\mathbf{K}))$  such that for all  $\lambda \ge \lambda_0$  where **K** is categorical in  $\lambda$ , there exists a type-full successful good<sup>+</sup>  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}$ .

PROOF. Combine Corollaries 8.6.16 and 8.6.21.

Now we reach a crucial point. For the purpose of a categoricity transfer, it would be enough to show that  $\mathbf{K}_{\neg^*p}$  above has arbitrarily large models, since this means that there are non-saturated models in every cardinal above  $\lambda$ . Unfortunately, even if  $\mathbf{K}$  is fully tame and short and has amalgamation, it is not easy to get a handle on  $\mathbf{K}_{\neg^*p}$ . For example, it is not clear if it has amalgamation or even if it is tame. In [**She09a**, Discussion III.12.40] Shelah claims to be able to show using enough instances of the weak generalized continuum hypothesis that  $\mathfrak{s} \upharpoonright \mathbf{K}_{\neg^*p}$  above has arbitrarily large models (this is probably how Claim 8.1.4 is proven) and this is the key to the proof of Fact 8.1.5.

We make the situation where  $\mathbf{K}_{\neg p}$  is well-behaved into a definition:

DEFINITION 8.5.13.  $\mathbf{K}$  is *nice* if:

- (1)  $\mathbf{K}$  has weak amalgamation.
- (2) For any  $M \in \mathbf{K}$  and any  $p \in \mathrm{gS}(M)$ ,  $\mathbf{K}_{\neg^* p}$  has weak amalgamation and if  $\mathbf{K}$  is ||M||-tame, then so is  $\mathbf{K}_{\neg^* p}$ .

Note that if **K** is a universal class, then  $\mathbf{K}_{\neg^* p}$  also is universal (using that **K** is fully ( $< \aleph_0$ )-tame and short, we can prove as in Proposition 8.5.9 that it is an AEC), hence **K** is nice! More generally:

PROPOSITION 8.5.14. If K weakly has primes, then K is nice.

PROOF. Weak amalgamation follows from the definition of weakly having primes. Now let  $M \in \mathbf{K}$  and  $p \in \mathrm{gS}(M)$ . Observe that  $\mathbf{K}_{\neg^* p}$  weakly has primes, because if  $N \in \mathbf{K}_{\neg^* p}$ ,  $N_0 \leq_{\mathbf{K}} N \upharpoonright L(\mathbf{K})$  is in  $\mathbf{K}$ , and  $M \leq_{\mathbf{K}} N_0$ , then the natural expansion of  $N_0$  is in  $\mathbf{K}_{\neg^* p}$ . Therefore  $\mathbf{K}_{\neg^* p}$  also has weak amalgamation. If in addition  $\mathbf{K}$  is ||M||-tame, then so is  $\mathbf{K}_{\neg^* p}$ : indeed if  $N \in \mathbf{K}_{\neg^* p}$ ,  $q_1, q_2 \in \mathrm{gS}(N)$ , and the two types are equal in  $\mathbf{K}$ , then since  $\mathbf{K}_{\neg^* p}$  weakly has primes there is a map witnessing equality of the types in  $\mathbf{K}_{\neg^* p}$  also. The following fact is the key to our argument. It was first proven under slightly stronger hypotheses by Will Boney [**Bon14a**]. The interesting consequence to us is that it gives a local criterion for a tame AEC to have arbitrarily large models.

FACT 8.5.15 (Corollary 5.6.10). ] If  $\mathfrak{s}$  is a good  $\lambda$ -frame on  $\mathbf{K}_{\lambda}$ ,  $\mathbf{K}$  is  $\lambda$ -tame and has amalgamation, then  $\mathfrak{s}$  extends to a good ( $\geq \lambda$ )-frame on  $\mathbf{K}_{\geq \lambda}$ . In particular,  $\mathbf{K}_{\geq \lambda}$  has no maximal models and is stable in every cardinal above  $\lambda$ .

THEOREM 8.5.16. Let  $\mathfrak{s}$  be a good  $\lambda$ -frame with underlying AEC **K**. Assume  $\mathfrak{s}$  is type-full, good<sup>+</sup>, successful, and  $\mathbf{K}_{\lambda}$  has primes. Assume also that **K** is categorical in  $\lambda$ ,  $\lambda$ -tame, and nice. The following are equivalent.

- (1)  $\mathbf{K}_{\lambda}$  is weakly uni-dimensional.
- (2) **K** is categorical in all  $\mu \geq \lambda$ .
- (3) **K** is categorical in some  $\theta > \lambda$ .

PROOF. Replacing **K** with  $\mathbf{K}_{\geq\lambda}$ , assume without loss of generality that  $\lambda = \mathrm{LS}(\mathbf{K})$  and  $\mathbf{K}_{\langle \mathrm{LS}(\mathbf{K})} = \emptyset$ . First note that **K** has amalgamation by Theorem 8.4.16. By Fact 8.5.15,  $\mathfrak{s}$  extends to a good ( $\geq \lambda$ )-frame on **K**. In particular, **K** has no maximal models and is stable in every cardinal. Moreover by Proposition 8.5.9,  $\mathbf{K}_{\neg^* p}$  is an AEC for all  $p \in \mathrm{gS}(M)$  and  $M \in \mathbf{K}$ .

If  $\mathbf{K}_{\lambda}$  is weakly uni-dimensional, then by Fact 8.5.5, **K** is categorical in  $\lambda^+$ . By Fact 8.5.6, **K** is categorical in all  $\mu \geq \lambda$ . So (1) implies (2). Of course, (2) implies (3). It remains to show (3) implies (1). We show the contrapositive.

Assume that **K** is *not* weakly uni-dimensional. Let  $M \in \mathbf{K}_{\lambda}$  and  $p \in \mathrm{gS}(M)$ be as given by Fact 8.5.10. Let  $\mathfrak{s}_{\neg^* p} := \mathfrak{s} \upharpoonright \mathbf{K}_{\neg^* p}$ , the restriction of  $\mathfrak{s}$  to models in  $\mathbf{K}_{\neg^* p}$ . Since **K** is nice,  $\mathbf{K}_{\neg^* p}$  has weak amalgamation and since **K** is also  $\lambda$ -tame,  $\mathbf{K}_{\neg^* p}$  is  $\lambda$ -tame. Since  $\mathfrak{s}_{\neg^* p}$  is a good  $\lambda$ -frame, Theorem 8.4.16 gives that  $\mathbf{K}_{\neg^* p}$  has amalgamation. By Fact 8.5.15,  $\mathbf{K}_{\neg^* p}$  has no maximal models and is stable in every cardinals. Now let  $\theta > \lambda$ . By stability, **K** has a saturated model of size  $\theta$ . Moreover since  $\mathbf{K}_{\neg^* p}$  has arbitrarily large models there must exist  $\hat{N} \in \mathbf{K}_{\neg^* p}$  of size  $\theta$ . By construction,  $\hat{N} \upharpoonright L(\mathbf{K})$  is not saturated of size  $\theta$ . Therefore **K** is not categorical in  $\theta$ .

We are now ready to prove a categoricity transfer in fully tame and short AECs with amalgamation (Theorem 8.0.3 from the abstract). We state one more fact:

FACT 8.5.17. If **K** is a LS(**K**)-tame AEC with amalgamation and no maximal models which is categorical in a  $\lambda \geq H_2$  (recall Definition 2.2.2) and the model of size  $\lambda$  is saturated, then **K** is categorical in  $H_2$ .

PROOF. By the proof of [She99, Theorem II.1.6] (or see [Bal09, Theorem 14.8]).  $\Box$ 

THEOREM 8.5.18. Let **K** be a fully LS(**K**)-tame and short AEC with amalgamation such that  $\mathbf{K}_{\geq H_2}$  has primes. If **K** is categorical in a  $\lambda > H_2$ , then **K** is categorical in all  $\lambda' \geq H_2$ .

PROOF. Without loss of generality,  $\mathbf{K}$  has joint embedding and no maximal models: we can start by splitting  $\mathbf{K}$  into disjoint parts, each of which has joint embedding, and then work with the unique part which has arbitrarily large models.

We start by observing that **K** is categorical in  $H_2$  by Fact 8.5.17 (note that the model of size  $\lambda$  is saturated by Facts 8.6.8 and 8.6.9). Now apply Theorem 8.5.12 (to  $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})^+}$ ) and Theorem 8.5.16.

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The only place where shortness is used above is to get the existence property for uniqueness triples (i.e. that the good frame is successful). The proof shows that it is enough to assume that for some  $\lambda$ ,  $\mathbf{K}_{>\lambda}$  is almost fully good, i.e. it has a nice-enough global independence relation (see 8.6.1 for a more precise definition). One can ask:

QUESTION 8.5.19. Can the full tameness and shortness hypothesis be weakened to just being  $LS(\mathbf{K})$ -tame?

We obtain a categoricity transfer for universal classes with amalgamation.

COROLLARY 8.5.20. Let K be a locally pseudo-universal AEC with amalgamation. If **K** is categorical in a  $\lambda > H_2$ , then **K** is categorical in all  $\lambda' \ge H_2$ .

**PROOF.** By Corollary 8.3.8,  $\mathbf{K}$  is fully  $LS(\mathbf{K})$ -tame and short. By Remark 8.5.3, K has primes. Now apply Theorem 8.5.18.  $\square$ 

In view of Theorem 8.5.18, a natural question is whether the existence of primes follows from the other hypotheses:

QUESTION 8.5.21. If K is fully tame and short, has amalgamation, and is categorical in unboundedly many cardinals, does there exists  $\lambda$  such that  $\mathbf{K}_{\geq \lambda}$  has primes?

Note that by [Bon14b], a positive answer would imply that Shelah's categoricity conjecture follows from the existence of a proper class of strongly compact cardinals. Moreover, it turns out that a converse is true. This was conjectured in earlier versions of this chapter, and the missing piece was proven in Chapter 12:

FACT 8.5.22. Let K be an almost fully good AEC (see Definition 8.6.2). For any  $\lambda > LS(\mathbf{K})^+$ ,  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  has primes.

Blackboxing Fact 8.5.22, we can give a proof of the converse of Theorem 8.5.18:

THEOREM 8.5.23. Let  $\mathbf{K}$  be a fully  $LS(\mathbf{K})$ -tame and short AEC with amalgamation. The following are equivalent:

(1)  $\mathbf{K}_{\geq H_2}$  has primes and is categorical in some  $\lambda > H_2$ (2)  $\mathbf{K}$  is categorical in all  $\lambda' \geq H_2$ .

**PROOF.** (1) implies (2) is Theorem 8.5.18. We show (2) implies (1). As in the proof of Theorem 8.5.18, we assume without loss of generality that K has joint embedding and no maximal models. By Corollary 8.6.16 (with  $\kappa, \theta$  there standing for LS(**K**)<sup>+</sup>,  $\lambda$  here), **K**<sup>\*</sup> := **K**<sup> $\mu$ -sat</sub> is almost fully good, where  $\mu := (2^{\text{LS}(\mathbf{K})})^{+5}$ . Now apply Fact 8.5.22 to the AEC **K**<sup>\*</sup> and use categoricity in all  $\lambda' \geq H_2$ .</sup>

REMARK 8.5.24. Using the threshold improvements of Chapter 14, we can replace  $H_2$  by  $H_1$  (and allow  $\lambda = H_1$  in (1)) in Theorem 8.5.23.

There is still an assumption of amalgamation in Theorem 8.5.18. Assuming the categoricity cardinals are sufficiently nice, this can be removed using the results of Section 8.4:

THEOREM 8.5.25. Let  $\mathbf{K}$  be a fully tame and short AEC with primes. If  $\mathbf{K}$  is categorical in cardinals of arbitrarily high cofinality, then **K** is categorical on a tail of cardinals.

PROOF. By Fact 8.4.5, **K** is eventually syntactically characterizable. By the definition of having primes, **K** has weak amalgamation. By Corollary 8.4.17, there exists  $\lambda$  such that  $\mathbf{K}_{>\lambda}$  has amalgamation. Now apply Theorem 8.5.18 to  $\mathbf{K}_{>\lambda}$ .

REMARK 8.5.26. Instead of categoricity in cardinals of arbitrarily high cofinality, it suffices to assume that  $\mathbf{K}$  is eventually syntactically characterizable and categorical in unboundedly many of cardinals.

We can replace the assumption on the categoricity cardinal by large cardinals. As pointed out in the introduction (Theorem 8.1.6), a strongly compact would be enough. Here we improve this to a measurable (but assume full tameness and shortness). This only gives amalgamation below the categoricity cardinal but we can then transfer amalgamation upward using the arguments in Section 8.4.

THEOREM 8.5.27. Let **K** be a fully LS(**K**)-tame and short AEC with primes. Let  $\kappa > \text{LS}(\mathbf{K})$  be a measurable cardinal. If **K** is categorical in *some*  $\lambda > h(h(\kappa))$ , then **K** is categorical in *all*  $\lambda' \ge h(h(\kappa))$ .

PROOF. By the main result of  $[\mathbf{SK96}]^{16} \mathbf{K}_{[\kappa,\lambda)}$  has amalgamation (and  $\mathbf{K}_{\geq\kappa}$  has no maximal models, using ultraproducts). Combining (the proofs of) Facts 8.6.8 and 8.6.9, there is a good  $\kappa^+$ -frame with underlying class  $\mathbf{K}_{\kappa^+}$ . By Theorem 8.4.16,  $\mathbf{K}_{\geq\kappa}$  has amalgamation. Now apply Theorem 8.5.18 to  $\mathbf{K}_{\geq\kappa}$ .

We can now prove Theorem 8.0.2 from the abstract.

COROLLARY 8.5.28. Let  $\mathbf{K}$  be a universal class (or just a locally pseudouniversal AEC, see Example 8.3.2.(1) and Remark 8.3.5).

- (1) If **K** is categorical in cardinals of arbitrarily high cofinality, then **K** is categorical on a tail of cardinals.
- (2) If  $\kappa > \text{LS}(\mathbf{K})$  is a measurable cardinal and  $\mathbf{K}$  is categorical in some  $\lambda > h(h(\kappa))$ , then  $\mathbf{K}$  is categorical in all  $\lambda' \ge h(h(\kappa))$ .

PROOF. Follow the proof of Corollary 8.5.20 to see that the assumptions of Theorems 8.5.25 and 8.5.27 respectively are satisfied.  $\Box$ 

## 8.6. Independence below the Hanf number

In this section, we give all the results needed for the proof of Theorem 8.5.12. We also define all the technical terms related to good frames used there. Good frames were introduced by Shelah in [She09a, Chapter II] but we use the notation and definitions in Chapter 6 (we also extensively use its results). The reader is invited to consult this paper for more motivation and background on the concepts used here.

The first definition is that of a *global* forking-like notion:

DEFINITION 8.6.1 (Definition 6.8.1).  $i = (\mathbf{K}, \downarrow)$  is a fully good independence relation if:

(1) **K** is an AEC with  $\mathbf{K}_{\langle \mathrm{LS}(\mathbf{K})} = \emptyset$  and  $\mathbf{K} \neq \emptyset$ .

- (2) K has amalgamation, joint embedding, and no maximal models.
- (3)  $\mathbf{K}$  is stable in all cardinals.

<sup>&</sup>lt;sup>16</sup>The result there is stated in terms of the class of models of an  $L_{\kappa,\omega}$  sentence. However, Boney [**Bon14b**] has pointed out that this applies as well when **K** is an AEC and  $\kappa > \text{LS}(\mathbf{K})$ , see in particular the discussion around Theorem 7.6 there.

- (4) i is a  $(< \infty, \ge LS(\mathbf{K}))$ -independence relation (see Definition 6.3.6). That is,  $\bot$  is a relation on quadruples (M, A, B, N) with  $M \le_{\mathbf{K}} N$  and  $A, B \subseteq$ |N| satisfying invariance, monotonicity, and normality. We write  $A \underset{M}{\downarrow} B$ instead of  $\bigcup (M, A, B, N)$ , and we also say  $gtp(\bar{a}/B; N)$  does not fork over M for ran  $\bar{a} \underset{M}{\downarrow} B$ .
- (5) i has base monotonicity, disjointness  $(A \underset{M}{\downarrow} B \text{ implies } A \cap B \subseteq |M|)$ , symmetry, uniqueness, extension, and the local character properties:
  - (a) If  $p \in gS^{\alpha}(M)$ , there exists  $M_0 \leq_{\mathbf{K}} M$  with  $||M_0|| \leq |\alpha| + LS(\mathbf{K})$  such that p does not fork over  $M_0$ .
  - (b) If  $\langle M_i : i \leq \delta \rangle$  is increasing continuous,  $p \in gS^{\alpha}(M_{\delta})$  and  $\mathrm{cf} \, \delta > \alpha$ , then there exists  $i < \delta$  such that p does not fork over  $M_i$ .
- (6) i has the left and right  $(\leq \mathrm{LS}(\mathbf{K}))$ -witness properties:  $A \stackrel{N}{\downarrow} B$  if and only if for all  $A_0 \subseteq A$  and  $B_0 \subseteq B$  with  $|A_0| + |B_0| \leq \mathrm{LS}(\mathbf{K})$ , we have that  $A_0 \stackrel{N}{\downarrow} B_0$ .
- (7) i has full model continuity: if for  $\ell < 4$ ,  $\langle M_i^{\ell} : i \leq \delta \rangle$  are increasing continuous such that for all  $i < \delta$ ,  $M_i^0 \leq_{\mathbf{K}} M_i^\ell \leq_{\mathbf{K}} M_i^3$  for  $\ell = 1, 2$  and  $M_i^1 \bigcup_{M_i^0} M_i^2$ , then  $M_\delta^1 \bigcup_{M_\delta^0} M_\delta^2$ .

We say that i is good if it has all the properties above except full model continuity. We say that K is [fully] good if there exists  $\bot$  such that  $(\mathbf{K}, \bot)$  is [fully] good.

We will use the following variation:

DEFINITION 8.6.2.  $i = (\mathbf{K}, \downarrow)$  is almost fully good if it satisfies Definition 8.6.1 except that only the following types are required to have a nonforking extension:

- (1) Types that do not fork over *saturated* models.
- (2) Type that do not fork over models of size  $LS(\mathbf{K})$ .
- (3) Types of length at most  $LS(\mathbf{K})$ .

As before, we say that **K** is almost fully good if there exists  $\downarrow$  such that  $(\mathbf{K}, \downarrow)$  is almost fully good. If we drop "fully" we mean that full model continuity need not hold.

In this terminology, we have:

FACT 8.6.3 (Theorem 6.15.1). Let **K** be a fully ( $< \kappa$ )-tame and short AEC with amalgamation.

If  $\kappa = \beth_{\kappa} > \mathrm{LS}(\mathbf{K})$ , and **K** is categorical in a  $\mu > \lambda_0 := (2^{\kappa})^{+5}$ , then  $\mathbf{K}_{\geq \lambda}$  is almost fully good, where we have set  $\lambda := \min(\mu, h(\lambda_0))$ .

A localization of fully good independence relation are Shelah's good  $\lambda$ -frames. Roughly speaking, we simply require the types to have length one and the models to have a fixed size  $\lambda$ . We only give the definition of a *type-full* good  $\lambda$ -frame here, since this is the one that we can build here. In **[She09a**, Section II.2], Shelah has a more general definition where he only requires a dense class of basic types to satisfy the properties of forking: this is also what we call a good  $\lambda$ -frame (without the "type-full") in this chapter, e.g. in Theorem 8.4.16. We use the definition in Definition 6.8.1 and refer to Remark 6.3.5 there for why this is equivalent (in the type-full case) to Shelah's definition in [She09a, Section II.2].

DEFINITION 8.6.4.  $\mathfrak{s} = (\mathbf{K}_{\mathfrak{s}}, \downarrow)$  is a type-full good  $\lambda$ -frame if:

- (1) There exists an AEC **K** with  $\lambda = LS(\mathbf{K})$ ,  $\mathbf{K}_{\lambda} = \mathbf{K}_{\mathfrak{s}}$ . Below, we require that all the models be in  $\mathbf{K}_{\mathfrak{s}}$ .
- (2)  $\mathbf{K}_{\mathfrak{s}} \neq \emptyset$ .
- (3)  $\mathbf{K}_{\mathfrak{s}}$  has amalgamation, joint embedding, and no maximal models.
- (4)  $\mathbf{K}_{\mathfrak{s}}$  is stable in  $\lambda$ .
- (5)  $\downarrow$  is a relation on quadruples  $(M_0, a, M, N)$  with  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  and  $a \in |N|$  satisfying invariance, monotonicity, and normality. As before, we write  $a \downarrow_{M_0}^N M$  instead of  $\downarrow(M_0, a, M, N)$ , and we also say gtp(a/M; N) does not fork over  $M_0$  for  $a \downarrow_{M_0}^N M$ .

(6)  $\mathfrak{s}$  has base monotonicity, disjointness, full symmetry (if  $a \underset{M_0}{\overset{N}{\downarrow}} M, b \in |M|$ , then there exists  $N' \geq_{\mathbf{K}} N$  and  $M'_0 \geq_{\mathbf{K}} M_0$  with  $M'_0 \leq_{\mathbf{K}} N', a \in |M'_0|$ , and  $b \underset{M_0}{\overset{N'}{\downarrow}} M'_0$ ), uniqueness, extension, and the local character property: If  $\langle M_i : i \leq \delta \rangle$  is increasing continuous,  $p \in \mathrm{gS}(M_\delta)$ , then there exists  $i < \delta$ such that p does not fork over  $M_i$ .

We define similarly "type-full good  $(\geq \lambda)$ -frame", where we allow the models in  $\mathbf{K}_{\mathfrak{s}}$  to have sizes in  $\mathbf{K}_{>\lambda}$  (but still work with types of length one).

NOTATION 8.6.5. When  $\mathbf{i} = (\mathbf{K}, \bigcup)$  is an almost good independence relation and  $\lambda \geq \mathrm{LS}(\mathbf{K})$ , we write  $\mathrm{pre}(\mathbf{i}^{\leq 1}) \upharpoonright \mathbf{K}_{\lambda}$  for the type-full good  $\lambda$ -frame obtained by restricting  $\bigcup$  to types of length one and models in  $\mathbf{K}_{\lambda}$ . Similarly for  $\mathrm{pre}(\mathbf{i}^{\leq 1}) \upharpoonright \mathbf{K}_{\geq \lambda}$ .

Assuming tameness and amalgamation, good frames can be built from a superstabilitylike condition (the superstability condition already appears implicitly in [**SV99**] and is developed further in [**Van06**, **Van13**, **GVV16**, **Van16a**] and Chapters 4, 6, 7, and 10). The construction of a good frame appears implicitly already in Chapter 4:

DEFINITION 8.6.6 (Superstability, see Definition 6.10.1).

- (1) For  $M, N \in \mathbf{K}$ , say  $M <_{\mathbf{K}}^{\mathrm{univ}} N$  (N is universal over M) if and only if  $M <_{\mathbf{K}} N$  and whenever we have  $M' \geq_{\mathbf{K}} M$  such that  $||M'|| \leq ||N||$ , then there exists  $f : M' \xrightarrow{M} N$ . Say  $M \leq_{\mathbf{K}}^{\mathrm{univ}} N$  if and only if M = N or  $M <_{\mathbf{K}}^{\mathrm{univ}} N$ .
- (2)  $p \in \mathbf{gS}(N)$   $\mu$ -splits over M if  $M \leq_{\mathbf{K}} N$ ,  $M \in \mathbf{K}_{\mu}$ , and there exists  $N_1, N_2 \in \mathbf{K}_{\mu}$  with  $M \leq_{\mathbf{K}} N_{\ell} \leq_{\mathbf{K}} N$ ,  $\ell = 1, 2$ , and an isomorphism  $f : N_1 \cong_M N_2$ , such that  $f(p \upharpoonright N_1) \neq p \upharpoonright N_2$ .
- (3) **K** is  $\mu$ -superstable if:
  - (a)  $LS(\mathbf{K}) \leq \mu$ .

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- (b) There exists  $M \in \mathbf{K}_{\mu}$  such that for any  $M' \in \mathbf{K}_{\mu}$  there is  $f : M' \to M$  with  $f[M'] <_{\mathbf{K}}^{\text{univ}} M$ .
- (c) If  $\langle M_i : i < \delta \rangle$  is increasing in  $\mathbf{K}_{\mu}$  such that  $i < \delta$  implies  $M_i <_{\mathbf{K}}^{\text{univ}}$  $M_{i+1}$  and  $p \in \mathrm{gS}(\bigcup_{i < \delta} M_i)$ , then there exists  $i < \delta$  such that p does not  $\mu$ -split over  $M_i$ .

DEFINITION 8.6.7. For  $\lambda$  a cardinal, let  $\mathbf{K}^{\lambda-\text{sat}}$  be the class of  $\lambda$ -saturated models in  $\mathbf{K}_{>\lambda}$ .

FACT 8.6.8. Assume K is  $\mu$ -superstable,  $\mu$ -tame, and has amalgamation. Then:

- (1) (Proposition 6.10.10) **K** is  $\mu'$ -superstable for all  $\mu' \ge \mu$ . In particular,  $\mathbf{K}_{\ge\mu}$  has joint embedding, no maximal models, and is stable in all cardinals.
- (2) (Corollary 10.6.10) If  $\lambda > \mu$ , then  $\mathbf{K}^{\lambda-\text{sat}}$  is an AEC with  $\text{LS}(\mathbf{K}^{\lambda-\text{sat}}) = \lambda$ .
- (3) (Corollary 10.6.14 and Corollary 5.6.10). For any  $\lambda > \mu$ , there exists a type-full good ( $\geq \lambda$ )-frame with underlying AEC  $\mathbf{K}^{\lambda-\text{sat}}$ .

From the analysis of Shelah and Villaveces in [**SV99**, Theorem 2.2.1], we obtain that superstability follows from categoricity (if the cofinality of the categoricity cardinal is high-enough, this appears as [**She99**, Lemma 6.3]). The version that we state here assumes amalgamation instead of GCH and is proven in Chapter 20.

FACT 8.6.9. Assume **K** has amalgamation and no maximal models. If **K** is categorical in a  $\theta > LS(\mathbf{K})$ , then **K** is  $LS(\mathbf{K})$ -superstable.

COROLLARY 8.6.10. Assume **K** is  $LS(\mathbf{K})$ -tame and has amalgamation and no maximal models. If **K** is categorical in a  $\theta > LS(\mathbf{K})$ , then there exists a type-full good ( $\geq LS(\mathbf{K})^+$ )-frame with underlying class  $\mathbf{K}^{\theta^+}$ -sat.

PROOF. Combine Facts 8.6.9, and 8.6.8.

It remains to see how to build a fully good (i.e. global) independence relation from just a local good frame. This is done using shortness, together with a property Shelah calls *successfulness* (we do not give the exact definition of uniqueness triple, the relation  $\leq_{\mathbf{K},+}^{\mathrm{NF}}$ , or the successor frame, as we have no use for it).

DEFINITION 8.6.11 (Definition III.1.1 in [She09a]). Let  $\mathfrak{s}$  be a type-full good  $\lambda$ -frame.

- (1)  $\mathfrak{s}$  is weakly successful if for any  $M \in \mathbf{K}_{\lambda}$  and any nonalgebraic  $p \in \mathrm{gS}(M)$ , there exists  $N \geq_{\mathbf{K}} M$  and  $a \in |N|$  such that  $p = \mathrm{gtp}(a/M; N)$  and (a, M, N) is a uniqueness triple (see [She09a, Definition II.5.3]).
- (2)  $\mathfrak{s}$  is *successful* if in addition the class  $(\mathbf{K}_{\lambda^+}^{\lambda^+\text{-sat}}, \leq_{\mathbf{K}_{\lambda^+}}^{\mathrm{NF}})$  (see [**JS13**, Definition 10.1.1]) is an AEC.
- (3) [She09a, Definition III.1.12]  $\mathfrak{s}$  is  $\omega$ -successful if for all  $n < \omega$ , the *n*th successor frame  $\mathfrak{s}^{+n}$  (see [She09a, Definition III.1.12]) is a type-full successful good  $\lambda$ -frame.

We can obtain an  $\omega$ -successful frame using existence of a sufficiently wellbehaved global independence relation:

FACT 8.6.12 (Theorem 6.11.21). Assume i is a  $(<\infty, \geq LS(\mathbf{K}))$ -independence relation on  $\mathbf{K}$  and  $\lambda > LS(\mathbf{K})$  is a cardinal such that<sup>17</sup>:

<sup>&</sup>lt;sup>17</sup>In Chapter 6, it is also assumed that  $\mathbf{K}^{\lambda^{+n}\text{-sat}}$  is an AEC for all  $n < \omega$ . However Fact 8.6.8 shows that this follows from the rest.

#### 8. CATEGORICITY IN UNIVERSAL CLASSES: PART I

- (1)  $\mathfrak{s} := \operatorname{pre}(\mathfrak{i}^{\leq 1})$  is a type-full good (> LS(**K**))-frame.
- (2) i has base monotonicity, uniqueness for types over models, and the left and right  $(\leq LS(\mathbf{K}))$ -witness properties.
- (3) i has the following local character property: for every  $n < \omega$ , if  $\mu :=$  $\lambda^{+(n+1)}$ , then for every increasing continuous  $\langle M_i : i \leq \mu \rangle$  and every  $p \in gS^{<\mu}(M_{\mu})$ , there exists  $i < \mu$  such that p does not fork (in the sense of  $\mathfrak{i}$ ) over  $M_i$ .

Then  $\mathfrak{s} \upharpoonright \mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  (the restriction of  $\mathfrak{s}$  to the class  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$ ) is an  $\omega$ -successful type-full good  $\lambda$ -frame.

In Chapter 6, we used  $(< \kappa)$ -satisfiability as i above. The downside is that we used that  $\kappa = \beth_{\kappa} > \mathrm{LS}(\mathbf{K})$ . Now we show we can use an independence relation induced by  $\mu$ -nonsplitting instead of ( $< \kappa$ )-satisfiability. We need one more fact:

FACT 8.6.13. Assume **K** has amalgamation and is stable in  $\mu \geq LS(\mathbf{K})$ . Let  $M \in \mathbf{K}_{\geq \mu}$  and let  $p \in \mathrm{gS}^{<\kappa}(M)$ . If  $\mu = \mu^{<\kappa}$ , then there exists  $M_0 \in \mathbf{K}_{\mu}$  with  $M_0 \leq_{\mathbf{K}} M$  such that p does not  $\mu$ -split over  $M_0$ .

PROOF. By [She99, Claim 3.3] (or [GV06b, Fact 4.6]), it is enough to show that  $|gS^{<\kappa}(N)| = \mu$  for every  $N \in \mathbf{K}_{\mu}$ . This holds by stability in  $\mu$  and [Bon17, Theorem 3.1]. 

LEMMA 8.6.14. Let  $\mathbf{K}$  be an AEC with amalgamation. Assume that  $\mathbf{K}$  is fully  $(< \kappa)$ -tame and short with  $\kappa \leq LS(\mathbf{K})^+$ . Assume further that **K** is  $LS(\mathbf{K})$ superstable. Let  $\lambda > LS(\mathbf{K})$  be such that  $\lambda = \lambda^{<\kappa}$ . Then there exists an  $\omega$ successful good  $\lambda^+$ -frame with underlying class  $\mathbf{K}_{\lambda^+}^{\lambda^+-\text{sat}}$ .

PROOF SKETCH. Define a  $(< \infty, \geq \lambda)$  independence relation  $\mathfrak{i} = (\mathbf{K}_{\mathfrak{i}}, \downarrow)$  as follows:

- $\mathbf{K}_{i} = \mathbf{K}^{\lambda \text{-sat}}$ .
- $p \in gS^{\alpha}(M)$  does not fork (in the sense of  $\mathfrak{i}$ ) over  $M_0 \leq_{\mathbf{K}} M$  if and only if

  - $\begin{array}{l} \ M_0, M \in \mathbf{K}^{\lambda \text{-sat}}. \\ \ \text{For every } I \subseteq \alpha \text{ with } |I| < \kappa, \text{ there exists } M'_0 \leq_{\mathbf{K}} M_0 \text{ in } \mathbf{K}_{\mu} \text{ such} \end{array}$ that  $p^I$  does not  $\mu$ -split over  $M'_0$ .

We claim that i satisfies the hypotheses of Fact 8.6.12 (where K there is  $K_{\geq\lambda}$ here and  $\lambda$  there is  $\lambda^+$  here). By Fact 8.6.13 and superstability, we have that i induces a LS(**K**)-generator for a weakly good ( $< \kappa$ )-independence relation (in the sense of Definition 6.7.3), as well as a  $LS(\mathbf{K})$ -generator for a good (< 1)independence relation (see Definition 6.8.5). It follows from Theorems 6.7.5, 6.8.9 that  $\mathfrak{s} := \operatorname{pre}(\mathfrak{i}^{\leq 1})$  is a type-full good  $(> \lambda)$ -frame and  $\mathfrak{i}^{<\kappa}$  (the restriction of  $\mathfrak{i}$ to types of length less than  $\kappa$ ) has base monotonicity, uniqueness for types over models, transitivity, and so that any type does not fork over a model of size  $LS(\mathbf{K})$ .

Now, it is easy to see using shortness that i also has uniqueness for types over models. By definition, it also has base monotonicity, transitivity, and the left  $(<\kappa)$ -witness property. Now from transitivity and the local character property mentioned in the previous paragraph, we get (Proposition 6.4.3.(6)) that i has the right ( $\leq LS(\mathbf{K})$ )-witness property. Thus all the hypotheses of Fact 8.6.12 are satisfied, so  $\mathfrak{s}$  is  $\omega$ -successful. 

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From an  $\omega$ -successful good  $\lambda$ -frame, we obtain the desired global independence relation:

FACT 8.6.15. Let  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \downarrow)$  be an  $\omega$ -successful good  $\lambda$ -frame which is categorical in  $\lambda$ . If **K** is fully ( $\langle \operatorname{cf} \lambda \rangle$ -tame and short and has amalgamation, then  $\mathbf{K}^{\lambda^{+3}\text{-sat}}$  is almost fully good.

PROOF. By Theorems 6.12.16, 6.13.6, and 6.14.15.

COROLLARY 8.6.16. Assume that **K** has amalgamation, no maximal models, and is fully  $(< \kappa)$ -tame and short, with  $\kappa \leq \mathrm{LS}(\mathbf{K})^+$  a regular cardinal. If **K** is categorical in a  $\theta > \mathrm{LS}(\mathbf{K})$ , then  $\mathbf{K}^{\lambda-\mathrm{sat}}$  is almost fully good, where  $\lambda := (\mathrm{LS}(\mathbf{K})^{<\kappa})^{+5}$ .

PROOF. By Fact 8.6.9, **K** is LS(**K**)-superstable. Let  $\mu := (\text{LS}(\mathbf{K})^{<\kappa})^+$ . By Fact 8.6.14 (with  $\lambda$  there standing for  $\mu$  here), there is an  $\omega$ -successful good  $\mu^+$ -frame with underlying class  $\mathbf{K}_{\mu^+}^{\mu^+\text{-sat}}$ . By Fact 8.6.15 (with  $\lambda$  there standing for  $\mu^+$  here),  $\mathbf{K}^{\mu^{+4}\text{-sat}}$  is almost fully good.

Note for future reference that in almost good AECs, uniqueness triples have an easier definition.

DEFINITION 8.6.17. Let  $\mathbf{i} = (\mathbf{K}, \downarrow)$  be an almost good independence relation. (a, M, N) is a *domination triple* if  $M \leq_{\mathbf{K}} N$ ,  $a \in |N| \setminus |M|$ , and for any  $N' \geq_{\mathbf{K}} N$  and any  $B \subseteq |N'|$ , if  $a \downarrow_{M}^{N'} B$ , then  $N \downarrow_{M}^{N'} B$ .

FACT 8.6.18 (Lemma 6.11.7). Let  $\mathbf{i} = (\mathbf{K}, \downarrow)$  be an almost good independence relation. Let  $\mu \geq \mathrm{LS}(\mathbf{K})$  and let  $\mathbf{s} := \mathrm{pre}(\mathbf{i}^{\leq 1}) \upharpoonright \mathbf{K}_{\mu}$ .

For  $M, N \in \mathbf{K}_{\mu}$ , (a, M, N) is a domination triple if and only it is a uniqueness triple in  $\mathfrak{s}$ .

We continue the proof of Theorem 8.5.12 by showing that the frame induced by an almost good independence relation is  $good^+$ , a technical property of frames:

DEFINITION 8.6.19 (Definition III.1.3 in [She09a]). Let  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \downarrow)$  be a typefull good  $\lambda$ -frame.  $\mathfrak{s}$  is good<sup>+</sup> if the following is *impossible*: There exists increasing continuous chains  $\langle M_i : i \leq \lambda^+ \rangle$ ,  $\langle N_i : i \leq \lambda^+ \rangle$ , a type  $p^* \in \mathrm{gS}(M_0)$ , and a sequence  $\langle a_i : i < \lambda^+ \rangle$  such that for all  $i < \lambda^+$ :

(1)  $M_{\lambda^+}$  is  $\lambda^+$ -saturated.

(2)  $M_i \leq_{\mathbf{K}} N_i$  and they are both in  $\mathbf{K}_{\lambda}$ .

- (3)  $a_{i+1} \in |M_{i+2}|$ .
- (4)  $gtp(a_{i+1}/M_{i+1}; M_{i+2})$  is a nonforking extension of  $p^*$ .
- (5)  $gtp(a_{i+1}/N_0; N_{i+2})$  forks over  $M_0$ .

PROPOSITION 8.6.20. If  $i = (\mathbf{K}, \downarrow)$  is an almost good independence relation, then  $\operatorname{pre}(i^{\leq 1}) \upharpoonright \mathbf{K}_{\operatorname{LS}(\mathbf{K})}$  is good<sup>+</sup>.

PROOF. Suppose  $\langle M_i : i \leq \lambda^+ \rangle$ ,  $\langle N_i : i \leq \lambda^+ \rangle$ ,  $\langle a_i : i < \lambda^+ \rangle$ , and  $p^*$  witness the failure of being good<sup>+</sup>. By local character, there exists  $i < \lambda^+$  such that  $N_0 \underset{M_i}{\bigcup} M_{\lambda^+}$ . By symmetry and monotonicity, we must have that  $a_{i+1} \underset{M_i}{\bigcup} N_0$ , i.e. gtp $(a_{i+1}/N_0; N_{i+2})$  does not fork over  $M_i$ . By transitivity and base monotonicity, gtp $(a_{i+1}/N_0; N_{i+2})$  does not fork over  $M_0$ , contradiction. COROLLARY 8.6.21. Assume  $\mathfrak{i} = (\mathbf{K}, \downarrow)$  is an almost good independence relation. Let  $\lambda > \mathrm{LS}(\mathbf{K})$  and let  $\mathfrak{s} := \mathrm{pre}(\mathfrak{i}^{\leq 1}) \upharpoonright \mathbf{K}^{\lambda \operatorname{-sat}}$ . Then  $\mathfrak{s}$  is  $\omega$ -successful and good<sup>+</sup>.

PROOF. By Fact 8.6.12 and Proposition 8.6.20 (applied to the restriction of i to  $\lambda$ -saturated models).

REMARK 8.6.22. In Section 8.5, we only need a (type-full) successful good<sup>+</sup> frame. Moreover Shelah proves in [She09a, Claim III.1.9] that if  $\mathfrak{s}$  is successful, then the successor frame  $\mathfrak{s}^+$  is good<sup>+</sup>, so why do we bother building an almost good independence relation? The reason is that we want a successful good<sup>+</sup>  $\lambda$ -frame when  $\lambda$  is a *limit* cardinal. Then if **K** is categorical in  $\lambda$  and has primes, the frame will have primes (no need to restrict to saturated models, where it is not clear whether primes exist even if the original class has primes), so the hypotheses of Theorem 8.5.16 will be satisfied

#### 8.7. Frames that are not weakly uni-dimensional

In this section, we give a proof of Fact 8.5.10. We work with the following hypotheses:

Hypothesis 8.7.1.

- (1)  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \downarrow)$  is a type-full successful good<sup>+</sup>  $\lambda$ -frame.
- (2)  $\mathbf{K}_{\lambda}$  has primes.
- (3) **K** is categorical in  $\lambda$ .

We will use the orthogonality calculus developed in [She09a, Chapter III].

DEFINITION 8.7.2 (Definition III.6.2 in [She09a]).

- (1) Let  $M \in \mathbf{K}_{\lambda}$  and let  $p, q \in \mathrm{gS}(M)$  be nonalgebraic. We say that p and q are weakly orthogonal if whenever (a, M, N) is a uniqueness triple with  $\mathrm{gtp}(a/M; N) = q$ , then p has a unique extension to  $\mathrm{gS}(N)$ . We say that p and q are orthogonal, written  $p \perp q$  if for every  $N \geq_{\mathbf{K}} M$ , the nonforking extensions to N p', q' of p and q respectively are weakly orthogonal.
- (2) Let  $M_{\ell} \in \mathbf{K}_{\lambda}$  and  $p_{\ell} \in \mathrm{gS}(M_{\ell})$  be nonalgebraic,  $\ell = 1, 2$ . We say that  $p_1$  and  $p_2$  are *orthogonal* if there exists  $N \geq_{\mathbf{K}} M_{\ell}$  such that the nonforking extensions to  $N p'_1, p'_2$  of  $p_1$  and  $p_2$  respectively are orthogonal.

FACT 8.7.3 (Claims III.6.7, III.6.8 in [She09a]). Let  $M \in \mathbf{K}_{\lambda}$  and  $p, q \in gS(M)$  be nonalgebraic.

- (1) [She09a, Claim III.6.3] p is weakly orthogonal to q if and only if there exists a uniqueness triple (a, M, N) such that gtp(a/M; N) = q and p has a unique extension to gS(N).
- (2) [She09a, Claim III.6.7.2]  $p \perp q$  if and only if  $q \perp p$ .
- (3) [She09a, Claim III.6.8.5] p and q are orthogonal if and only if they are weakly orthogonal.

We will also use the following without comments:

FACT 8.7.4 (Claim III.3.7 in [She09a]). If (a, M, N) is a prime triple, then it is a uniqueness triple.

Some orthogonality calculus gives us a useful description of the types in  $\mathbf{K}_{\neg^* p}$  (recall Definition 8.5.7).

LEMMA 8.7.5. Let  $M \in \mathbf{K}_{\lambda}$  and let  $p \in \mathrm{gS}(M)$  be nonalgebraic. Let  $N \in \mathbf{K}_{\neg^* p}$ be of size  $\lambda$  such that the map  $a \mapsto c_a^N$  is the identity (so  $M \leq_{\mathbf{K}} N \upharpoonright L(\mathbf{K})$ ). For any  $N_0 \leq_{\mathbf{K}} N \upharpoonright L(\mathbf{K})$  with  $M \leq_{\mathbf{K}} N_0$  and any  $q \in \mathrm{gS}(N_0; N), p \perp q$ .

PROOF. Let p' be the nonforking extension of p to  $N_0$ . By Fact 8.7.3, it is enough to show that p' is weakly orthogonal to q. Let  $(a, N_0, N')$  be a prime triple such that  $gtp(a/N_0; N') = q$  and  $N' \leq_{\mathbf{K}} N$  (exists since we are assuming that  $\mathbf{K}_{\lambda}$ has primes). Then since p has a unique extension to N it has a unique extension to N', which must be the nonforking extension so p' also has a unique extension to N'. By Fact 8.7.4,  $(a, N_0, N')$  is a uniqueness triple and by Fact 8.7.3 again, this suffices to conclude that p' and q are weakly orthogonal.

The next lemma justifies the "uni-dimensional" terminology: if the class is *not* uni-dimensional, then there are two orthogonal types.

LEMMA 8.7.6. If  $\mathbf{K}_{\lambda}$  is not weakly uni-dimensional, there exists  $M \in \mathbf{K}_{\lambda}$  and types  $p, q \in gS(M)$  such that  $p \perp q$ .

PROOF. Assume  $\mathbf{K}_{\lambda}$  is not weakly uni-dimensional. This means that there exists  $M <_{\mathbf{K}} M_{\ell}, \ell = 1, 2, \text{ all in } \mathbf{K}_{\lambda}$  such that for any  $c \in |M_2| \setminus |M|$ ,  $\operatorname{gtp}(c/M; M_2)$  has a unique extension to  $\operatorname{gS}(M_1)$ . Pick any  $c \in |M_2| \setminus |M|$  and let  $p := \operatorname{gtp}(c/M; M_2)$ . Then there is a natural expansion of  $M_1$  to  $\mathbf{K}_{\neg^* p}$ . So pick any  $d \in |M_1| \setminus |M|$  and let  $q := \operatorname{gtp}(d/M; M_1)$ . By Lemma 8.7.5,  $p \perp q$ , as desired.

We can now prove Fact 8.5.10. We restate it here for convenience:

FACT 8.7.7. If  $\mathbf{K}_{\lambda}$  is not weakly uni-dimensional, then there exists  $M \in \mathbf{K}_{\lambda}$ and  $p \in \mathrm{gS}(M)$  such that  $\mathfrak{s} \upharpoonright \mathbf{K}_{\neg^* p}$  (the restriction of  $\mathfrak{s}$  to the models in  $\mathbf{K}_{\neg^* p}$ ) is a type-full good  $\lambda$ -frame.

PROOF. Assume  $\mathbf{K}_{\lambda}$  is not weakly uni-dimensional. By Lemma 8.7.6, there exists  $M \in \mathbf{K}_{\lambda}$  and types  $p, q \in \mathrm{gS}(M)$  such that  $p \perp q$ .

Let  $\mathfrak{s}_{\neg^* p} := \mathfrak{s} \upharpoonright \mathbf{K}_{\neg^* p}$ . We check that it is a type-full good  $\lambda$ -frame. For ease of notation, we identify a model  $N \in \mathbf{K}_{\neg^* p}$  and its reduct to  $\mathbf{K}$ . For  $N \geq_{\mathbf{K}} M$ , we write  $p_N$  for the nonforking extension of p to gS(N), and similarly for  $q_N$ .

- $\mathbf{K}_{\neg *p}$  is not empty, since (the natural expansion of) M is in it.
- $(\mathbf{K}_{\neg^* p})_{\lambda}$  is an AEC in  $\lambda$  (that is, its models of size  $\lambda$  behave like an AEC, see [She09a, Definition II.1.18]) by the proof of Proposition 8.5.9.
- Forking has many of the usual properties: monotonicity, invariance, disjointness, local character, continuity, and transitivity all trivially follow from the definition of K<sub>¬\*p</sub>.
- Forking has the uniqueness property: Let  $N \in \mathbf{K}_{\neg^* p}$  have size  $\lambda$ . Without loss of generality  $M \leq_{\mathbf{K}} N$ . Let  $N' \geq_{\mathbf{K}} N$  be in  $\mathbf{K}_{\neg p}$  of size  $\lambda$  and let  $r_1, r_2 \in \mathrm{gS}(N')$  be nonforking over N and such that  $r_1 \upharpoonright N = r_2 \upharpoonright N$ . Say  $r_{\ell} = \mathrm{gtp}(a_{\ell}/N'; N_{\ell})$ . Now in  $\mathbf{K}$ ,  $r_1 = r_2$ , and since  $\mathbf{K}_{\lambda}$  has primes, the equality is witnessed by an embedding  $f : M_1 \xrightarrow{N} N_2$ , with  $M_1 \leq_{\mathbf{K}} N_1$ .

Since  $N_1 \in \mathbf{K}_{\neg p}$ ,  $M_1 \in \mathbf{K}_{\neg p}$ , and so  $r_1 = r_2$  also in  $\mathbf{K}_{\neg p}$  (this is similar to the proof of Proposition 8.5.14).

• Forking has the extension property. Let  $N \in \mathbf{K}_{\neg^* p}$  have size  $\lambda$ . Without loss of generality,  $M \leq_{\mathbf{K}} N$ . Let  $r \in \mathrm{gS}(N)$  be nonalgebraic and let  $N' \geq_{\mathbf{K}} N$  be in  $\mathbf{K}_{\neg^* p}$  of size  $\lambda$ . Let  $r' \in \mathrm{gS}(N')$  be the nonforking extension of r to N' (in  $\mathbf{K}$ ). Let (a, N', N'') be a prime triple such that  $\operatorname{gtp}(a/N'; N'') = r'$ . By Lemma 8.7.5,  $r \perp p$ . Thus r' is weakly orthogonal to  $p_{N'}$  and hence  $p_{N''}$  is the unique extension of  $p_{N'}$  to N''. Now if p' is an extension of p to  $\operatorname{gS}(N'')$ , then  $p' \upharpoonright N' = p_{N'}$  as  $N' \in \mathbf{K}_{\neg^* p}$ , so  $p' = p_{N''}$  by the previous sentence. This shows that  $N'' \in \mathbf{K}_{\neg^* p}$ , so as  $r' \in \operatorname{gS}(N'; N'')$ , r' is a Galois type in  $\mathbf{K}_{\neg^* p}$ , as desired.

- $\mathbf{K}_{\neg^* p}$  has  $\lambda$ -amalgamation: because  $(\mathbf{K}_{\neg^* p})_{\lambda}$  has the type extension property and weak  $\lambda$ -amalgamation (as  $\mathbf{K}_{\lambda}$ , and hence  $(\mathbf{K}_{\neg^* p})_{\lambda}$ , has primes, see the proof of Proposition 8.5.14), thus one can apply Theorem 8.4.14.
- K<sub>¬\*p</sub> has λ-joint embedding: since any model contains a copy of M, this is a consequence of λ-amalgamation over M.
- K<sub>¬\*p</sub> is stable in λ: because K<sub>¬\*p</sub> has "fewer" Galois types than K, and K is stable in λ.
- $(\mathbf{K}_{\neg^* p})_{\lambda}$  has no maximal models: This is where we use the negation of weakly uni-dimensional. Let  $N \in \mathbf{K}_{\neg^* p}$  be of size  $\lambda$  and without loss of generality assume  $M \leq_{\mathbf{K}} N$ . Recall from above that there is a nonalgebraic type  $q \in \mathrm{gS}(M)$  such that  $p \perp q$ . Let  $q_N$  be the nonforking extension of q to N and let (a, N, N') be a prime triple such that  $q = \mathrm{gtp}(a/N; N')$ . As in the proof of the extension property,  $N' \in \mathbf{K}_{\neg^* p}$ . Moreover as  $a \in |N'| \setminus |N|$ ,  $N <_{\mathbf{K}} N'$ , as needed.
- $\mathfrak{s}_{\neg^* p}$  is type-full: because  $\mathfrak{s}$  is.
- $\mathfrak{s}_{\neg^*p}$  has full symmetry: Assume  $a \stackrel{N}{\downarrow} N_1$ , for  $N_0, N_1, N \in \mathbf{K}_{\neg^*p}, M \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N_1 \leq_{\mathbf{K}} N$ , and  $a \in |N|$ . Let  $b \in |N_1|$ . Without loss of generality,  $a \notin |N_1|$  (if  $a \in |N_1|$ , then  $a \in |N_0|$  by disjointness and as  $b \stackrel{N}{\downarrow} N_0, N_0$ and N witness the full symmetry). By full symmetry in  $\mathfrak{s}$ , there exists  $N'_0, N' \in \mathbf{K}$  such that  $N \leq_{\mathbf{K}} N', N_0 \leq_{\mathbf{K}} N'_0 \leq_{\mathbf{K}} N'$ , and  $b \stackrel{N}{\downarrow} N'_0$  (note that the first use of  $\downarrow$  was in  $\mathfrak{s}_{\neg^*p}$  and the second in  $\mathfrak{s}$ , but since the first is just the restriction of the first to models in  $\mathbf{K}_{\neg^*p}$ , we do not make the difference). Now let  $N''_0$  be such that  $N_0 \leq_{\mathbf{K}} N''_0 \leq_{\mathbf{K}} N'_0$  and  $(a, N_0, N''_0)$  is a prime triple. Since  $r = \operatorname{gtp}(a/N_0; N''_0) = \operatorname{gtp}(a/N_0; N)$  is orthogonal to p (by Lemma 8.7.5), we have that  $N''_0 \in \mathbf{K}_{\neg^*p}$ . By monotonicity,  $b \stackrel{N'}{\downarrow} N''_0$ . Now let  $(b, N''_0, N'')$  be a prime triple with  $N'' \leq_{\mathbf{K}} N'$ . By the same argument as before,  $N'' \in \mathbf{K}_{\neg^*p}$  and by monotonicity,  $b \stackrel{N''}{\downarrow} N''_0$ . Since all the models are in  $\mathbf{K}_{\neg^*p}$ , this shows that the nonforking happens in  $\mathfrak{s}_{\neg^*p}$ ,

We have checked all the properties and therefore  $\mathfrak{s}_{\neg^* p}$  is a type-full good  $\lambda$ -frame.

#### 8.8. Independence in universal classes

We investigate the properties of independence in universal classes (more generally in AECs admitting intersections). Recall that Theorem 6.15.6 showed that a fully tame and short AEC with amalgamation categorical in unboundedly many cardinals eventually admits a well-behaved independence notion. We want to specialize this result to AECs admitting intersections and prove more properties of forking there. Here, we prove that the independence relation satisfies the axioms of Chapter 3 (partially answering Question 3.7.1 there). Moreover it has a finite character property (Theorem 8.8.7) and can be extended to an independence relation over sets (Theorem 8.8.14). A simple corollary is the disjoint amalgamation property (Corollary 8.8.6).

While none of the results are used in this chapter, we believe they shed further light on how the existence a closure operator helps in the structural analysis of an AEC. Since several classes of interests to algebraists admit intersections, we believe the existence of a well-behaved independence notion there is likely to have further applications.

By Fact 8.6.3 or Corollary 8.6.16, it is reasonable to assume:

Hypothesis 8.8.1.

- (1) K locally admits intersections.
- (2)  $\mathbf{i} = (\mathbf{K}, \downarrow)$  is an almost fully good independence relation (see Definition 8.6.1).

Our goal is to prove that i is actually fully good, i.e. extension holds. Note that if we knew that  $\mathbf{K}$  was categorical above the Löwenheim-Skolem-Tarski number, we could use the categoricity transfer of Section 8.5. However here we do not make any categoricity assumption and our approach is easier: we study how the closure operator interacts with independence. The key lemma is:

LEMMA 8.8.2. If 
$$A \stackrel{N}{\underset{M_0}{\downarrow}} B$$
, then  $\operatorname{cl}^N(A) \stackrel{N}{\underset{M_0}{\downarrow}} \operatorname{cl}^N(B)$ .

PROOF. By normality, without loss of generality  $|M_0| \subseteq A, B$ . Using symmetry, it is enough to show that  $A \underset{M_0}{\downarrow} \operatorname{cl}^N(B)$ . By the witness property and finite character of the closure operator, we can assume without loss of generality that  $|A| \leq \operatorname{LS}(\mathbf{K})$ . Therefore by extension there exists  $N' \geq_{\mathbf{K}} N$  and  $M \geq_{\mathbf{K}} M_0$  such that  $M \leq_{\mathbf{K}} N'$ , M contains B, and  $A \underset{M_0}{\downarrow} M$ . By definition,  $\operatorname{cl}^N(B) = \operatorname{cl}^{N'}(B)$  is contained in M, so  $A \underset{M_0}{\downarrow} \operatorname{cl}^N(B)$ , so  $A \underset{M_0}{\downarrow} \operatorname{cl}^N(B)$ .

An abstract way of stating Lemma 8.8.2 is via domination triples (recall Definition 8.6.17).

LEMMA 8.8.3. Let  $M \leq_{\mathbf{K}} N$  and let  $a \in |N| \setminus |M|$ . Then  $(a, M, cl^N(\{a\} \cup |M|))$  is a domination triple.

PROOF. Directly from Lemma 8.8.2.

In our framework, domination triples are the same as the uniqueness triples of [She09a, Definition II.5.3] by Fact 8.6.18, thus we get:

THEOREM 8.8.4. i has extension. Hence it is a fully good independence relation.

PROOF. Let  $\mu \geq LS(\mathbf{K})$  and let  $\mathfrak{s} := \operatorname{pre}(\mathfrak{i}^{\leq 1} \upharpoonright \mathbf{K}_{\mu})$ . By Lemma 8.8.3 and Fact 8.6.18  $\mathfrak{s}$  has the so-called existence property for uniqueness triples (see [She09a, Definition II.5.3]). By Section II.6 of [She09a] (and the results of Section 6.12)  $\mathfrak{s}$  induces an independence relation  $\mathfrak{i}'$  for types of length at most  $\mu$  over models of size  $\mu$  that is well-behaved (i.e. it has all of the properties of a fully good independence relation except full model continuity and disjointness). By the canonicity of such relations (see the proofs of Corollary 3.5.18 and Theorem 3.6.12),  $\mathfrak{i}'$  must be the same as  $\mathfrak{i}^{\leq \mu} \upharpoonright \mathbf{K}_{\mu}$ , the restriction of  $\mathfrak{i}$  to size  $\mu$ . Thus for all  $\mu \geq LS(\mathbf{K})$ ,  $\mathfrak{i}$  has extension for types of length at most  $\mu$  over models of size  $\mu$ . By the proof of Lemma 6.14.13, this suffices to conclude that  $\mathfrak{i}$  has extension.

REMARK 8.8.5. The proof shows that instead of the AEC admitting intersections, it is enough to assume that for each  $\mu$ , the restriction of i to a good frame in  $\mu$  has the existence property for uniqueness triples. Unfortunately the proof in Section 6.11 only works when the frame is restricted to the saturated models of size  $\mu$ .

COROLLARY 8.8.6. K has disjoint amalgamation.

PROOF. Because i has existence, extension and disjointness.  $\Box$ 

Another consequence of having a closure operator is:

THEOREM 8.8.7 (Finite character of independence).  $A \stackrel{N}{\underset{M_0}{\downarrow}} B$  if and only if for all finite  $A_0 \subseteq A$  and  $B_0 \subseteq B$ ,  $A_0 \stackrel{N}{\underset{M_0}{\downarrow}} B_0$ . That is, i has the  $(<\aleph_0)$ -witness property.

PROOF. By symmetry it is enough to show that if  $A_0 \stackrel{N}{\underset{M_0}{\downarrow}} B$  for all finite  $A_0 \subseteq A$ , then  $A \stackrel{N}{\underset{M_0}{\downarrow}} B$ . For each finite  $A_0 \subseteq A$ , let  $M_{A_0} := \operatorname{cl}^N(|M_0| \cup A_0)$ . Let  $M := \operatorname{cl}^N(|M_0| \cup B)$ . By Lemma 8.8.2,  $M_{A_0} \stackrel{N}{\underset{M_0}{\downarrow}} M$  for each finite  $A_0 \subseteq A$ . Let  $M_A := \operatorname{cl}^N(|M_0| \cup A)$ . It is easy to see that  $\langle M_{A_0} \mid A_0 \in [A]^{<\aleph_0} \rangle$  is a directed system with union  $M_A$ . Therefore by full model continuity,  $M_A \stackrel{N}{\underset{M_0}{\downarrow}} M$ , and so  $A \stackrel{N}{\underset{M_0}{\downarrow}} B$ .  $\Box$ 

REMARK 8.8.8. One can check that  $(\mathbf{K}, \leq_{\mathbf{K}}, \downarrow, \mathrm{cl})$  satisfies the axiomatic framework AxFri<sub>1</sub> from [She09b, Chapter V.B].

For the next two results, we drop our hypotheses.

THEOREM 8.8.9. Let  $\mathbf{K}$  be a fully (<  $LS(\mathbf{K})$ )-tame and short AEC with amalgamation. Assume further that  $\mathbf{K}$  locally admits intersections.

If **K** is categorical in a  $\mu \ge h(LS(\mathbf{K}))$ , then there exists  $\lambda < h(LS(\mathbf{K}))$  such that  $\mathbf{K}_{\ge\lambda}$  is fully good. Moreover the independence relation has the  $(<\aleph_0)$ -witness property.

PROOF. Combine Corollary 8.6.16, Theorem 8.8.4, and Theorem 8.8.7.  $\Box$ 

REMARK 8.8.10. If **K** is not categorical but only superstable (see Definition 8.6.6), then we can generalize the result (using Theorem 6.15.1) provided that for all  $\lambda$ ,  $\mathbf{K}^{\lambda-\text{sat}}$  (the class of  $\lambda$ -saturated models in **K**) locally admits intersections.

**8.8.1. Set bases.** We end by showing that it is possible to extend the independence relation to define forking not only over models but also over sets. In the terminology of [**HL02**], **K** is simple (note that the paper gives an example due to Shelah of a class that has a fully good independence relation, yet is not simple).

For our arguments to work, we have to assume that **K** admits intersections, i.e. not just locally. To see that this is not a big loss, recall that if **K** is categorical in unboundedly many cardinals and has amalgamation, then the models in the categoricity cardinals are saturated, so for  $M \in LS(\mathbf{K})$ ,  $\mathbf{K}_M$  will also be categorical in unboundedly many cardinals.

Hypothesis 8.8.11. K admits intersections.

DEFINITION 8.8.12. Let  $N \in \mathbf{K}$  and  $A, B, C \subseteq |N|$ . Define  $B \stackrel{N}{\downarrow} C$  to hold if

and only if  $\operatorname{cl}^N(AB) \underset{\operatorname{cl}^N(A)}{\overset{N}{\downarrow}} \operatorname{cl}^N(AC)$ .

We define properties such as invariance, monotonicity, etc. just as for the modelbased version of independence.

REMARK 8.8.13. When  $A \leq_{\mathbf{K}} N$ , this agrees with the previous definition of independence.

Theorem 8.8.14.

- (1)  $\downarrow$  has invariance, left and right monotonicity, base monotonicity, and normality.
- (2)  $\downarrow$  has symmetry, finite character (i.e. the ( $< \aleph_0$ )-witness property), existence and transitivity.
- (3)  $\downarrow$  has extension.
- (4) Let  $N \in \mathbf{K}$  and let  $\langle B_i : i < \delta \rangle$  be an increasing chain of sets. Let  $B_{\delta} := \bigcup_{i < \delta} B_i$  and assume  $B_{\delta} \subseteq |N|$ . Let  $p \in \mathrm{gS}^{\alpha}(B; N)$ . If  $\mathrm{cf} \, \delta > \alpha$ , then there exists  $i < \delta$  such that p does not fork over  $B_i$ .
- (5) If  $p \in gS^{\alpha}(B; N)$ , there exists  $A \subseteq B$  such that p does not fork over A and  $|A| < |\alpha|^+ + \aleph_0$ .

Proof.

- (1) Easy.
- (2) Easy.
- (3) By transitivity and extension of i.
- (4) By local character for i.
- (5) By finite character, it is enough to show it when  $\alpha < \omega$ . Work by induction on  $\lambda := |B|$ . If  $\lambda < \aleph_0$ , take A = B and use the existence property. If  $\lambda \ge \aleph_0$ , write  $B = \bigcup_{i < \lambda} B_i$ , where  $|B_i| < \lambda$  for all  $i < \lambda$ . By the previous result, there exists  $i < \lambda$  such that p does not fork over  $B_i$ . Now apply the induction hypothesis and transitivity.

REMARK 8.8.15. Thus in this framework types of finite length really do not fork over a finite set. This removes the need for a special chain version of local character (i.e. if  $\langle M_i : i \leq \delta \rangle$  is increasing continuous,  $p^{<\omega} \in gS(M_{\delta})$ , there exists  $i < \delta$  such that p does not fork over  $M_i$ ).

# CHAPTER 9

# Equivalent definitions of superstability in tame abstract elementary classes

This chapter is based on  $[\mathbf{GV}]$  and is joint work with Rami Grossberg. We thank Will Boney and a referee for feedback that helped improve the presentation of the chapter.

#### Abstract

In the context of abstract elementary classes (AECs) with a monster model, several possible definitions of superstability have appeared in the literature. Among them are no long splitting chains, uniqueness of limit models, and solvability. Under the assumption that the class is tame and stable, we show that (asymptotically) no long splitting chains implies solvability and uniqueness of limit models implies no long splitting chains. Using known implications, we can then conclude that all the previously-mentioned definitions (and more) are equivalent:

COROLLARY 9.0.16. Let  $\mathbf{K}$  be a tame AEC with a monster model. Assume that  $\mathbf{K}$  is stable in a proper class of cardinals. The following are equivalent:

- (1) For all high-enough  $\lambda$ , **K** has no long splitting chains.
- (2) For all high-enough  $\lambda$ , there exists a good  $\lambda$ -frame on a skeleton of  $\mathbf{K}_{\lambda}$ .
- (3) For all high-enough  $\lambda$ , **K** has a unique limit model of cardinality  $\lambda$ .
- (4) For all high-enough  $\lambda$ , **K** has a superlimit model of cardinality  $\lambda$ .
- (5) For all high-enough  $\lambda$ , the union of any increasing chain of  $\lambda$ -saturated models is  $\lambda$ -saturated.
- (6) There exists  $\mu$  such that for all high-enough  $\lambda$ , **K** is  $(\lambda, \mu)$ -solvable.

This gives evidence that there is a clear notion of superstability in the framework of tame AECs with a monster model.

# 9.1. Introduction

In the context of classification theory for abstract elementary classes (AECs), a notion analogous to the first-order notion of *stability* exists: let us say that an AEC **K** is *stable in*  $\lambda$  if **K** has at most  $\lambda$ -many Galois types over every model of cardinality  $\lambda$  (a justification for this definition is Fact 2.4.15, showing that it is equivalent, under tameness, to failure of the order property). However it has been unclear what a parallel notion to superstability might be. Recall that for first-order theories we have:

FACT 9.1.1. Let T be a first-order complete theory. The following are equivalent:

(1) T is stable in every cardinal  $\lambda \geq 2^{|T|}$ .

- (2) For all infinite cardinals λ, the union of an increasing chain of λ-saturated models is λ-saturated.
- (3)  $\kappa(T) = \aleph_0$  and T is stable.
- (4) T has a saturated model of cardinality  $\lambda$  for every  $\lambda \geq 2^{|T|}$ .
- (5) T is stable and  $D^n[\bar{x} = \bar{x}, L(T), \infty] < \infty$ .
- (6) There does not exists a set of formulas  $\Phi = \{\phi_n(\bar{x}; \bar{y}_n) \mid n < \omega\}$  such that  $\Phi$  can be used to code the structure  $(\omega^{\leq \omega}, <, <_{lex})$

(1)  $\implies$  (2) and (1)  $\iff$  ( $\ell$ ) for  $\ell \in \{3, 4, 5, 6\}$  all appear in Shelah's book [She90]. Albert and Grossberg [AG90, 13.2] established (2)  $\implies$  (6).

In the last 30 years, in the context of classification theory for non elementary classes, several notions that generalize that of first-order superstability have been considered. See papers by Grossberg, Shelah, VanDieren and Villaveces: [GS86a, Gro88], [She99], [SV99], [Van06, Van13], [GVV16], and Chapters 4 and 6 of this thesis. Reasons for developping a superstability theory in the non-elementary setup include the aesthetic appeal (guided by motivation from the first-order case) and recent applications such as Shelah's eventual categoricity conjecture in universal classes, Chapters 8 and 16 or the fact that (in an AEC with a monster model) the model in a categoricity cardinal is saturated (Chapter 17).

In [She99, p. 267] Shelah states that part of the program of classification theory for AECs is to show that all the various notions of first-order saturation (limit, superlimit, or model-homogeneous, see Section 9.2.2) are equivalent under the assumption of superstability. A possible definition of superstability is *solvability* (see Definition 9.2.14), which appears in the introduction to [She09a] and is hailed as a true counterpart to first-order superstability. Full justification is delayed to [Sheb] but [She09a, Chapter IV] already uses it. Other definitions of superstability analogous to the ones in Fact 9.1.1 can also be formulated. The main result of this chapter is to show that, at least in tame AECs with a monster model, several definitions of superstability that previously appeared in the literature are equivalent (see the preliminaries for precise definitions of some of the concepts appearing below). Many of the implications have already been proven in earlier papers, but here we complete the loop by proving two theorems. Before stating them, some notation will be helpful:

NOTATION 9.1.2 (4.24(5) in [Bal09]). Given a fixed AEC K, set  $H_1 := \beth_{(2^{LS}(K))^+}$ .

THEOREM 9.3.18. Let **K** be an LS(**K**)-tame AEC with a monster model. There exists  $\chi < H_1$  such that for any  $\mu \geq \chi$ , if **K** is stable in  $\mu$ , there is a saturated model of cardinality  $\mu$ , and every limit model of cardinality  $\mu$  is  $\chi$ -saturated, then **K** has no long splitting chains in  $\mu$ .

THEOREM 9.4.9. Let **K** be an LS(**K**)-tame AEC with a monster model. There exists  $\chi < H_1$  such that for any  $\mu \ge \chi$ , if **K** is stable in  $\mu$  and has no long splitting chains in  $\mu$  then **K** is uniformly  $(\mu', \mu')$ -solvable, where  $\mu' := (\beth_{\omega+2}(\mu))^+$ .

These two theorems prove (3) implies (1) and (1) implies (6) of our main corollary, proven in detail after the proof of Corollary 9.5.5.

COROLLARY 9.1.3 (Main Corollary). Let  $\mathbf{K}$  be a LS( $\mathbf{K}$ )-tame AEC with a monster model. Assume that  $\mathbf{K}$  is stable in some cardinal greater than or equal to LS( $\mathbf{K}$ ). The following are equivalent:

#### 9.1. INTRODUCTION

- (1) There exists  $\mu_1 < H_1$  such that for every  $\lambda \ge \mu_1$ , **K** has no long splitting chains in  $\lambda$ .
- (2) There exists  $\mu_2 < H_1$  such that for every  $\lambda \ge \mu_2$ , there is a good  $\lambda$ -frame on a skeleton of  $\mathbf{K}_{\lambda}$  (see Section 9.2.3).
- (3) There exists  $\mu_3 < H_1$  such that for every  $\lambda \ge \mu_3$ , **K** has a unique limit model of cardinality  $\lambda$ .
- (4) There exists  $\mu_4 < H_1$  such that for every  $\lambda \ge \mu_4$ , **K** has a superlimit model of cardinality  $\lambda$ .
- (5) There exists  $\mu_5 < H_1$  such that for every  $\lambda \ge \mu_5$ , the union of any increasing chain of  $\lambda$ -saturated models is  $\lambda$ -saturated.
- (6) There exists  $\mu_6 < H_1$  such that for every  $\lambda \ge \mu_6$ , **K** is  $(\lambda, \mu_6)$ -solvable.
- Moreover any of the above conditions also imply:
- (7) There exists  $\mu_7 < H_1$  such that for every  $\lambda \ge \mu_7$ , **K** is stable in  $\lambda$ .

REMARK 9.1.4. The main corollary has a global assumption of stability (in some cardinal). While stability is implied by some of the equivalent conditions (e.g. by (2) or (6)) other conditions may be vacuously true if stability fails (e.g. (1)). Thus in order to simplify the exposition we just require stability outright.

REMARK 9.1.5. In the context of the main corollary, if  $\mu_1 \geq LS(\mathbf{K})$  is such that **K** is stable in  $\mu_1$  and has no long splitting chains in  $\mu_1$ , then for any  $\lambda \geq \mu_1$ , **K** is stable in  $\lambda$  and has no long splitting chains in  $\lambda$  (see Fact 9.2.3). In other words, superstability defined in terms of no long splitting chains transfers up.

REMARK 9.1.6. In (3), one can also require the following strong version of uniqueness of limit models: if  $M_0, M_1, M_2 \in \mathbf{K}_{\lambda}$  and both  $M_1$  and  $M_2$  are limit over  $M_0$ , then  $M_1 \cong_{M_0} M_2$  (i.e. the isomorphism fixes the base). This is implied by (2): see Fact 9.2.11.

REMARK 9.1.7. At the time this chapter was first circulated (July 2015), we did not know whether (7) implied the other conditions. This has now been proven and appears in Chapter 19.

Note that in Corollary 9.1.3, we can let  $\mu$  be the maximum of the  $\mu_{\ell}$ 's and then each property will hold above  $\mu$ . Interestingly however, the proof of Corollary 9.1.3 does *not* tell us that the *least* cardinals  $\mu_{\ell}$  where the corresponding properties holds are all equal. In fact, it uses tameness heavily to move from one cardinal to the next and uses e.g. that one equivalent definition holds below  $\lambda$  to prove that another definition holds at  $\lambda$ . Showing equivalence of these definitions cardinal by cardinals, or even just showing that the least cardinals where the properties hold are all equal seems much harder. We also show that we can ask only for each property to hold in *a single high-enough* cardinals below  $H_1$  (but again the least such cardinal may not be the same for each property, see Corollary 9.5.5). In general, we suspect that the problem of computing the minimal value of the cardinals  $\mu_{\ell}$  could play a role similar to the computation of the first stability cardinal for a first-order theory (which led to the development of forking, see e.g. the introduction of [**GIL02**]).

We discuss earlier work. Shelah [She09a, Chapter II] introduced good  $\lambda$ -frames (a local axiomatization of first-order forking in a superstable theory, see more in Section 9.2.4) and attempts to develop a theory of superstability in this context. He proves for example the uniqueness of limit models (see Fact 9.2.11, so (2) implies (3) in the main theorem is due to Shelah) and (with strong assumptions,

see below) the fact that the union of a chain (of length strictly less than  $\lambda^{++}$ ) of saturated models of cardinality  $\lambda^+$  is saturated [She09a, II.8]. From this he deduces the existence of a good  $\lambda^+$ -frame on the class of  $\lambda^+$ -saturated models of **K** and goes on to develop a theory of prime models, regular types, independent sequences, etc. in [She09a, Chapter III]. The main issue with Shelah's work is that it does not make any global model-theoretic hypotheses (such as tameness or even just amalgamation) and hence often relies on set-theoretic assumptions as well as strong local model-theoretic hypotheses (few models in several cardinals). For example, Shelah's construction of a good frame in the local setup [She09a, II.3.7] uses categoricity in two successive cardinals, few models in the next, as well as several diamond-like principles.

By making more global hypotheses, building a good frame becomes easier and can be done in ZFC (see Chapter 4 or [She09a, Chapter IV]). Recently, assuming a monster model and tameness (a locality property of types introduced by VanDieren and Grossberg, see Definition 2.2.23), progress have been made in the study of superstability defined in terms of no long splitting chains. Specifically, Theorem 4.5.6 proved (1) implies (7). Partial progress in showing (1) implies (2) is made in Chapters 4 and 6 but the missing piece of the puzzle, that (1) implies (5), is proven in Chapter 7. From these results, it can be deduced that (1) implies (2)-(5)(see Theorem 7.6.1). Shelah has shown that (2) implies (3), see Fact 9.2.11. Some implications between variants of (3), (4) and (5) are also straightforward (see Fact 9.2.5), though one has to be careful about where the class is stable (the existence of a limit model of cardinality  $\lambda$  implies stability in  $\lambda$ , but not the fact that the union of a chain of  $\lambda$ -saturated models is  $\lambda$ -saturated). In the proof of Corollary 9.5.5, we end up using a single technical condition,  $(3^*)$ , asserting that limit models have a certain degree of saturation. It is quite easy to see that (3), (4), and (5) all imply  $(3^*)$ . Finally, (6) directly implies (4) from its definition (see Section 9.2.5).

Thus as noted before the main contributions of this chapter are (3) (or really (3<sup>\*</sup>)) implies (1) and (1) implies (6). In Theorem 9.5.4 it is shown that, assuming a monster model and tameness, solvability in *some* high-enough cardinal implies solvability in *all* high-enough cardinals. Note that Shelah asks (inspired by the analogous question for categoricity) in [She09a, Question N.4.4] what the solvability spectrum can be (in an arbitrary AEC). Theorem 9.5.4 provides a partial answer under the additional assumptions of a monster model and tameness. The proof notices that a powerful results of Shelah and Villaveces [SV99] (deriving no long splitting chains from categoricity) can be adapted to our setup (see Fact 9.5.1 and Corollary 9.5.2). Shelah also asks [She09a, Question N.4.5] about the super-limit spectrum. In our context, we can show that if there is a high-enough *stability* cardinal  $\lambda$  with a superlimit model, then **K** has a superlimit on a tail of cardinals (see Corollary 9.5.5). We do not know if the hypothesis that  $\lambda$  is a stability cardinal is needed (see Question 9.5.7).

Since this chapter was first circulated (July 2015), several related results have been proven. VanDieren [Van16a, Van16b] gives some relationships between versions of (3) and (5) in a single cardinal (with (1) as a background assumption). This is done without assuming tameness, using very different technologies than in this chapter. This work is applied to the tame context in Chapter 10, showing for example that (1) implies (3) holds cardinal by cardinal. Chapter 19 studies the model theory of strictly stable tame AECs, establishing in particular that stability on a tail implies no long splitting chains (see Remark 9.1.7).

We do not know how to prove analogs to the last two properties of Fact 9.1.1. Note also that, while the analogous result is known for stability (see Fact 2.4.15), we do not know whether no long splitting chains should hold below the Hanf number:

QUESTION 9.1.8. Let **K** be a LS(**K**)-tame AEC with a monster model. Assume that there exists  $\mu \geq \text{LS}(\mathbf{K})$  such that **K** is stable in  $\mu$  and has no long splitting chains in  $\mu$ . Is the least such  $\mu$  below  $H_1$ ?

The background required to read this chapter is a solid knowledge of AECs (for example Chapters 4-12 of Baldwin's book [**Bal09**] or the upcoming [**Gro**]). We rely on the first ten sections of Chapter 6, as well as on the material in Chapters 2 and 7.

At the beginning of Sections 9.3 and 9.4, we make *global* hypotheses that hold until the end of the section (unless said otherwise). This is to make the statements of several technical lemmas more readable. We will repeat the global hypotheses in the statement of major theorems.

#### 9.2. Preliminaries

We assume familiarity with a basic text on AECs such as [Bal09] or [Gro] and refer the reader to the preliminaries of Chapter 2 for more details and motivations on the concepts used in this chapter.

We use **K** (boldface) to denote a class of models together with an ordering (written  $\leq_{\mathbf{K}}$ ). We will often abuse notation and write for example  $M \in \mathbf{K}$ . When it becomes necessary to consider only a class of models without an ordering, we will write K (no boldface).

Throughout all this chapter, **K** is a fixed AEC. Most of the time, **K** will have amalgamation, joint embedding, and arbitrarily large models. In this case we say that **K** has a monster model. We also often assume that **K** is  $LS(\mathbf{K})$ -tame (this means that Galois types are determined by their restrictions of size  $LS(\mathbf{K})$ , see Definition 2.2.23). Note that if **K** is  $\chi$ -tame for  $\chi > LS(\mathbf{K})$ , the class  $\mathbf{K}' := \mathbf{K}_{\geq \chi}$ will be an  $LS(\mathbf{K}')$ -tame AEC. Thus we might as well directly assume that **K** is  $LS(\mathbf{K})$ -tame.

9.2.1. Superstability and no long splitting chains. A definition of superstability analogous to  $\kappa(T) = \aleph_0$  in first-order model theory has been studied in AECs (see [SV99, GVV16, Van06, Van13] and Chapters 4 and 6). Since it is not immediately obvious what forking should be in that framework, the more rudimentary independence relation of  $\lambda$ -splitting is used in the definition. Since in AECs, types over models are much better behaved than types over sets, it does not make sense in general to ask for every type to not split over a finite set<sup>1</sup>. Thus we require that every type over the union of a chain does not split over a model in the chain. For technical reasons (it is possible to prove that the condition follows from categoricity), we require the chain to be increasing with respect to universal extension. Definition 9.2.1 rephrases (1) in Corollary 9.1.3:

 $<sup>^{1}\</sup>mathrm{But}$  see Theorem 8.8.14 where a notion of forking over set is constructed from categoricity in a universal class.

DEFINITION 9.2.1. Let  $\lambda \geq \mathrm{LS}(K)$ . We say **K** has no long splitting chains in  $\lambda$  if for any limit  $\delta < \lambda^+$ , any increasing  $\langle M_i : i < \delta \rangle$  in  $\mathbf{K}_{\lambda}$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \delta$ , any  $p \in \mathrm{gS}(\bigcup_{i < \delta} M_i)$ , there exists  $i < \delta$  such that p does not  $\lambda$ -split over  $M_i$ .

REMARK 9.2.2. The condition in Definition 9.2.1 first appears in [She99, Question 6.1]. In [Bal09, 15.1], it is written as<sup>2</sup>  $\kappa(\mathbf{K}, \lambda) = \aleph_0$ . We do not adopt this notation, since it blurs out the distinction between forking and splitting, and does not mention that only a certain type of chains are considered. A similar notation is in Definition 6.3.16: **K** has no long splitting chains in  $\lambda$  if and only if  $\kappa_1(\mathbf{i}_{\lambda-ns}(\mathbf{K}_{\lambda}), <_{univ}) = \aleph_0$ .

In tame AECs with a monster model, no long splitting chains transfers upward:

FACT 9.2.3 (Proposition 6.10.10). Let **K** be an AEC with a monster model and let  $LS(\mathbf{K}) \leq \lambda \leq \mu$ . If **K** is stable in  $\lambda$  and has no long splitting chains in  $\lambda$ , then **K** is stable in  $\mu$  and has no long splitting chains in  $\mu$ .

**9.2.2. Definitions of saturated.** The search for a good definition of "saturated" in AECs is central. We quickly review various possible notions and cite some basic facts about them, including basic implications.

Implicit in the definition of no long splitting chains is the notion of a *limit model*. It plays a central role in the study of AECs that do not necessarily have amalgamation [**SV99**] (their study in this context was continued in [**Van06**, **Van13**]). We use the notation and basic definitions from [**GVV16**]. The two basic facts about limit models (in an AEC with a monster model) are:

- (1) Existence: If **K** is stable in  $\lambda$ , then for every M and every limit  $\delta < \lambda^+$  there is a  $(\lambda, \delta)$ -limit over M.
- (2) Uniqueness: Any two limit models of the same length are isomorphic.

Uniqueness of limit models that are *not* of the same cofinality is a key concept which is equivalent to superstability in first-order model theory.

A second notion of saturation can be defined using Galois types (when **K** has a monster model): for  $\lambda > \text{LS}(\mathbf{K})$ , say M is  $\lambda$ -saturated if every type over a  $\leq_{\mathbf{K}}$ substructure of M of size less than  $\lambda$  is realized inside M. We will write  $\mathbf{K}^{\lambda$ -sat for the class of  $\lambda$ -saturated models in **K**.

A third notion of saturation appears in  $[She87a, 3.1(1)]^3$ . The idea is to encode a generalization of the fact that a union of saturated models should be saturated.

DEFINITION 9.2.4. Let  $M \in \mathbf{K}$  and let  $\lambda \geq \mathrm{LS}(\mathbf{K})$ . *M* is called *superlimit in*  $\lambda$  *if:* 

- (1)  $M \in \mathbf{K}_{\lambda}$ .
- (2) M is "properly universal": For any  $N \in \mathbf{K}_{\lambda}$ , there exists  $f : N \to M$  such that  $f[N] <_{\mathbf{K}} M$ .
- (3) Whenever  $\langle M_i : i < \delta \rangle$  is an increasing chain in  $\mathbf{K}_{\lambda}$ ,  $\delta < \lambda^+$  and  $M_i \cong M$  for all  $i < \delta$ , then  $\bigcup_{i < \delta} M_i \cong M$ .

The following local implications between the three definitions are known:

<sup>&</sup>lt;sup>2</sup>Of course, the  $\kappa$  notation has a long history, appearing first in [She70].

 $<sup>^{3}\</sup>mathrm{We}$  use the definition in  $[\mathbf{She09a},$  N.2.4(4)] which requires in addition that the model be universal.

#### 9.2. PRELIMINARIES

FACT 9.2.5 (Local implications). Assume that **K** has a monster model. Let  $\lambda \geq LS(\mathbf{K})$  be such that **K** is stable in  $\lambda$ .

- (1) If  $\chi \in [LS(\mathbf{K})^+, \lambda]$  is regular, then any  $(\lambda, \chi)$ -limit model is  $\chi$ -saturated.
- (2) If  $\lambda > \text{LS}(\mathbf{K})$  and  $\lambda$  is regular, then  $M \in \mathbf{K}_{\lambda}$  is saturated if and only if M is  $(\lambda, \lambda)$ -limit.
- (3) If  $\lambda > LS(\mathbf{K})$ , then any two limit models of size  $\lambda$  are isomorphic if and only if every limit model of size  $\lambda$  is saturated.
- (4) If  $M \in \mathbf{K}_{\lambda}$  is superlimit, then for any limit  $\delta < \lambda^+$ , M is  $(\lambda, \delta)$ -limit and (if  $\lambda > \mathrm{LS}(\mathbf{K})$ ) saturated.
- (5) Assume that  $\lambda > \text{LS}(\mathbf{K})$  and there exists a saturated model M of size  $\lambda$ . Then M is superlimit if and only if in  $\mathbf{K}_{\lambda}$ , the union of any increasing chain (of length strictly less than  $\lambda^+$ ) of saturated models is saturated.

PROOF. (1), (2), and (3) are straightforward from the basic facts about limit models and the uniqueness of saturated models. (4) is by [**Dru13**, 2.3.10] and the previous parts. (5) then follows.  $\Box$ 

REMARK 9.2.6. (3) is stated for  $\lambda$  regular in [**Dru13**, 2.3.12] but the argument above shows that it holds for any  $\lambda$ .

**9.2.3.** Skeletons. The notion of a skeleton was introduced in Section 6.5 and is meant to be an axiomatization of a subclass of saturated models of an AEC. It is mentioned in (2) of the main corollary.

Recall the definition of an abstract class, due to Grossberg [**Gro**] (or see Definition 2.2.7): it is a pair  $\mathbf{K}' = (K', \leq_{\mathbf{K}'})$  so that K' is a class of  $\tau$ -structures in a fixed vocabulary  $\tau = \tau(\mathbf{K}')$ , closed under isomorphisms, and  $\leq_{\mathbf{K}'}$  is a partial order on K' which respects isomorphisms and extends the  $\tau$ -substructure relation.

DEFINITION 9.2.7 (Definition 6.5.3). A skeleton of an abstract class  $\mathbf{K}^*$  is an abstract class  $\mathbf{K}'$  such that:

- (1)  $K' \subseteq K^*$  and for  $M, N \in \mathbf{K}', M \leq_{\mathbf{K}'} N$  implies  $M \leq_{\mathbf{K}^*} N$ .
- (2)  $\mathbf{K}'$  is dense in  $\mathbf{K}^*$ : For any  $M \in \mathbf{K}^*$ , there exists  $M' \in \mathbf{K}'$  such that  $M \leq_{\mathbf{K}^*} M'$ .
- (3) If  $\alpha$  is a (not necessarily limit) ordinal and  $\langle M_i : i < \alpha \rangle$  is a strictly  $\leq_{\mathbf{K}^*}$ increasing chain in  $\mathbf{K}'$ , then there exists  $N \in \mathbf{K}'$  such that  $M_i \leq_{\mathbf{K}'} N$ and  $M_i \neq N$  for all  $i < \alpha$ .

EXAMPLE 9.2.8. Let  $\lambda \geq LS(\mathbf{K})$ . Assume that  $\mathbf{K}$  is stable in  $\lambda$ , has amalgamation and no maximal models in  $\lambda$ . Let K' be the class of limit models of size  $\lambda$  in  $\mathbf{K}$ . Then  $(K', \leq_{\mathbf{K}})$  (or even K' ordered with "being equal or universal over") is a skeleton of  $\mathbf{K}_{\lambda}$ .

REMARK 9.2.9. If  $\mathbf{K}'$  is a skeleton of  $\mathbf{K}_{\lambda}$  and  $\mathbf{K}'$  itself generates an AEC, then  $M \leq_{\mathbf{K}'} N$  if and only if  $M, N \in \mathbf{K}'$  and  $M \leq_{\mathbf{K}} N$ . This is because of the third clause in the definition of a skeleton (used with  $\alpha = 2$ ) and the coherence axiom.

We can define notions such as amalgamation and Galois types for any abstract class (see the preliminaries of Chapter 2). The properties of a skeleton often correspond to properties of the original AEC:

<sup>&</sup>lt;sup>4</sup>Note that if  $\alpha$  is limit this follows.

FACT 9.2.10. Let  $\lambda \geq LS(\mathbf{K})$  and assume that  $\mathbf{K}$  has amalgamation in  $\lambda$ . Let  $\mathbf{K}'$  be a skeleton of  $\mathbf{K}_{\lambda}$ .

- (1) For P standing for having no maximal models in  $\lambda$ , being stable in  $\lambda$ , or having joint embedding in  $\lambda$ , **K** has P if and only if **K**' has P.
- (2) Assume that **K** has joint embedding in  $\lambda$  and for every limit  $\delta < \lambda^+$  and every  $N \in \mathbf{K}'$  there exists  $N' \in \mathbf{K}'$  which is  $(\lambda, \delta)$ -limit over N (in the sense of  $\mathbf{K}'$ ).
  - (a) Let  $M, M_0 \in \mathbf{K}'$  and let  $\delta < \lambda^+$  be a limit ordinal. Then M is  $(\lambda, \delta)$ -limit over  $M_0$  in the sense of  $\mathbf{K}'$  if and only M is  $(\lambda, \delta)$ -limit over  $M_0$  in the sense of  $\mathbf{K}$ .
  - (b) K' has no long splitting chains in λ if and only if K has no long splitting chains in λ.

PROOF. (1) is by Proposition 6.5.8. As for (2a), (2b), note first that the hypotheses of (2) imply (by (1)) that **K** is stable in  $\lambda$  and has no maximal models in  $\lambda$ . In particular, limit models of size  $\lambda$  exist in **K**.

Let us prove (2a). If M is  $(\lambda, \delta)$ -limit over  $M_0$  in the sense of  $\mathbf{K}'$ , then it is straightforward to check that the chain witnessing it will also witness that M is  $(\lambda, \delta)$ -limit over  $M_0$  in the sense of  $\mathbf{K}$ . For the converse, observe that by assumption there exists a  $(\lambda, \delta)$ -limit M' over  $M_0$  in the sense of  $\mathbf{K}'$ . Furthermore, by what has just been observed M' is also limit in the sense of  $\mathbf{K}$ , hence by uniqueness of limit models of the same length,  $M' \cong_{M_0} M$ . Therefore M is also  $(\lambda, \delta)$ -limit over  $M_0$ in the sense of  $\mathbf{K}'$ . The proof of (2b) is similar, see Lemma 6.6.7.

**9.2.4.** Good frames. Good frames are a local axiomatization of forking in a first-order superstable theories. They are introduced in [She09a, Chapter II]. We will use Definition 6.8.1 which is weaker and more general than Shelah's, as it does not require the existence of a superlimit (as in [JS13]). As opposed to Chapter 6, we allow good frames that are *not* type-full: we only require the existence of a set of well-behaved basic types satisfying some density property (see [She09a, Chapter II] for more). Note however that Remark 9.5.6 says that in the context of the main theorem the existence of a good frame implies the existence of a *type-full* good frame (possibly over a different class).

In Definition 6.8.1, the underlying class of the good frame consists only of models of size  $\lambda$ . Thus when we say that there is a good  $\lambda$ -frame on a class  $\mathbf{K}'$ , we mean the underlying class of the good frame is  $\mathbf{K}'$ , and the axioms of good frames will require that  $\mathbf{K}'$  generates a non-empty AEC with amalgamation in  $\lambda$ , joint embedding in  $\lambda$ , no maximal models in  $\lambda$ , and stability in  $\lambda$ .

The only facts that we will use about good frames are:

FACT 9.2.11. Let  $\lambda \geq \mathrm{LS}(\mathbf{K})$ . If there is a good  $\lambda$ -frame on a skeleton of  $\mathbf{K}_{\lambda}$ , then **K** has a unique limit model of size  $\lambda$ . Moreover, for any  $M_0, M_1, M_2 \in \mathbf{K}_{\lambda}$ , if both  $M_1$  and  $M_2$  are limit over  $M_0$ , then  $M_1 \cong_{M_0} M_2$  (i.e. the isomorphism fixes  $M_0$ ).

PROOF. Let  $\mathbf{K}'$  be the skeleton of  $\mathbf{K}_{\lambda}$  which is the underlying class of the good  $\lambda$ -frame. By [She09a, II.4.8] (see [Bon14a, 9.2] for a detailed proof),  $\mathbf{K}'$  has a unique limit model of size  $\lambda$  (and the moreover part holds for  $\mathbf{K}'$ ). By Fact 9.2.10(2a), this must also be the unique limit model of size  $\lambda$  in  $\mathbf{K}$  (and the moreover part holds in  $\mathbf{K}$  too).

FACT 9.2.12. Assume that **K** has a monster model and is  $LS(\mathbf{K})$ -tame. If  $\mu < H_1$  is such that **K** is stable in  $\mu$  and has no long splitting chains in  $\mu$ , then there exists  $\lambda_0 < H_1$  such that for all  $\lambda \geq \lambda_0$ , there is a good  $\lambda$ -frame on  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$ . Moreover,  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  is a skeleton of  $\mathbf{K}_{\lambda}$ , **K** is stable in  $\lambda$ , any  $M \in \mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  is superlimit, and the union of any increasing chain of  $\lambda$ -saturated models is  $\lambda$ -saturated.

PROOF. First assume that **K** has no long splitting chains in LS(**K**) and is stable in LS(**K**). By Theorem 7.6.1, there exists  $\lambda_0 < \beth_{(2^{\mu^+})^+}$  such that for any  $\lambda \geq \lambda_0$ , any increasing chain of  $\lambda$ -saturated models is  $\lambda$ -saturated and there is a good  $\lambda$ -frame on  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$ . That any  $M \in \mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  is a superlimit (Fact 9.2.5(5)) and  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  is a skeleton of  $\mathbf{K}_{\lambda}$  easily follows, and stability in  $\lambda$  is given (for example) by Fact 9.2.10(1).

Now by Theorem 7.5.17, we more precisely have that if **K** has no long splitting chains in  $\mu$  and is stable in  $\mu$  then the same conclusion holds with  $\beth_{(2^{\mu^+})^+}$  replaced by  $H_1$ .

**9.2.5.** Solvability. Solvability appears as a possible definition of superstability for AECs in [She09a, Chapter IV]. The definition uses Ehrenfeucht-Mostowski models and we assume the reader has some familiarity with them, see for example [Bal09, Section 6.2] or [She09a, IV.0.8].

DEFINITION 9.2.13.

- (1) A countable set  $\Phi = \{p_n : n < \omega\}$  is proper for linear orders if the  $p_n$ 's are an increasing sequence of *n*-variable quantifier-free types in a fixed vocabulary  $\tau(\Phi)$  which are satisfied by a sequence of indiscernibles. As usual, such a set  $\Phi$  determines an EM-functor from linear orders into  $\tau(\Phi)$ -structures, mapping a linear order I to  $\text{EM}(I, \Phi)$  and taking suborders to substructures.
- (2) [She09a, IV.0.8] For  $\mu \geq \text{LS}(\mathbf{K})$ , let  $\Upsilon_{\mu}[\mathbf{K}]$  be the set of  $\Phi$  proper for linear orders with  $\tau(\mathbf{K}) \subseteq \tau(\Phi)$ ,  $|\tau(\Phi)| \leq \mu$ , and such that the  $\tau(\mathbf{K})$ reduct  $\text{EM}_{\tau(\mathbf{K})}(I, \Phi)$  is a functor from linear orders into members of  $\mathbf{K}$  of cardinality at most  $|I| + \mu$ . Such a  $\Phi$  is called an *EM blueprint for*  $\mathbf{K}$ .

DEFINITION 9.2.14. Let  $LS(\mathbf{K}) \leq \mu \leq \lambda$ .

- (1) [She09a, IV.1.4(1)] We say that  $\Phi$  witnesses  $(\lambda, \mu)$ -solvability if:
  - (a)  $\Phi \in \Upsilon_{\mu}[\mathbf{K}].$
  - (b) If I is a linear order of size  $\lambda$ , then  $\text{EM}_{\tau(\mathbf{K})}(I, \Phi)$  is superlimit in  $\lambda$  for **K**, see Definition 9.2.4.
    - **K** is  $(\lambda, \mu)$ -solvable if there exists  $\Phi$  witnessing  $(\lambda, \mu)$ -solvability.
- (2) **K** is uniformly  $(\lambda, \mu)$ -solvable if there exists  $\Phi$  such that for all  $\lambda' \geq \lambda$ ,  $\Phi$  witnesses  $(\lambda', \mu)$ -solvability.

FACT 9.2.15 (IV.0.9 in [She09a]). Let **K** be an AEC and let  $\mu \ge LS(\mathbf{K})$ . Then **K** has arbitrarily large models if and only if  $\Upsilon_{\mu}[\mathbf{K}] \neq \emptyset$ .

We give some more manageable definitions of solvability ((3) is the one we will use). Shelah already mentions one of them on [She09a, p. 61] (but does not prove it is equivalent).

LEMMA 9.2.16. Let  $LS(\mathbf{K}) \leq \mu \leq \lambda$ . The following are equivalent.

(1) **K** is [uniformly]  $(\lambda, \mu)$ -solvable.

- (2) There exists  $\tau' \supseteq \tau(\mathbf{K})$  with  $|\tau'| \leq \mu$  and  $\psi \in \mathbb{L}_{\mu^+,\omega}(\tau')$  such that:
  - (a)  $\psi$  has arbitrarily large models.
  - (b) [For all  $\lambda' \ge \lambda$ ], if  $M \models \psi$  and  $||M|| = \lambda$  [ $||M|| = \lambda'$ ], then  $M \upharpoonright \tau(\mathbf{K})$  is in **K** and is superlimit.
- (3) There exists  $\tau' \supseteq \tau(\mathbf{K})$  and an AEC  $\mathbf{K}'$  with  $\tau(\mathbf{K}') = \tau'$ ,  $LS(\mathbf{K}') \leq \mu$  such that:
  - (a)  $\mathbf{K}'$  has arbitrarily large models.
  - (b) [For all  $\lambda' \geq \lambda$ ], if  $M \in \mathbf{K}'$  and  $||M|| = \lambda$  [ $||M|| = \lambda'$ ], then  $M \upharpoonright \tau(\mathbf{K})$  is in **K** and is superlimit.

# Proof.

- (1) implies (2): Let  $\Phi$  witness  $(\lambda, \mu)$ -solvability and write  $\Phi = \{p_n \mid n < \omega\}$ . Let  $\tau' := \tau(\Phi) \cup \{P, <\}$ , where P, < are symbols for a unary predicate and a binary relation respectively. Let  $\psi \in \mathbb{L}_{\mu^+,\omega}(\tau')$  say:
  - (1) (P, <) is a linear order.
  - (2) For all  $n < \omega$  and all  $x_0 < \cdots < x_{n-1}$  in  $P, x_0 \ldots x_{n-1}$  realizes  $p_n$ .
  - (3) For all y, there exists  $n < \omega$ ,  $x_0 < \cdots < x_{n-1}$  in P, and  $\rho$  an n-ary term of  $\tau(\Phi)$  such that  $y = \rho(x_0, \ldots, x_{n-1})$ .

Then if  $M \models \psi$ ,  $M \upharpoonright \tau = \mathrm{EM}_{\tau(\mathbf{K})}(P^M, \Phi)$  (and by solvability if  $||M|| = \lambda$  then M is superlimit in  $\mathbf{K}$ ). Conversely, if  $M = \mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi)$ , we can expand M to a  $\tau'$ -structure M' by letting  $(P^{M'}, <^{M'}) := (I, <)$ . Thus  $\psi$  is as desired.

- (2) implies (3): Given  $\tau'$  and  $\psi$  as given by (2), Let  $\Psi$  be a fragment of  $\overline{\mathbb{L}}_{\mu^+,\omega}(\tau')$  containing  $\psi$  of size  $\mu$  and let  $\mathbf{K}'$  be  $\operatorname{Mod}(\psi)$  ordered by  $\preceq_{\Psi}$ . Then  $\mathbf{K}'$  is as desired for (3).
- (3) implies (1): Directly from Fact 9.2.15.

# 9.3. Forking and averages in stable AECs

In the introduction to his book [She09a, p. 61], Shelah asserts (without proof) that in the first-order context solvability (see Section 9.2.5) is equivalent to superstability. We aim to give a proof (see Corollary 9.5.3) and actually show (assuming amalgamation, stability, and tameness) that solvability is equivalent to any of the definitions in the main theorem. First of all, if there exists  $\mu$  such that **K** is  $(\lambda, \mu)$ -solvable for all high-enough  $\lambda$ , then in particular **K** has a superlimit in all high-enough  $\lambda$ , so we obtain (4) in the main corollary. We work toward a converse. The proof is similar to that in [BGS99]: we aim to code saturated models using their characterization with average of sequences (the crucial result for this is Lemma 9.3.16). In this section, we use the theory of averages in AECs (as developed by Shelah in [She09b, Chapter V.A] and by Boney and the author in Chapter 7) to give a new characterization of forking (Lemma 9.3.12). We also prove the key result for (5) implies (1) in the main corollary (Theorem 9.3.18). All throughout, we assume:

Hypothesis 9.3.1.

- (1) **K** has a monster model  $\mathfrak{C}$  (we work inside it).
- (2)  $\mathbf{K}$  is  $LS(\mathbf{K})$ -tame.
- (3)  $\mathbf{K}$  is stable in some cardinal greater than or equal to  $LS(\mathbf{K})$ .

We set  $\kappa := \mathrm{LS}(\mathbf{K})^+$  and work in the setup of Section 7.4. In particular we think of Galois types of size  $\mathrm{LS}(\mathbf{K})$  as formulas and think of bigger Galois types as set of such formulas. That is, we work inside the Galois Morleyization of  $\mathbf{K}$  (see Definition 2.3.3). We encourage the reader to be ready to flip back to both Chapters 2 and 7, since we will cite from there freely and use basic notation and terminology ( $\chi$ -convergent,  $\chi$ -based, ( $\chi_0, \chi_1, \chi_2$ )-Morley,  $\mathrm{Av}_{\chi}(\mathbf{I}/A)$  etc.) often without even an explicit citation. We will say that  $p \in \mathrm{gS}^{<\kappa}(M)$  does not syntactically split over  $M_0 \leq_{\mathbf{K}} M$  if it does not split in the syntactic sense of Definition 7.4.7 (that is, it does not split in the usual first-order sense when we think of Galois types of size  $\mathrm{LS}(\mathbf{K})$  as formulas). Note that several results from Chapter 7 that we quote assume ( $< \mathrm{LS}(\mathbf{K})$ )-tameness (defined in terms of Galois types over sets). However, as argued in the proof of Fact 9.2.12,  $\mathrm{LS}(\mathbf{K})$ -tameness suffices.

We will define several other cardinals  $\chi_0 < \chi'_0 < \chi_1 < \chi'_1 < \chi_2$  (see Notation 9.3.4, 9.3.9, and 9.3.10). The reader can simply see them as "high-enough" cardinals with reasonable closure properties. If  $\chi_0$  is chosen reasonably, we will have  $\chi_2 < H_1$ .

The letters **I**, **J** will denote sequences of tuples of length strictly less than  $\kappa$ . We will use the same conventions as in Section 7.4. Note that the sequences may be indexed by arbitrary linear orders.

By Facts 2.4.15 and [She99, I.4.5(3)] (recalling that there is a global assumption of stability in this section), we have:

FACT 9.3.2. There exists  $\chi_0 < H_1$  such that **K** does not have the LS(**K**)-order property of length  $\chi_0$ .

Another property of  $\chi_0$  is the following more precise version of Fact 2.4.15 (see Chapter 2 on how to translate Shelah's syntactic version to AECs):

FACT 9.3.3 (Theorem V.A.1.19 in [She09b]). If  $\lambda = \lambda^{\chi_0}$ , then **K** is stable in  $\lambda$ . In particular, **K** is stable in  $\chi'_0$ .

The following notation will be convenient:

NOTATION 9.3.4. Let  $\chi_0$  be any regular cardinal such that  $\chi_0 \geq 2^{\text{LS}(\mathbf{K})}$  and **K** does not have the LS(**K**)-order property of length  $\chi_0^+$ . For a cardinal  $\lambda$ , let  $\gamma(\lambda) := (2^{2^{\lambda}})^+$ . We write  $\chi'_0 := \gamma(\chi_0)$ .

REMARK 9.3.5. By Fact 9.3.2, one can take  $\chi_0 < H_1$ . In that case also  $\chi'_0 < H_1$ . For the sake of generality, we do *not* require that  $\chi_0$  be least with the property above.

Recall (Theorem 7.4.21) that if **I** is a  $(\chi_0^+, \chi_0^+, \gamma(\chi_0))$ -Morley sequence, then **I** is  $\chi$ -convergent. We want to use this to relate average and forking:

DEFINITION 9.3.6. Let  $M_0, M \in \mathbf{K}^{(\chi'_0)^+\text{-sat}}$  be such that  $M_0 \leq_{\mathbf{K}} M$ . Let  $p \in \mathrm{gS}(M)$ . We say that p does not fork over  $M_0$  if there exists  $M'_0 \in \mathbf{K}_{\chi'_0}$  such that  $M'_0 \leq_{\mathbf{K}} M_0$  and p does not  $\chi'_0$ -split over  $M'_0$ .

We will use without comments:

FACT 9.3.7. Forking has the following properties:

(1) Invariance under isomorphisms and monotonicity: if  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$  are all  $(\chi'_0)^+$ -saturated and  $p \in \mathrm{gS}(M_2)$  does not fork over  $M_0$ , then  $p \upharpoonright M_1$  does not fork over  $M_0$  and p does not fork over  $M_1$ .

- (2) Set local character: if  $M \in \mathbf{K}^{(\chi'_0)^+\text{-sat}}$  and  $p \in \mathrm{gS}(M)$ , there exists  $M_0 \in \mathbf{K}^{(\chi'_0)^+\text{-sat}}$  of size  $(\chi'_0)^+$  such that  $M_0 \leq_{\mathbf{K}} M$  and p does not fork over  $M_0$ .
- (3) Transitivity: Assume  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$  are all  $(\chi'_0)^+$ -saturated and  $p \in \mathrm{gS}(M_2)$ . If p does not fork over  $M_1$  and  $p \upharpoonright M_1$  does not fork over  $M_0$ , then p does not fork over  $M_0$ .
- (4) Uniqueness: If  $M_0 \leq_{\mathbf{K}} M$  are all  $(\chi'_0)^+$ -saturated and  $p, q \in \mathrm{gS}(M)$  do not fork over  $M_0$ , then  $p \upharpoonright M_0 = q \upharpoonright M_0$  implies p = q. Moreover p does not  $\lambda$ -split over  $M_0$  for any  $\lambda \geq (\chi'_0)^+$ .
- (5) Local extension over saturated models: If  $M_0 \leq_{\mathbf{K}} M$  are both saturated,  $\|M_0\| = \|M\| \geq (\chi'_0)^+, p \in \mathrm{gS}(M_0)$ , there exists  $q \in \mathrm{gS}(M)$  such that q extends p and does not fork over  $M_0$ .

PROOF. Use Theorem 6.7.5. The generator used is the one given by Proposition 7.4(2) there. For the moreover part of uniqueness, use Lemma 3.4.2 (and Proposition 3.3.12).  $\Box$ 

Note that the extension property need not hold in general. However if the class has no long splitting chains we have:

FACT 9.3.8. If **K** has no long splitting chains in  $\chi'_0$ , then:

- (1) (Theorem 6.8.9 or Theorem 4.7.1) Forking has:
  - (a) The extension property: If  $M_0 \leq_{\mathbf{K}} M$  are  $(\chi'_0)^+$ -saturated and  $p \in gS(M_0)$ , then there exists  $q \in gS(M)$  extending p and not forking over  $M_0$ .
  - (b) The chain local character property: If  $\langle M_i : i < \delta \rangle$  is an increasing chain of  $(\chi'_0)^+$ -saturated models and  $p \in \mathrm{gS}(\bigcup_{i < \delta} M_i)$ , then there exists  $i < \delta$  such that p does not fork over  $M_i$ .
- (2) (Theorem 7.5.16) For any  $\lambda > (\chi'_0)^+$ ,  $\mathbf{K}^{\lambda\text{-sat}}$  is an AEC with  $\mathrm{LS}(\mathbf{K}^{\lambda\text{-sat}}) = \lambda$ .

For notational convenience, we "increase"  $\chi_0$ :

NOTATION 9.3.9. Let  $\chi_1 := (\chi'_0)^{++}$ . Let  $\chi'_1 := \gamma(\chi_1)$ .

We obtain a characterization of forking that adds to those proven in Section 6.9. A form of it already appears in [She09a, IV.4.6]. Again, we define more cardinal parameters:

NOTATION 9.3.10. Let  $\chi_2 := \beth_{\omega}(\chi_0)$ .

REMARK 9.3.11. We have that  $\chi_0 < \chi'_0 < \chi_1 < \chi'_1 < \chi_2$ , and  $\chi_2 < H_1$  if  $\chi_0 < H_1$ .

LEMMA 9.3.12. Let  $M_0, M$  be  $\chi_2$ -saturated with  $M_0 \leq_{\mathbf{K}} M$ . Let  $p \in gS(M)$ . The following are equivalent:

- (1) p does not fork over  $M_0$ .
- (2)  $p \upharpoonright M_0$  has a nonforking extension to gS(M) and there exists  $M'_0 \leq_{\mathbf{K}} M_0$ with  $||M'_0|| < \chi_2$  such that p does not syntactically split over  $M'_0$ .
- (3)  $p \upharpoonright M_0$  has a nonforking extension to gS(M) and there exists  $\mu \in [\chi_0^+, \chi_2)$ and **I** a  $(\mu, \mu, \gamma(\mu)^+)$ -Morley sequence for p, with all the witnesses inside  $M_0$ , such that  $\operatorname{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$ .

REMARK 9.3.13. When **K** has no long splitting chains in  $\chi'_0$ , forking has the extension property (Fact 9.3.8) so the first part of (2) and (3) always hold. However in Theorem 9.3.18 we apply Lemma 9.3.12 in the strictly stable case (i.e. **K** may only be *stable* in  $\chi'_0$  and not have no long splitting chains there).

We recall more definitions and facts before giving the proof of Lemma 9.3.12:

FACT 9.3.14 (V.A.1.12 in [She09b]). If  $p \in gS(M)$  and M is  $\chi_0^+$ -saturated, there exists  $M_0 \in \mathbf{K}_{\leq \chi_0}$  with  $M_0 \leq_{\mathbf{K}} M$  such that p does not syntactically split over  $M_0$ .

FACT 9.3.15. Let  $M_0 \leq_{\mathbf{K}} M$  be both  $(\chi'_1)^+$ -saturated. Let  $\mu := ||M_0||$ . Let  $p \in gS(M)$  and let  $\mathbf{I}$  be a  $(\mu^+, \mu^+, \gamma(\mu))$ -Morley sequence for p over  $M_0$  with all the witnesses inside M. Then if p does not syntactically split or does not fork over  $M_0$ , then  $\operatorname{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$ .

PROOF. For syntactic splitting, this is Lemma 7.4.25. The Lemma is actually more general and the proof of Theorem 7.5.16 shows that this also works for forking.  $\Box$ 

PROOF OF LEMMA 9.3.12. Before starting, note that if  $\mu < \chi_2$ , then **K** is stable in  $2^{\mu+\chi_0} < \chi_2$  by Fact 9.3.3. Thus there are unboundedly many stability cardinals below  $\chi_2$ , so we have "enough space" to build Morley sequences.

- (1) implies (2): By Fact 9.3.14, we can find  $M'_0 \leq_{\mathbf{K}} M_0$  such that  $p \upharpoonright M_0$ does not syntactically split over  $M'_0$  and  $||M'_0|| \leq \chi_1$ . Taking  $M'_0$  bigger, we can assume  $M'_0$  is  $\chi_1$ -saturated and  $p \upharpoonright M_0$  does not fork over  $M'_0$ . Thus by transitivity p does not fork over  $M'_0$ . Let  $\mathbf{I}$  be a  $(\chi_1^+, (\chi'_1)^+, (\chi'_1)^+)$ -Morley sequence for  $p \upharpoonright M_0$  over  $M'_0$  inside  $M_0$ . By Theorem 7.4.21,  $\mathbf{I}$  is  $\chi'_1$ -convergent. By Lemma 7.4.20,  $\mathbf{I}$  is  $\chi'_1$ -based on  $M'_0$ . Note also that  $\mathbf{I}$ is a  $(\chi_1^+, (\chi'_1)^+, (\chi'_1)^+)$ -Morley sequence for p over  $M'_0$  and by Fact 9.3.15,  $\operatorname{Av}_{\chi'_1}(\mathbf{I}/M_0) = p$  so as  $\mathbf{I}$  is  $\chi'_1$ -based on  $M'_0$ , p does not syntactically split over  $M'_0$ .
- (2) implies (3): As in the proof of (1) implies (2) (except  $\chi_1$  could be bigger).
- (3) implies (2): By Theorem 7.4.21, **I** is  $\gamma(\mu)$ -convergent. Pick any  $\mathbf{J} \subseteq \mathbf{I}$ of length  $\gamma(\mu)$  and use Lemma 7.4.10 to find  $M'_0 \leq_{\mathbf{K}} M_0$  of size  $\gamma(\mu)$  such that **J** is  $\gamma(\mu)$ -based on  $M'_0$ . Since also **J** is  $\gamma(\mu)$ -convergent, we have that **I** is  $\gamma(\mu)$ -based on  $M'_0$ . Thus  $\operatorname{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$  does not syntactically split over  $M'_0$ .
- (2) implies (1): Without loss of generality, we can choose  $M'_0$  to be such that  $p \upharpoonright M_0$  also does not fork over  $M'_0$ . Let  $\mu := ||M'_0|| + \chi_0$ . Build a  $(\mu^+, \mu^+, \gamma(\mu))$ -Morley sequence **I** for p over  $M'_0$  inside  $M_0$ . If q is the nonforking extension of  $p \upharpoonright M_0$  to M, then **I** is also a Morley sequence for q over  $M'_0$  so by the proof of (1) implies (2) we must have  $\operatorname{Av}_{\gamma(\mu)}(\mathbf{I}/M) = q$ , but also  $\operatorname{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$ , since p does not syntactically split over  $M'_0$  (Fact 9.3.15). Thus p = q.

The next result is a version of [She90, III.3.10] in our context. It is implicit in the proof of Theorem 7.4.27.

LEMMA 9.3.16. Let  $M \in \mathbf{K}^{\chi_2\text{-sat}}$ . Let  $\lambda \geq \chi_2$  be such that **K** is stable in unboundedly many  $\mu < \lambda$ . The following are equivalent.

- (1) M is  $\lambda$ -saturated.
- (2) If  $q \in gS(M)$  is not algebraic and does not syntactically split over  $M_0 \leq_{\mathbf{K}} M$  with  $||M_0|| < \chi_2$ , there exists a  $((||M_0|| + \chi_0)^+, (||M_0|| + \chi_0)^+, \lambda)$ -Morley sequence for p over  $M_0$  inside M.

PROOF. (1) implies (2) is trivial using saturation. Now assume (2). Let  $p \in$  gS(N),  $||N|| < \lambda$ ,  $N \leq_{\mathbf{K}} M$ . We show that p is realized in M. Let  $q \in$  gS(M) extend p. If q is algebraic, we are done so assume it is not. Let  $M_0 \leq_{\mathbf{K}} M$  have size  $(\chi'_1)^+$  such that q does not fork over  $M_0$ . By Lemma 9.3.12, we can increase  $M_0$  if necessary so that q does not syntactically split over  $M_0$  and  $\mu := ||M_0|| \geq \chi_0$ . Now by (2), there exists a  $(\mu^+, \mu^+, \lambda)$ -Morley sequence  $\mathbf{I}$  for q over  $M_0$  inside M. Now by Fact 9.3.15,  $\operatorname{Av}_{\gamma(\mu)}(\mathbf{I}/M) = q$ . Thus  $\operatorname{Av}_{\gamma(\mu)}(\mathbf{I}/N) = p$ . By Lemma 7.4.6 and the hypothesis of stability in unboundedly many cardinals below  $\lambda$ , p is realized by an element of  $\mathbf{I}$  and hence by an element of M.

We end by showing that if high-enough limit models are sufficiently saturated, then no long splitting chains holds. A similar argument already appears in the proof of [She09a, IV.4.10]. We start with a more local version,

LEMMA 9.3.17. Let  $\lambda \geq \chi_2$ . Let  $\delta < \lambda^+$  be a limit ordinal and let  $\langle M_i : i < \delta \rangle$  be an increasing chain of saturated models in  $\mathbf{K}_{\lambda}$ . Let  $M_{\delta} := \bigcup_{i < \delta} M_i$ . If  $M_{\delta}$  is  $\chi_2$ -saturated, then for any  $p \in \mathrm{gS}(M_{\delta})$ , there exists  $i < \delta$  such that p does not fork over  $M_i$ .

PROOF. Without loss of generality,  $\delta$  is regular. If  $\delta \geq \chi_2$ , by set local character (Fact 9.3.7(2)), there exists  $M'_0$  of size  $\chi_1$  such that p does not fork over  $M'_0$  and  $M'_0 \leq_{\mathbf{K}} M_{\delta}$ , so pick  $i < \delta$  such that  $M'_0 \leq_{\mathbf{K}} M_i$  and use monotonicity.

Now assume  $\delta < \chi_2$ . By assumption, we have that  $M_{\delta}$  is  $\chi_2$ -saturated. We also have that p does not fork over  $M_{\delta}$  (by set local character) so by Lemma 9.3.12, there exists  $\mu \in [\chi_0^+, \chi_2)$  and  $\mathbf{I}$  a  $(\mu, \mu, \gamma(\mu)^+)$ -Morley sequence for p with all the witnesses inside  $M_{\delta}$  such that  $\operatorname{Av}_{\gamma(\mu)}(\mathbf{I}/M_{\delta}) = p$ . Since  $M_{\delta}$  is  $\chi_2$ -saturated (and there are unboundedly many stability cardinals below  $\chi_2$ ), we can increase  $\mathbf{I}$  if necessary to assume that  $|\mathbf{I}| \geq \chi_2$ . Write  $\mathbf{I}_i := |M_i| \cap \mathbf{I}$ . Since  $\delta < \chi_2$ , there must exists  $i < \delta$  such that  $|\mathbf{I}_i| \geq \chi_2$ . Note that  $\mathbf{I}_i$  is a  $(\mu, \mu, \chi_2)$ -Morley sequence for p. Because  $\mathbf{I}$  is  $\gamma(\mu)$ -convergent and  $|\mathbf{I}_i| \geq \chi_2 > \gamma(\mu)$ ,  $\operatorname{Av}_{\gamma(\mu)}(\mathbf{I}_i/M_{\delta}) = p$ . Letting  $M' \geq_{\mathbf{K}} M_{\delta}$  be a saturated model of size  $\lambda$  and using local extension over saturated models (Fact 9.3.7(5)),  $p \upharpoonright M_i$  has a nonforking extension to  $\operatorname{gS}(M')$  and hence to  $\operatorname{gS}(M_{\delta})$ . By Lemma 9.3.12, p does not fork over  $M_i$ , as desired.

THEOREM 9.3.18. Assume that  $\mathbf{K}$  has a monster model, is  $LS(\mathbf{K})$ -tame, and stable in some cardinal greater than or equal to  $LS(\mathbf{K})$ .

Let  $\chi_0 \geq \mathrm{LS}(\mathbf{K})$  be such that  $\mathbf{K}$  does not have the  $\mathrm{LS}(\mathbf{K})$ -order property of length  $\chi_0$ , and let  $\chi_2 := \beth_{\omega}(\chi_0)$ . Let  $\lambda \geq \chi_2$  be such that  $\mathbf{K}$  is stable in  $\lambda$  and there exists a saturated model of cardinality  $\lambda$ . If every limit model of cardinality  $\lambda$  is  $\chi_2$ -saturated, then  $\mathbf{K}$  has no long splitting chains in  $\lambda$ .

PROOF. Let  $\mathbf{K}'$  be  $K_{\lambda}^{\chi_2\text{-sat}}$  ordered by being equal or universal over. Note that, by stability in  $\lambda$ ,  $\mathbf{K}'$  is a skeleton of  $\mathbf{K}_{\lambda}$  (see Definition 9.2.7). Moreover since every limit model of cardinality  $\lambda$  is  $\chi_2$ -saturated, for any limit  $\delta < \lambda^+$ , one can build an increasing continuous chain  $\langle M_i : i \leq \delta \rangle$  in  $\mathbf{K}_{\lambda}$  such that for all  $i \leq \delta$ ,  $M_i$  is  $\chi_2$ -saturated and (when  $i < \delta$ )  $M_{i+1}$  is universal over  $M_i$ . Therefore limit models exist in  $\mathbf{K}'$ , so the assumptions of Fact 9.2.10(2b) are satisfied. So it is enough to see that  $\mathbf{K}'$  (not  $\mathbf{K}$ ) has no long splitting chains in  $\lambda$ .

Let  $\delta < \lambda^+$  be limit and let  $\langle M_i : i < \delta \rangle$  be an increasing chain of models in  $\mathbf{K}'$ , with  $M_{i+1}$  universal over  $M_i$  for all  $i < \delta$ . Let  $M_\delta := \bigcup_{i < \delta} M_i$ . By assumption,  $M_\delta$ is  $\chi_2$ -saturated. By uniqueness of limit models of the same length, we can assume without loss of generality that  $M_{i+1}$  is saturated for all  $i < \delta$ .

Let  $p \in gS(M_{\delta})$ . By Lemma 9.3.17 (applied to  $\langle M_{i+1} : i < \delta \rangle$ ), there exists  $i < \delta$  such that p does not fork over  $M_i$ . By the moreover part of Fact 9.3.7(4), p does not  $\lambda$ -split over  $M_i$ , as desired.

# 9.4. No long splitting chains implies solvability

From now on we assume no long splitting chains:

Hypothesis 9.4.1.

- (1) Hypothesis 9.3.1, and we fix cardinals  $\chi_0 < \chi'_0 < \chi_1 < \chi'_1 < \chi_2$  as defined in Notation 9.3.4, 9.3.9, and 9.3.10. Note that by Fact 9.3.3 **K** is stable in  $\chi'_0$ .
- (2) **K** has no long splitting chains in  $\chi'_0$ .

In Notation 9.4.3, we will define another cardinal  $\chi$  with  $\chi_2 < \chi$ . If  $\chi_0 < H_1$ , we will also have that  $\chi < H_1$ .

Note that no long splitting chains in  $\chi'_0$  and stability in  $\chi'_0$  implies (Fact 9.2.3) that **K** is stable in all  $\lambda \geq \chi'_0$ . Further, forking is well-behaved in the sense of Fact 9.3.8. This implies that Morley sequences are closed under unions (here we use that they are indexed by arbitrary linear orders, as opposed to just well-orderings). Recall that we say  $\mathbf{I} \frown \langle N_i : i \leq \delta \rangle$  is a Morley sequence when **I** is a sequence of elements and the  $N_i$ 's are an increasing chain of sufficiently saturated models witnessing that **I** is Morley, see Definition 7.4.14 for the details.

LEMMA 9.4.2. Let  $\langle I_{\alpha} : \alpha < \delta \rangle$  be an increasing (with respect to substructure) sequence of linear orders and let  $I_{\delta} := \bigcup_{\alpha < \delta} I_{\alpha}$ . Let  $M_0, M$  be  $\chi_2$ -saturated such that  $M_0 \leq_{\mathbf{K}} M$ . Let  $\mu_0, \mu_1, \mu_2$  be such that  $\chi_2 < \mu_0 \leq \mu_1 \leq \mu_2, p \in \mathrm{gS}(M)$  and for  $\alpha < \delta$ , let  $\mathbf{I}_{\alpha} := \langle a_i : i \in I_{\alpha} \rangle$  together with  $\langle N_i^{\alpha} : i \in I_{\alpha} \rangle$  be  $(\mu_0, \mu_1, \mu_2)$ -Morley for p over  $M_0$ , with  $N_i^{\alpha} \leq_{\mathbf{K}} N_i^{\beta} \leq_{\mathbf{K}} M$  for all  $\alpha \leq \beta < \delta$  and  $i \in I_{\alpha}$ . For  $i \in I_{\alpha}$ , let  $N_i^{\delta} := \bigcup_{\beta \in [\alpha, \delta)} N_i^{\beta}$ . Let  $\mathbf{I}_{\delta} := \langle a_i : i \in I_{\delta} \rangle$ .

If p does not fork over  $M_0$ , then  $\mathbf{I}_{\delta} \frown \langle N_i^{\delta} : i \in I_{\delta} \rangle$  is  $(\mu_0, \mu_1, \mu_2)$ -Morley for p over  $M_0$ .

PROOF. By Lemma 9.3.12, p does not syntactically split over  $M_0$ . Therefore the only problematic clauses in Definition 7.4.14 are (4) and (7). Let's check (4): let  $i \in I_{\delta}$ . By hypothesis,  $\bar{a}_i$  realizes  $p \upharpoonright N_i^{\alpha}$  for all sufficiently high  $\alpha < \delta$ . By local character of forking, there exists  $\alpha < \delta$  such that  $gtp(\bar{a}_i/N_i^{\delta})$  does not fork over  $N_i^{\alpha}$ . Since  $gtp(\bar{a}_i/N_i^{\delta}) \upharpoonright N_i^{\alpha} = p \upharpoonright N_i^{\alpha}$  and p does not fork over  $M_0 \leq_{\mathbf{K}} N_i^{\alpha}$ , we must have by uniqueness that  $p \upharpoonright N_i^{\delta} = gtp(\bar{a}_i/N_i^{\delta})$ . The proof of (7) is similar.  $\Box$ 

For convenience, we make  $\chi_2$  even bigger:

NOTATION 9.4.3. Let  $\chi := \gamma(\chi_2)$  (recall from Notation 9.3.4 that  $\gamma(\chi_2) = (2^{2^{\chi_2}})^+$ ). A Morley sequence means a  $(\chi_2^+, \chi_2^+, \chi)$ -Morley sequence.

REMARK 9.4.4. By Remark 9.3.11, we still have  $\chi < H_1$  if  $\chi_0 < H_1$ .

We are finally in a position to prove solvability (in fact even uniform solvability). We will use condition (3) in Lemma 9.2.16.

DEFINITION 9.4.5. We define a class of models K' and a binary relation  $\leq_{\mathbf{K}'}$ on K' (and write  $\mathbf{K}' := (K', \leq_{\mathbf{K}'})$ ) as follows.

• K' is a class of  $\tau' := \tau(\mathbf{K}')$ -structures, with:

$$\tau' := \tau(\mathbf{K}) \cup \{N_0, N, F, R\}$$

where:

- $N_0$  and R are binary relations symbols.
- -N is a ternary relation symbol.
- F is a binary function symbol.
- A  $\tau'$ -structure M is in K' if and only if:
  - (1)  $M \upharpoonright \tau(\mathbf{K}) \in \mathbf{K}^{\chi\text{-sat}}$ .
  - (2)  $R^M$  is a linear ordering of |M|. We write I for this linear ordering.
  - (2) For 5 all  $a \in |M|$  and all  $i \in I$ ,  $N^M(a,i) \leq_{\mathbf{K}} M \upharpoonright \tau(\mathbf{K})$  (where we see  $N^{M}(a, i)$  as an  $\tau(\mathbf{K})$ -structure; in particular,  $N^{M}(a, i) \in \mathbf{K}$ ; it will follow from (4b) that the  $N^{M}(a,i)$ 's are increasing with i,  $N_0^M(a) \leq_{\mathbf{K}} N^M(a, i)$ , and  $N_0^M(a)$  is saturated of size  $\chi_2$ .
  - (4) There exists a map  $a \mapsto p_a$  from |M| onto the non-algebraic Galois types (of length one) over  $M \upharpoonright \tau(\mathbf{K})$  such that for all  $a \in |M|$ :

    - (a)  $p_a$  does not fork<sup>6</sup> over  $N_0^M(a)$ . (b)  $\langle F^M(a,i) : i \in I \rangle \frown \langle N^M(a,i) : i \in I \rangle$  is a Morley sequence for  $p_a$  over  $N_0^M(a)$ .
- $M \leq_{K'} M'$  if and only if:
  - (1)  $M \subseteq M'$ .
  - (2)  $M \upharpoonright \tau(\mathbf{K}) \leq_{\mathbf{K}} M' \upharpoonright \tau(\mathbf{K}).$
  - (3) For all  $a \in |M|$ ,  $N_0^M(a) = N_0^{M'}(a)$ .

We show in Lemma 9.4.7 that  $\mathbf{K}'$  is an AEC, but first let us see that this suffices:

LEMMA 9.4.6. Let  $\lambda \geq \chi$ .

- (1) If  $M \in \mathbf{K}_{\lambda}$  is saturated, then there exists an expansion M' of M to  $\tau'$ such that  $M' \in \mathbf{K}'$ .
- (2) If  $M' \in \mathbf{K}'$  has size  $\lambda$ , then  $M' \upharpoonright \tau(\mathbf{K})$  is saturated.

# Proof.

(1) Let  $R^{M'}$  be a well-ordering of |M| of type  $\lambda$ . Identify |M| with  $\lambda$ . By stability, we can fix a bijection  $p \mapsto a_p$  from gS(M) onto |M|. For each  $p \in \mathrm{gS}(M)$  which is not algebraic, fix  $N_p \leq_{\mathbf{K}} M$  saturated such that p does not fork over  $N_p$  and  $||N_p|| = \chi_2$ . Then use saturation to build  $\langle a_p^i : i < \lambda \rangle \land \langle N_p^i : i < \lambda \rangle$  Morley for p over  $N_p$  (inside M). Let  $N_0^{M'}(a_p) := N_p, N^{M'}(a_p, i) := N_p^i, F^{M'}(a, i) := a_p^i$ . For p algebraic, pick

<sup>&</sup>lt;sup>5</sup>For a binary relation Q we write Q(a) for  $\{b \mid Q(a, b)\}$ , similarly for a ternary relation.

 $<sup>^{6}</sup>$ Note that by Lemma 9.3.12 this also implies that it does not syntactically split over some  $M_0' \leq_{\mathbf{K}} N_0^M(a)$  with  $\|M_0'\| < \chi_2$ 

 $p_0 \in \mathrm{gS}(M)$  nonalgebraic and let  $N_0^{M'}(a_p) := N_0^{M'}(a_{p_0}), \ N^{M'}(a_{p_0}) := N^{M'}(a_{p_0}), \ F^{M'}(a_p) := F^{M'}(a_{p_0}).$ (2) By Lemma 9.3.16.

LEMMA 9.4.7.  $\mathbf{K}'$  is an AEC with  $LS(\mathbf{K}') = \chi$ .

PROOF. It is straightforward to check that  $\mathbf{K}'$  is an abstract class with coherence. Moreover:

- <u>**K**' satisfies the chain axioms</u>: Let  $\langle M_i : i < \delta \rangle$  be increasing in **K**'. Let  $M_{\delta} := \bigcup_{i < \delta} M_i$ .
  - $-M_0 \leq_{\mathbf{K}'} M_{\delta}$ , and if  $N \geq_{\mathbf{K}'} M_i$  for all  $i < \delta$ , then  $N \geq_{\mathbf{K}'} M_{\delta}$ : Straightforward.
  - $-M_{\delta} \in \mathbf{K}': M_{\delta} \upharpoonright \tau(\mathbf{K})$  is  $\chi$ -saturated by Fact 9.3.8. Moreover,  $R^{M_{\delta}}$  is clearly a linear ordering of  $M_{\delta}$ . Write  $I_i$  for the linear ordering  $(M_i, R_i)$ . Condition 3 in the definition of  $\mathbf{K}'$  is also easily checked. We now check Condition 4.

Let  $a \in |M_{\delta}|$ . Fix  $i < \delta$  such that  $a \in |M_i|$ . Without loss of generality, i = 0. By hypothesis, for each  $i < \delta$ , there exists  $p_a^i \in \mathrm{gS}(M_i \upharpoonright \tau(\mathbf{K}))$  not algebraic such that  $\langle F^{M_i}(a,j) \mid j \in I_i \rangle \frown \langle N^{M_i}(a,j) \mid j \in I_i \rangle$  is a Morley sequence for  $p_a^i$  over  $N_0^{M_i}(a) = N_0^{M_0}(a)$ . Clearly,  $p_a^i \upharpoonright N_0^{M_0}(a) = p_a^0 \upharpoonright N_0^{M_0}(a)$  for all  $i < \delta$ . Moreover by assumption  $p_a^i$  does not fork over  $N_0^{M_0}$ . Thus for all  $i < j < \delta$ ,  $p_a^i \upharpoonright M_i = p_a^i \upharpoonright M_i$ . By extension and uniqueness, there exists  $p_a \in \mathrm{gS}(M_\delta \upharpoonright \tau(\mathbf{K}))$  that does not fork over  $N_0^{M_0}(a)$  and we have  $p_a \upharpoonright M_i = p_a^i$  for all  $i < \delta$ . Now by Lemma 9.4.2,  $\langle F^{M_\delta}(a,j) \mid j \in I_\delta \rangle \frown \langle N^{M_\delta}(a,j) \mid j \in I_\delta \rangle$  is a Morley sequence for  $p_a$  over  $N_0^{M_0}(a)$ .

Moreover, the map  $a \mapsto p_a$  is onto the nonalgebraic Galois types over  $M_{\delta} \upharpoonright \tau(\mathbf{K})$ : let  $p \in \mathrm{gS}(M_{\delta} \upharpoonright \tau(\mathbf{K}))$  be nonalgebraic. Then there exists  $i < \delta$  such that p does not fork over  $M_i$ . Let  $a \in |M_i|$  be such that  $\langle F^{M_i}(a,j) \mid j \in I_i \rangle \frown \langle N^{M_i}(a,j) \mid j \in I_i \rangle$  is a Morley sequence for  $p \upharpoonright M_i$  over  $N_0^{M_i}(a)$ . It is easy to check it is also a Morley sequence for p over  $N_0^{M_i}(a)$ . By uniqueness of the nonforking extension, we get that the extended Morley sequence is also Morley for p, as desired.

•  $LS(\mathbf{K}') = \chi$ : An easy closure argument.

THEOREM 9.4.8. **K** is uniformly  $(\chi, \chi)$ -solvable.

PROOF. By Lemma 9.4.7,  $\mathbf{K}'$  is an AEC with  $\mathrm{LS}(\mathbf{K}') = \chi$ . Now combine Lemma 9.4.6 and Lemma 9.2.16. Note that by Fact 9.3.8, for each  $\lambda \geq \chi$  there is a saturated model of size  $\lambda$ , and it is also a superlimit.

For the convenience of the reader, we give a more quotable version of Theorem 9.4.8. For the next results, we drop Hypothesis 9.4.1.

THEOREM 9.4.9. Assume that **K** has a monster model, is  $LS(\mathbf{K})$ -tame, and is stable in some cardinal greater than or equal to  $LS(\mathbf{K})$ . There exists  $\chi < H_1$  such that for any  $\mu \geq \chi$ , if **K** is stable in  $\mu$  and has no long splitting chains in  $\mu$  then **K** is uniformly  $(\mu', \mu')$ -solvable, where  $\mu' := (\beth_{\omega+2}(\mu))^+$ .

PROOF. Hypothesis 9.3.1 holds. Let  $\chi < H_1$  be such that **K** does not have the LS(**K**)-order property of length  $\chi$  (see Fact 9.3.2).

Let  $\mu \geq \chi$  be such that **K** is stable in  $\mu$  and has no long splitting chains in  $\mu$ . We apply Theorem 9.4.8 by letting  $\chi_0$  in Notation 9.3.4 stand for  $\mu$  here. By Fact 9.2.3, **K** is stable in  $\mu_1$  and has no long splitting chains in  $\mu_1$  for every  $\mu_1 \geq \mu$ , thus Hypothesis 9.4.1 holds. Moreover  $\chi_2$  in Notation 9.3.10 corresponds to  $\beth_{\omega}(\mu)$  here, and  $\chi$  in Notation 9.4.3 corresponds to  $\mu'$  here. Thus the result follows from Theorem 9.4.8.

COROLLARY 9.4.10. Assume that **K** has a monster model and is LS(**K**)-tame. If there exists  $\mu < H_1$  such that **K** is stable in  $\mu$  and has no long splitting chains in  $\mu$ , then there exists  $\mu' < H_1$  such that **K** is uniformly  $(\mu', \mu')$ -solvable.

PROOF. Let  $\mu < H_1$  be such that **K** is stable in  $\mu$  and has no long splitting chains in  $\mu$ . Fix  $\chi < H_1$  as given by Theorem 9.4.9. Without loss of generality,  $\mu \leq \chi$ . By Fact 9.2.3, **K** is stable in  $\chi$  and has no long splitting chains in  $\chi$ , so apply the conclusion of Theorem 9.4.9.

## 9.5. Superstability below the Hanf number

In this section, we prove the main corollary. In fact, we prove a stronger version that instead of asking for the properties to hold on a tail asks for them to hold only in a single high-enough cardinal. Toward this end, we start by explaining why no long splitting chains follows from categoricity in a high-enough cardinal. In fact, categoricity can be replaced by solvability. All the ingredients for this result are contained in [**SV99**] and this specific form has only appeared recently (see Chapter 20). Note also that Shelah states a similar result in [**She99**, 5.5] but his definition of superstability is different.

FACT 9.5.1 (The ZFC Shelah-Villaveces theorem). Let **K** be an AEC with arbitrarily large models and amalgamation<sup>7</sup> in LS(**K**). Let  $\lambda > \text{LS}(\mathbf{K})$  be such that  $\mathbf{K}_{<\lambda}$  has no maximal models. If **K** is  $(\lambda, \text{LS}(\mathbf{K}))$ -solvable, then **K** is stable in LS(**K**) and has no long splitting chains in LS(**K**).

COROLLARY 9.5.2. Let **K** be an AEC with a monster model. Let  $\lambda > LS(\mathbf{K})$ . If **K** is categorical in  $\lambda$ , then **K** is stable in  $\mu$  and has no long splitting chains in  $\mu$  for all  $\mu \in [LS(\mathbf{K}), \lambda)$ .

PROOF. By Fact 9.5.1 applied to  $\mathbf{K}_{\geq \mu}$  for each  $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$ . Note that, since **K** has arbitrarily large models, categoricity in  $\lambda$  implies  $(\lambda, \mathrm{LS}(\mathbf{K}))$ -solvability.  $\Box$ 

We conclude that solvability is equivalent to superstability in the first-order case:

COROLLARY 9.5.3. Let T be a first-order theory and let **K** be the AEC of models of T ordered by elementary substructure. Let  $\mu \ge |T|$ . The following are equivalent:

(1) T is stable in all  $\lambda \ge \mu$ .

- (2) **K** is  $(\lambda, \mu)$ -solvable, for some  $\lambda > \mu$ .
- (3) **K** is uniformly  $(\mu, \mu)$ -solvable.

<sup>&</sup>lt;sup>7</sup>In [SV99], this is replaced by the generalized continuum hypothesis (GCH).

PROOF SKETCH. (3) implies (2) is trivial. (2) implies (1) is by Corollary 9.5.2 together with Fact 9.2.3). Finally, (1) implies (3) is as in the proof of Theorem 9.4.9.  $\Box$ 

We can also use the ZFC Shelah-Villaveces theorem to prove the following interesting result, showing that the solvability spectrum satisfies an analog of Shelah's categoricity conjecture in tame AECs (Shelah asks what the behavior of the solvability spectrum should be in [She09a, Question N.4.4]).

THEOREM 9.5.4. Assume that **K** has a monster model and is LS(**K**)-tame. There exists  $\chi < H_1$  such that for any  $\mu \ge \chi$ , if **K** is  $(\lambda, \mu)$ -solvable for some  $\lambda > \mu$ , then **K** is uniformly  $(\mu', \mu')$ -solvable, where  $\mu' := (\beth_{\omega+2}(\mu))^+$ .

PROOF. Let  $\chi < H_1$  be as given by Theorem 9.4.9. Let  $\mu \ge \chi$  and fix  $\lambda > \mu$  such that **K** is solvable in  $\lambda$ . By Fact 9.5.1, **K** is stable in  $\mu$  and has no long splitting chains in  $\mu$ . Now apply Theorem 9.4.9.

We are now ready to prove the stronger version of the main corollary where the properties hold only in a single high-enough cardinal below  $H_1$  (but the cardinal may be different for each property).

COROLLARY 9.5.5. Assume that **K** has a monster model, is  $LS(\mathbf{K})$ -tame, and is stable in some cardinal greater than or equal to  $LS(\mathbf{K})$ . Then there exists  $\chi \in (LS(\mathbf{K}), H_1)$  such that the following are equivalent:

- (1)<sup>-</sup> For some  $\lambda_1 \in [\chi, H_1)$ , **K** is stable in  $\lambda_1$  and has no long splitting chains in  $\lambda_1$ .
- (2)<sup>-</sup> For some  $\lambda_2 \in [\chi, H_1)$ , there is a good  $\lambda_2$ -frame on a skeleton of  $\mathbf{K}_{\lambda_2}$ .
- (3)<sup>-</sup> For some  $\lambda_3 \in [\chi, H_1)$ , **K** has a unique limit model of cardinality  $\lambda_3$ .
- (4)<sup>-</sup> For some  $\lambda_4 \in [\chi, H_1)$ , **K** is stable in  $\lambda_4$  and has a superlimit model of cardinality  $\lambda_4$ .
- (5)<sup>-</sup> For some  $\lambda_5 \in [\chi, H_1)$ , the union of any increasing chain of  $\lambda_5$ -saturated models is  $\lambda_5$ -saturated.
- (6)<sup>-</sup> For some  $\lambda_6 \in [\chi, H_1)$ , for some  $\mu < \lambda_6$ , **K** is  $(\lambda_6, \mu)$ -solvable.

REMARK 9.5.6. In  $(2)^-$ , we do not assume that the good frame is type-full (i.e. it may be that there exists some nonalgebraic types which are not basic, so fork over their domain). However if  $(1)^-$  holds, then the proof of  $(1)^-$  implies  $(2)^-$  (Fact 9.2.12) actually builds a *type-full* frame. Therefore, in the presence of tameness, the existence of a good frame implies the existence of a *type-full* good frame (in a potentially much higher cardinal, and over a different class).

PROOF OF COROLLARY 9.5.5. By Fact 2.4.15, **K** does not have the LS(**K**)order property. By Fact 9.3.2, there exists  $\chi_0 < H_1$  such that **K** does not have the LS(**K**)-order property of length  $\chi_0$ . Let  $\chi := \beth_{\omega} (\chi_0 + \text{LS}(\mathbf{K}))$ .

We will use the following auxiliary condition, which is a weakening of  $(3)^-$  (the problem is that we do not quite know that  $(5)^-$  implies  $(3)^-$  as **K** might not be stable in  $\lambda_5$ ):

(3)\* For some  $\lambda_3^* \in [\chi, H_1)$ , **K** is stable in  $\lambda_3^*$ , has a saturated model of cardinality  $\lambda_3^*$ , and every limit model of cardinality  $\lambda_3^*$  is  $\chi$ -saturated.

We will prove the following claims, which put together give us what we want: <u>Claim 1</u>:  $(1)^- \Leftrightarrow (6)^-$ .

 $\underline{\text{Claim } 2}: (3)^* \Rightarrow (1)^-.$ 

<u>Claim 3</u>: For  $\ell \in \{1, 2, 3, 4, 5\}, (\ell)^- \Rightarrow (3)^*$ . Proof of Claim 1: By Theorem 9.4.9 and Fact 9.5.1. †<sub>Claim 1</sub> <u>Proof of Claim 2</u>: This is Theorem 9.3.18, where  $\chi_2$  there stands for  $\chi$  here. <sup>†</sup>Claim 2 <u>Proof of Claim 3</u>: It is enough to prove the following subclaims: Subclaim 1:  $(1)^- \Rightarrow (2)^- \Rightarrow (3)^-$ . Subclaim 2:  $(4)^- \Rightarrow (3)^-$ . Subclaim 3:  $(3)^- \Rightarrow (3)^*$ . Subclaim 4:  $(5)^- \Rightarrow (3)^*$ . Proof of Subclaim 1: By Fact 9.2.12. <sup>†</sup>Subclaim 1 Proof of Subclaim 2: By Fact 9.2.5(4). <sup>†</sup>Subclaim 2 Proof of Subclaim 3: By Fact 9.2.5(3). <sup>†</sup>Subclaim 3 <u>Proof of Subclaim 4</u>: Let  $\lambda_3^* \in [\lambda_5, H_1)$  be a regular stability cardinal. Then **K** has a saturated model of cardinality  $\lambda_3^*$ , and from  $(5)^-$  it is easy to see that any limit model of cardinality  $\lambda_3^*$  is  $\lambda_5$ -saturated, hence  $\chi$ -saturated.  $\dagger_{\text{Subclaim 4}}$ 

We can now prove the main result of this chapter (Corollary 9.1.3):

PROOF OF COROLLARY 9.1.3. Let  $\chi$  be as given by Corollary 9.5.5. By Fact 2.4.15, there exists unboundedly-many regular stability cardinals in  $(\chi, H_1)$ . This implies that for  $\ell \in \{1, 2, 3, 4, 5, 6\}$ ,  $(\ell)$  (from Corollary 9.1.3) implies  $(\ell)^-$  (from Corollary 9.5.5). Moreover  $(1)^-$  implies both (1) and (7) by Fact 9.2.3. Since Corollary 9.5.5 tells us that  $(\ell_1)^-$  is equivalent to  $(\ell_2)^-$  for  $\ell_1, \ell_2 \in \{1, 2, 3, 4, 5, 6\}$ , it follows that  $(\ell_1)$  is equivalent to  $(\ell_2)$  as well, and (7) is implied by any of these conditions.

QUESTION 9.5.7. Is stability in  $\lambda_4$  needed in condition (4)<sup>-</sup> of Corollary 9.5.5? That is, can one replace the condition with:

 $(4)^{--}$  For some  $\lambda_4 \in [\chi, \theta)$ , **K** has a superlimit model of cardinality  $\lambda_4$ .

The answer is positive when  $\mathbf{K}$  is an elementary class [She12, 3.1].

#### 9.6. Future work

While we managed to prove that some analogs of the conditions in Fact 9.1.1 are equivalent, much remains to be done.

For example, one may want to make precise what the analog to (5) and (6) in 9.1.1 should be in tame AECs. One possible definition for (6) could be:

DEFINITION 9.6.1. Let  $\lambda, \mu > \text{LS}(\mathbf{K})$ . We say that  $\mathbf{K}$  has the  $(\lambda, \mu)$ -tree property provided there exists  $\{p_n(\mathbf{x}; \mathbf{y}_n) \mid n < \omega\}$  Galois-types over models of size less than  $\mu$  and  $\{M_\eta \mid \eta \in {}^{\leq \omega}\lambda\}$  such that for all  $n < \omega, \nu \in {}^n\lambda$  and every  $\eta \in {}^{\omega}\lambda$ :

 $\langle M_n, M_\nu \rangle \models p_n \iff \nu$  is an initial segment of  $\eta$ .

We say that **K** has the *tree property* if it has it for all high-enough  $\mu$  and all high-enough  $\lambda$  (where the "high-enough" quantifier on  $\lambda$  can depend on  $\mu$ ).

We can ask whether no long splitting chains (or any other reasonable definition of superstability) implies that  $\mathbf{K}$  does not have the tree property, or at least obtain

many models from the tree property as in **[GS86a]**. This is conjectured in **[She99**] (see the remark after Claim 5.5 there).

As for the D-rank in 9.1.1(5), perhaps a simpler analog would be the U-rank defined in terms of  $(< \kappa)$ -satisfiability in [**BG**, 7.2] (another candidate for a rank is Lieberman's *R*-rank, see [**Lie13**]). By [**BG**, 7.9], no long splitting chains implies that the U-rank is bounded but we do not know how to prove the converse. Perhaps it is possible to show that  $U[p] = \infty$  implies the tree property.

# CHAPTER 10

# Symmetry in abstract elementary classes with amalgamation

This chapter is based on [VV17] and is joint work with Monica VanDieren. We thank the referees for reports which helped improve the presentation of this paper.

## Abstract

This chapter is part of a program initiated by Saharon Shelah to extend the model theory of first order logic to the non-elementary setting of abstract elementary classes (AECs). An abstract elementary class is a semantic generalization of the class of models of a complete first order theory with the elementary substructure relation. We examine the symmetry property of splitting (previously isolated by VanDieren) in AECs with amalgamation that satisfy a local definition of superstability.

The key results are a downward transfer of symmetry and a deduction of symmetry from failure of the order property. These results are then used to prove several structural properties in categorical AECs, improving classical results of Shelah who focused on the special case of categoricity in a successor cardinal.

We also study the interaction of symmetry with tameness, a locality property for Galois (orbital) types. We show that superstability and tameness together imply symmetry. This sharpens some results from Chapter 7.

### 10.1. Introduction

The guiding conjecture for the classification of abstract elementary classes (AECs) is Shelah's categoricity conjecture. For an introduction to AECs and Shelah's cateogicity conjecture, see [Bal09].

Although most progress towards Shelah's categoricity conjecture has been made under the assumption that the categoricity cardinal is a successor, e.g. [She99, GV06a, Bon14b], in Chapters 8 and 16, we prove a categoricity transfer theorem without assuming that the categoricity cardinal is a successor, but assuming that the class is universal, Chapters 8 and 16 (other partial results not assuming categoricity in a successor cardinal are in Chapter 6 and [She09a, Chapter IV]). In this chapter, we work in a more general framework than Chapters 8, 16. We assume the amalgamation property and no maximal models and deduce new structural results without having to assume that the categoricity cardinal is a successor, or even has "high-enough" cofinality.

Beyond Shelah's categoricity conjecture, a major focus in developing a classification theory for AECs has been to find an appropriate generalization of first-order superstability. Approximations isolated in [She99] and [SV99] have provided a mechanism for proving categoricity transfer results (see also [**GV06a**], Chapters 8, 16). In Chapter IV of [**She09a**], Shelah introduced *solvability* and claims it should be the true definition of superstability in AECs (see Discussion 2.9 in the introduction to [**She09a**]). It seems, however, that under the assumption that the class has amalgamation, a more natural definition is a version of " $\kappa(T) = \aleph_0$ ", first considered by Shelah and Villaveces in [**SV99**]. In Chapter 9, it is shown that this definition is equivalent to many others (including solvability and the existence of a *good frame*, a local notion of independence), provided that the AEC satisfies a locality property for types called *tameness* [**GV06b**].

Without tameness, progress has been made in the study of structural consequences of the Shelah-Villaveces definition of superstability such as the uniqueness of limit models (e.g. [**GVV16**]) or the property that the union of saturated models is saturated (Chapter 7 or [**Van16b**]). Recently in [**Van16a**], VanDieren isolated a symmetry property for splitting that turns out to be closely related to the uniqueness of limit models.

**10.1.1. Transferring symmetry.** In this chapter we prove a downward transfer theorem for this symmetry property. This allows us to gain insight into all of the aspects of superstability mentioned above.

THEOREM 10.1.1. Let **K** be an AEC. Suppose  $\lambda$  and  $\mu$  are cardinals so that  $\lambda > \mu \ge \text{LS}(\mathbf{K})$  and **K** is superstable in every  $\chi \in [\mu, \lambda]$ . Then  $\lambda$ -symmetry implies  $\mu$ -symmetry.

Theorem 10.1.1 (proven at the end of Section 10.3) improves Theorem 2 of [Van16b] which transfers symmetry from  $\mu^+$  to  $\mu$ . We also clarify the relationship between  $\mu$ -symmetry (as a property of  $\mu$ -splitting) and the symmetry property in good frames (see Section 10.4). The latter is older and has been studied in the literature: the work of Shelah in [She01a] led to [She09a, Theorem 3.7], which gives conditions under which a good frame (satisfying a version of symmetry) exists (but uses set-theoretic axioms beyond ZFC and categoricity in two successive cardinals). One should also mention [She09a, Theorem IV.4.10] which builds a good frame (in ZFC) from categoricity in a high-enough cardinal. Note, however, the cardinal is very high and the underlying class of the frame is a smaller class of Ehrenfeucht-Mostowski models, although this can be fixed by taking an even larger cardinal.

It was observed in Theorem 3.5.13 that Shelah's proof of symmetry of first-order forking generalizes naturally to give that the symmetry property of any reasonable global independence notion follows from the assumption of no order property. This is used in Chapter 4 to build a good frame from tameness and categoricity (the results there are improved in Chapters 6 and 7). As for symmetry *transfers*, Boney [**Bon14a**] has shown how to transfer symmetry of a good frame *upward* using tameness for types of length two. This was later improved to tameness for types of length one with a more conceptual proof in Chapter 5.

Theorem 10.1.1 differs from these works in a few ways. First, we do not assume tameness nor set-theoretic assumptions, and we do not work within the full strength of a frame or with categoricity (only with superstability). Also, we obtain a *downward* and not an upward transfer. The arguments of this chapter use *towers* whereas the aforementioned result of Chapter 5 use independent sequences to transfer symmetry upward. 10.1.2. Symmetry and superstability. Another consequence of our work is a better understanding of the relationship between superstability and symmetry. It was claimed in an early version of  $[\mathbf{GVV16}]$  that  $\mu$ -superstability directly implies the uniqueness of limit models of size  $\mu$  but an error was later found in the proof. Here we show that this *is* true provided we have enough instances of superstability:

THEOREM 10.5.4. Let **K** be an AEC and let  $\mu \geq \text{LS}(\mathbf{K})$ . If **K** is superstable in all  $\mu' \in [\mu, \beth_{(2^{\mu})^+})$ , then **K** has  $\mu$ -symmetry.

The main idea is to imitate the proof of the aforementioned Theorem 3.5.13 to get the order property from failure of symmetry. However we do not have as much global independence as there so the proof here is quite technical.

10.1.3. Implications in categorical AECs. As a corollary of Theorem 10.5.4, we obtain several applications to categorical AECs. A notable contribution of this chapter is an improvement on a 1999 result of Shelah (see [She99, Theorem 6.5]):

FACT 10.1.2. Let **K** be an AEC with amalgamation and no maximal models. Let  $\lambda$  and  $\mu$  be cardinals such that  $cf(\lambda) > \mu > LS(\mathbf{K})$ . If **K** is categorical in  $\lambda$ , then any limit model of size  $\mu$  is saturated.

Shelah claims in a remark immediately following his result that this can be generalized to show that for  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$ , if  $M_1$  and  $M_2$  are limit over  $M_0$ , then  $M_1 \cong_{M_0} M_2$  (that is, the isomorphism also fixes  $M_0$ ). This is however not what his proof gives (see the discussion after Theorem 10.17 in [**Bal09**]). Here we finally prove this stronger statement. Moreover, we can replace the hypothesis that  $\mathrm{cf}(\lambda) > \mu$  by  $\lambda \geq \beth_{(2^{\mu})^+}$ . That is, it is enough to ask for  $\lambda$  to be high-enough (but of arbitrary cofinality):

COROLLARY 10.7.3.. Let **K** be an AEC with amalgamation and no maximal models. Let  $\lambda$  and  $\mu$  are cardinals so that  $\lambda > \mu \ge \text{LS}(\mathbf{K})$  and assume that **K** is categorical in  $\lambda$ . If either  $\text{cf}(\lambda) > \mu$  or  $\lambda \ge \beth_{(2^{\mu})^+}$ , then whenever  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$ are such that both  $M_1$  and  $M_2$  are limit models over  $M_0$ , we have that  $M_1 \cong_{M_0} M_2$ .

This gives a proof (assuming amalgamation, no maximal models, and a highenough categoricity cardinal) of the (in)famous [**SV99**, Theorem 3.3.7], where a gap was identified in the VanDieren's Ph.D. thesis. The gap was fixed assuming categoricity in  $\mu^+$  in [**Van06**, **Van13**] (see also the exposition in [**GVV16**]). In [**BG**, Corollary 6.18], this was improved to categoricity in an arbitrary  $\lambda > \mu$ provided that  $\mu$  is big-enough and the class satisfies strong locality assumptions (full tameness and shortness and the extension property for coheir). In Theorem 4.7.11, only tameness was required but the categoricity in any  $\lambda \ge \beth_{(2^{\mu})^+}$ in Theorem 7.6.1. Here assuming tameness we will improve this to categoricity in any  $\lambda > \text{LS}(\mathbf{K})$  (see Corollary 10.7.10).

In general, we obtain that an AEC with amalgamation categorical in a highenough cardinal has several structural properties that were previously only known for AECs categorical in a cardinal of high-enough *cofinality*, or even just in a successor. COROLLARY 10.1.3. Let **K** be an AEC with amalgamation. Let  $\lambda > \mu \ge LS(\mathbf{K})$ and assume that **K** is categorical in  $\lambda$ . Let  $\mu \ge LS(\mathbf{K})$ . If **K** is categorical in a  $\lambda > \mu$ , then:

- (1) (see Corollary 10.7.4) If  $\lambda \geq \beth_{(2^{\mu})^+}$  and  $\mu > \text{LS}(\mathbf{K})$ , then the model of size  $\lambda$  is  $\mu$ -saturated.
- (2) (see Corollary 10.7.9) If  $\mu \geq \beth_{(2^{\text{LS}(\mathbf{K})})^+}$  and the model of size  $\lambda$  is  $\mu^+$ -saturated, then there exists a type-full good  $\mu$ -frame with underlying class the saturated models in  $\mathbf{K}_{\mu}$ .

This improves several classical results from Shelah's milestone study of categorical AECs with amalgamation [She99]:

- Corollay 10.1.3.(1) partially answers Baldwin [Bal09, Problem D.1.(2)] which asked if in any AEC with amalgamation categorical in a high-enough cardinal, then model in the categoricity cardinal is saturated.
- Corollay 10.1.3.(2) partially answers the question in [She99, Remark 4.9.(1)] of whether there is a parallel to forking in categorical AECs with amalgamation. It also improves on Theorem 4.7.4, which assumed categoricity in a successor (and a higher Hanf number bound).
- As part of the proof of Corollay 10.1.3.(2), we derive *weak tameness* (i.e. tameness over saturated models) from categoricity in a big-enough cardinal (this is Corollary 10.7.5). It was previously only known how to do so assuming that the categoricity cardinal has high-enough cofinality [She99, Main Claim II.2.3].

We deduce a downward categoricity transfer in AECs with amalgamation (see also Corollary 10.7.7):

COROLLARY 10.7.8. Let **K** be an AEC with amalgamation. Let  $LS(\mathbf{K}) < \mu = \Box_{\mu} < \lambda$ . If **K** is categorical in  $\lambda$ , then **K** is categorical in  $\mu$ .

This improves on Theorem 6.10.16 where the result is stated with the additional assumption of  $(< \mu)$ -tameness.

10.1.4. Implications in tame AECs. This chapter also combines our results with tameness: in Section 10.6, we improve Hanf number bounds for several consequences of superstability. With Will Boney, we have shown (Theorem 7.6.1) that  $\mu$ -superstability and  $\mu$ -tameness imply that for all high-enough  $\lambda$ , limit models of size  $\lambda$  are unique (in the strong sense discussed above), unions of chains of  $\lambda$ -saturated models are saturated, and there exists a type-full good  $\lambda$ -frame. We transfer this behavior downward using our symmetry transfer theorem to get that the latter result is actually true starting from  $\lambda = \mu^+$ , and the former starting from  $\lambda = \mu$ :

COROLLARY 10.1.4. Let  $\mu \geq LS(\mathbf{K})$ . If **K** is  $\mu$ -superstable and  $\mu$ -tame, then:

- (1) (see Corollary 10.6.9) If  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$  are such that both  $M_1$  and  $M_2$  are limit models over  $M_0$ , then  $M_1 \cong_{M_0} M_2$ .
- (2) (see Corollary 10.6.10) For any λ > μ, the union of an increasing chain of λ-saturated models is λ-saturated.
- (3) (see Corollary 10.6.14) There exists a type-full good  $\mu^+$ -frame with underlying class the saturated models in  $\mathbf{K}_{\mu^+}$ .

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In fact,  $\mu$ -tameness along with  $\mu$ -superstability already implies  $\mu$ -symmetry. Many assumptions weaker than tameness (such as the existence of a good  $\mu^+$ -frame, see Theorem 10.4.15) suffice to obtain such a conclusion.

10.1.5. Notes. A word on the background needed to read this chapter: It is assumed that the reader has a solid knowledge of AECs (including the material in [Bal09]). Some familiarity with good frames, in particular the material of Chapter 4 would be very helpful. In addition to classical results, e.g. in [She99], the chapter uses heavily the results of [Van16a, Van16b] on limit models and the symmetry property of splitting. It also relies on the construction of a good frame in Chapter 4. At one point we also use the canonicity theorem for good frames (Theorem 6.9.7).

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All throughout this chapter, we assume the amalgamation property:

HYPOTHESIS 10.2.1. K is an AEC with amalgamation.

For convenience, we fix a big-enough monster model  $\mathfrak{C}$  and work inside  $\mathfrak{C}$ . This is possible since by Remark 10.2.9, we will have the joint embedding property in addition to the amalgamation property for models of the relevant cardinalities. At some point, we will also use the following fact whose proof is folklore (see e.g. Proposition 6.10.13).

FACT 10.2.2. Assume that **K** has joint embedding in some  $\lambda \geq LS(\mathbf{K})$ . Then there exists  $\chi < \beth_{(2^{LS}(\mathbf{K}))^+}$  and an AEC  $\mathbf{K}^*$  such that:

- (1)  $\mathbf{K}^* \subseteq \mathbf{K}$  and  $\mathbf{K}^*$  has the same strong substructure relation as  $\mathbf{K}$ .
- (2)  $LS(\mathbf{K}^*) = LS(\mathbf{K}).$
- (3)  $\mathbf{K}^*$  has amalgamation, joint embedding, and no maximal models.
- (4)  $\mathbf{K}_{\geq\min(\lambda,\chi)} = (\mathbf{K}^*)_{\geq\min(\lambda,\chi)}.$

Many of the pre-requisite definitions and notations used in this chapter can be found in [**GVV16**]. Here we recall the more specialized concepts that we will be using explicitly.

We write  $gtp(\bar{a}/M)$  for the Galois type of the sequence  $\bar{a}$  over M (and we write gS(M) for the set of all Galois types over M). While the reader can think of  $gtp(\bar{a}/M)$  as the orbit of  $\bar{a}$  under the action of  $Aut_M(\mathfrak{C})$ ,  $gtp(\bar{a}/M)$  is really defined as the equivalence class of the triple  $(\bar{a}, M, \mathfrak{C})$  under a certain equivalence relation (see for example [**Gro02**, Definition 6.4]). This allows us to define the restriction of a Galois type to any strong substructure of its domain, as well as its image under any automorphism of  $\mathfrak{C}$  (and by extension any **K**-embedding whose domain contains the domain of the type).

With that remark in mind, we can state the definition of non-splitting, a notion of independence from [She99, Definition 3.2].

DEFINITION 10.2.3. A type  $p \in gS(N)$  does not  $\mu$ -split over M if and only if for any  $N_1, N_2 \in \mathbf{K}_{\mu}$  such that  $M \leq_{\mathbf{K}} N_{\ell} \leq_{\mathbf{K}} N$  for  $\ell = 1, 2$ , and any  $f : N_1 \cong_M N_2$ , we have  $f(p \upharpoonright N_1) = p \upharpoonright N_2$ 

We will use the definition of universality from [Van06, Definition I.2.1]:

DEFINITION 10.2.4. Let  $M, N \in \mathbf{K}$  be such that  $M \leq_{\mathbf{K}} N$ . We say that N is  $\mu$ -universal over M if for any  $M' \geq_{\mathbf{K}} M$  with  $||M'|| \leq \mu$ , there exists  $f: M' \xrightarrow{M} N$ . We say that N is universal over M if N is ||M||-universal over M.

A fundamental concept in the study of superstable AECs is the notion of a *limit model*, first introduced in [She99]. We only give the definition here and refer the reader to [GVV16] for more history and motivation.

DEFINITION 10.2.5. Let  $\mu \geq LS(\mathbf{K})$  and let  $\alpha < \mu^+$  be a limit ordinal. Let  $M \in \mathbf{K}_{\mu}$ . We say that N is  $(\mu, \alpha)$ -limit over M (or a  $(\mu, \alpha)$ -limit model over M) if there exists a strictly increasing continuous chain  $\langle M_i : i \leq \alpha \rangle$  in  $\mathbf{K}_{\mu}$  such that  $M_0 = M$ ,  $M_{\alpha} = N$ , and  $M_{i+1}$  is universal over  $M_i$  for all  $i < \alpha$ . We say that N is limit over M (or a limit model over M) if it is  $(\mu, \beta)$ -limit over M for some  $\beta < \mu^+$ . Finally, we say that N is limit if it is limit over  $N_0$  for some  $N_0 \in \mathbf{K}_{\parallel N \parallel}$ .

Towers were introduced in Shelah and Villaveces [**SV99**] as a tool to prove the uniqueness of limit models. A tower is an increasing sequence of length  $\alpha$  of limit models, denoted by  $\overline{M} = \langle M_i \in \mathbf{K}_{\mu} \mid i < \alpha \rangle$ , along with a sequence of designated elements  $\overline{a} = \langle a_i \in M_{i+1} \setminus M_i \mid i+1 < \alpha \rangle$  and a sequence of designated submodels  $\overline{N} = \langle N_i \mid i+1 < \alpha \rangle$  for which  $N_i \leq_{\mathbf{K}} M_i$ ,  $\operatorname{tp}(a_i/M_i)$  does not  $\mu$ -split over  $N_i$ , and  $M_i$  is universal over  $N_i$  (see Definition I.5.1 of [Van06]).

Now we recall a bit of terminology regarding towers. The collection of all towers  $(\bar{M}, \bar{a}, \bar{N})$  made up of models of cardinality  $\mu$  and sequences indexed by  $\alpha$  is denoted by  $\mathbf{K}^*_{\mu,\alpha}$ . For  $(\bar{M}, \bar{a}, \bar{N}) \in \mathbf{K}^*_{\mu,\alpha}$ , if  $\beta < \alpha$  then we write  $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright \beta$  for the tower made of the subsequences  $\bar{M} \upharpoonright \beta = \langle M_i \mid i < \beta \rangle$ ,  $\bar{a} \upharpoonright \beta = \langle a_i \mid i + 1 < \beta \rangle$ , and  $\bar{N} \upharpoonright \beta = \langle N_i \mid i + 1 < \beta \rangle$ . We sometimes abbreviate the tower  $(\bar{M}, \bar{a}, \bar{N})$  by  $\mathcal{T}$ .

DEFINITION 10.2.6. For towers  $(\overline{M}, \overline{a}, \overline{N})$  and  $(\overline{M}', \overline{a}', \overline{N}')$  in  $\mathbf{K}^*_{\mu,\alpha}$ , we say

$$(\bar{M}, \bar{a}, \bar{N}) \le (\bar{M}', \bar{a}', \bar{N}')$$

if for all  $i < \alpha$ ,  $M_i \leq_{\mathbf{K}} M'_i$ ,  $\bar{a} = \bar{a}'$ ,  $\bar{N} = \bar{N}'$  and whenever  $M'_i$  is a proper extension of  $M_i$ , then  $M'_i$  is universal over  $M_i$ . If for each  $i < \alpha$ ,  $M'_i$  is universal over  $M_i$  we will write  $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}', \bar{a}', \bar{N}')$ .

In order to transfer symmetry from  $\lambda$  to  $\mu$  we will need to consider a generalization of these towers where the models  $M_i$  and  $N_i$  may have different cardinalities. Fix  $\lambda \geq \mu \geq \text{LS}(\mathbf{K})$  and  $\alpha$  a limit ordinal  $\langle \mu^+$ . We will write  $\mathbf{K}^*_{\lambda,\alpha,\mu}$  for the collection of towers of the form  $(\overline{M}, \overline{a}, \overline{N})$  where  $\overline{M} = \langle M_i \mid i < \alpha \rangle$  is a sequence of models each of cardinality  $\lambda$  and  $\overline{N} = \langle N_i \mid i + 1 < \alpha \rangle$  is a sequence of models of cardinality  $\mu$ . We require that for  $i < \alpha$ ,  $M_i$  is  $\mu$ -universal over  $N_i$  and  $\text{tp}(a_i/M_i)$ does not  $\mu$ -split over  $N_i$ .

In a natural way we order these towers by the following adaptation of Definition 10.2.6.

DEFINITION 10.2.7. Let  $\lambda \geq \chi \geq \mu \geq \mathrm{LS}(\mathbf{K})$  be cardinals and fix  $\alpha < \mu^+$  an ordinal. For towers  $(\bar{M}, \bar{a}, \bar{N}) \in \mathbf{K}^*_{\lambda,\alpha,\mu}$  and  $(\bar{M}', \bar{a}', \bar{N}') \in \mathbf{K}^*_{\chi,\alpha,\mu}$ , we say

$$(\bar{M}, \bar{a}, \bar{N}) <^{\mu}_{\mathbf{K}} (\bar{M}', \bar{a}', \bar{N}')$$

if for all  $i < \alpha$ ,  $M_i \leq_{\mathbf{K}} M'_i$ ,  $\bar{a} = \bar{a}'$ ,  $\bar{N} = \bar{N}'$ , and there is  $\theta < \lambda^+$  so that  $M'_i$  is a  $(\lambda, \theta)$ -limit model witnessed by a sequence  $\langle M_i^{\gamma} | \gamma < \theta \rangle$  with  $M_i <_{\mathbf{K}} M_0^{\gamma}$ .

Note that Definition 10.2.6 is defined only on towers in  $\mathbf{K}^*_{\mu,\alpha}$  and is slightly weaker from the ordering  $<^{\mu}_{\mathbf{K}}$  when restricted to  $\mathbf{K}^*_{\mu,\alpha}$ . In particular, the models  $M'_i$  in the tower  $(\bar{M}', \bar{a}, \bar{N})$  <-extending  $(\bar{M}, \bar{a}, \bar{N})$  are only required to be universal over  $M_i$  and limit. It is not necessary that  $M'_i$  is limit over  $M_i$  as we require if  $(\bar{M}', \bar{a}, \bar{N}) <^{\mu}_{\mathbf{K}} (\bar{M}, \bar{a}, \bar{N})$ .
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Towers are particularly suited for superstable abstract elementary classes, in which they are known to exist and in which the union of an increasing chain of towers will be a tower. The definition below is already implicit in [SV99] and has since then been studied in many papers, e.g. [Van06, GVV16] and Chapters 6, 7, 9. We will use Definition 6.10.1:

DEFINITION 10.2.8. **K** is  $\mu$ -superstable (or superstable in  $\mu$ ) if:

- (1)  $\mu \geq \text{LS}(\mathbf{K})$ .
- (2)  $\mathbf{K}_{\mu}$  is nonempty, has joint embedding, and no maximal models.
- (3) **K** is stable in  $\mu$ . That is,  $|gS(M)| \le \mu$  for all  $M \in \mathbf{K}_{\mu}$ . Some authors call this "Galois-stable," and:
- (4)  $\mu$ -splitting in **K** satisfies the "no long splitting chains" property: For any limit ordinal  $\alpha < \mu^+$ , for every sequence  $\langle M_i \mid i < \alpha \rangle$  of models of cardinality  $\mu$  with  $M_{i+1}$  universal over  $M_i$  and for every  $p \in$  $gS(\bigcup_{i < \alpha} M_i)$ , there exists  $i < \alpha$  such that p does not  $\mu$ -split over  $M_i$ .

REMARK 10.2.9. By our global hypothesis of amalgamation (Hypothesis 10.2.1), if **K** is  $\mu$ -superstable, then  $\mathbf{K}_{\geq \mu}$  has joint embedding.

REMARK 10.2.10. By the weak transitivity property of  $\mu$ -splitting (Proposition 4.3.7,  $\mu$ -superstability implies the following continuity property (which is sometimes also stated as part of the definition): For any limit ordinal  $\alpha < \mu^+$ , for every sequence  $\langle M_i \mid i < \alpha \rangle$  of models of cardinality  $\mu$  with  $M_{i+1}$  universal over  $M_i$  and for every  $p \in \mathrm{gS}(\bigcup_{i < \alpha} M_i)$ , if  $p \upharpoonright M_i$  does not  $\mu$ -split over  $M_0$  for all  $i < \alpha$ , then p does not  $\mu$ -split over  $M_0$ . We will use this freely.

PROPOSITION 10.2.11. Let  $\mu \geq LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{\mu}$  has joint embedding and  $\mathbf{K}$  is stable in  $\mu$ . The following are equivalent:

- (1) **K** has a model of size  $\mu^+$ .
- (2)  $\mathbf{K}_{\mu}$  is nonempty and has no maximal models.
- (3) **K** has a limit model of size  $\mu$ .
- (4) There exists  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$  such that  $M_0 <_{\mathbf{K}} M_1 <_{\mathbf{K}} M_2$  and  $M_1$  is universal over  $M_0$ .

PROOF. (1) implies (2) is by joint embedding. (2) implies (3) implies (4) is straightforward. We show (4) implies (1). Assume  $M_0 <_{\mathbf{K}} M_1 <_{\mathbf{K}} M_2$  and  $M_1$  is universal over  $M_0$ . It is enough to show that  $M_2$  has a proper extension. By universality, there exists  $f: M_2 \xrightarrow[M_0]{} M_1$ . Now extend f to  $g: M'_2 \cong_{M_0} M_2$ . Since  $M_1 <_{\mathbf{K}} M_2$ ,  $M_2 <_{\mathbf{K}} M'_2$ , as desired.

The main results of this chapter involve the concept of symmetry over limit models and its equivalent formulation involving towers which was identified in **[Van16a]**:

DEFINITION 10.2.12. We say that an abstract elementary class exhibits symmetry for non- $\mu$ -splitting if whenever models  $M, M_0, N \in \mathbf{K}_{\mu}$  and elements a and b satisfy the conditions 1-4 below, then there exists  $M^b$  a limit model over  $M_0$ , containing b, so that  $\operatorname{tp}(a/M^b)$  does not  $\mu$ -split over N. See Figure 1.

- (1) M is universal over  $M_0$  and  $M_0$  is a limit model over N.
- (2)  $a \in M \setminus M_0$ .
- (3)  $tp(a/M_0)$  is non-algebraic and does not  $\mu$ -split over N.



FIGURE 1. A diagram of the models and elements in the definition of symmetry. We assume the type tp(b/M) does not  $\mu$ -split over  $M_0$  and  $tp(a/M_0)$  does not  $\mu$ -split over N. Symmetry implies the existence of  $M^b$  a limit model over  $M_0$  containing b, so that  $tp(a/M^b)$  does not  $\mu$ -split over N.

## (4) $\operatorname{tp}(b/M)$ is non-algebraic and does not $\mu$ -split over $M_0$ .

We end by recalling a few results of VanDieren showing the importance of the symmetry property:

FACT 10.2.13 (Theorem 2 in [Van16a]). If **K** is  $\mu$ -superstable and the union of any chain (of length less than  $\mu^{++}$ ) of saturated models of size  $\mu^{+}$  is saturated, then **K** has  $\mu$ -symmetry.

Many of the results on symmetry rely on the equivalent formulation of  $\mu$ -symmetry in terms of reduced towers.

DEFINITION 10.2.14. A tower  $(\bar{M}, \bar{a}, \bar{N}) \in \mathbf{K}^*_{\mu,\alpha}$  is reduced if it satisfies the condition that for every <-extension  $(\bar{M}', \bar{a}, \bar{N}) \in \mathbf{K}^*_{\mu,\alpha}$  of  $(\bar{M}, \bar{a}, \bar{N})$  and for every  $i < \alpha, M'_i \cap (\bigcup_{j < \alpha} M_j) = M_i$ .

DEFINITION 10.2.15. A tower  $(\overline{M}, \overline{a}, \overline{N}) \in \mathbf{K}^*_{\mu,\alpha}$  is continuous if for any limit  $i < \alpha, M_i = \bigcup_{j < i} M_j$ .

FACT 10.2.16 (Theorem 3 in [Van16a]). Assume K is  $\mu$ -superstable. The following are equivalent:

(1) **K** has  $\mu$ -symmetry.

(2) Any reduced tower in  $\mathbf{K}^*_{\mu,\alpha}$  is continuous.

It was previously established (in [SV99] or more explicitly in [GVV16]) that the continuity of reduced towers gives uniqueness of limit models:

FACT 10.2.17. Assume **K** is  $\mu$ -superstable. If any reduced tower in  $\mathbf{K}^*_{\mu,\alpha}$  is continuous (or equivalently by Fact 10.2.16 if **K** has  $\mu$ -symmetry), then for any  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$ , if  $M_1$  and  $M_2$  are limit over  $M_0$ , then  $M_1 \cong_{M_0} M_2$ .

Symmetry also has implications to chains of saturated models. For  $\lambda > LS(\mathbf{K})$ , write  $\mathbf{K}^{\lambda\text{-sat}}$  for the class of  $\lambda$ -saturated models in  $\mathbf{K}_{\geq\lambda}$ . We also define  $\mathbf{K}^{0\text{-sat}} := \mathbf{K}$ . Using this notation, we have:

FACT 10.2.18 (Theorem 22 in [Van16b]). Assume **K** is  $\mu$ -superstable,  $\mu^+$ -superstable, and every limit model in  $\mathbf{K}_{\mu^+}$  is saturated. Then  $\mathbf{K}^{\mu^+\text{-sat}}$  is an AEC with  $\mathrm{LS}(\mathbf{K}^{\mu^+\text{-sat}}) = \mu^+$ .

REMARK 10.2.19. By Fact 10.2.17, the hypotheses of Fact 10.2.18 hold if **K** is  $\mu$ -superstable,  $\mu^+$ -superstable, and has  $\mu^+$ -symmetry.

We will also use the following easy lemma:

LEMMA 10.2.20. Let  $\lambda$  be a limit cardinal and let  $\lambda_0 < \lambda$ . Assume that for all  $\mu \in [\lambda_0, \lambda)$ ,  $\mathbf{K}^{\mu\text{-sat}}$  is an AEC with  $\mathrm{LS}(\mathbf{K}^{\mu\text{-sat}}) = \mu$ . Then  $\mathbf{K}^{\lambda\text{-sat}}$  is an AEC with  $\mathrm{LS}(\mathbf{K}^{\lambda\text{-sat}}) = \lambda$ .

PROOF. That  $\mathbf{K}^{\lambda\text{-sat}}$  is closed under chains is easy to check. To see  $\mathrm{LS}(\mathbf{K}^{\lambda\text{-sat}}) = \lambda$ , let  $M \in \mathbf{K}^{\lambda\text{-sat}}$  and let  $A \subseteq |M|$ . Without loss of generality,  $\chi := |A| \ge \lambda$ . Let  $\delta := \mathrm{cf}(\lambda)$  and let  $\langle \lambda_i : i < \delta \rangle$  be an increasing sequence of cardinals with limit  $\lambda$ . Build  $\langle M_i : i \le \delta \rangle$  increasing continuous in  $\mathbf{K}_{\chi}$  such that for all  $i < \delta$ ,  $M_{i+1}$  is  $\lambda_i^+$ -saturated and  $A \subseteq |M_0|$ . This is possible by assumption. Then  $M_{\delta}$  is  $\lambda_i^+$ -saturated for all  $i < \delta$ , hence is  $\lambda$ -saturated. Thus it is as needed.  $\Box$ 

# 10.3. Transferring symmetry

In this section we prove Theorem 10.1.1 which is key to the results in the following sections. We start with a few observations which will allow us to extend the tower machinery from [**GVV16**] and [**Van16b**] to include towers composed of models of different cardinalities. In particular, we derive an extension property for towers of different cardinalities, Lemma 10.3.7. This will allow us to adapt the arguments from [**Van16b**] to prove Theorem 10.1.1.

We start with a study of chains where each model indexed by a successor is universal over its predecessor:

PROPOSITION 10.3.1. Suppose that  $\lambda \geq \mathrm{LS}(\mathbf{K})$  is a cardinal. Assume that  $\mathbf{K}$  is stable in  $\lambda$  with no maximal models of cardinality  $\lambda$ . Let  $\theta$  be a limit ordinal. Assume  $\langle M_i \in \mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})} | i < \theta \rangle$  is a strictly increasing and continuous sequence of models so that for all  $i < \theta$ ,  $M_{i+1}$  is universal over  $M_i$ . If  $M := \bigcup_{i < \theta} M_i$  has size  $\lambda$ , then M is a  $(\lambda, \theta)$ -limit model over some model containing  $M_0$ .

PROOF. By cardinality considerations,  $\theta < \lambda^+$ . Replacing  $\theta$  by  $cf(\theta)$  if necessary, we can assume without loss of generality that  $\theta$  is regular. By  $\lambda$ -stability and the assumption that **K** has no maximal models of cardinality  $\lambda$ , we can fix a  $(\lambda, \theta)$ -limit model  $M^*$  witnessed by  $\langle M_i^* \mid i < \theta \rangle$  with  $M_0 \leq_{\mathbf{K}} M_0^*$ . If there exists  $i < \theta$  such that  $M_i \in \mathbf{K}_{\lambda}$ , then the sequence  $\langle M_j \mid j \in [i, \theta) \rangle$  witnesses that M is  $(\lambda, \theta)$ -limit and  $M_0 \leq_{\mathbf{K}} M_i$ ; so assume that  $\lambda > \mathrm{LS}(\mathbf{K})$  and  $M_i \in K_{<\lambda}$  for all  $i < \theta$ . Then we must have that  $\theta = cf(\lambda)$ . If  $\lambda$  is a successor, we must have that  $\theta = \lambda$  and we obtain the result from [**Van16b**, Proposition 14]; so assume  $\lambda$  is limit. For  $i < \theta$ , let  $\lambda_i := ||M_i||$ .

Fix  $\langle a_{\alpha} \mid \alpha < \lambda \rangle$  an enumeration of  $M^*$ . Using the facts that  $M_{i+1}$  is universal over  $M_i$  and that  $M_{i+1}^*$  is universal over  $M_i^*$ , we can build an isomorphism  $f: M \cong$  $M^*$  inductively by defining an increasing and continuous sequence of **K**-embeddings  $f_i$  so that  $f_i: M_i \to M_i^*$ ,  $f_0 = \mathrm{id}_{M_0}$ , and  $\{a_{\alpha} \mid \alpha < \lambda_i\} \subseteq \mathrm{ran}(f_{i+1})$ .  $\Box$  We will use the following generalization of the weak transitivity property of  $\mu$ -splitting proven in Proposition 4.3.7. The difference here is that the models are allowed to be of size bigger than  $\mu$ .

PROPOSITION 10.3.2. Let  $\mu \geq \mathrm{LS}(\mathbf{K})$  be such that  $\mathbf{K}$  is stable in  $\mu$ . Let  $M_0 \leq_{\mathbf{K}} M_1 <_{\mathbf{K}} M'_1 \leq_{\mathbf{K}} M_2$  all be in  $\mathbf{K}_{\geq \mu}$ . Assume that  $M'_1$  is universal over  $M_1$ . Let  $p \in \mathrm{gS}(M_2)$ . If  $p \upharpoonright M'_1$  does not  $\mu$ -split over  $M_0$  and p does not  $\mu$ -split over some  $N \in \mathbf{K}_{\mu}$  with  $N \leq_{\mathbf{K}} M_1$ , then p does not  $\mu$ -split over  $M_0$ .

PROOF. Note that by definition of  $\mu$ -splitting,  $M_0 \in \mathbf{K}_{\mu}$ . Thus by making N larger if necessary we can assume that  $M_0 \leq_{\mathbf{K}} N$ . By basic properties of universality we have that  $M'_1$  is universal over N, hence without loss of generality  $M_1 = N$ . In particular,  $M_1 \in \mathbf{K}_{\mu}$ . By stability, build  $M''_1 \in \mathbf{K}_{\mu}$  universal over  $M_1$  such that  $M_1 <_{\mathbf{K}} M''_1 \leq_{\mathbf{K}} M'_1$ . By monotonicity,  $p \upharpoonright M''_1$  does not  $\mu$ -split over  $M_0$ . Thus without loss of generality also  $M''_1 \in \mathbf{K}_{\mu}$ . By definition of  $\mu$ -splitting, it is enough to check that  $p \upharpoonright M'_2$  does not  $\mu$ -split over  $M_0$  for all  $M'_2 \in \mathbf{K}_{\mu}$  with  $M'_2 \leq_{\mathbf{K}} M_2$ . Thus without loss of generality again  $M_2 \in \mathbf{K}_{\mu}$ . Now use the weak transitivity property of  $\mu$ -splitting (Proposition 4.3.7).

We use the previous proposition to extend the continuity property of  $\mu$ -splitting to models of size bigger than  $\mu$ . This is very similar to the argument in [She09a, Claim II.2.11].

PROPOSITION 10.3.3. Let  $\mu \geq LS(\mathbf{K})$  and assume that **K** is  $\mu$ -superstable.

Suppose  $\langle M_i \in \mathbf{K}_{\geq \mu} \mid i < \delta \rangle$  is an increasing sequence of models so that, for all  $i < \delta$ ,  $M_{i+1}$  is universal over  $M_i$ . Let  $p \in \mathrm{gS}(\bigcup_{i < \delta} M_i)$ . If  $p \upharpoonright M_i$  does not  $\mu$ -split over  $M_0$  for each  $i < \delta$ , then p does not  $\mu$ -split over  $M_0$ .

PROOF. Without loss of generality,  $\delta = cf(\delta)$ . Let  $M_{\delta} := \bigcup_{i < \delta} M_i$ . There are two cases to check. If  $\delta > \mu$ , then by [She99, Claim 3.3], there exists  $N \in \mathbf{K}_{\mu}$  with  $N \leq_{\mathbf{K}} M_{\delta}$  such that p does not  $\mu$ -split over N. Pick  $i < \delta$  such that  $N \leq_{\mathbf{K}} M_i$ . Then p does not  $\mu$ -split over  $M_i$ . By Proposition 10.3.2 (where  $(M_0, M_1M'_1, M_2, N)$ there stand for  $(M_0, M_i, M_{i+1}, M_{\delta}, N)$  here), p does not  $\mu$ -split over  $M_0$ .

Suppose then that  $\delta \leq \mu$  and for sake of contradiction that  $M^*$  of cardinality  $\mu$  witnesses the splitting of p over  $M_0$ , i.e.  $p \upharpoonright M^* \mu$ -splits over  $M_0$ . We can find  $\langle M_i^* \in \mathbf{K}_{\mu} \mid i < \delta \rangle$  an increasing resolution of  $M^*$  so that  $M_i^* \leq_{\mathbf{K}} M_i$  for all  $i < \delta$ . By monotonicity of splitting, stability in  $\mu$ , and the fact that each  $M_{i+1}$  is universal over  $M_i$ , we can increase  $M^*$ , if necessary, to arrange that  $M_{i+1}^*$  is universal over  $M_i^*$ . Since  $p \upharpoonright M_i$  does not  $\mu$ -split over  $M_0$ , monotonicity of non-splitting implies that  $p \upharpoonright M_i^*$  does not  $\mu$ -split over  $M_0$ . Then, by  $\mu$ -superstability  $p \upharpoonright M^*$ -does not  $\mu$ -split over  $M_0$ . This contradicts our choice of  $M^*$ .

We adapt the proof of the extension property for non- $\mu$ -splitting ([Van06, Theorem I.4.10]) to handle models of different sizes under the additional assumption of superstability in the size of the bigger model. The conclusion can also be achieved using the assumption of tameness instead of superstability (since  $\mu$ -splitting and  $\lambda$ -splitting coincide if **K** is  $\mu$ -tame and  $\mu \leq \lambda$ , see Proposition 3.3.12).

PROPOSITION 10.3.4. Fix cardinals  $\lambda > \mu \ge LS(\mathbf{K})$ . Suppose that  $\mathbf{K}$  is  $\mu$ -stable and  $\lambda$ -superstable.

Let  $M \in \mathbf{K}_{\mu}$  and  $M^{\lambda}, M' \in \mathbf{K}_{\lambda}$  be such that  $M \leq_{\mathbf{K}} M^{\lambda} \leq_{\mathbf{K}} M'$  and  $M^{\lambda}$ is limit over some model containing M. Let  $p \in gS(M^{\lambda})$  be such that p does not  $\mu$ -split over M. Then there exists  $q \in gS(M')$  extending p so that q does not  $\mu$ -split over M. Moreover q is algebraic if and only if p is.

PROOF. Let  $\theta < \lambda^+$  and  $\langle M_i^{\lambda} \mid i < \theta \rangle$  witness that  $M^{\lambda}$  is  $(\lambda, \theta)$ -limit with  $M \leq_{\mathbf{K}} M_0^{\lambda}$ . Write  $p := \operatorname{gtp}(a/M^{\lambda})$ . By  $\lambda$ -superstability there exists  $i < \theta$  so that p does not  $\lambda$ -split over  $M_i^{\lambda}$ . Since  $M_{i+2}^{\lambda}$  is universal over  $M_{i+1}^{\lambda}$  there exists  $f : M' \xrightarrow[M_{i+1}]{\rightarrow} M_{i+2}^{\lambda}$ . Extend f to  $g \in \operatorname{Aut}_{M_{i+1}^{\lambda}}(\mathfrak{C})$ . Let  $q := g^{-1}(p) \upharpoonright M' =$  $gtp(q^{-1}(a)/M')$ . Note that q is nonalgebraic if p is nonalgebraic (the converse will follow once we have shown that q extends p). By monotonicity, invariance, and our assumption that p does not  $\mu$ -split over M, we can conclude that q does not  $\mu$ -split over N. By similar reasoning also q does not  $\lambda$ -split over  $M_i^{\lambda}$ . In particular  $gtp(g^{-1}(a)/M^{\lambda}) = q \upharpoonright M^{\lambda}$  does not  $\lambda$ -split over  $M_i^{\lambda}$ . Since g fixes  $M_{i+1}^{\lambda}$ , we know that  $g^{-1}(a)$  realizes  $p \upharpoonright M_{i+1}^{\lambda}$ . Therefore, we get by the uniqueness of non- $\lambda$ splitting extensions that  $q \upharpoonright M^{\lambda} = \operatorname{gtp}(f^{-1}(a)/M^{\lambda}) = \operatorname{gtp}(a/M^{\lambda}) = p$ . This shows that q extends p, as desired.  $\square$ 

We can now prove an extension property for towers in  $\mathbf{K}^*_{\lambda,\alpha,\mu}$ .

LEMMA 10.3.5. Let  $\lambda$  and  $\mu$  be cardinals satisfying  $\lambda \geq \mu \geq \text{LS}(\mathbf{K})$ . Assume that **K** is superstable in  $\mu$  and in  $\lambda$ . For any  $(M, \bar{a}, N) \in \mathbf{K}^*_{\lambda,\alpha,\mu}$ , there exists  $(\overline{M}', \overline{a}, \overline{N}) \in \mathbf{K}^*_{\lambda, \alpha, \mu}$  so that:

$$(\bar{M}, \bar{a}, \bar{N}) <^{\mu}_{\mathbf{K}} (\bar{M}', \bar{a}, \bar{N})$$

**PROOF.** If  $\lambda = \mu$ , the result follows from infinitely many (for example  $cf(\lambda)$ ) many) applications of **GVV16**, Lemma 5.3] which is the extension property for towers. If  $\lambda > \mu$ , the result follows similarly from the proof of the extension property for towers using Proposition 10.3.4. 

We also have a continuity property:

LEMMA 10.3.6. Let  $\mu \geq \text{LS}(\mathbf{K})$  be such that **K** is  $\mu$ -superstable. Let  $\langle \lambda_i : i < \delta \rangle$ be an increasing sequence of cardinals with  $\lambda_0 \geq \mu$ . Let  $\langle (\bar{M}^i, \bar{a}, \bar{N}) \in \mathbf{K}^*_{\lambda_i, \alpha, \mu} \mid i < 0$  $\delta$  be a sequence of towers such that  $(\bar{M}^i, \bar{a}, \bar{N}) <^{\mu}_{\mathbf{K}} (\bar{M}^{i+1}, \bar{a}, \bar{N})$  for all  $i < \delta$ .

Let  $\overline{M}^{\delta}$  be the sequence composed of models of the form  $M^{\delta}_{\beta} := \bigcup_{i < \delta} M^{i}_{\beta}$  for  $\beta < \alpha. \text{ Let } \lambda := \sum_{i < \delta} \lambda_i.$ Then  $(\bar{M}^{\delta}, \bar{a}, \bar{N}) \in \mathbf{K}_{\lambda, \alpha, \mu} * \text{ and } (\bar{M}^i, \bar{a}, \bar{N}) <^{\mu}_{\mathbf{K}} (\bar{M}^{\delta}, \bar{a}, \bar{N}) \text{ for all } i < \delta.$ 

**PROOF.** Working by induction on  $\delta$ , we can assume without loss of generality that the sequence of tower is continuous. That is, for each  $\beta < \alpha$  and limit  $i < \delta$ ,  $M^i_{\beta} = \bigcup_{j < i} M^j_{\beta}$ . Of course, it is enough to show that  $(\bar{M}^0, \bar{a}, \bar{N}) <^{\mu}_{\mathbf{K}} (\bar{M}^{\delta}, \bar{a}, \bar{N})$ . Let  $\beta < \alpha$ . There are two things to check:  $M_{\beta}^{\lambda}$  is a limit model over a model that contains  $M^0_\beta$ , and  $gtp(a_\beta/M^\lambda_\beta)$  does not  $\mu$ -split over  $N_\beta$ . Proposition 10.3.1 confirms that  $M^{\lambda}_{\beta}$  is a  $(\lambda, \delta)$ -limit model over some model containing  $M^{0}_{\beta}$ . Because each  $(M^i, \bar{a}, \bar{N})$  is a tower, we know that  $gtp(a_\beta/M_\beta^i)$  does not  $\mu$ -split over  $N_\beta$ . This allows us to apply Proposition 10.3.3 to conclude that  $gtp(a_{\beta}/M_{\beta}^{\lambda})$  does not  $\mu$ -split over  $N_{\beta}$ .  $\square$ 

We conclude an extension property for towers of different sizes

LEMMA 10.3.7. Let  $\kappa$ ,  $\lambda$  and  $\mu$  be cardinals satisfying  $\lambda \geq \kappa \geq \mu \geq \text{LS}(\mathbf{K})$ . Assume that **K** is superstable in  $\mu$  and in every  $\chi \in [\kappa, \max(\kappa^+, \lambda))$ 

Let  $(\overline{M}^{\kappa}, \overline{a}, \overline{N}) \in \mathbf{K}^*_{\kappa, \alpha, \mu}$ .

(1) There exists  $(\bar{M}, \bar{a}, \bar{N}) \in \mathbf{K}^*_{\lambda, \alpha, \mu}$  so that

$$(\bar{M}^{\kappa}, \bar{a}, \bar{N}) <^{\mu}_{\mathbf{K}} (\bar{M}, \bar{a}, \bar{N}).$$

(2) If in addition **K** is  $\lambda$ -superstable, then there exists a sequence  $\langle N_{\beta}^{\lambda} | \beta < \alpha \rangle$  so that  $N_i \leq N_{\beta}^{\lambda}$  for all  $\beta < \alpha$  and  $(\bar{M}, \bar{a}, \bar{N}^{\lambda}) \in \mathbf{K}^*_{\lambda, \alpha}$ .

PROOF. We prove the first statement in the lemma by induction on  $\lambda$ . If  $\lambda = \kappa$ , this is given by Lemma 10.3.5. Now assume that  $\lambda > \kappa$ . Fix an increasing continuous sequence  $\langle \lambda_i \mid i < \operatorname{cf}(\lambda) \rangle$  which is cofinal in  $\lambda$  and so that  $\lambda_0 = \kappa$  (if  $\lambda = \chi^+$  is a successor we can take  $\lambda_i = \chi$  for all  $i < \lambda$ ). We build a sequence  $\langle (\bar{M}^{\lambda_i}, \bar{a}, \bar{N}) \in \mathbf{K}^*_{\lambda_i, \alpha, \mu} \mid i < \operatorname{cf}(\lambda) \rangle$  which is increasing (that is,  $(\bar{M}^{\lambda_i}, \bar{a}, \bar{N}) <_{\mathbf{K}}^{\mu}$  $(\bar{M}^{\lambda_{i+1}}, \bar{a}, \bar{N})$  for all  $i < \operatorname{cf}(\lambda)$ ) and continuous (in the obvious sense, see Lemma 10.3.6). This is possible by the induction hypothesis. Now by Lemma 10.3.6, the union of the chain of towers (defined there) is as desired.

For part (2), recall from Definition 10.2.7 that for each  $\beta < \alpha$ ,  $M_{\beta}$  is a limit model over some model containing  $M_{\beta}^{\kappa}$ . Let  $\langle M_{\beta,i}^{*} \in \mathbf{K}_{\lambda} \mid i < \theta_{\beta} \rangle$  witness this By  $\lambda$ -superstability, for each  $\beta < \alpha$ , there exists  $i_{\beta} < \theta_{\beta}$  so that  $gtp(a_{\beta}/M_{\beta}^{\lambda})$  does not  $\lambda$ -split over  $M_{\beta,i_{\beta}}^{*}$ . By our choice of  $M_{\beta,0}^{*}$  containing  $M_{\beta}^{\kappa}$ , and consequently  $N_{\beta}^{\kappa}$ , we can take  $N_{\beta}^{\lambda} := M_{\beta,i_{\beta}}^{*}$ .

We now begin the proof of Theorem 10.1.1. The structure of the proof is similar to the proof of Theorem 2 of [Van16b]; only here we work with towers in  $\mathbf{K}^*_{\lambda,\alpha,\mu}$  as opposed to only towers in  $\mathbf{K}^*_{\mu,\alpha}$ .

PROOF OF THEOREM 10.1.1. Suppose for the sake of contradiction that **K** does not have symmetry for  $\mu$ -non-splitting. By Fact 10.2.16 and our  $\mu$ -superstability assumption, **K** has a reduced discontinuous tower in  $\mathbf{K}^*_{\mu,\alpha}$  for some  $\alpha < \mu^+$ . Let  $\alpha$  be the minimal ordinal for which there is a reduced, discontinuous tower in  $\mathbf{K}^*_{\mu,\alpha}$ . By [**GVV16**, Lemma 5.7], we may assume that  $\alpha = \delta + 1$  for some limit ordinal  $\delta$ . Fix  $\mathcal{T} = (\bar{M}, \bar{a}, \bar{N}) \in \mathbf{K}^*_{\mu,\alpha}$  a reduced discontinuous tower with  $b \in M_{\delta} \setminus \bigcup_{\beta < \alpha} M_{\beta}$ .

Let  $I := \operatorname{cf}(\lambda)$ . By Lemma 10.3.7, we can build an increasing and continuous chain of towers  $\langle \mathcal{T}^i \mid i \in I \rangle$  extending  $\mathcal{T} \upharpoonright \delta$ . If  $\lambda = \kappa^+$  for some  $\kappa$ , then select each  $\mathcal{T}^i \in \mathbf{K}^*_{\kappa,\delta,\mu}$ . If  $\lambda$  is a limit cardinal, fix  $\langle \lambda_i \mid i < \operatorname{cf}(\lambda) \rangle$  to be an increasing and continuous sequence of cardinals cofinal in  $\lambda$ , with  $\lambda_0 > \mu$  and choose  $\mathcal{T}^i \in \mathbf{K}^*_{\lambda_i,\delta,\mu}$ . Let  $\mathcal{T}^{\lambda} := \bigcup_{i \in I} \mathcal{T}^i$ .

Notice that by Lemma 10.3.6, and our assumptions on the towers  $\mathcal{T}^i$ , we can conclude that  $\mathcal{T}^{\lambda} \in \mathbf{K}^*_{\lambda,\delta,\mu}$  and  $\mathcal{T}^{\lambda}$  extends  $\mathcal{T} \upharpoonright \delta$ . In particular, for each  $\beta < \alpha$ ,

(2) 
$$\operatorname{tp}(a_{\beta}/M_{\beta}^{\lambda})$$
 does not  $\mu$ -split over  $N_{\beta}$ .

Furthermore by the second part of Lemma 10.3.7 we can find  $N^{\lambda}_{\beta}$  so that the tower defined by  $(\bar{M}^{\lambda}, \bar{a}, \bar{N}^{\lambda})$  is in  $\mathbf{K}^{*}_{\lambda,\delta}$  and each  $M^{\lambda}_{\beta}$  is a limit over  $N^{\lambda}_{\beta}$ . We can extend this to a tower of length  $\delta + 1$  by appending to  $(\bar{M}^{\lambda}, \bar{a}, \bar{N}^{\lambda})$  a model  $M^{\lambda}_{\delta}$  of cardinality  $\lambda$  containing  $\bigcup_{\beta < \delta} M^{\lambda}_{\beta}$  and  $M^{\delta}$ . Call this tower  $\mathcal{T}^{b}$ , since it contains b. By  $\lambda$ -symmetry and Fact 10.2.16, we know that all reduced towers in  $\mathbf{K}^{*}_{\lambda,\alpha}$ 

By  $\lambda$ -symmetry and Fact 10.2.16, we know that all reduced towers in  $\mathbf{K}^*_{\lambda,\alpha}$  are continuous. Therefore  $\mathcal{T}^b$  is not reduced. However, by the density of reduced

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towers [**GVV16**, Theorem 5.6], we can find a reduced, continuous extension of  $\mathcal{T}^b$  in  $\mathbf{K}^*_{\lambda,\delta+1}$ . By  $\lambda$ -applications of this theorem, we may assume that for each  $\beta < \alpha$ , the model indexed by  $\beta$  in this reduced tower is a  $(\lambda, \mathrm{cf}(\lambda))$ -limit over  $M^{\lambda}_{\beta}$ . Refer to this tower as  $\mathcal{T}^*$ . See Fig. 2.



FIGURE 2. The towers in the proof of Theorem 10.1.1

CLAIM 10.3.8. For every  $\beta < \alpha$ ,  $\operatorname{tp}(a_{\beta}/M_{\beta}^*)$  does not  $\mu$ -split over  $N_{\beta}$ .

PROOF. Since  $M_{\beta}^{\lambda}$  and  $M_{\beta}^{*}$  are both limit models over  $N_{\beta}^{\lambda}$ , by  $\lambda$ -symmetry and Fact 10.2.17, there exists  $f: M_{\beta}^{\lambda} \cong_{N_{\beta}^{\lambda}} M_{\beta}^{*}$ . Since  $\mathcal{T}^{*}$  is a tower extending  $\mathcal{T}^{b}$ , we know  $\operatorname{tp}(a_{\beta}/M_{\beta}^{*})$  does not  $\lambda$ -split over  $N_{\beta}^{\lambda}$ . Therefore by the definition of nonsplitting, it must be the case that  $\operatorname{tp}(f(a_{\beta})/M_{\beta}^{*}) = \operatorname{tp}(a_{\beta}/M_{\beta}^{*})$ . From this equality of types we can fix  $g \in \operatorname{Aut}_{M_{\beta}^{*}}(\mathfrak{C})$  with  $g(f(a_{\beta})) = a_{\beta}$ . An application of  $g \circ f$  to (2) yields the statement of the claim.  $\Box$ 

We can now complete the proof of Theorem 10.1.1. By the continuity of  $\mathcal{T}^*$ there exists  $\beta < \delta$  so that  $b \in M^*_{\beta}$ . We can then use  $\mathcal{T}^*$  to construct a tower  $\hat{\mathcal{T}}$  in  $\mathbf{K}^*_{\mu,\delta+1}$  extending  $\mathcal{T}$  so that  $b \in \hat{M}_{\beta}$  contradicting our assumption that  $\mathcal{T}$ was reduced. This is possible by the downward Löwenheim property of abstract elementary classes,  $\mu$ -stability, universality of the models in  $\mathcal{T}^*$ , monotonicity of non- $\mu$ -splitting, and Claim 10.3.8. Similar to the proof of [Van16a, Theorem 2] we can use Lemma 10.3.7 to derive symmetry from categoricity. More precisely, it is enough to assume that all the models in the top cardinal have enough saturation.

THEOREM 10.3.9. Suppose  $\lambda$  and  $\mu$  are cardinals so that  $\lambda > \mu \ge LS(\mathbf{K})$ .

If **K** is superstable in every  $\chi \in [\mu, \lambda)$ , and all the models of size  $\lambda$  are  $\mu^+$ -saturated, then **K** has  $\mu$ -symmetry.

PROOF. Suppose that **K** does not satisfy  $\mu$ -symmetry. Then by Fact 10.2.16 there is a reduced discontinuous tower in  $\mathbf{K}^*_{\mu,\alpha}$ . As in the proof of Theorem 10.1.1, we can find a discontinuous reduced tower  $\mathcal{T} \in \mathbf{K}^*_{\mu,\alpha}$  with  $\alpha = \delta + 1$  with the witness of discontinuity  $b \in M_{\delta} \setminus \bigcup_{\beta < \delta} M_{\beta}$ . As in the proof of Theorem 10.1.1, we can use Lemma 10.3.7 (note that we only use the first part so not assuming  $\lambda$ -superstability is okay) to find a tower  $\mathcal{T}^{\lambda} \in \mathbf{K}^*_{\lambda,\mu,\delta}$  extending  $\mathcal{T} \upharpoonright \delta$ .

By our assumption that all the models of size  $\lambda$  are  $\mu^+$ -saturated,  $\operatorname{tp}(b/\bigcup_{\beta<\delta}M_\beta)$ is realized in  $\bigcup_{\beta<\delta}M_\beta^{\lambda}$ . Let b' and  $\beta'<\delta$  be such that  $b'\models\operatorname{tp}(b/\bigcup_{\beta<\delta}M_\beta)$  and  $b'\in M_{\beta'}^{\lambda}$ . Fix  $f\in\operatorname{Aut}_{\bigcup_{\beta<\delta}M_\beta}(\mathfrak{C})$  so that f(b')=b. Notice that  $\mathcal{T}^b:=f(\mathcal{T}^{\lambda})$  is a tower in  $\mathbf{K}^*_{\lambda,\delta,\mu}$  extending  $\mathcal{T} \upharpoonright \delta$  with  $b\in M_{\beta'}^b$ .

We can now use the downward Löwenheim-Skolem property of abstract elementary classes, stability in  $\mu$ ,  $\mu^+$ -saturation of models of cardinality  $\lambda$ , and monotonicity of non- $\mu$ -splitting to construct from  $\mathcal{T}^b$  a discontinuous tower in  $\mathbf{K}^*_{\mu,\alpha}$  extending  $\mathcal{T}$  so that b appears in the model indexed by  $\beta'$  in the tower. This will contradict our choice of  $\mathcal{T}$  being reduced.

REMARK 10.3.10. Instead of assuming that all the models of size  $\lambda$  are  $\mu^+$ saturated, it is enough to assume the following weaker property. For any  $\delta < \mu^+$ and any increasing chain  $\langle M_i : i < \delta \rangle$  in  $\mathbf{K}_{\lambda}$  of  $(< \lambda, \operatorname{cf}(\lambda))$ -limit models (i.e. for each  $i < \delta$ , there exists a resolution of  $M_i \langle M_i^j \in \mathbf{K}_{<\lambda} : j < \operatorname{cf}(\lambda) \rangle$  such that  $M_i^{j+1}$ is universal over  $M_i^j$  for each  $j < \operatorname{cf}(\lambda)$ ,  $\bigcup_{i < \delta} M_i$  is  $\mu^+$ -saturated.

# 10.4. A hierarchy of symmetry properties

We discuss the relationship between the symmetry property of Definition 10.2.12 and other symmetry properties previously defined in the literature, especially the symmetry property in the definition of a good  $\mu$ -frame. This expands on the short remark after Definition 3 of [**Van16a**] and on Corollary 2 there. It will be convenient to use the following terminology. A minor variation (where "limit over" is replaced by "universal over") appears in Definition 4.3.8.

DEFINITION 10.4.1. Let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  be models in  $\mathbf{K}_{\mu}$ . We say a type  $p \in \mathrm{gS}(N)$  explicitly does not  $\mu$ -fork over  $(M_0, M)$  if:

(1) M is limit over  $M_0$ .

(2) p does not  $\mu$ -split over  $M_0$ .

We say that p does not  $\mu$ -fork over M if there exists  $M_0$  so that p explicitly does not  $\mu$ -fork over  $(M_0, M)$ .

REMARK 10.4.2. Assuming  $\mu$ -superstability, the relation "p does not  $\mu$ -fork over M" is very close to defining an independence notion with the properties of forking in a first-order superstable theory (i.e. a good  $\mu$ -frame, see below). In fact using tameness it can be used to do precisely that, see Chapter 4 or Theorem 10.6.4. Moreover forking in any categorical good  $\mu$ -frame has to be  $\mu$ -forking, see Fact 10.4.10.

We now give several variations on  $\mu$ -symmetry. We will show that variation (1) is equivalent to (2) which implies (3) which implies (4). Moreover variation (1) is equivalent to the  $\mu$ -symmetry of Definition 10.2.12 and variation (4) is equivalent to the symmetry property of good frames. We do not know if any of the implications can be reversed, or even if all the variations already follow from superstability (see Question 10.4.14).

For clarity, we have highlighted the differences between each property.

DEFINITION 10.4.3. Let  $\mu \geq LS(\mathbf{K})$ .

- (1) **K** has uniform  $\mu$ -symmetry if for any limit models  $N, M_0, M$  in  $\mathbf{K}_{\mu}$  where M is limit over  $M_0$  and  $M_0$  is limit over N, if  $\underline{\operatorname{gtp}}(b/M)$  does not  $\mu$ -split over  $M_0$ ,  $a \in |M|$ , and  $\underline{\operatorname{gtp}}(a/M_0)$  explicitly does not  $\mu$ -fork over  $(N, M_0)$ , there exists  $M_b \in \mathbf{K}_{\mu}$  containing b and limit over  $M_0$  so that  $\underline{\operatorname{gtp}}(a/M_b)$  explicitly does not  $\mu$ -fork over  $(N, M_0)$ .
- (2) K has weak uniform μ-symmetry if for any limit models N, M<sub>0</sub>, M in K<sub>μ</sub> where M is limit over M<sub>0</sub> and M<sub>0</sub> is limit over N, if gtp(b/M) does not μ-fork over M<sub>0</sub>, a ∈ |M|, and gtp(a/M<sub>0</sub>) explicitly does not μ-fork over (N, M<sub>0</sub>), there exists M<sub>b</sub> ∈ K<sub>μ</sub> containing b and limit over M<sub>0</sub> so that gtp(a/M<sub>b</sub>) explicitly does not μ-fork over (N, M<sub>0</sub>). See Figure 3.
- (3) **K** has non-uniform  $\mu$ -symmetry if for any limit models  $M_0, M$  in  $\mathbf{K}_{\mu}$ where M is limit over  $M_0$ , if  $\underline{\operatorname{gtp}}(b/M)$  does not  $\mu$ -split over  $M_0$ ,  $a \in |M|$ , and  $\underline{\operatorname{gtp}}(a/M_0)$  does not  $\mu$ -fork over  $M_0$ , there exists  $M_b \in \mathbf{K}_{\mu}$  containing b and limit over  $M_0$  so that  $\underline{\operatorname{gtp}}(a/M_b)$  does not  $\mu$ -fork over  $M_0$ .
- (4) **K** has weak non-uniform  $\mu$ -symmetry if for any limit models  $M_0, M$  in  $\mathbf{K}_{\mu}$  where M is limit over  $M_0$ , if gtp(b/M) does not  $\mu$ -fork over  $M_0$ ,  $a \in |M|$ , and  $gtp(a/M_0)$  does not  $\mu$ -fork over  $M_0$ , there exists  $M_b \in \mathbf{K}_{\mu}$  containing b and limit over  $M_0$  so that  $gtp(a/M_b)$  does not  $\mu$ -fork over  $M_0$ .



FIGURE 3. A diagram of the models and elements in the definition of weak uniform  $\mu$ -symmetry. We require that gtp(b/M) does not  $\mu$ -fork over  $M_0$  in the weak version, so there exists  $M'_0$  such that  $M_0$  is limit over  $M'_0$  and gtp(b/M) does not  $\mu$ -split over  $M'_0$ 

The difference between the uniform and non-uniform variations is in the conclusion: in the uniform case, we start with  $gtp(a/M_0)$  which explicitly does not  $\mu$ -fork over  $(N, M_0)$  and get get  $gtp(a/M_b)$  explicitly does not  $\mu$ -fork over  $(N, M_0)$ . Thus both types do not  $\mu$ -split over N. In the non-uniform case, we start with  $gtp(a/M_0)$  which does not  $\mu$ -fork over  $M_0$ , hence explicitly does not  $\mu$ -fork over  $(N, M_0)$  for some N, but we only get that  $gtp(a/M_b)$  does not  $\mu$ -fork over  $M_0$ , so it explicitly does not  $\mu$ -fork over  $(N', M_0)$ , for some N' potentially different from N.

The difference between weak and non-weak is in the starting assumption: in the weak case, we assume that gtp(b/M) does not  $\mu$ -fork over  $M_0$ , hence there exists  $M'_0$  so that  $M_0$  is limit over  $M'_0$  and gtp(b/M) does not  $\mu$ -split over  $M'_0$ . In the non-weak case, we assume only that gtp(b/M) does not  $\mu$ -split over  $M_0$ . Even under  $\mu$ -superstability, it is open whether this implies that there must exist a smaller  $M'_0$  so that gtp(b/M) does not  $\mu$ -split over  $M'_0$ . The problem is that  $\mu$ -splitting need not satisfy the transitivity property, see the discussion after Definition 4.3.8.

Using the monotonicity property of  $\mu$ -splitting, we get the easy implications:

PROPOSITION 10.4.4. Let  $\mu \geq LS(\mathbf{K})$ . If **K** has uniform  $\mu$ -symmetry, then it has non-uniform  $\mu$ -symmetry and weak uniform  $\mu$ -symmetry. If **K** has non-uniform  $\mu$ -symmetry, then it has weak non-uniform  $\mu$ -symmetry.

Playing with the definitions and monotonicity of  $\mu$ -splitting (noting that cases ruled out by Definition 10.2.12 such as  $a \in |M_0|$  are easy to handle), we also have:

PROPOSITION 10.4.5. **K** has uniform  $\mu$ -symmetry if and only if it has  $\mu$ -symmetry (in the sense of Definition 10.2.12).

Surprisingly, uniform symmetry and weak uniform symmetry are also equivalent assuming superstability. We will use the characterization of symmetry in terms of reduced towers provided by Fact 10.2.16.

LEMMA 10.4.6. If **K** is  $\mu$ -superstable, then weak uniform  $\mu$ -symmetry is equivalent to uniform  $\mu$ -symmetry.

PROOF. By Proposition 10.4.4, uniform symmetry implies weak uniform symmetry. Now assuming weak uniform symmetry, the proof of  $(1) \Rightarrow (2)$  of Fact 10.2.16 still goes through. The point is that whenever we consider gtp(b/M) in the proof,  $M = \bigcup_{i < \delta} M_i$  for some increasing continuous  $\langle M_i : i < \delta \rangle$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \delta$ , and we simply use that by superstability gtp(b/M) does not  $\mu$ -split over  $M_i$  for some  $i < \delta$ . However we also have that gtp(b/M) explicitly does not  $\mu$ -fork over  $(M_i, M_{i+1})$ .

Therefore reduced towers are continuous, and hence by Fact 10.2.16 K has  $\mu$ -symmetry (and so by Proposition 10.4.5 uniform  $\mu$ -symmetry).

How do these definitions compare to the symmetry property in good  $\mu$ -frames? Recall [She09a, Definition II.2.1] that a good  $\mu$ -frame is a triple  $\mathfrak{s} = (\mathbf{K}_{\mu}, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  where:

- (1)  $\mathbf{K}$  is an AEC.
- (2) For each  $M \in \mathbf{K}_{\mu}$ , gS<sup>bs</sup>(M) (called the set of *basic types* over M) is a set of nonalgebraic Galois types over M satisfying (among others) the *density property*: if  $M <_{\mathbf{K}} N$  are in  $\mathbf{K}_{\mu}$ , there exists  $a \in |N| \setminus |M|$  such that gtp $(a/M; N) \in \mathrm{gS}^{\mathrm{bs}}(M)$ .

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(3)  $\downarrow$  is an (abstract) independence relation on types of length one over models in  $\mathbf{K}_{\mu}$  satisfying several basic properties (that we will not list here) of first-order forking in a superstable theory.

REMARK 10.4.7. We will *not* use the axiom (B) [She09a, Definition II.2.1] requiring the existence of a superlimit model of size  $\mu$ . In fact many papers (e.g. [JS13]) define good frames without this assumption.

As in [She09a, Definition II.6.35], we say that a good  $\mu$ -frame  $\mathfrak{s}$  is type-full if for each  $M \in \mathbf{K}_{\mu}$ , gS<sup>bs</sup>(M) consists of all the nonalgebraic types over M. For simplicity, we focus on type-full good frames in this chapter. Given a type-full good  $\mu$ -frame  $\mathfrak{s} = (\mathbf{K}_{\mu}, \downarrow, \text{gS}^{\text{bs}})$  and  $M_0 \leq_{\mathbf{K}} M$  both in  $\mathbf{K}_{\mu}$ , we say that a nonalgebraic type  $p \in \text{gS}(M)$  does not  $\mathfrak{s}$ -fork over  $M_0$  if it does not fork over  $M_0$  according to the abstract independence relation  $\downarrow$  of  $\mathfrak{s}$ . We say that a good  $\mu$ -frame  $\mathfrak{s}$  is on  $\mathbf{K}_{\mu}$ if its underlying class is  $\mathbf{K}_{\mu}$ .

The existence of a good  $\mu$ -frame gives quite a lot of information about the class.

FACT 10.4.8. Assume there is a good  $\mu$ -frame on  $\mathbf{K}_{\mu}^{\lambda\text{-sat}}$ , for  $\lambda \leq \mu$  (so in particular, unions of chains of  $\lambda$ -saturated models are  $\lambda$ -saturated). Then:

- (1) For any  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$  such that  $M_1$  and  $M_2$  are limit over  $M_0, M_1 \cong_{M_0} M_2$ .
- (2) **K** is  $\mu$ -superstable.

PROOF. The first part is [She09a, Lemma II.4.8] (or see [Bon14a, Theorem 9.2]). Note that by the usual back and forth argument (as made explicit in the proof of Theorem 7.6.1), any limit model is isomorphic to a limit model where the models witnessing it are  $\lambda$ -saturated. The second part is because:

- By definition of a good  $\mu$ -frame,  $\mu \geq LS(\mathbf{K})$ ,  $\mathbf{K}_{\mu}$  is nonempty, has amalgamation, joint embedding, and no maximal models.
- By [She09a, Claim II.4.2.(1)], **K** is stable in μ.
- By the uniqueness property of \$\mathbf{s}\$-forking, if a type does not \$\mathbf{s}\$-fork over M<sub>0</sub> (where \$\mathbf{s}\$ is a good \$\mu\$-frame on \$\mathbf{K}\_{\mu}\$), then it does not \$\mu\$-split over M<sub>0</sub> (see Lemma 3.4.2). Thus we obtain the "no long splitting chains" condition in Definition 10.2.8 when the M<sub>i</sub>s are \$\lambda\$-saturated, and as noted above (or in Proposition 6.10.6) we can do a back and forth argument to get that no long splitting chains holds even when the members of the chain are not saturated.

Among the axioms a good  $\mu$ -frame must satisfy is the symmetry axiom:

DEFINITION 10.4.9. The symmetry axiom for a good  $\mu$ -frame  $\mathfrak{s} = (\mathbf{K}_{\mu}, \bigcup, \mathrm{gS}^{\mathrm{bs}})$ is the following statement: For any  $M_0 \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{\mu}$ , if  $\mathrm{gtp}(b/M)$  does not  $\mathfrak{s}$ -fork over  $M_0$  and  $a \in |M| \setminus |M_0|$  is so that  $\mathrm{gtp}(a/M_0) \in \mathrm{gS}^{\mathrm{bs}}(M_0)$ , there exists  $M_b \in \mathbf{K}_{\mu}$ containing b and extending  $M_0$  so that  $\mathrm{gtp}(a/M_b)$  does not  $\mathfrak{s}$ -fork over  $M_0$ .

Note that the good frame axioms imply that **K** has amalgamation in  $\mu$ , so for this definition (and for simplicity only) we work inside a saturated model  $\mathfrak{C}$  of size  $\mu^+$ .

Since the symmetry properties of Definition 10.4.3 are all over limit models only, we will discuss only frames whose models are the limit models. By Fact 10.4.8,

such frames are categorical (that is, their underlying class has a single model up to isomorphism). This is not a big loss since most known general constructions of a good  $\mu$ -frame (e.g. [She09a, Theorem II.3.7], Theorem 4.1.3) assume categoricity in  $\mu$ . In the known constructions when categoricity in  $\mu$  is not assumed (such as in Corollary 6.10.19), it holds that the union of a chain of  $\mu$ -saturated model is  $\mu$ -saturated, so we can simply restrict the frame to the saturated models of size  $\mu$ .

We will use Theorem 6.9.7 that categorical good  $\mu$ -frames are canonical:

FACT 10.4.10 (The canonicity theorem for categorical good frames). Let  $\mathfrak{s} = (\mathbf{K}_{\mu}, \bigcup, \mathrm{gS}^{\mathrm{bs}})$  be a categorical good  $\mu$ -frame. Let  $p \in \mathrm{gS}^{\mathrm{bs}}(M)$  and let  $M_0 \leq_{\mathbf{K}} M$  be in  $\mathbf{K}_{\mu}$ . Then p does not  $\mathfrak{s}$ -fork over  $M_0$  if and only if p does not  $\mu$ -fork over  $M_0$  (recall Definition 10.4.1).

REMARK 10.4.11. The statement of Theorem 6.9.7 uses the definition of  $\mu$ -forking with "universal over" instead of "limit over," but the proof goes through also for the "limit over" definition.

REMARK 10.4.12. The proof of the second part of Fact 10.4.8 and Fact 10.4.10 do not use the symmetry axiom (but the first part of Fact 10.4.8 does).

Using the canonicity theorem, we obtain:

THEOREM 10.4.13. Let  $\mathfrak{s}$  be a type-full categorical good  $\mu$ -frame on  $\mathbf{K}_{\mu}$ , except that we do not assume that it satisfies the symmetry axiom. The following are equivalent:

- (1)  $\mathfrak{s}$  satisfies the symmetry axiom (Definition 10.4.9).
- (2) **K** has weak non-uniform  $\mu$ -symmetry (Definition 10.4.3.(4)).

PROOF. By Fact 10.4.10 (and Remark 10.4.12),  $\mu$ -forking and  $\mathfrak{s}$ -forking coincide. Now replace  $\mathfrak{s}$ -forking by  $\mu$ -forking in the symmetry axiom and expand the definition.

One can ask whether weak non-uniform symmetry can be replaced by the uniform version:

QUESTION 10.4.14. Assume there is a type-full categorical good  $\mu$ -frame on  $\mathbf{K}_{\mu}$ . Does  $\mathbf{K}$  have  $\mu$ -symmetry? More generally, if  $\mathbf{K}$  is  $\mu$ -superstable, does  $\mathbf{K}$  have  $\mu$ -symmetry?

We will show (Corollary 10.6.9) that the answer is positive if **K** is  $\mu$ -tame. Still much less suffices:

THEOREM 10.4.15. If **K** is  $\mu$ -superstable and has a good  $\mu^+$ -frame on  $\mathbf{K}_{\mu^+}^{\lambda-\text{sat}}$  for some  $\lambda \leq \mu^+$ , then **K** has  $\mu$ -symmetry.

PROOF. By Fact 10.4.8, all limit models in  $\mathbf{K}_{\mu^+}$  are saturated and  $\mathbf{K}$  is  $\mu^+$ -superstable. By Fact 10.2.18, the remark following it, and Fact 10.2.13,  $\mathbf{K}$  has  $\mu$ -symmetry.

We end this section with a partial answer to Question 10.4.14 assuming that the good frame satisfies several additional technical properties of frames introduced by Shelah (see [She09a, Definitions III.1.1, III.1.3]). For this result amalgamation (Hypothesis 10.2.1) is not necessary.

COROLLARY 10.4.16. Assume there is a successful good<sup>+</sup>  $\mu$ -frame with underlying class  $\mathbf{K}_{\mu}$ . Then **K** has  $\mu$ -symmetry.

PROOF. Let  $\mathfrak{s}$  be a successful good<sup>+</sup>  $\mu$ -frame with underlying class  $\mathbf{K}_{\mu}$ . By [She09a, Claim II.6.36], we can assume without loss of generality that  $\mathfrak{s}$  is type-full (note that by Theorem 3.6.12 there can be only one such type-full frame). By Fact 10.4.8,  $\mathbf{K}$  is  $\mu$ -superstable. By [She09a, III.1.6, III.1.7, III.1.8], there is a good  $\mu^+$ -frame  $\mathfrak{s}^+$  on  $\mathbf{K}_{\mu^+}^{\mu^+$ -sat}. By Theorem 10.4.15,  $\mathbf{K}$  has  $\mu$ -symmetry.

## 10.5. Symmetry from no order property

In this section, we give another way to derive symmetry. The idea is to imitate the argument from Theorem 3.5.13, but we first have to obtain enough properties of independence. We will work with  $\mu$ -forking (Definition 10.4.1). We start by improving Proposition 10.3.4.

PROPOSITION 10.5.1 (Extension property of forking). Let  $\mathrm{LS}(\mathbf{K}) \leq \mu \leq \lambda$ . Let  $M \leq_{\mathbf{K}} N$  be in  $K_{[\mu,\lambda]}$ . Let  $p \in \mathrm{gS}(M)$  be such that p explicitly does not  $\mu$ -fork over  $(M_0, M)$ . If  $\mathbf{K}$  is superstable in every  $\chi \in [\mu, \lambda]$ , then there exists  $q \in \mathrm{gS}(N)$  extending p and explicitly not  $\mu$ -forking over  $(M_0, M)$ . Moreover q is algebraic if and only if p is.

PROOF. By induction on ||N||. Let *a* realize *p*. If ||N|| = ||M|| this is given by Proposition 10.3.4 (if  $||M|| = ||N|| = \mu$ , this is [Van06, Theorem I.4.10]). If ||M|| < ||N||, build  $\langle N_i \in \mathbf{K}_{||M||+|i|} : i \leq ||N|| \rangle$  increasing continuous such that  $N_0 = M, N_{i+1}$  is limit over  $N_i$ , and  $gtp(a/N_i)$  explicitly does not  $\mu$ -fork over  $(M_0, M)$ . This is possible by the induction hypothesis and the continuity property of splitting (Proposition 10.3.3). Now  $N_{\lambda}$  is ||N||-universal over  $N_0 = M$ , so let  $f : N \xrightarrow{M} N_{\lambda}$ . Let  $q := f^{-1}(gtp(a/f[N]))$ . It is easy to check that q is as desired.  $\Box$ 

Recall the definition of the order property in AECs from Definition 2.4.3. We will use two important facts: that it is enough to look at length of the order property up to the Hanf number (Fact 2.4.7) and that the order property implies instability (Fact 2.4.11).

The following lemma appears in some more abstract form in Lemma 3.5.6. The lemma says that if we assume that p does not  $\mu$ -fork over M, then in the definition of non-splitting (Definition 10.2.3) we can replace the  $N_{\ell}$  by arbitrary sequences in N of length at most  $\mu$ . In the proof of Lemma 10.5.3, this will be used for sequences of length one.

LEMMA 10.5.2. Let  $\mu \geq \text{LS}(\mathbf{K})$ . Let  $M \in \mathbf{K}_{\mu}$  and  $N \in \mathbf{K}_{\geq \mu}$  be such that  $M \leq_{\mathbf{K}} N$ . Assume that  $\mathbf{K}$  is stable in  $\mu$ . If  $p \in \text{gS}(N)$  does not  $\mu$ -fork over M (Definition 10.4.1), a realizes p, and  $\bar{b}_1, \bar{b}_2 \in {}^{\leq \mu}|N|$  are such that  $\text{gtp}(\bar{b}_1/M) = \text{gtp}(\bar{b}_2/M)$ , then  $\text{gtp}(a\bar{b}_1/M) = \text{gtp}(a\bar{b}_2/M)$ .

PROOF. Pick  $N_0 \in \mathbf{K}_{\mu}$  containing  $\bar{b}_1 \bar{b}_2$  with  $M \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$ . Then  $p \upharpoonright N_0$ does not  $\mu$ -fork over M. Replacing N by  $N_0$  if necessary, we can assume without loss of generality that  $N \in \mathbf{K}_{\mu}$ . By definition of  $\mu$ -forking, there exists  $M_0 \in$  $\mathbf{K}_{\mu}$  such that  $M_0 \leq_{\mathbf{K}} M$  and p does not  $\mu$ -split over  $M_0$ . By the extension and uniqueness property for  $\mu$ -splitting there exists N' extending N of cardinality  $\mu$ so that N' is universal over both N and M, and  $\operatorname{gtp}(a/N')$  does not  $\mu$ -split over  $M_0$ . Since  $\operatorname{gtp}(\overline{b}_1/M) = \operatorname{gtp}(\overline{b}_2/M)$  and since N' is universal over N, we can find  $f: N \xrightarrow{M} N'$  so that  $f(\overline{b}_1) = \overline{b}_2$ . Since  $\operatorname{gtp}(a/N')$  does not  $\mu$ -split over  $M_0$  we know  $\operatorname{gtp}(f(a)/f(N)) = \operatorname{gtp}(a/f(N))$ . By our choice of f this implies that there exists  $g \in \operatorname{Aut}_{f(N)}(\mathfrak{C})$  so that  $g(f(a)) = a, g \upharpoonright M = \operatorname{id}_M$ , and  $g(\overline{b}_2) = \overline{b}_2$ . In other words  $\operatorname{gtp}(f(a)\overline{b}_2/M) = \operatorname{gtp}(a\overline{b}_2/M)$ . Moreover  $f^{-1}$  witnesses that  $\operatorname{gtp}(a\overline{b}_1/M) = \operatorname{gtp}(f(a)\overline{b}_2/M)$ , which we have seen is equal to  $\operatorname{gtp}(a\overline{b}_2/M)$ .

The next lemma shows that failure of symmetry implies the order property. The proof is similar to that of Theorem 3.5.13, the difference is that we use Lemma 10.5.2 and the equivalence between symmetry and weak uniform symmetry (Lemma 10.4.6).

LEMMA 10.5.3. Let  $\lambda > \mu \geq \text{LS}(\mathbf{K})$ . Assume that **K** is superstable in every  $\chi \in [\mu, \lambda)$ . If **K** does *not* have  $\mu$ -symmetry, then it has the  $\mu$ -order property of length  $\lambda$ .

PROOF. By Lemma 10.4.6, **K** does not have weak uniform  $\mu$ -symmetry. We first pick witnesses to that fact. Pick limit models  $N, M_0, M \in \mathbf{K}_{\mu}$  such that M is limit over  $M_0$  and  $M_0$  is limit over N. Pick b such that  $\operatorname{gtp}(b/M)$  does not  $\mu$ -fork over  $M_0, a \in |M|$ , and  $\operatorname{gtp}(a/M_0)$  explicitly does not  $\mu$ -fork over  $(N, M_0)$ , and there does not exist  $M_b \in \mathbf{K}_{\mu}$  containing b and limit over  $M_0$  so that  $\operatorname{gtp}(a/M_b)$  explicitly does not  $\mu$ -fork over  $(N, M_0)$ . We will show that  $\mathfrak{C}$  has the  $\mu$ -order property of length  $\lambda$ .

We build increasing continuous  $\langle N_{\alpha} : \alpha < \lambda \rangle$  and  $\langle a_{\alpha}, b_{\alpha}, N'_{\alpha} : \alpha < \lambda \rangle$  by induction so that for all  $\alpha < \lambda$ :

- (1)  $N_{\alpha}, N'_{\alpha} \in \mathbf{K}_{\mu+|\alpha|}.$
- (2)  $N_0$  is limit over M and  $b \in |N_0|$ .
- (3)  $\operatorname{gtp}(a_{\alpha}/M_0) = \operatorname{gtp}(a/M_0)$  and  $a_{\alpha} \in |N'_{\alpha}|$ .
- (4)  $gtp(b_{\alpha}/M) = gtp(b/M)$  and  $b_{\alpha} \in |N_{\alpha+1}|$ .
- (5)  $N'_{\alpha}$  is limit over  $N_{\alpha}$  and  $N_{\alpha+1}$  is limit over  $N'_{\alpha}$ .
- (6)  $gtp(a_{\alpha}/N_{\alpha})$  explicitly does not  $\mu$ -fork over  $(N, M_0)$  and  $gtp(b_{\alpha}/N'_{\alpha})$  does not  $\mu$ -fork over  $M_0$ .

This is possible. Let  $N_0$  be any model in  $\mathbf{K}_{\mu}$  containing M and a and limit over  $\overline{M}$ . At  $\alpha$  limits, let  $N_{\alpha} := \bigcup_{\beta < \alpha} N_{\beta}$ . Now assume inductively that  $N_{\beta}$ has been defined for  $\beta \leq \alpha$ , and  $a_{\beta}$ ,  $b_{\beta}$ ,  $N'_{\beta}$  have been defined for  $\beta < \alpha$ . By extension for splitting, find  $q \in gS(N_{\alpha})$  that explicitly does not  $\mu$ -fork over  $(N, M_0)$ and extends  $gtp(a/M_0)$ . Let  $a_{\alpha}$  realize q and pick  $N'_{\alpha}$  limit over  $N_{\alpha}$  containing  $a_{\alpha}$ . Now by extension again, find  $q' \in gS(N'_{\alpha})$  that does not  $\mu$ -fork over  $M_0$  and extends gtp(b/M). Let  $b_{\alpha}$  realize q' and pick  $N_{\alpha+1}$  limit over  $N'_{\alpha}$  containing  $b_{\alpha}$ .

This is enough. We show that for  $\alpha, \beta < \lambda$ :

- (1)  $\operatorname{gtp}(a_{\alpha}b/M_0) \neq \operatorname{gtp}(ab/M_0)$
- (2) If  $\beta < \alpha$ ,  $\operatorname{gtp}(ab/M_0) \neq \operatorname{gtp}(a_\alpha b_\beta/M_0)$ .
- (3) If  $\beta \ge \alpha$ ,  $\operatorname{gtp}(ab/M_0) = \operatorname{gtp}(a_\alpha b_\beta/M_0)$ .

For (1), observe that  $b \in |N_0| \subseteq |N_\alpha|$  and  $gtp(a_\alpha/N_\alpha)$  explicitly does not  $\mu$ -fork over  $(N, M_0)$ . Therefore by monotonicity  $N_\alpha$  witnesses that there exists  $N_b \in \mathbf{K}_\mu$  containing b and limit over  $M_0$  so that  $gtp(a_\alpha/M_b)$  explicitly does not  $\mu$ -fork over  $(N, M_0)$ . By failure of symmetry and invariance, we must have that  $gtp(a_\alpha b/M_0) \neq gtp(ab/M_0)$ .

For (2), suppose  $\beta < \alpha$ . We know that  $gtp(a_{\alpha}/N_{\alpha})$  does not  $\mu$ -fork over  $M_0$ . Since  $\beta < \alpha$ ,  $b, b_{\beta} \in |N_{\alpha}|$  and  $gtp(b/M) = gtp(b_{\beta}/M)$ , we must have by Lemma 10.5.2 that  $gtp(a_{\alpha}b/M_0) = gtp(a_{\alpha}b_{\beta}/M_0)$ . Together with (1), this implies  $gtp(ab/M_0) \neq gtp(a_{\alpha}b_{\beta}/M_0)$ . This is really where we use the equivalence between uniform  $\mu$ -symmetry and weak uniform  $\mu$ -symmetry: if we only had failure of uniform  $\mu$ -symmetry, then we would only know that gtp(b/M) does not  $\mu$ -split over  $M_0$ , so would be unable to use Lemma 10.5.2.

To see (3), suppose  $\beta \geq \alpha$  and recall that (by (1))  $\operatorname{gtp}(ab/M_0) = \operatorname{gtp}(a_\beta b/M_0)$ . We also have that  $\operatorname{gtp}(b_\beta/N'_\beta)$  does not  $\mu$ -fork over  $M_0$ . Moreover  $\operatorname{gtp}(a/M_0) = \operatorname{gtp}(a_\alpha/M_0)$ , and  $a, a_\alpha \in N'_\beta$ . By Lemma 10.5.2 again,  $\operatorname{gtp}(ab_\beta/M_0) = \operatorname{gtp}(a_\alpha b_\beta/M_0)$ . This gives us that  $\operatorname{gtp}(ab/M_0) = \operatorname{gtp}(a_\alpha b_\beta/M_0)$ .

Now let d be an enumeration of  $M_0$  and for  $\alpha < \lambda$ , let  $\bar{c}_{\alpha} := a_{\alpha}b_{\alpha}d$ . Then (2) and (3) together tell us that the sequence  $\langle \bar{c}_{\alpha} \mid \alpha < \lambda \rangle$  witnesses the  $\mu$ -order property of length  $\lambda$ .

We conclude that symmetry follows from enough instances of superstability.

THEOREM 10.5.4. Let  $\mu \geq LS(\mathbf{K})$ . Then there exists  $\lambda < h(\mu)$  such that if  $\mathbf{K}$  is superstable in every  $\chi \in [\mu, \lambda)$ , then  $\mathbf{K}$  has  $\mu$ -symmetry.

PROOF. If **K** is unstable in  $2^{\mu}$ , then we can set  $\lambda := (2^{\mu})^+$  and get a vacuously true statement; so assume that **K** is stable in  $2^{\mu}$ . By Facts 2.4.11 and 2.2.25(1), **K** does not have the  $\mu$ -order property. By Fact 2.4.7, there exists  $\lambda < h(\mu)$  such that **K** does not have the  $\mu$ -order property of length  $\lambda$ . By Lemma 10.5.3, it is as desired.

REMARK 10.5.5. How can one obtain many instances of superstability as in the hypothesis of Theorem 10.5.4? One way is categoricity, see Fact 10.7.1. Another way is to start with one instance of superstability and transfer it up using tameness, see Fact 10.6.7.

### 10.6. Symmetry and tameness

Tameness is a locality property for types introduced by Grossberg and VanDieren in [**GV06b**] and used to prove Shelah's eventual categoricity conjecture from a successor in [**GV06c**]. It has also played a key roles in the proof of several other categoricity transfers, for example [**Bon14b**] or Chapters 8, 16.

DEFINITION 10.6.1 (Tameness). Let  $\mu \geq \mathrm{LS}(\mathbf{K})$ . **K** is  $\mu$ -tame if for every  $M \in \mathbf{K}$  and every  $p, q \in \mathrm{gS}(M)$ , if  $p \neq q$ , then there exists  $M_0 \in \mathbf{K}_{\leq \mu}$  with  $M_0 \leq_{\mathbf{K}} M$  such that  $p \upharpoonright M_0 \neq q \upharpoonright M_0$ .

In this section, we study the combination of tameness (and its relatives, see below) with superstability. In Section 10.7, we will combine tameness and categoricity.

10.6.1. Weak tameness. We will start by studying a weaker, more local, variation that appears already in [She99]. We use the notation in [Bal09, Definition 11.6].

DEFINITION 10.6.2 (Weak tameness). Let  $\chi, \mu$  be cardinals with  $LS(\mathbf{K}) \leq \chi \leq \mu$ . **K** is  $(\chi, \mu)$ -weakly tame if for any saturated  $M \in \mathbf{K}_{\mu}$ , any  $p, q \in gS(M)$ , if  $p \neq q$ , there exists  $M_0 \in \mathbf{K}_{\chi}$  with  $M_0 \leq_{\mathbf{K}} M$  and  $p \upharpoonright M_0 \neq q \upharpoonright M_0$ .

Tameness says that type over *any* models are determined by their small restrictions. Weak tameness says that only types over *saturated* models have this property.

While there is no known example of an AEC that is weakly tame but not tame, it is known that weak tameness follows from categoricity in a suitable cardinal (but the corresponding result for non-weak tameness is open, see [**GV06a**, Conjecture 1.5]): this appears as [**She99**, Main Claim II.2.3] and a simplified argument is in [**Bal09**, Theorem 11.15].

FACT 10.6.3. Let  $\lambda > \mu \ge h(\mathrm{LS}(\mathbf{K}))$ . Assume that **K** is categorical in  $\lambda$ , and the model of cardinality  $\lambda$  is  $\mu^+$ -saturated. Then there exists  $\chi < h(\mathrm{LS}(\mathbf{K}))$  such that **K** is  $(\chi, \mu)$ -weakly tame.

It was shown in Chapter 4 (and further improvements in Section 6.10 and Chapter 7 that tameness can be combined with superstability to build a good frame at a high-enough cardinal. At a meeting in the winter of 2015 in San Antonio, VanDieren asked whether weak tameness could be used instead. This is not a generalization for the sake of generalization because weak tameness (but not tameness) is known to follow from categoricity. We can answer in the affirmative:

THEOREM 10.6.4. Let  $\lambda > \mu \ge LS(\mathbf{K})$ . Assume that  $\mathbf{K}$  is superstable in every  $\chi \in [\mu, \lambda]$  and has  $\lambda$ -symmetry.

If **K** is  $(\mu, \lambda)$ -weakly tame, then there exists a type-full good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$ .

PROOF. First observe that limit models in  $\mathbf{K}_{\lambda}$  are unique (by Fact 10.2.17), hence saturated. By Theorem 10.1.1,  $\mathbf{K}$  has  $\chi$ -symmetry for every  $\chi \in [\mu, \lambda]$ . By Fact 10.2.18, for every  $\chi \in [\mu, \lambda)$ ,  $\mathbf{K}^{\chi^+$ -sat}, the class of  $\chi^+$ -saturated models in  $\mathbf{K}_{\geq \chi^+}$  is an AEC with  $\mathrm{LS}(\mathbf{K}^{\chi^+$ -sat}) = \chi^+. Therefore by Lemma 10.2.20  $\mathbf{K}^{\lambda\text{-sat}}$ is an AEC with  $\mathrm{LS}(\mathbf{K}^{\lambda\text{-sat}}) = \lambda$ . By the  $\lambda$ -superstability assumption,  $\mathbf{K}_{\lambda}^{\lambda\text{-sat}}$  is nonempty, has amalgamation, no maximal models, and joint embedding. It is also stable in  $\lambda$ . We want to define a type-full good  $\lambda$ -frame  $\mathfrak{s}$  on  $\mathbf{K}_{\lambda}^{\lambda\text{-sat}}$ . We define forking in the sense of  $\mathfrak{s}$  ( $\mathfrak{s}$ -forking) as follows: For  $M \leq_{\mathbf{K}} N$  saturated of size  $\lambda$ , a non-algebraic  $p \in \mathrm{gS}(N)$  does not  $\mathfrak{s}$ -fork over M if and only if it does not  $\mu$ -fork over M.

Now most of the axioms of good frames are verified in Section 4.4, the only properties that remain to be checked are extension, uniqueness, and symmetry. Extension is by Proposition 10.5.1, and uniqueness is by uniqueness of splitting in  $\mu$  ([Van06, I.4.12]) and the weak tameness assumption. As for symmetry, we know that  $\lambda$ -symmetry holds, hence by Proposition 10.4.4, Proposition 10.4.5, and Theorem 10.4.13 the symmetry property of good frame follows.

REMARK 10.6.5. If  $\lambda = \mu^+$  above, then the hypotheses reduce to "**K** is superstable in  $\mu$  and  $\mu^+$  and **K** has  $\mu^+$ -symmetry".

We can combine this construction with the results of Section 10.5:

COROLLARY 10.6.6. Let  $\lambda > \mu \geq \text{LS}(\mathbf{K})$ . Assume that **K** is superstable in every  $\chi \in [\mu, h(\lambda))$ . If **K** is  $(\mu, \lambda)$ -weakly tame, then there exists a type-full good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$ .

PROOF. Combine Theorem 10.6.4 and Theorem 10.5.4.

10.6.2. Global tameness. For the rest of this section, we will work with global non-weak tameness. Superstability is studied together with amalgamation and tameness in many of the chapters of this thesis (for example Chapters 4, 6, 7, and 9). Recall the following upward transfer of superstability:

FACT 10.6.7 (Proposition 6.10.10). Assume **K** is  $\mu$ -superstable and  $\mu$ -tame. Then for all  $\mu' \geq \mu$ , **K** is  $\mu'$ -superstable. In particular,  $\mathbf{K}_{\geq \mu}$  has no maximal models and is stable in all cardinals.

An application of tameness and superstability is to chains of saturated models. Recall from Section 10.4 that  $\mathbf{K}^{\lambda-\text{sat}}$  denotes the class of  $\lambda$ -saturated models in  $\mathbf{K}_{\geq\lambda}$ . We would like to give conditions under which  $\mathbf{K}^{\lambda-\text{sat}}$  is an AEC – in particular unions of chains of  $\lambda$ -saturated models are  $\lambda$ -saturated. From superstability and tameness, it is known that one eventually obtains this behavior:

FACT 10.6.8 (Theorem 7.6.1). Assume **K** is  $\mu$ -superstable and  $\mu$ -tame. Then there exists  $\lambda_0 < \beth_{(2^{\mu^+})^+}$  such that for any  $\lambda \geq \lambda_0$ ,  $\mathbf{K}^{\lambda\text{-sat}}$  is an AEC with  $\mathrm{LS}(\mathbf{K}^{\lambda\text{-sat}}) = \lambda$ .

We can use this to show that superstability implies symmetry in tame AECs (obtaining another partial answer to Question 10.4.14). We also give another, more self-contained proof that does not rely on Fact 10.6.8.

COROLLARY 10.6.9. If K is  $\mu$ -superstable and  $\mu$ -tame, then K has  $\mu$ -symmetry.

FIRST PROOF. First observe that by Fact 10.6.7, **K** is superstable in every  $\mu' \ge \mu$ . By Fact 10.6.8, there exists  $\lambda_0 \ge \mu$  such that  $\mathbf{K}^{\lambda_0^+\text{-sat}}$  is an AEC. Therefore the hypotheses of Fact 10.2.13 are satisfied, so **K** has  $\lambda_0$ -symmetry. By Theorem 10.1.1, **K** has  $\mu$ -symmetry.

SECOND PROOF. As in the first proof, **K** is superstable in every  $\mu' \ge \mu$ . By Theorem 10.5.4, **K** has  $\mu$ -symmetry.

Thus we obtain an improvement on the Hanf number of Fact 10.6.8:

COROLLARY 10.6.10. Assume **K** is  $\mu$ -superstable and  $\mu$ -tame. For every  $\lambda > \mu$ ,  $\mathbf{K}^{\lambda$ -sat} is an AEC with  $\mathrm{LS}(\mathbf{K}^{\lambda}$ -sat) =  $\lambda$ .

PROOF. By Fact 10.6.7 and Corollary 10.6.9, **K** is  $\lambda$ -superstable and has  $\lambda$ -symmetry for any  $\lambda > \mu$ . By Fact 10.2.18,  $\mathbf{K}^{\mu^+\text{-sat}}$  is an AEC with  $\mathrm{LS}(\mathbf{K}^{\mu^+\text{-sat}}) = \mu^+$ . We can replace  $\mu^+$  with any successor  $\lambda > \mu$ . To take care of limit cardinals  $\lambda$ , use Lemma 10.2.20.

Note that Corollary 10.6.10 is an improvement on Fact 10.6.8 and its second proof does not rely on Fact 10.6.8. However beyond Fact 10.6.8, the arguments of Chapter 7 (in particular the use of averages) have other applications (see for example the proof of solvability in Theorem 9.4.9).

We can also say more on another result of Chapter 7: Theorem 7.5.17 implies that, assuming  $\mu$ -superstability, there is a  $\lambda_0 \ge \mu$  such that if  $\langle M_i : i < \delta \rangle$  is a chain of  $\lambda_0$ -saturated models where  $\delta \ge \lambda_0$  and  $M_{i+1}$  is universal over  $M_i$ , then  $\bigcup_{i < \delta} M_i$  is saturated. We can improve this too:

COROLLARY 10.6.11. Assume **K** is  $\mu$ -superstable and  $\mu$ -tame. Let  $\delta$  be a limit ordinal and  $\langle M_i : i < \delta \rangle$  is increasing in  $\mathbf{K}_{\geq \mu}$  and  $M_{i+1}$  is universal over  $M_i$  for all  $i < \delta$ . Let  $M_{\delta} := \bigcup_{i < \delta} M_i$ . If  $||M_{\delta}|| > \mathrm{LS}(\mathbf{K})$ , then  $M_{\delta}$  is saturated.

PROOF. By Proposition 10.3.1,  $M_{\delta}$  is a  $(\lambda, cf(\delta))$ -limit model, where  $\lambda = ||M_{\delta}||$ . By Fact 10.6.7, **K** is  $\lambda$ -superstable. By Corollary 10.6.9, **K** has  $\lambda$ -symmetry. By Fact 10.2.17,  $M_{\delta}$  is saturated.

One can ask whether Corollary 10.6.10 can be improved further by also getting the conclusion for  $\lambda = \mu$ . If  $\mu = \text{LS}(\mathbf{K})$ , it is not clear that LS(K)-saturated models are the right notion so perhaps the right question is to be framed in terms of a superlimit. Recall from [She09a, Definition N.2.2.4] that a superlimit model is a universal model M with a proper extension so that if  $\langle M_i : i < \delta \rangle$  is an increasing chain with  $M \cong M_i$  for all  $i < \delta$ , then (if  $\delta < ||M||^+$ ),  $M \cong \bigcup_{i < \delta} M_i$ . Note that, assuming  $\mu$ -superstability and uniqueness of limit models of size  $\mu$ , it is easy to see that the existence of a superlimit of size  $\mu$  is equivalent to the statement that the union of an increasing chain of limit models in  $\mu$  (of length less than  $\mu^+$ ) is limit.

QUESTION 10.6.12. Assume **K** is  $\mu$ -tame and there is a type-full good  $\mu$ -frame on  $\mathbf{K}_{\mu}$  (or just that **K** is  $\mu$ -superstable). Is there a superlimit model of size  $\mu$ ?

We now turn to good frames and show that, assuming tameness, the statement of Theorem 10.6.4 can be simplified. Recall that previous work gives a condition under which good frames can be constructed from tameness:

FACT 10.6.13 (Theorem 6.10.8). Assume **K** is  $\mu$ -superstable and  $\mu$ -tame. If for any  $\delta < \mu^+$ , any chain of length  $\delta$  of saturated models in  $\mathbf{K}_{\mu^+}$  has a saturated union, then there is a type-full good  $\mu^+$ -frame with underlying class  $\mathbf{K}_{\mu^+}^{\mu^+}$ .

Combining this with Fact 10.6.8 it was proven in Chapter 7 that  $\mu$ -superstability and  $\mu$ -tameness implies the existence of a good  $\lambda$ -frame on the saturated models of size  $\lambda$ , for some high-enough  $\lambda > \mu$ . Now we show that we can take  $\lambda = \mu^+$ . We again give two proofs: one uses Theorem 10.6.4 and the other relies on Fact 10.6.13.

COROLLARY 10.6.14. If **K** is  $\mu$ -superstable and  $\mu$ -tame, then there is a type-full good  $\mu^+$ -frame with underlying class  $\mathbf{K}_{\mu^+}^{\mu^+-\text{sat}}$ .

FIRST PROOF. Combine Fact 10.6.13 and Corollary 10.6.10.  $\hfill \Box$ 

SECOND PROOF. By Fact 10.6.7, **K** is superstable in every  $\mu' \ge \mu$ . Now apply Corollary 10.6.6 (with  $\lambda$  there standing for  $\mu^+$  here).

REMARK 10.6.15. To obtain a type-full good  $\lambda$ -frame for  $\lambda > \mu^+$ , we can either make a slight change to the second proof of Corollary 10.6.14, or use the upward frame transfer of [**Bon14a**], Chapter 5.

#### 10.7. Symmetry and categoricity

Theorem 10.1.1 has several applications to categorical AECs. We will use the following result, an adaptation of an argument of Shelah and Villaveces [SV99, Theorem 2.2.1] to settings with amalgamation, see Chapter 9:

FACT 10.7.1 (The Shelah-Villaveces theorem). Assume that **K** has no maximal models. Let  $\mu \geq LS(K)$ . If **K** is categorical in a  $\lambda > \mu$ , then **K** is  $\mu$ -superstable.

COROLLARY 10.7.2. Assume that **K** has no maximal models. Suppose  $\lambda$  and  $\mu$  are cardinals so that  $\lambda > \mu \ge LS(\mathbf{K})$  and assume that **K** is categorical in  $\lambda$ . Then **K** is  $\mu$ -superstable and it has  $\mu$ -symmetry if at least one of the following conditions hold:

- (1) The model of size  $\lambda$  is  $\mu^+$ -saturated.
- (2)  $\lambda \ge h(\mu)$ .

**PROOF.** By Fact 10.7.1, **K** is  $\chi$ -superstable in every  $\chi \in [\mu, \lambda)$ . Now:

- (1) If the model of size  $\lambda$  is  $\mu^+$ -saturated, then by Theorem 10.3.9, **K** has  $\mu$ -symmetry.
- (2) If  $\lambda \ge h(\mu)$ , then by Theorem 10.5.4, **K** has  $\mu$ -symmetry.

As announced in the introduction, we can combine Corollary 10.7.2 with Fact 10.2.17 to improve on [She99, Theorem 6.5]. The following result also improves on Corollary 18 of [Van16b], by removing the successor assumption in the categoricity cardinal and obtaining uniqueness of limit models in much smaller cardinalities as well.

COROLLARY 10.7.3. Assume that **K** has no maximal models. Suppose  $\lambda$  and  $\mu$  are cardinals so that  $\lambda > \mu \ge \text{LS}(\mathbf{K})$  and assume that **K** is categorical in  $\lambda$ . If either  $\text{cf}(\lambda) > \mu$  or  $\lambda \ge h(\mu)$ , then **K** has uniqueness of limit models of cardinality  $\mu$ . That is, if  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$  are such that both  $M_1$  and  $M_2$  are limit models over  $M_0$ , then  $M_1 \cong_{M_0} M_2$ .

PROOF. Categoricity in  $\lambda$ , the assumption that  $cf(\lambda) > \mu$ , and Fact 10.7.1 imply that the model of cardinality  $\lambda$  is  $\mu^+$ -saturated. We can now apply Corollary 10.7.2, to get that **K** is  $\mu$ -superstable and has  $\mu$ -symmetry. Then Fact 10.2.17 finishes the proof.

Once we have obtained symmetry from a high-enough categoricity cardinal, we can deduce that the model in the categoricity cardinal has some saturation:

COROLLARY 10.7.4. Let  $\mu > \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}$  is categorical in a  $\lambda \geq \sup_{\mu_0 < \mu} h(\mu_0^+)$ . Then the model of size  $\lambda$  is  $\mu$ -saturated.

PROOF. By Fact 10.2.2, we can assume without loss of generality that **K** has no maximal models. We check that the model of size  $\lambda$  is  $\mu_0^+$ -saturated for every  $\mu_0 \in [\text{LS}(\mathbf{K}), \mu)$ . Fix such a  $\mu_0$ . By Corollary 10.7.2, **K** is  $\mu_0$ -superstable,  $\mu_0^+$ superstable, and has  $\mu_0^+$ -symmetry. By Fact 10.2.18  $\mathbf{K}^{\mu_0^+}$ -sat, the class of  $\mu_0^+$ saturated models in  $\mathbf{K}_{\geq \mu_0^+}$ , is an AEC with Löwenheim-Skolem number  $\mu_0^+$ . Since it has arbitrarily large models, it must have a model of size  $\lambda$ , which is unique by categoricity.

We conclude that categoricity in a high-enough cardinal implies some amount of weak tameness:

COROLLARY 10.7.5. Let  $\mu \geq \text{LS}(\mathbf{K})$ . Let  $\lambda \geq h(\mu^+)$ . If **K** is categorical in  $\lambda$ , then there exists  $\chi < H_1$  such that **K** is  $(\chi, \mu)$ -weakly tame.

PROOF. By Corollary 10.7.4, the model of size  $\lambda$  is  $\mu^+$ -saturated. Now apply Fact 10.6.3.

We can derive a downward categoricity transfer. We will use the notation from Chapter 14 of [**Bal09**]: we write  $H_1$  for  $h(LS(\mathbf{K}))$  and  $H_2$  for  $h(H_1) = h(h(LS(\mathbf{K})))$ . We will use the following fact, given by the proof of [**Bal09**, Theorem 14.9] (originally [**She99**, II.1.6]):

FACT 10.7.6. If **K** is categorical in a  $\lambda > H_2$ , **K** is  $(\chi, H_2)$ -weakly tame for some  $\chi < H_1$ , and the model of size  $\lambda$  is  $\chi$ -saturated, then **K** is categorical in  $H_2$ .

COROLLARY 10.7.7. If **K** is categorical in a  $\lambda \ge h(H_2^+)$ , then **K** is categorical in  $H_2$ .

PROOF. By Corollary 10.7.5, there exists  $\chi < H_1$  such that **K** is  $(\chi, H_2)$ -weakly tame. By Corollary 10.7.4, the model of size  $\lambda$  is  $\chi$ -saturated. Now apply Fact 10.7.6.

We obtain in particular:

COROLLARY 10.7.8. Let  $\mu = \beth_{\mu} > LS(\mathbf{K})$ . If **K** is categorical in some  $\lambda > \mu$ , then **K** is categorical in  $\mu$ .

PROOF. Without loss of generality (Fact 10.2.2), **K** has no maximal models. Applying Corollary 10.7.7 to  $\mathbf{K}_{\geq\mu_0}$  for each  $\mu_0 < \mu$ , we get that **K** is categorical in unboundedly many  $\mu_0 < \mu$ . By (for example) Fact 10.7.1, **K** is stable in every  $\mu_0 < \mu$ . Thus the models in the categoricity cardinals below  $\mu$  are saturated, hence every model of size  $\mu$  is also saturated.

We can also build a good frame assuming categoricity in a high-enough cardinal (of arbitrary cofinality).

COROLLARY 10.7.9. Let  $\mu \geq H_1$ . Assume that **K** is categorical in a  $\lambda > \mu$ . If the model of size  $\lambda$  is  $\mu^+$ -saturated (e.g. if  $cf(\lambda) > \mu$  or by Corollary 10.7.4 if  $\lambda \geq h(\mu^+)$ ), then there exists a type-full good  $\mu$ -frame with underlying class  $\mathbf{K}_{\mu}^{\mu-\text{sat}}$ .

PROOF. By Fact 10.2.2, we can assume without loss of generality that **K** has no maximal models. By Fact 10.6.3, there exists  $\chi < H_1$  such that **K** is  $(\chi, \mu)$ weakly tame. By Corollary 10.7.2, **K** has  $\chi'$ -symmetry and is  $\chi'$ -superstable for every  $\chi' \in [\chi, \mu]$ . Now apply Theorem 10.6.4 with  $(\mu, \lambda)$  there standing for  $(\chi, \mu)$ here.

The Hanf number of Corollary 10.7.9 can be improved if we assume that the AEC is tame. We state a more general corollary summing up our results in tame categorical AECs:

COROLLARY 10.7.10. Assume that **K** has no maximal models and is  $LS(\mathbf{K})$ -tame. If **K** is categorical in a  $\lambda > LS(\mathbf{K})$ , then:

- (1) For any  $\mu \geq \text{LS}(\mathbf{K})$ , **K** has uniqueness of limit models in  $\mu$ : if  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$  are such that both  $M_1$  and  $M_2$  are limit over models  $M_0$ , then  $M_1 \cong_{M_0} M_2$ .
- (2) For any  $\mu > \text{LS}(\mathbf{K})$ ,  $\mathbf{K}^{\mu\text{-sat}}$  is an AEC with  $\text{LS}(\mathbf{K}^{\mu\text{-sat}}) = \mu$  and there exists a type-full good  $\mu$ -frame with underlying class  $\mathbf{K}^{\mu\text{-sat}}_{\mu}$ .

PROOF. By Fact 10.7.1, **K** is superstable in LS(**K**). By Fact 10.6.7, **K** is superstable in every  $\mu \geq \text{LS}(\mathbf{K})$ . By Corollary 10.6.9, **K** has symmetry in every  $\mu \geq \text{LS}(\mathbf{K})$ . The first part now follows from Fact 10.2.17 and the second from Corollary 10.6.10 and Corollary 10.6.14 (together with Remark 10.6.15).

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REMARK 10.7.11. By Chapter 5, we can transfer the type-full good  $LS(\mathbf{K})^+$ frame on  $\mathbf{K}_{LS(\mathbf{K})^+}^{LS(\mathbf{K})^+}$  given by the previous corollary to a type-full good ( $\geq LS(\mathbf{K})^+$ )frame with underlying class  $\mathbf{K}^{LS(\mathbf{K})^+}$ . That is, the nonforking relation of the frame is can be extended to types over all  $LS(\mathbf{K})^+$ -saturated models.

# CHAPTER 11

# Shelah's eventual categoricity conjecture in tame AECs with primes

This chapter is based on [**Vasf**]. I thank Tapani Hyttinen for his comments on the categoricity conjecture for homogeneous model theory, as well as the referee for several thorough reports that greatly helped improve the presentation and focus of this chapter.

#### Abstract

A new case of Shelah's eventual categoricity conjecture is established:

THEOREM 11.0.12. Let **K** be an AEC with amalgamation. Write  $H_2 := \Box_{\left(2^{2} (2^{\text{LS}(\mathbf{K})})^+\right)^+}$ . Assume that **K** is  $H_2$ -tame and  $\mathbf{K}_{\geq H_2}$  has primes over sets of

the form  $M \cup \{a\}$ . If **K** is categorical in some  $\lambda > H_2$ , then **K** is categorical in all  $\lambda' \ge H_2$ .

The result had previously been established when the stronger locality assumptions of full tameness and shortness are also required.

An application of the method of proof of Theorem 11.0.12 is that Shelah's categoricity conjecture holds in the context of homogeneous model theory (this was known, but our proof gives new cases):

THEOREM 11.0.13. Let D be a homogeneous diagram in a first-order theory T. If D is categorical in a  $\lambda > |T|$ , then D is categorical in all  $\lambda' \ge \min(\lambda, \beth_{(2^{|T|})^+})$ .

# 11.1. Introduction

Shelah's eventual categoricity conjecture is a major force in the development of classification theory for abstract elementary classes  $(AECs)^1$ .

CONJECTURE 11.1.1 (Shelah's eventual categoricity conjecture, N.4.2 in [She09a]). An AEC categorical in a high-enough cardinal is categorical on a tail of cardinals.

In Chapter 8, we established the conjecture for universal classes with the amalgamation property<sup>2</sup> (a universal class is a class of models closed under isomorphisms, substructures, and unions of  $\subseteq$ -increasing chains, see [She87b]). The proof starts by noting that universal classes satisfy tameness: a locality property introduced in VanDieren's 2002 Ph.D. thesis (the relevant chapter appears in [GV06b]).

 $<sup>^{1}</sup>$ For a history, see the introduction of Chapter 8. We assume here that the reader is familiar with the basics of AECs as presented in e.g. [**Bal09**].

 $<sup>^{2}</sup>$ After the initial submission of the paper this chapter is based on, we managed to remove the amalgamation hypothesis (Chapter 16).

FACT 11.1.2 ([**Bonc**]). Any universal class  $\mathbf{K}$  is<sup>3</sup> LS( $\mathbf{K}$ )-tame.

The proof generalizes to give a stronger locality property introduced in [Bon14b]:

DEFINITION 11.1.3. Let **K** be an AEC and let  $\chi \geq \text{LS}(\mathbf{K})$  be an infinite cardinal. **K** is *fully*  $\chi$ -tame and short if for any  $M \in \mathbf{K}$ , any ordinal  $\alpha$ , and any Galois types  $p, q \in \text{gS}^{\alpha}(M)$  of length  $\alpha$ , p = q if and only if  $p^{I} \upharpoonright M_{0} = q^{I} \upharpoonright M_{0}$  for any  $M_{0} \in \mathbf{K}_{\leq \chi}$  with  $M_{0} \leq_{\mathbf{K}} M$  and any  $I \subseteq \alpha$  with  $|I| \leq \chi$ .

FACT 11.1.4. Any universal class  $\mathbf{K}$  is fully  $LS(\mathbf{K})$ -tame and short.

Another important property of universal classes used in the proof of Shelah's eventual categoricity conjecture (Corollary 8.5.20) is that they have primes. The definition is due to Shelah and appears in [She09a, III.3]. For the convenience of the reader, we include it here:

DEFINITION 11.1.5. Let  $\mathbf{K}$  be an AEC.

- (1) We say a triple (a, M, N) represents a Galois type p if p = gtp(a/M; N). In particular,  $M \leq_{\mathbf{K}} N$  and  $a \in |N|$ .
- (2) A prime triple is a triple (a, M, N) representing a nonalgebraic Galois type p such that for every  $N' \in \mathbf{K}$ ,  $a' \in |N'|$ , if  $p = \operatorname{gtp}(a'/M; N')$  then there exists  $f: N \xrightarrow{M} N'$  so that f(a) = a'.
- (3) We say that **K** has primes if for every  $M \in \mathbf{K}$  and every nonalgebraic  $p \in gS(M)$ , there exists a prime triple representing p.
- (4) We define localizations such as " $\mathbf{K}_{\lambda}$  has primes" in the natural way.

By taking the closure of  $|M| \cup \{a\}$  under the functions of N, we get:

FACT 11.1.6 (Remark 8.5.3). Any universal class has primes.

The proof of the eventual categoricity conjecture for universal classes with amalgamation in Chapter 8 generalizes to give:

FACT 11.1.7 (Theorem 8.5.18). Fully tame and short AECs that have amalgamation and primes satisfy Shelah's eventual categoricity conjecture.

Many results only use the assumption of tameness (for example [**GV06b**, **GV06c**, **GV06a**, **BKV06**, **Lie11b**] and in Chapters 4 and 7), while others use full tameness and shortness (e.g. [**BG**] or Chapter 6, but it is also unclear whether it is really needed there, see Question 6.15.4).

It is natural to ask whether shortness can be removed from Fact 11.1.7. We answer in the affirmative: Tame AECs with primes and amalgamation satisfy Shelah's eventual categoricity conjecture.

MAIN THEOREM 11.3.8. Let **K** be an AEC with amalgamation. Assume that **K** is  $H_2$ -tame and  $\mathbf{K}_{\geq H_2}$  has primes. If **K** is categorical in some  $\lambda > H_2$ , then **K** is categorical in all  $\lambda' \geq H_2$ .

This improves Theorem 8.5.18 which assumed full  $LS(\mathbf{K})$ -tameness and shortness (so the improvement is on two counts: "full tameness and shortness" is replaced by "tameness" and "LS( $\mathbf{K}$ )" is replaced by " $H_2$ "). Compared to Grossberg and

 $<sup>^{3}</sup>$ While the main idea of the proof is due to Will Boney, the fact that it applies to universal classes is due to the author. A full proof of Fact 11.1.2 appears as Theorem 8.3.6.

VanDieren's upward transfer [**GV06a**], we do *not* require categoricity in a successor cardinal, but we *do* require the categoricity cardinal to be at least  $H_2$  and more importantly ask for the AEC to have primes.

Let us give a rough picture of the proof of both Theorem 11.3.8 and the earlier Theorem 8.5.18. We will then explain where exactly the two proofs differ. The first step of the proof is to find a sub-AEC  $\mathbf{K}'$  of  $\mathbf{K}$  (typically a class of saturated models or just a tail: in the case of Theorem 11.3.8 we will have  $\mathbf{K}' = \mathbf{K}_{\geq H_2}$ ) which is "well-behaved" in the sense of admitting a good-enough notion of independence. Typically, the first step does not use primes. The second step is to show that in  $\mathbf{K}'$ , categoricity in *some*  $\lambda > \mathrm{LS}(\mathbf{K}')$  implies categoricity in *all*  $\lambda' > \mathrm{LS}(\mathbf{K}')$ . This uses orthogonality calculus and the existence of prime models. The third step pulls back this categoricity transfer to  $\mathbf{K}$ .

Shelah has developed orthogonality calculus in the context of what he calls successful good<sup>+</sup>  $\lambda$ -frames [**She09a**, III.6]. It is known (from Chapter 6) that one can build such a frame using categoricity, amalgamation, and full tameness and shortness so this is how **K**' from the previous paragraph was chosen in Chapter 8. The orthogonality calculus part was just quoted from Shelah (although we did provide some proofs for the convenience of the reader). It is not known how to build a successful good<sup>+</sup>  $\lambda$ -frame using just categoricity, amalgamation, and *tameness*.

In this chapter, we develop orthogonality calculus in the setup of good  $\lambda$ -frames with primes (i.e. we get rid of the successful good<sup>+</sup> hypothesis). Note that it is easier to build good frames than to build successful ones (see Chapter 4 and Corollary 10.6.14). In particular, this can be done with just amalgamation, categoricity, and tameness (the threshold cardinals are also lower than in the construction of a successful good frame).

To develop orthogonality calculus in good frames with primes, we change Shelah's definition of orthogonality: Shelah's definition uses the so-called uniqueness triples, which may not exist here. This chapter's definition uses prime triples instead and shows that the proofs needed for the categoricity transfer still go through. This is the main difference between this chapter and Chapter 8. In some places, new arguments are provided. For example, Lemma 11.2.4, saying that a definition of orthogonality in terms of "for all" is equivalent to one in terms of "there exists", has a different proof than Shelah's.

Let us justify the assumptions of Theorem 11.3.8. First of all, why do we ask for  $\lambda > H_2$  and not e.g.  $\lambda > H_1$  or even  $\lambda > \text{LS}(\mathbf{K})$ ? The reason is that the argument uses categoricity in *two* cardinals, so we appeal to a downward categoricity transfer implicit in [She99, II.1.6] which proves (without using primes) that classes as in the hypothesis of Theorem 11.3.8 must be categorical in  $H_2$ . If we know that the class is categorical in two cardinals already, then we can work above  $\text{LS}(\mathbf{K})$  (provided of course we adjust the levels at which tameness and primes occur). This is Theorem 11.3.4. Moreover if we know that for some  $\chi < \lambda$ , the class of  $\chi$ -saturated models of  $\mathbf{K}$  has primes, then we can also lower the Hanf number from  $H_2$  to  $H_1$  (see Theorem 11.3.10).

Let us now discuss the structural assumptions on **K**. Many classes occurring in practice have amalgamation. Grossberg conjectured [**Gro02**, 2.3] that eventual amalgamation should follow from categoricity and, assuming that the class is *eventually syntactically characterizable* (see Section 8.4), it does assuming the other assumptions: tameness and having primes. We now focus on these two assumptions.

A wide variety of AECs are tame (see e.g. the introduction to  $[\mathbf{GV06b}]$  or the upcoming survey  $[\mathbf{BVd}]$ ), and many classes studied by algebraists have primes (one example are AECs which admit intersections, i.e. whenever  $N \in \mathbf{K}$  and  $A \subseteq |N|$ , we have that  $\bigcap \{M \leq_{\mathbf{K}} N \mid A \subseteq |M|\} \leq_{\mathbf{K}} N$ . See  $[\mathbf{BS08}]$  or Section 8.2). Tameness is conjectured (see  $[\mathbf{GV06a}, \text{Conjecture 1.5}]$ ) to follow from categoricity and of course, the existence of prime models plays a key role in many categoricity transfer results including Morley's categoricity theorem and Shelah's generalization to excellent classes  $[\mathbf{She83a}, \mathbf{She83b}]$ . Currently, no general way<sup>4</sup> of building prime models in AECs is known except by going through the machinery of excellence  $[\mathbf{She09a}, \mathbf{Chapter III}]$ . It is unknown whether excellence follows from categoricity.

In the special case of homogeneous model theory, it is easier to build prime models<sup>5</sup>. Let **K** be a class of models of a homogeneous diagram categorical in a  $\lambda > H_2$ . Clearly, **K** has amalgamation and is fully LS(**K**)-tame and short. By stability and [**She70**, Section 5], the class of  $H_2$ -saturated models of **K** has primes. The proof of Theorem 11.3.8 first argues without using primes that **K** is categorical in  $H_2$ . Hence the class of  $H_2$ -saturated models of **K** is just the class  $\mathbf{K}_{\geq H_2}$ , so it has primes. We apply Theorem 11.3.8 to obtain the eventual categoricity conjecture for homogeneous model theory. Actually Theorem 11.3.8 is not needed for that result: Theorem 8.5.18 suffices. However we can also improve on the Hanf number  $H_2$  and obtain Theorem 11.0.13 from the abstract:

THEOREM 11.4.22. Let D be a homogeneous diagram in a first-order theory T. If D is categorical in some  $\lambda > |T|$ , then D is categorical in all  $\lambda' \ge \min(\lambda, h(|T|))$ .

When T is countable, a stronger result has been established by Lessmann [Les00]: categoricity in some uncountable cardinal implies categoricity in *all* uncountable cardinals. When T is uncountable, the eventual categoricity conjecture for homogeneous model theory is implicit in [She70, Section 7] and was also given a proof by Hyttinen [Hyt98]. More precisely, Hyttinen prove that categoricity in some  $\lambda > |T|$  with  $\lambda \neq \aleph_{\omega}(|T|)$  implies categoricity in all  $\lambda' \ge \min(\lambda, h(|T|))$ . Our proof of Theorem 11.4.22 is new and also covers the case  $\lambda = \aleph_{\omega}(|T|)$ . We do not know whether a similar result also holds in the framework of finitary AECs (there the categoricity conjecture has been solved for tame and *simple*<sup>6</sup> finitary AECs with countable Löwenheim-Skolem number [HK06]<sup>7</sup>).

A continuation of the present chapter is in Chapter 14, where orthogonality calculus is developed inside good frames that do not necessarily have primes. We establish there that the analog of Theorem 11.4.22 (i.e. the threshold is  $H_1$ ) holds in any LS(**K**)-tame AEC with amalgamation and primes.

 $<sup>^{4}</sup>$ We discuss homogeneous model theory and more generally finitary AECs later.

 $<sup>^5\</sup>mathrm{We}$  thank Rami Grossberg for asking us if the methods of Chapter 8 could be adapted to this context.

<sup>&</sup>lt;sup>6</sup>In this context, stable does not imply simple.

<sup>&</sup>lt;sup>7</sup>The argument is similar to the proof of Morley's categoricity theorem.

#### 11.2. Orthogonality with primes

In [She09a, III.6], Shelah develops a theory of orthogonality for good frames. In addition to the existence of primes, his assumptions include that the good frame is successful good<sup>+</sup> (see [She09a, III.1]) so in particular it expands to an independence relation NF for models in  $\mathbf{K}_{\lambda}$ . While successfulness follows from full tameness and shortness (Theorem 6.11.13), it is not clear if it follows from tameness only, so we do not adopt this assumption. Instead we will assume only that the good frame has primes.<sup>8</sup>

The proof of Fact 11.1.7 uses Shelah's theory of orthogonality to prove a technical statement on good frames being preserved when doing a certain change of AEC [She09a, III.12.39]. We show that this statements still holds if we do not assume successfulness but only the existence of primes (see Theorem 11.2.7). Along the way, we develop orthogonality calculus in good frames with primes. To do so, we change Shelah's definition of orthogonality from [She09a, III.6.2] to use prime triples instead of uniqueness triples and check that [She09a, III.12.39] can still be proven using this new definition of orthogonality.

We assume that the reader is familiar with Sections 8.5 and 8.7. We also assume that the reader is familiar with the basics of good frames as presented in [She09a, II.2]. As in [She09a, II.6.35], we say that a good  $\lambda$ -frame  $\mathfrak{s}$  is *type-full* if the basic types consist of *all* the nonalgebraic types over M. For simplicity, we focus on type-full good frames here. We say that a good  $\lambda$ -frame  $\mathfrak{s}$  is on  $\mathbf{K}_{\lambda}$  if its underlying class is  $\mathbf{K}_{\lambda}$ . We say that  $\mathfrak{s}$  is *categorical* if  $\mathbf{K}$  is categorical in  $\lambda$  and we say that it *has primes* if  $\mathbf{K}_{\lambda}$  has primes (where we localize Definition 11.1.5 in the natural way).

All throughout, we assume:

HYPOTHESIS 11.2.1.  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \bigcup, gS^{bs})$  is a categorical type-full good  $\lambda$ -frame which has primes. We work inside  $\mathfrak{s}$ .

Hypothesis 11.2.1 is reasonable: By Fact 11.3.2, categorical good frames exist assuming categoricity, amalgamation, and tameness. As for assuming the existence of primes, this is an hypothesis of our main theorem (Theorem 11.3.8) and we have tried to justify it in the introduction. Se also Fact 11.4.6, which shows how to obtain the existence of primes in the setup of homogeneous model theory.

The definition of orthogonality is similar to [She09a, III.6.2]: the only difference is that uniqueness triples are replaced by prime triples. In Shelah's context, this gives an equivalent definition (see [She09a, III.6.3]).

DEFINITION 11.2.2. Let  $M \in \mathbf{K}_{\lambda}$  and let  $p, q \in \mathrm{gS}(M)$  be nonalgebraic. We say that p is weakly orthogonal to q and write  $p \perp_{\mathrm{wk}} q$  if for all prime triples (b, M, N)representing q (i.e.  $q = \mathrm{gtp}(b/M; N)$ , see Definition 11.1.5(1)), we have that p has a unique extension to  $\mathrm{gS}(N)$ .

We say that p is orthogonal to q (written  $p \perp q$ ) if for every  $N \in \mathbf{K}_{\lambda}$  with  $N \geq_{\mathbf{K}} M, p' \perp_{wk} q'$ , where p', q' are the nonforking extensions to N of p and q respectively.

<sup>&</sup>lt;sup>8</sup>Recently, Will Boney and the author have shown (see Chapter 18) that the  $\aleph_{n-3}$ -good frame in the Hart-Shelah example is *not* (weakly) successful. However it is categorical and has primes (because the Hart-Shelah example admits intersections). Thus the setup of this chapter is strictly weaker than Shelah's.

For  $p_{\ell} \in gS(M_{\ell})$  nonalgebraic,  $\ell = 1, 2, p_1 \perp p_2$  if and only if there exists  $N \geq_{\mathbf{K}} M_{\ell}, \ell = 1, 2$  such that the nonforking extensions to  $N p'_1$  and  $p'_2$  of  $p_1$  and  $p_2$  respectively are orthogonal.

REMARK 11.2.3. Formally, the definition of orthogonality depends on the frame but  $\mathfrak{s}$  will always be fixed.

The next basic lemma says that we can replace the "for all" in Definition 11.2.2 by "there exists". This corresponds to [She09a, III.6.3], but the proof is different.

LEMMA 11.2.4. Let  $M \in \mathbf{K}_{\lambda}$  and  $p, q \in \mathrm{gS}(M)$  be nonalgebraic. Then  $p \perp_{\mathrm{wk}} q$  if and only if there exists a prime triple (b, M, N) representing q such that p has a unique extension to  $\mathrm{gS}(N)$ .

PROOF. The left to right direction is straightforward. Now assume (b, M, N) is a prime triple representing q such that p has a unique extension to gS(N). Let  $(b_2, M, N_2)$  be another prime triple representing q. We want to see that p has a unique extension to  $gS(N_2)$ . Let  $p_2 \in gS(N_2)$  be an extension of p. By primeness of  $(b_2, M, N_2)$ , there exists  $f : N_2 \xrightarrow[M]{} N$  such that  $f(b_2) = b$ .

We have that  $f(p_2)$  is an element of  $gS(f[N_2])$  and  $f[N_2] \leq_{\mathbf{K}} N$ , so using amalgamation pick  $p'_2 \in gS(N)$  extending  $f(p_2)$ . Now as f fixes M,  $f(p_2)$  extends p, so  $p'_2$  extends p. Since by assumption p has a unique extension to gS(N),  $p'_2$ must be this unique extension, and in particular  $p'_2$  does not fork over M. By monotonicity,  $f(p_2)$  does not fork over M. By invariance,  $p_2$  does not fork over M. This shows that  $p_2$  must be the unique extension of p to  $gS(N_2)$ , as desired.

We now show that weak orthogonality is the same as orthogonality. Recall (Hypothesis 11.2.1) that we are assuming categoricity in  $\lambda$ . In particular, all the models of size  $\lambda$  are superlimit<sup>9</sup>. Thus we can use the following property, which Shelah proves for superlimit models  $M, N \in \mathbf{K}_{\lambda}$ :

FACT 11.2.5 (The conjugation property, III.1.21 in [She09a]). Let  $M \leq_{\mathbf{K}} N$  be in  $\mathbf{K}_{\lambda}$ ,  $\alpha < \lambda$ , and let  $(p_i)_{i < \alpha}$  be types in gS(N) that do not fork over M. Then there exists  $f : N \cong M$  such that  $f(p_i) = p_i \upharpoonright M$  for all  $i < \alpha$ .

LEMMA 11.2.6 (III.6.8(5) in [She09a]). For  $M \in \mathbf{K}_{\lambda}$ ,  $p, q \in \mathrm{gS}(M)$  nonalgebraic,  $p \perp q$  if and only if  $p \perp q$ .

PROOF. Clearly if  $p \perp q$  then  $p \perp_{wk} q$ . Conversely assume  $p \perp_{wk} q$  and let  $N \geq_{\mathbf{K}} M$ . Let p', q' be the nonforking extensions to N of p, q respectively. We want to show that  $p' \perp_{wk} q'$ . By the conjugation property, there exists  $f : N \cong M$  such that f(p') = p and f(q) = q'. Since weak orthogonality is invariant under isomorphism,  $p' \perp_{wk} q'$ .

We have arrived to the main theorem of this section. This generalizes [She09a, III.12.39] (a full proof of which appears in Fact 8.7.7) which assumes in addition that

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<sup>&</sup>lt;sup>9</sup>Recall [She09a, N.2.4(4)] that  $M \in \mathbf{K}_{\lambda}$  is superlimit if it is universal in  $\mathbf{K}_{\lambda}$ , has a proper extension, and whenever  $\delta < \lambda^+$  is limit,  $\langle M_i : i < \delta \rangle$  is increasing with  $M \cong M_i$  for all  $i < \delta$ , then  $M \cong \bigcup_{i < \delta} M_i$ . Directly from the definition, one checks that for any AEC  $\mathbf{K}$  and any  $\lambda \geq \mathrm{LS}(\mathbf{K})$ , if  $\mathbf{K}$  is categorical in  $\lambda$  and has no maximal models in  $\lambda$  (so in particular if there is a categorical good  $\lambda$ -frame on  $\mathbf{K}_{\lambda}$ ), then the model of cardinality  $\lambda$  is superlimit.

 $\mathfrak s$  is successful and good<sup>+</sup>. For the convenience of the reader, we repeat Hypothesis 11.2.1.

THEOREM 11.2.7. Let  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \, \bigcup, \, \mathrm{gS}^{\mathrm{bs}})$  be a categorical good  $\lambda$ -frame which has primes. If  $\mathbf{K}_{\lambda}$  is not weakly uni-dimensional (see [She09a, III.2.2(6)]), then there exists  $M \in \mathbf{K}_{\lambda}$  and  $p \in \mathrm{gS}(M)$  such that  $\mathfrak{s} \upharpoonright \mathbf{K}_{\neg^* p}$  (the expansion of  $\mathfrak{s}$  to  $\mathbf{K}_M$  restricted to the models in  $\mathbf{K}_{\neg^* p}$ , see Definitions 8.2.18 and 8.5.7) is a type-full good  $\lambda$ -frame with primes.

PROOF. Exactly the same as in Fact 8.7.7, except that we replace uniqueness triples with prime triples, and use Lemmas 11.2.4 and 11.2.6 wherever appropriate.  $\Box$ 

Assuming tameness and existence of primes above  $\lambda$ , we can conclude an equivalence between uni-dimensionality and categoricity. Once again, we repeat Hypothesis 11.2.1.

THEOREM 11.2.8. Assume that **K** is an AEC categorical in  $\lambda$  which has a (type-full) good  $\lambda$ -frame. If  $\mathbf{K}_{\geq \lambda}$  has primes and is  $\lambda$ -tame, then the following are equivalent:

- (1)  $\mathbf{K}$  is weakly uni-dimensional (see [She09a, III.2.2(6)]).
- (2) **K** is categorical in all  $\mu > \lambda$ .
- (3) **K** is categorical in some  $\mu > \lambda$ .

PROOF. Exactly as in the proof of Theorem 8.5.16, except that we use Theorem 11.2.7 (and replace uniqueness triples with prime triples).  $\Box$ 

REMARK 11.2.9. For the proof of Theorem 11.2.8 (and the other categoricity transfer theorems of this chapter), the symmetry property of good frames is not needed.

## 11.3. Categoricity transfers in AECs with primes

In this section, we prove Theorem 9.0.16 from the abstract. We first recall that the existence of good frames follow from categoricity, amalgamation, and tameness. We use the following notation:

NOTATION 11.3.1. For **K** an AEC with amalgamation and  $\lambda > \text{LS}(\mathbf{K})$ , we write  $\mathbf{K}^{\lambda-\text{sat}}$  for the class of  $\lambda$ -saturated models in  $\mathbf{K}_{>\lambda}$ .

FACT 11.3.2. Let **K** be a LS(**K**)-tame AEC with amalgamation and no maximal models. Let  $\lambda$  and  $\mu$  be cardinals such that both  $\lambda$  and  $\mu$  are strictly bigger than LS(**K**). If **K** is categorical in  $\mu$ , then:

- (1)  $\mathbf{K}$  is stable in every cardinal.
- (2)  $\mathbf{K}^{\lambda-\text{sat}}$  is an AEC with  $\text{LS}(\mathbf{K}^{\lambda-\text{sat}}) = \lambda$ .
- (3) There exists a categorical type-full good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$ .

PROOF. By the Shelah-Villaveces theorem (see Chapter 20), **K** is  $LS(\mathbf{K})$ superstable (see for example Definition 6.10.1), in particular it is stable in  $LS(\mathbf{K})$ . Now we start to use  $LS(\mathbf{K})$ -tameness. By Corollary 10.6.10,  $\mathbf{K}^{\lambda-\text{sat}}$  is an AEC with  $LS(\mathbf{K}^{\lambda-\text{sat}}) = \lambda$ . By Theorem 6.10.8, there is a type-full good  $\lambda$ -frame with underlying class  $\mathbf{K}^{\lambda-\text{sat}}_{\lambda}$  (and in particular stable in  $\lambda$ ) By uniqueness of saturated models,  $\mathbf{K}^{\lambda-\text{sat}}$  is categorical in  $\lambda$ . We obtain a categoricity transfer for tame AECs with primes categorical in two cardinals. First we prove a more general lemma:

LEMMA 11.3.3. Let **K** be a LS(**K**)-tame AEC with amalgamation and arbitrarily large models. Let  $\lambda$  and  $\mu$  be cardinals such that LS(**K**)  $< \lambda < \mu$ .

If **K** is categorical in  $\mu$  and  $\mathbf{K}^{\lambda\text{-sat}}$  has primes, then  $\mathbf{K}^{\lambda\text{-sat}}$  is categorical in all  $\mu' \geq \lambda$ .

PROOF. By partitioning **K** into disjoint AECs, each of which has joint embedding (see for example [**Bal09**, 16.14]) and working inside the unique piece that is categorical in  $\mu$ , we can assume without loss of generality that **K** has joint embedding. Because **K** has arbitrarily large models, **K** also has no maximal models.

By Fact 11.3.2, there is a categorical type-full good  $\lambda$ -frame  $\mathfrak{s}$  with underlying class  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$ . Now apply Theorem 11.2.8 to  $\mathfrak{s}$  and  $\mathbf{K}^{\lambda-\text{sat}}$ .

THEOREM 11.3.4. Let **K** be a LS(**K**)-tame AEC with amalgamation and arbitrarily large models. Let  $\lambda$  and  $\mu$  be cardinals such that LS(**K**)  $< \lambda < \mu$ . Assume that **K**<sub>> $\lambda$ </sub> has primes.

If **K** is categorical in both  $\lambda$  and  $\mu$ , then **K** is categorical in all  $\mu' \geq \lambda$ .

PROOF. By categoricity,  $\mathbf{K}^{\lambda\text{-sat}} = \mathbf{K}_{>\lambda}$ . Now apply Lemma 11.3.3.

REMARK 11.3.5. What if  $\lambda = LS(\mathbf{K})$ ? Then it is open whether **K** has a good  $LS(\mathbf{K})$ -frame (see the discussion in Section 4.3). If it does, then we can use Theorem 11.2.8.

We present two transfers from categoricity in a single cardinal. The first uses the following downward transfer which follows from the proof of [**Bal09**, 14.9] (an exposition of [**She99**, II.1.6]).

FACT 11.3.6. Let **K** be an AEC with amalgamation and no maximal models. If **K** is categorical in a  $\lambda > H_2$  (recall Definition 2.2.2) and the model of size  $\lambda$  is  $H_2^+$ -saturated, then **K** is categorical in  $H_2$ .

To get the optimal tameness bound, we will use<sup>10</sup>:

FACT 11.3.7 (Corollary 10.7.9). Let **K** be an AEC with amalgamation and no maximal models. Let  $\mu \geq H_1$  and assume that **K** is categorical in a  $\lambda > \mu$  so that the model of size  $\lambda$  is  $\mu^+$ -saturated. Then there exists a categorical type-full good  $\mu$ -frame with underlying class  $\mathbf{K}_{\mu}^{\mu-\text{sat}}$ .

THEOREM 11.3.8. Let **K** be an AEC with amalgamation. Assume that **K** is  $H_2$ -tame and  $\mathbf{K}_{\geq H_2}$  has primes. If **K** is categorical in some  $\lambda > H_2$ , then **K** is categorical in all  $\lambda' \geq H_2$ .

PROOF. As in the proof of Lemma 11.3.3, we can assume without loss of generality that **K** has no maximal models. By Fact 11.3.2 (applied to the AEC  $\mathbf{K}_{\geq H_2}$ ), **K** is in particular stable in  $\lambda$ , hence the model of size  $\lambda$  is saturated. By Fact 11.3.6, **K** is categorical in  $H_2$ . By Fact 11.3.7, there is a categorical type-full good  $H_2$ -frame  $\mathfrak{s}$  with underlying class  $\mathbf{K}_{H_2}^{H_2\text{-sat}}$ . By categoricity in  $H_2$ ,  $\mathbf{K}^{H_2\text{-sat}} = \mathbf{K}_{\geq H_2}$ . Now apply Theorem 11.2.8 to  $\mathfrak{s}$ .

<sup>&</sup>lt;sup>10</sup>For a simpler proof of Theorem 11.3.8 from slightly stronger assumptions, replace " $H_{2}$ -tame" by " $\chi$ -tame for some  $\chi < H_2$ . Then in the proof one can use Fact 11.3.2 together with Theorem 11.3.4, both applied to the class  $\mathbf{K}_{\geq \chi}$ .

We give a variation on Theorem 11.3.8 which gives a lower Hanf number but assumes that classes of saturated models have primes. We will use the following consequence of the omitting type theorem for AECs [She99, II.1.10] (or see [Bal09, 14.3]):

FACT 11.3.9. Let **K** be an AEC with amalgamation. Let  $\lambda \geq \chi > LS(\mathbf{K})$  be cardinals. Assume that all the models of size  $\lambda$  are  $\chi$ -saturated. Then all the models of size at least min $(\lambda, \sup_{\chi_0 < \chi} h(\chi_0))$  are  $\chi$ -saturated.

THEOREM 11.3.10. Let **K** be a LS(**K**)-tame AEC with amalgamation and arbitrarily large models. Let  $\lambda > \text{LS}(\mathbf{K})^+$  be such that **K** is categorical in  $\lambda$  and let  $\chi \in (\text{LS}(\mathbf{K}), \lambda)$  be such that  $\mathbf{K}^{\chi\text{-sat}}$  has primes. Then **K** is categorical in all  $\lambda' \geq \min(\lambda, \sup_{\chi_0 < \chi} h(\chi_0))$ .

PROOF. As in the proof of Lemma 11.3.3, we may assume that **K** has no maximal models. By Lemma 11.3.3,  $\mathbf{K}^{\chi\text{-sat}}$  is categorical in all  $\lambda' \geq \chi$ . By Fact 11.3.2, **K** is stable in  $\lambda$ , so the model of size  $\lambda$  is saturated, hence  $\chi$ -saturated. By Fact 11.3.9, all the models of size at least  $\lambda'_0 := \min(\lambda, \sup_{\chi_0 < \chi} h(\chi_0))$  are  $\chi$ -saturated. In other words,  $\mathbf{K}_{\geq \lambda'_0} = \mathbf{K}^{\chi\text{-sat}}_{\geq \lambda'_0}$ . Since  $\mathbf{K}^{\chi\text{-sat}}$  is categorical in all  $\lambda' \geq \chi$ , **K** is categorical in all  $\lambda' \geq \lambda'_0$ .

REMARK 11.3.11. Theorem 11.3.8 and Theorem 11.3.10 have different strengths. It could be that we know our AEC **K** has primes but it is unclear that  $\mathbf{K}^{\chi\text{-sat}}$  has primes for any  $\chi$ . For example, **K** could be a universal class (or more generally an AEC admitting intersections). In this case we can use Theorem 11.3.8. On the other hand we may not know that **K** has primes but we could know how to build primes in  $\mathbf{K}^{\chi\text{-sat}}$  (for example **K** could be an elementary class or more generally a class of homogeneous models, see the next section). There Theorem 11.3.10 applies.

#### 11.4. Categoricity in homogeneous model theory

We use the results of the previous section to obtain Shelah's categoricity conjecture for homogeneous model theory, a nonelementary framework extending classical first-order model theory. It was introduced in [She70]. The idea is to look at a class of models of a first-order theory omitting a set of types and assume that this class has a very nice (sequentially homogeneous) monster model. We quote from the presentation in [GL02] but all the results on homogeneous model theory that we use initially appeared in either [She70] or [HS00].

The following definitions appear in [**GL02**]. They differ from (but are equivalent to) Shelah's original definitions from [**She70**].

DEFINITION 11.4.1. Fix a first-order theory T.

- (1) A set of T-types D is a diagram in T if it has the form  $\{\operatorname{tp}(\bar{a}/\emptyset; M) \mid \bar{a} \in {}^{<\omega}A\}$  for a model M of T.
- (2) A model M of T is a D-model if  $D(M) := \{ \operatorname{tp}(\bar{a}/\emptyset; M) \mid \bar{a} \in {}^{<\omega}|M| \} \subseteq D.$
- (3) For D a diagram of T, we let  $\mathbf{K}_D$  be the class of D-models of T, ordered with elementary substructure.
- (4) For M a model of T, we write  $S_D^{\leq \omega}(A; M)$  for the set of types of finite tuples over A which are realized in some D-model N with  $N \leq M$ .

DEFINITION 11.4.2. Let T be a first-order theory and D a diagram in T. A model M of T is  $(D, \lambda)$ -homogeneous if it is a D-model and for every  $N \succeq M$ , every  $A \subseteq |M|$  with  $|A| < \lambda$ , every  $p \in S_D^{<\omega}(A; N)$  is realized in M.

DEFINITION 11.4.3. We say a diagram D in T is homogeneous if for every  $\lambda$  there exists a  $(D, \lambda)$ -homogeneous model of T.

We are not aware of any source explicitly stating the facts below, but they are straightforward to check, so we omit the proof. They will be used without mention.

**PROPOSITION 11.4.4.** For D a homogeneous diagram in T:

- (1)  $\mathbf{K}_D$  is an AEC with  $\mathrm{LS}(\mathbf{K}_D) = |T|$ .
- (2) **K** has amalgamation, no maximal models, and is fully  $LS(\mathbf{K})$ -tame and short (in fact syntactic and Galois types coincide).
- (3) For  $\lambda > |T|$ , a *D*-model *M* is  $(D, \lambda)$ -homogeneous if and only if  $M \in \mathbf{K}_D^{\lambda-\text{sat}}$ .

Note that in this framework it also makes sense to talk about the |T|-saturated models, so we let:

DEFINITION 11.4.5. Let  $\mathbf{K}_D^{|T|\text{-sat}}$  be the class of (D, |T|)-homogeneous models, ordered by elementary substructure.

To apply the results of the previous section, we must give conditions under which  $\mathbf{K}_{D}^{\chi\text{-sat}}$  has primes. This is implicit in [She70, Section 5]:

FACT 11.4.6. Let D be a homogeneous diagram in T. If  $\mathbf{K}_D$  is stable in  $\chi \geq \mathrm{LS}(\mathbf{K})$  then  $\mathbf{K}_D^{\chi-\mathrm{sat}}$  has primes.

PROOF. By [She70, 5.11(1)] (with  $\mu, \lambda$  there standing for  $\chi, \chi$  here; in particular  $2^{\mu} > \lambda$ ), D satisfies a property Shelah calls  $(P, \chi, 1)$  (a form of density of isolated types, see [She70, 5.4]). By the proof of [She70, 5.2(1)] and [She70, 5.3(1)] there, this implies that the class  $\mathbf{K}_D^{\chi$ -sat has primes.

We immediately obtain:

THEOREM 11.4.7. If a homogeneous diagram D in a first-order theory T is categorical in a  $\lambda > |T|^+$ , then it is categorical in all  $\lambda' \ge \min(\lambda, h(|T|))$ .

PROOF. Note that  $\mathbf{K}_D$  is stable in all cardinals by Fact 11.3.2. So we can combine Fact 11.4.6 and Theorem 11.3.10.

This proves Theorem 11.0.13 in the abstract modulo a small wrinkle: the case  $\lambda = |T|^+$ . One would like to use the categoricity transfer of Grossberg and VanDieren [**GV06a**] but they assume that **K** is categorical in a successor  $\lambda > \text{LS}(\mathbf{K})^+$  since otherwise it is in general unclear whether there is a superlimit (see footnote 9) in  $\text{LS}(\mathbf{K})$  (one can get around this difficulty if  $\text{LS}(\mathbf{K}) = \aleph_0$ , see [**Les05**]). However in the case of homogeneous model theory we can show that there *is* a superlimit, completing the proof. The key is that under stability, (D, |T|)-homogeneous models are closed under unions of chains. This is claimed without proof by Shelah in [**She75c**, 1.15]. We give a proof here which imitates the first-order proof of Harnik [**Har75**]. Still it seems that a fair amount of forking calculus has to be developed first. All throughout, we assume:

HYPOTHESIS 11.4.8. D is a homogeneous diagram in a first-order theory T. We work inside a  $(D, \bar{\kappa})$ -homogeneous model  $\mathfrak{C}$  for  $\bar{\kappa}$  a very big cardinal. In particular, all sets are assumed to be D-sets (see [**GL02**, 2.1(2)]).

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The following can be seen as a first approximation for forking in the homogeneous context. It was used by Shelah to prove the stability spectrum theorem in this framework (see Fact 11.4.11). We will not use the exact definition, only its consequences.

DEFINITION 11.4.9 (4.1 in [She70]). A type  $p \in S_D^{<\omega}(A)$  strongly splits over  $B \subseteq A$  if there exists an indiscernible sequence  $\langle \bar{a}_i : i < \omega \rangle$  over B and a formula  $\phi(\bar{x}, \bar{y})$  such that  $\phi(\bar{x}, \bar{a}_0) \in p$  and  $\neg \phi(\bar{x}, \bar{a}_1) \in p$ .

DEFINITION 11.4.10.  $\kappa(D)$  is the minimal cardinal  $\kappa$  such that for all A and all  $p \in S_D^{<\omega}(A)$ , there exists  $B \subseteq A$  with  $|B| < \kappa$  so that p does not strongly split over B.

The following is due to Shelah [She70, 4.4]. See also [GL02, 4.11, 4.14, 4.15]:

FACT 11.4.11. If D is stable in  $\lambda_0 \ge |T|$ , then  $\kappa(D) < \infty$  and for  $\lambda \ge \lambda_0$ , D is stable in  $\lambda$  if and only if  $\lambda = \lambda^{<\kappa(D)}$ .

We can define forking using strong splitting:

DEFINITION 11.4.12 (3.1 in [**HS00**]). For  $A \subseteq B$ ,  $p \in S_D^{<\omega}(B)$  does not fork over A if there exists  $A_0 \subseteq A$  such that:

- (1)  $|A_0| < \kappa(D)$ .
- (2) For every set C, there exists  $q \in S_D^{\leq \omega}(B \cup C)$  such that q extends p and q does not strongly split over  $A_0$ .

Assuming that the base has a certain degree of saturation, forking behaves well:

FACT 11.4.13. Assume that D is stable in  $\lambda \geq |T|$ . Let M be  $(D, \lambda)$ -homogeneous and let  $A \subseteq B \subseteq C$  be sets.

- (1) (Monotonicity) For  $p \in S_D^{<\omega}(C)$ , if p does not fork over A, then  $p \upharpoonright B$  does not fork over A and p does not fork over B.
- (2) (Extension-existence) For any  $p \in S_D^{<\omega}(M)$ , there exists  $q \in S_D^{<\omega}(M \cup B)$  that extends p and does not fork over M. Also, q is algebraic if and only if p is. Moreover if  $p \in S_D^{<\omega}(M)$  does not strongly split over  $A_0 \subseteq |M|$ , then p does not fork over  $A_0$ .
- (3) (Uniqueness) If  $p, q \in S_D^{<\omega}(M \cup B)$  both do not fork over M and are such that  $p \upharpoonright M = q \upharpoonright M$ , then p = q.
- (4) (Transitivity) For any  $p \in S_D^{<\omega}(M \cup B)$ , if p does not fork over M and  $p \upharpoonright M$  does not fork over  $A_0 \subseteq |M|$ , then p does not fork over  $A_0$ .
- (5) (Symmetry) If  $tp(\bar{b}/M\bar{a})$  does not fork over M, then  $tp(\bar{a}/M\bar{b})$  does not fork over M.
- (6) (Local character) For any  $p \in S_D^{<\omega}(M)$ , there exists  $A_0 \subseteq |M|$  such that  $|A_0| < \kappa(D)$  and p does not fork over  $A_0$ . Moreover, for any  $\langle M_i : i < \delta \rangle$  increasing chain of  $(D, \lambda)$ -homogeneous models, if  $p \in S_D^{<\omega}(\bigcup_{i < \delta} M_i)$  and cf  $\delta \ge \kappa(D)$ , then there exists  $i < \delta$  and  $A_0 \subseteq |M_i|$  such that  $|A_0| < \kappa(D)$  and p does not fork over  $A_0$ .

PROOF. We use freely that (by [**HS00**, 1.9(iv)]) a  $(D, \lambda)$ -homogeneous model is an *a*-saturated model in the sense of [**HS00**, 1.8(ii)]. Monotonicity is [**HS00**, 3.2.(i)], extension-existence is given by [**HS00**, 3.2.(ii), (v), (vi)] and the definitions of  $\kappa(D)$  and forking. Uniqueness is [**HS00**, 3.4], transitivity is [**HS00**, 3.5.(iv)], and symmetry is [**HS00**, 3.6]. For local character, we prove the moreover part and the first part follows by taking  $M_i := M$  for all  $i < \delta$ . Let  $M_\delta := \bigcup_{i < \delta} M_i$ . Without loss of generality,  $\delta = \operatorname{cf} \delta \ge \kappa(D)$ . By definition of  $\kappa(D)$ , there exists  $A_0 \subseteq |M_\delta|$  such that  $|A_0| < \kappa(D)$  and p does not strongly split over  $A_0$ . By cofinality consideration, there exists  $i < \delta$  such that  $A_0 \subseteq |M_i|$ . By the moreover part of extension-existence, for all  $j \in [i, \delta)$ ,  $p \upharpoonright M_j$  does not fork over  $A_0$ . By [**HS00**, 3.5.(i)], it follows that pdoes not fork over  $M_i$ , and therefore by transitivity over  $A_0$ .

We will use the machinery of indiscernibles and averages. Note that by [**GL02**, 3.4, 3.12], indiscernible sequences are indiscernible sets under stability. We will use this freely. The following directly follows from the definition of strong splitting:

FACT 11.4.14 (5.3 in [GL02]). Assume that D is stable. For all infinite indiscernible sequences I over a set A and all elements b, there exists  $J \subseteq I$  with  $|J| < \kappa(D)$  such that  $I \setminus J$  is indiscernible over  $A \cup \{b\}$ .

DEFINITION 11.4.15. For I an indiscernible sequence of cardinality at least  $\kappa(D)$ , let  $\operatorname{Av}(I/A)$  be the set of formulas  $\phi(\bar{x}, \bar{a})$  with  $\bar{a} \in {}^{<\omega}A$  such that for at least  $\kappa(D)$ -many elements  $\bar{b}$  of I,  $\models \phi[\bar{b}, \bar{a}]$ .

FACT 11.4.16 (5.5 in [**GL02**]). If D is stable and I is an indiscernible sequence of cardinality at least  $\kappa(D)$ , then  $\operatorname{Av}(I/A) \in S_D^{<\omega}(A)$ .

FACT 11.4.17. Assume that D is stable.

Let  $A \subseteq B$  and let  $p \in S_D^{<\omega}(B)$ . If p does not fork over A,  $|A| < \kappa(D)$ , and p is nonalgebraic, then there exists an indiscernible set I over A with  $|I| \ge \kappa(D)$  such that  $\operatorname{Av}(I/M) = p$ .

PROOF. This follows from [**HS00**, 3.9]. We have to check that  $p \upharpoonright A$  has unboundedly-many realizations, but this is easy using the extension-existence property of forking (Fact 11.4.13) and the assumption that p is nonalgebraic.

We can conclude:

THEOREM 11.4.18. Let  $\lambda \geq |T|$ . Assume that D is stable in some  $\mu \leq \lambda$ . Let  $\delta$  be a limit ordinal with  $\mathrm{cf} \ \delta \geq \kappa(D)$  and let  $\langle M_i : i < \delta \rangle$  be an increasing sequence of  $(D, \lambda)$ -homogeneous models. Then  $\bigcup_{i < \delta} M_i$  is  $(D, \lambda)$ -homogeneous.

PROOF. By cofinality consideration, we can assume without loss of generality that  $\delta = \operatorname{cf} \delta$  and  $\lambda > \delta$ . Also without loss of generality,  $\lambda$  is regular. Let  $M_{\delta} := \bigcup_{i < \delta} M_i$ . Let  $A \subseteq |M_{\delta}|$  have size less than  $\lambda$  and let  $p \in S_D^{<\omega}(A)$ . Let  $q \in S_D^{<\omega}(M_{\delta})$  be an extension of p and assume for sake of contradiction that p is not realized in  $M_{\delta}$ . By the moreover part of local character (Fact 11.4.13), there exists  $i < \delta$  and  $B \subseteq |M_i|$  such that  $|B| < \kappa(D)$  and q does not fork over B. By making A slightly bigger we can assume without loss of generality that  $B \subseteq A$ .

Since p is not realized in  $M_{\delta}$ , q is nonalgebraic. By Fact 11.4.17, there exists an indiscernible set I over B with  $\operatorname{Av}(I/M_{\delta}) = q$ . Enlarging I if necessary,  $|I| = \lambda$ . Since  $M_{i+1}$  is  $(D, \lambda)$ -homogeneous, we can assume without loss of generality that  $I \subseteq |M_{i+1}|$ . By Fact 11.4.14 used |A|-many times (recall  $|A| < \lambda$ ), there exists  $I_0 \subseteq I$  with  $|I_0| = \lambda$  and  $I_0$  indiscernible over A. Then  $\operatorname{Av}(I_0/M_{\delta}) = \operatorname{Av}(I/M_{\delta}) = q$ so  $p = \operatorname{Av}(I_0/A)$ . By definition of average, if  $\phi(\bar{x}, \bar{a}) \in p$ , there exists  $\bar{b} \in I_0$  such that  $\models \phi[\bar{b}, \bar{a}]$ . By indiscernibility over A, this is true for any  $\bar{b} \in I_0$ , hence any element of  $I_0$  realizes p.

REMARK 11.4.19. When  $\lambda > |T|$  and  $\kappa(D) = \aleph_0$ , Theorem 11.4.18 generalizes to superstable tame AECs with amalgamation (see Chapter 7 and the more recent Corollary 10.6.10). We do not know whether there is a generalization of Theorem 11.4.18 to AECs when  $\lambda = LS(\mathbf{K})$  (see also Question 10.6.12).

In homogeneous model theory, superstability follows from categoricity:

LEMMA 11.4.20. If a homogeneous diagram D in a first-order theory T is categorical in a  $\lambda > |T|$ , then  $\kappa(D) = \aleph_0$ .

**PROOF.** By Fact 11.3.2 (applied to  $\mathbf{K} := \mathbf{K}_D$ , recall Proposition 11.4.4), D is stable in all cardinals and in particular in  $\mu := \aleph_{\omega}(|T|)$ . Since  $\mu^{\aleph_0} > \mu$ , Fact 11.4.11 gives  $\kappa(D) = \aleph_0$ .

Note that Lemma 11.4.20 was known when  $\lambda \neq \aleph_{\omega}(|T|)$  (see [Hyt98, Theorem 3]). The case  $\lambda = \aleph_{\omega}(|T|)$  is new (in fact, once Lemma 11.4.20 is proven for  $\lambda = \aleph_{\alpha}(|T|)$ , Hyttinen's argument for transferring categoricity [Hyt98, 14.(ii)] goes through).

The referee asked if Lemma 11.4.20 had an easier proof using tools specific to homogeneous model theory. An easy proof of Lemma 11.4.20 when  $\lambda \neq \aleph_{\omega}(|T|)$  runs as follows: By a standard Ehrenfeucht-Mostowski (EM) model argument of Morley (see for example [Bal09, 8.21]), D is stable in every  $\mu \in [|T|, \lambda)$ . If  $\lambda > \aleph_{\omega}(|T|)$ , then D is stable in  $\mu := \aleph_{\omega}(|T|)$  and  $\mu^{\aleph_0} > \mu$  so by the stability spectrum theorem (Fact 11.4.11), we must have that  $\kappa(D) = \aleph_0$ . If  $\lambda < \aleph_{\omega}(|T|)$ ,  $\lambda$  is a successor and we can use other EM model tricks. Only the case  $\lambda = \aleph_{\omega}(|T|)$  remains but to deal with it, we are not aware of any tools specific to the homogeneous setup. The proof above is in effect an application of a result of Shelah and Villaveces (see [SV99, 2.2.1] and Chapter 20) and an upward stability transfer of the author (Theorem 4.5.6).

We can conclude with a proof of Theorem 11.0.13 from the abstract. When  $\lambda = |T|^+$ , we could appeal to [**GV06a**] but prefer to prove a more general statement using primes:

THEOREM 11.4.21. If a homogeneous diagram D in a first-order theory T is categorical in a  $\lambda > |T|$ , then the class  $\mathbf{K}_{D}^{|T|-\text{sat}}$  of its (D, |T|)-homogeneous models is categorical in all  $\lambda' \geq |T|$ . In particular, if D is also categorical in |T|, then D is categorical in all  $\lambda' \geq |T|$ 

PROOF. Let  $\mathbf{K} := \mathbf{K}_D$  be the class of *D*-models of *T*. By Proposition 11.4.4, **K** is a LS(**K**)-tame AEC (where LS(**K**) = |T|) with amalgamation and no maximal models. Furthermore **K** is categorical in  $\lambda$ . By Lemma 11.4.20,  $\kappa(D) = \aleph_0$ . By Theorem 11.4.18, the union of any increasing chain of (D, |T|)-homogeneous models is (D, |T|)-homogeneous. Moreover, there is a unique (D, |T|)-homogeneous model of cardinality |T| (see e.g. [GL02, 5.9]). So we get that:

- (1)  $\mathbf{K}_D^{[T]\text{-sat}}$  is an AEC with  $\mathrm{LS}(\mathbf{K}_D^{[T]\text{-sat}}) = \mathrm{LS}(\mathbf{K})$ . (2)  $\mathbf{K}_D^{[T]\text{-sat}}$  has amalgamation, no maximal models, and is  $\mathrm{LS}(\mathbf{K})\text{-tame}$ . (3)  $\mathbf{K}_D^{[T]\text{-sat}}$  is categorical in  $\mathrm{LS}(\mathbf{K})$  and  $\lambda$ .

Thus the last sentence in the statement of the theorem follows from uniqueness of homogeneous models. Let us prove the first. By Fact 11.4.13, nonforking induces a type-full good |T|-frame on the class  $(\mathbf{K}_D^{|T|-\text{sat}})_{|T|}$ . By Fact 11.4.6,  $\mathbf{K}_D^{|T|-\text{sat}}$  has primes. Now apply Theorem 11.2.8.

THEOREM 11.4.22. If a homogeneous diagram D in a first-order theory T is categorical in a  $\lambda > |T|$ , then it is categorical in all  $\lambda' \ge \min(\lambda, h(|T|))$ .

PROOF. By Theorem 11.4.21,  $\mathbf{K}_D^{|T|\text{-sat}}$  is categorical in all  $\lambda' \geq |T|$ . In particular by categoricity in  $\lambda$ ,  $\left(\mathbf{K}_D^{|T|\text{-sat}}\right)_{\geq\lambda} = (\mathbf{K}_D)_{\geq\lambda}$ , so  $\mathbf{K}_D$  is categorical in all  $\lambda' \geq \lambda$ . To see that  $\mathbf{K}_D$  is categorical in all  $\lambda' \geq h(|T|)$ , use Theorem 11.4.7 (or just directly Fact 11.3.9).
# CHAPTER 12

# Building prime models in fully good abstract elementary classes

This chapter is based on [Vasa]. I thank a referee for suggestions that helped refocus the topic of the paper and improve its presentation.

# Abstract

We show how to build primes models in classes of saturated models of abstract elementary classes (AECs) having a well-behaved independence relation:

THEOREM 12.0.23. Let **K** be an almost fully good AEC that is categorical in  $LS(\mathbf{K})$  and has the  $LS(\mathbf{K})$ -existence property for domination triples. For any  $\lambda > LS(\mathbf{K})$ , the class of Galois saturated models of **K** of size  $\lambda$  has prime models over every set of the form  $M \cup \{a\}$ .

This generalizes an argument of Shelah, who proved the result when  $\lambda$  is a successor cardinal.

#### 12.1. Introduction

Prime models (over sets) are a crucial ingredient in the proof of Morley's categoricity theorem [Mor65]. Morley's construction gives a *primary model*: a model whose universe can be enumerated so that the type of each element is isolated over the previous ones. This construction can be generalized to certain non-elementary context such as homogeneous model theory [She70] and even finitary abstract elementary classes [HK06].

In general abstract elementary classes (AECs), it seems that the construction breaks down due to the lack of even rudimentary compactness: it is not clear how to define a usable generalization of the notion of an isolated type. Shelah [**She09a**, Section III.4] works around this difficulty by assuming that the class satisfies an axiomatization of superstable forking for its models of size  $\lambda$  (in Shelah's terminology, **K** has a successful good  $\lambda$ -frame) and uses *domination* to build for every saturated M of size  $\lambda^+$  and every element a a saturated model N containing  $M \cup \{a\}$  and prime in the class of saturated models of **K** size  $\lambda^+$ . Here, saturation is defined in terms of Galois (orbital) types.

Shelah shows [She09a, Chapter II] that the assumption of existence of a successful good  $\lambda$ -frame follows from strong *local* hypotheses: categoricity in  $\lambda$ ,  $\lambda^+$ , a medium number of models in  $\lambda^{++}$ , and set-theoretic hypotheses such as  $2^{\lambda} < 2^{\lambda^+} < 2^{\lambda^{++}}$ . In Chapters 4 and 6, we showed that successful good frames can also be built assuming that the class satisfies *global hypotheses*: amalgamation, categoricity in some high-enough cardinal, and a locality property called full tameness and shortness. It is known that amalgamation and the locality property both

follow from categoricity and a large cardinal axiom [MS90, Bon14b]. The global hypotheses actually enable us to build the global generalization of a successful good  $\lambda$ -frame: what we call an almost fully good independence relation (see Definition 12.2.2). In this chapter, we show that Shelah's construction of prime models generalizes to this global setup and  $\lambda^+$  can be replaced by a limit cardinal. Thus we obtain a general construction of primes (in an appropriate class of saturated models) that works assuming only the existence of a well-behaved independence notion (this is Theorem 12.0.23 from the abstract).

Chapter 8 showed that assuming the global hypotheses above, existence of primes over every set of the form  $M \cup \{a\}$  implies categoricity on a tail of cardinals. Unfortunately, we cannot use the construction of prime models of this chapter to deduce a new categoricity transfer in the global framework: the catch is that we only get existence of primes in the subclass of saturated models of **K**: Given Mand a with M saturated, we obtain N saturated that is prime over  $M \cup \{a\}$  in the class of saturated models. That is (roughly<sup>1</sup>), if N' contains  $M \cup \{a\}$  and is saturated, then there exists a **K**-embedding  $f : N \to N'$  fixing M and a. We cannot conclude anything if M is not saturated (even if we know it is  $\lambda$ -saturated for some  $\lambda < ||M||$ ). This contrasts sharply with the more uniform results from the first-order context (in a totally transcendental theory, a prime model exists over every set) and finitary AECs [**HK11**, Lemma 5.4] (in a simple  $\aleph_0$ -stable finitary AEC with the extension property, an f-primary model exists over every set).

We can however use the construction of this chapter to obtain that in the global framework, categoricity on a tail of cardinals implies the existence of primes. This gives a converse to Theorem 8.0.2. The full proof appears in Theorem 8.5.23. We state the result again here as an additional motivation:

COROLLARY 12.1.1. Let **K** be a fully LS(**K**)-tame and short AEC with amalgamation. Write  $H_1 := \beth_{(2^{\text{LS}(\mathbf{K})})^+}$ . Assume that **K** is categorical in *some* cardinal  $\lambda \geq H_1$ . The following are equivalent:

- (1) **K** is categorical in all  $\lambda' \geq H_1$ .
- (2)  $\mathbf{K}_{\geq H_1}$  has primes over all sets of the form  $M \cup \{a\}$ .

The background required to read this chapter is a basic knowledge of AECs (for example Chapters 4-12 of Baldwin's book [**Bal09**]). Some familiarity with good frames and their generalizations (in particular the beginning of Chapter 6, Section 6.11, and Shelah's construction of primes [**She09a**, Section III.4]) would be helpful but we state all the necessary definitions here. We rely on a few results from [**Jar16**] and Chapters 6, 10 but they are used as black boxes: little understanding of the material there is needed.

# 12.2. Background

We give some background on independence that will be used in the next section. We assume familiarity with the basics of AECs as laid out in e.g. [**Bal09**] or the forthcoming [**Gro**]. We will use the notation from the preliminaries of Chapter 2. All throughout this section, we assume:

HYPOTHESIS 12.2.1. K is an AEC with amalgamation.

<sup>&</sup>lt;sup>1</sup>the precise statement uses Galois types, see Definition 12.2.13.

#### 12.2. BACKGROUND

This will mostly be assumed throughout the chapter (Hypothesis 12.3.1 implicitly implies it by Definition 12.2.2.(1b)). Note however that assuming high-enough categoricity and a large cardinal axiom, it will hold on a tail [Bon14b].

We will work use a global forking-like independence notion that has the basic properties of forking in a superstable first-order theory. This is a stronger notion than Shelah's good frame [She09a, Chapter II] because in good frames forking is only defined for types of length one. We invite the reader to consult Chapter 6 for more explanations and motivations on global and local independence notions.

DEFINITION 12.2.2 (Definition 6.8.1 and Definition 8.6.2).  $\mathbf{i} = (\mathbf{K}, \mathbf{\downarrow})$  is an almost fully good independence relation if:

- (1)  $\mathbf{K}$  is an AEC satisfying the following structural assumptions:
  - (a)  $\mathbf{K}_{\leq \mathrm{LS}(K)} = \emptyset$  and  $\mathbf{K} \neq \emptyset$ .
  - (b) **K** has amalgamation, joint embedding, and no maximal models.
  - (c) **K** is stable in all cardinals.
- (2) (a) i is a  $(< \infty, \ge LS(K))$ -independence relation (see Definition 6.3.6). That is,  $\downarrow$  is a relation on quadruples (M, A, B, N) with  $M \leq_{\mathbf{K}} N$ and  $A, B \subseteq |N|$  satisfying invariance, monotonicity, and normal-

ity. We write  $A \underset{M}{\downarrow} B$  instead of  $\downarrow (M, A, B, N)$ , and we also say

 $gtp(\bar{a}/B; N)$  does not fork over M for ran  $\bar{a} \bigcup_{M}^{N} B$ .

- (b) i has base monotonicity, disjointness  $(A \bigcup_{M}^{N} B \text{ implies } A \cap B \subseteq |M|)$ , symmetry, uniqueness (whenever  $M \leq_{\mathbf{K}} N$  and  $p, q \in \mathrm{gS}^{<\infty}(N)$  do not fork over M and are such that  $p \upharpoonright M = q \upharpoonright M$ , then p = q), and the local character properties:
  - (i) If  $p \in gS^{\alpha}(M)$ , there exists  $M_0 \leq_{\mathbf{K}} M$  with  $||M_0|| \leq |\alpha| +$ LS(K) such that p does not fork over  $M_0$ .
  - (ii) If  $\langle M_i : i \leq \delta \rangle$  is increasing continuous,  $p \in gS^{\alpha}(M_{\delta})$  and cf  $\delta > \alpha$ , then there exists  $i < \delta$  such that p does not fork over  $M_i$ .
- (c) i has the following weakening of the extension property: for any  $M \leq_{\mathbf{K}} N$  and any  $p \in \mathrm{gS}^{\alpha}(M)$ , there exists  $q \in \mathrm{gS}^{\alpha}(N)$  that extends p and does not fork over M provided at least one of the following conditions hold:
  - (i) *M* is saturated.
  - (ii)  $||M|| = \mathrm{LS}(\mathbf{K}).$
  - (iii)  $\alpha \leq \text{LS}(\mathbf{K})$ .
- (d) i has the left and right  $(\leq LS(K))$ -witness properties:  $A \stackrel{N}{\downarrow} B$  if and only if for all  $A_0 \subseteq A$  and  $B_0 \subseteq B$  with  $|A_0| + |B_0| \leq LS(K)$ , we have that  $A_0 \stackrel{N}{\downarrow} B_0$ .
- (e) i has full model continuity: For all limit ordinals  $\delta$ , if for  $\ell < 4$ ,  $\langle M_i^\ell: i\le \delta\rangle$  are increasing continuous such that for all  $i<\delta,\,M_i^0\le_{\bf K}$  $M_i^{\ell} \leq_{\mathbf{K}} M_i^3 \text{ for } \ell = 1, 2 \text{ and } M_i^1 \bigcup_{\substack{M_i^0 \\ M_i^0}}^{M_i^3} M_i^2, \text{ then } M_{\delta}^1 \bigcup_{\substack{M_{\delta}^0 \\ M_{\delta}^0}}^{M_{\delta}^3} M_{\delta}^2.$

We say that **K** is almost fully good if there exists  $\downarrow$  such that  $(K, \downarrow)$  is almost fully good<sup>2</sup>.

REMARK 12.2.3. In Theorem 6.15.1, it was shown that AECs with amalgamation that satisfy a locality condition (full tameness and shortness) and are categorical in a high-enough cardinal are (on a tail) almost fully good. The threshold cardinals were improved in Section 8.6. We use the name "almost" fully good because we do not assume the full extension property, only the weakening above. The problem is that it is not known how to get the full extension property with the aforementioned hypotheses (see the discussion in Section 6.15).

We will use (in the proof of Fact 12.3.3) that almost fully good AECs are tame (a locality property for types introduced by Grossberg and VanDieren [**GV06b**]). Recall that **K** is  $\mu$ -tame (for  $\mu \geq \text{LS}(\mathbf{K})$ ) if for any distinct  $p, q \in \text{gS}(M)$  there exists  $M_0 \in \mathbf{K}_{\leq \text{LS}(\mathbf{K})}$  with  $M_0 \leq_{\mathbf{K}} M$  such that  $p \upharpoonright M_0 \neq q \upharpoonright M_0$ . Using local character and uniqueness (see e.g. [**GK**, p. 15]) we have:

FACT 12.2.4. If  $\mathbf{K}$  is almost fully good, then  $\mathbf{K}$  is  $LS(\mathbf{K})$ -tame.

We will also make use of limit models (see [**GVV16**] for history and motivation). We will use a global definition, where we permit the limit model and the base to have different sizes. This extra generality is used: in (4) in Lemma 12.3.4,  $M_i^{\ell}$  and  $M_{i+1}^{\ell}$  may have different sizes.

DEFINITION 12.2.5. Let  $M_0 \leq_{\mathbf{K}} M$  be models in  $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$ . M is limit over  $M_0$  if there exists a limit ordinal  $\delta$  and a strictly increasing continuous sequence  $\langle N_i : i \leq \delta \rangle$  such that  $N_0 = M_0$ ,  $N_{\delta} = M$ , and for all  $i < \delta$ ,  $N_{i+1}$  is universal over  $N_i$ .

We say that M is *limit* if it is limit over some  $M' \leq_{\mathbf{K}} M$ .

We will use the following notation to describe classes of saturated models:

DEFINITION 12.2.6. For  $\lambda > LS(\mathbf{K})$ ,  $\mathbf{K}^{\lambda\text{-sat}}$  is the class of  $\lambda$ -saturated (according to Galois types) models in  $\mathbf{K}_{\geq \lambda}$ . We order  $\mathbf{K}^{\lambda\text{-sat}}$  with the strong substructure relation induced from  $\mathbf{K}$ .

In an almost fully good AEC, classes of  $\lambda$ -saturated models are well-behaved and limit models are saturated. This is a combination of results of Shelah [She09a, Chapter II] and VanDieren [Van16b], and is key in the construction of prime models of the next section.

FACT 12.2.7. Assume that  $\mathbf{K}$  is almost fully good. Then:

- (1) If  $M, N \in \mathbf{K}$  are limit and ||M|| = ||N||, then  $M \cong N$ . In particular (if  $||M|| > \mathrm{LS}(\mathbf{K})$ ), M and N are saturated.
- (2) For any  $\lambda > \text{LS}(\mathbf{K})$ ,  $\mathbf{K}^{\lambda\text{-sat}}$  is an AEC with  $\text{LS}(\mathbf{K}^{\lambda\text{-sat}}) = \lambda$ . Moreover,  $\mathbf{K}^{\lambda\text{-sat}}$  is almost fully good.

PROOF. Let i be an almost fully good independence relation on K.

(1) Let  $M, N \in \mathbf{K}$  be limit models with  $\lambda := \|M\| = \|N\|$ . First, by a back and forth argument we can assume that M is limit over some  $M_0 \in \mathbf{K}_{\lambda}$ and N is limit over some  $N_0 \in \mathbf{K}_{\lambda}$  (see Proposition 10.3.1). Next, note that the restriction of  $\mathbf{i}$  to types of length one and models of size  $\lambda$  induces

<sup>&</sup>lt;sup>2</sup>the relation  $\downarrow$  is in fact unique, see Chapter 3.

#### 12.2. BACKGROUND

a good  $\lambda$ -frame (see [She09a, Chapter II]). The result now follows from [She09a, Lemma II.4.8] (see [Bon14a, Theorem 9.2] for a detailed proof).

(2) We prove the result when λ is a successor cardinal, and the result for λ limit easily follows (the moreover part is also straightforward to check). So assume that λ = μ<sup>+</sup>. Note that K is μ-superstable in the sense of [Van16b, Definition 5] (because nonforking implies nonsplitting, see the proof of Fact 10.4.8). Similarly, K is μ<sup>+</sup>-superstable. By the first part, limit models of cardinality μ<sup>+</sup> are unique. Therefore we can apply [Van16b, Theorem 22] which tells us that the union of an increasing chain of μ<sup>+</sup>-saturated models is μ<sup>+</sup>-saturated. That LS(K<sup>μ<sup>+</sup>-sat</sub>) = μ<sup>+</sup> follows from stability and the other axioms of an AEC are straightforward to check.</sup>

REMARK 12.2.8. Fact 12.2.7.(2) is an improvement on the threshold cardinal in Chapter 7 (where it is shown that that  $\mathbf{K}^{\lambda-\text{sat}}$  is an AEC for all  $\lambda \geq \beth_{(2^{\text{LS}}(\mathbf{K}))^+}$ ).

Domination will be the notion replacing isolation in this chapter's construction of prime models:

DEFINITION 12.2.9. Let  $\mathbf{i} = (\mathbf{K}, \downarrow)$  be an almost fully good independence relation. (a, M, N) is a *domination triple* if  $M \leq_{\mathbf{K}} N$ ,  $a \in |N| \setminus |M|$ , and for any  $N' \geq_{\mathbf{K}} N$  and any  $B \subseteq |N'|$ , if  $a \downarrow_{M}^{N'} B$ , then  $N \downarrow_{M}^{L'} B$ .

REMARK 12.2.10. This is a variation on Shelah's uniqueness triples [She09a, Definition II.5.3]. In fact by Lemma 6.11.7, uniqueness triples and domination triples coincide in our framework (this will be used in the proof of Fact 12.3.3, although an understanding of the exact definition of uniqueness triples is not needed for this chapter).

A key property is the existence property for domination triples<sup>3</sup>:

DEFINITION 12.2.11. Let  $\mathbf{i} = (\mathbf{K}, \downarrow)$  be an almost fully good independence relation and let  $\lambda \geq \mathrm{LS}(\mathbf{K})$ . We say that  $\mathbf{i}$  has the  $\lambda$ -existence property for domination triples if for every  $M \in \mathbf{K}_{\lambda}$  and every nonalgebraic  $p \in \mathrm{gS}(M)$ , there exists a domination triple (a, M, N) so that  $p = \mathrm{gtp}(a/M; N)$ .

The existence property for domination triples is a reasonable hypothesis: if the independence relation does not have it, we can restrict ourselves to a subclass of saturated models (see the moreover part of Fact 12.2.7.(2)).

FACT 12.2.12 (Lemma 6.11.12). Let  $\mathbf{i} = (\mathbf{K}, \bigcup)$  be an almost fully good independence relation. For every  $\lambda > \mathrm{LS}(\mathbf{K})$ ,  $\mathbf{i} \upharpoonright \mathbf{K}^{\lambda-\mathrm{sat}}$  (the restriction of  $\mathbf{i}$  to  $\lambda$ -saturated models) has the  $\lambda$ -existence property for domination triples.

Finally, we recall the definition of prime models in the framework of abstract elementary classes. This does not need amalgamation and is due to Shelah [She09a, Section III.3]. While it is possible to define what it means for a model to be prime over an arbitrary set (see Definition 8.5.1), here we focus on primes over sets of

<sup>&</sup>lt;sup>3</sup>This is analogous to Shelah's definition of a *weakly successful* good  $\lambda$ -frame [She09a, Definition III.1.1] which means the frame has the existence property for uniqueness triples.

the form  $M \cup \{a\}$ . The technical point in the definition is that since we are not working inside a monster model, how  $M \cup \{a\}$  is embedded matters. Thus we use a formulation in terms of Galois types: instead of saying that N is prime over  $M \cup \{a\}$ , we say that (a, M, N) is a *prime triple*:

DEFINITION 12.2.13. Let **K** be an AEC (not necessarily with amalgamation).

- (1) A prime triple is (a, M, N) such that  $M \leq_{\mathbf{K}} N$ ,  $a \in |N| \setminus |M|$  and for every  $N' \in \mathbf{K}$ ,  $a' \in |N'|$  such that gtp(a/M; N) = gtp(a'/M; N'), there exists  $f: N \xrightarrow{M} N'$  so that f(a) = a'.
- (2) We say that **K** has primes if for  $M \in \mathbf{K}$  and every nonalgebraic  $p \in gS(M)$ , there exists a prime triple representing p, i.e. there exists a prime triple (a, M, N) so that p = gtp(a/M; N).
- (3) We define localizations such as " $\mathbf{K}_{\lambda}$  has primes" or " $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  has primes" in the natural way (in the second case, we ask that all models in the definition be saturated).

#### 12.3. Building primes over saturated models

We show that in almost fully good AECs, there exists primes among the saturated models (see Definition 12.2.13). For models of successor size, this is shown in [She09a, Claim III.4.9] (or in [Jar] with slightly weaker hypotheses). We generalizes Shelah's proof to limit sizes here. This is the core of the chapter. Throughout this section, we assume:

Hypothesis 12.3.1.

- (1) **K** is an almost fully good AEC, as witnessed by  $i = (\mathbf{K}, \downarrow)$ .
- (2)  $\mathbf{K}$  is categorical in  $LS(\mathbf{K})$ .
- (3) i has the LS(K)-existence property for domination triples (see Definition 12.2.11).

We consider theses hypotheses reasonable: Remark 12.2.3 gives conditions under which an AEC is almost fully good and Fact 12.2.12 shows that we can then restrict it to a subclass of saturated models to obtain the existence property for domination triples and categoricity in  $LS(\mathbf{K})$ .

Note that Hypothesis 12.3.1.(3) is used in the proof of Fact 12.3.3. We do not know whether it follows from the other two hypotheses.

We start by showing that domination triples are closed under unions. This is a key consequence of full model continuity.

LEMMA 12.3.2. Let  $\langle M_i : i < \delta \rangle$ ,  $\langle N_i : i < \delta \rangle$  be increasing and assume that  $(a, M_i, N_i)$  are domination triples for all  $i < \delta$ . Then  $(a, \bigcup_{i < \delta} M_i, \bigcup_{i < \delta} N_i)$  is a domination triple.

PROOF. For ease of notation, we work inside a monster model  $\mathfrak{C}$  and write  $\underset{M}{\mathfrak{C}} A \underset{M}{\downarrow} B$  for  $A \underset{M}{\downarrow} B$ . Let  $M_{\delta} := \bigcup_{i < \delta} M_i$ ,  $N_{\delta} := \bigcup_{i < \delta} N_i$ . Assume that  $a \underset{M_{\delta}}{\downarrow} N$  with  $M_{\delta} \leq_{\mathbf{K}} N$  (by extension for types of length one, we can assume this without loss of generality). By local character, for all sufficiently large  $i < \delta$ ,  $a \underset{M_i}{\downarrow} N$ . By definition of domination triples,  $N_i \underset{M_i}{\downarrow} N$ . By full model continuity,  $N_{\delta} \underset{M_{\delta}}{\downarrow} N$ .

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The conclusion of the next fact is a key step in Shelah's construction of a successor frame in [She09a, Chapter II]. The fact says that if  $M^0 \leq_{\mathbf{K}} M^1$  are of the same successor size, then their resolutions satisfy a natural independence property on a club. In the framework of this chapter, this is due to Jarden [Jar16]. To give the reader a feeling for the difficulties encountered, we first explain in the proof how the (straightforward) first-order argument fails to generalize.

FACT 12.3.3. For every  $\mu \geq \mathrm{LS}(\mathbf{K})$ , for every  $M^0 \leq_{\mathbf{K}} M^1$  both in  $\mathbf{K}_{\mu^+}$ , if  $\langle M_i^{\ell} : i < \mu^+ \rangle$  are increasing continuous resolutions of  $M^{\ell}$  and all are limit models<sup>4</sup> in  $\mathbf{K}_{\mu}$ ,  $\ell = 0, 1$ , then the set of  $i < \mu^+$  so that  $M^0 \bigcup_{M_i^0}^{M^1} M_i^1$  is a club.

PROOF. Let us first see how the first-order argument would go. By local character, for every  $i < \mu^+$ , there exists  $j_i < \mu^+$  such that  $M_i^1 \underset{M_{j_i}^0}{\overset{M^0}{\rightarrow}} M^0$ . Pick  $i^* < \mu^+$ such that  $j_i < i^*$  for every  $i < i^*$ . Using symmetry and the fact that forking is witnessed by a formula (this is where we use the first-order theory), it is then straightforward to see that  $M^0 \underset{M_{i^*}}{\overset{M^1}{\rightarrow}} M_{i^*}^1$ . Thus  $i^*$  has the desired property, and the argument shows we can find a closed unbounded subset of such  $i^*$ . Here however we do not have that forking is witnessed by a formula, or even a finite set (we only have the LS(**K**)-witness property, see Definition 12.2.2.(2d)).

Full model continuity (Definition 12.2.2.(2e)) seems to be the replacement we are looking for, but in the argument above we do *not* have that  $M_{j_i}^0 \leq_{\mathbf{K}} M_i^1$  so cannot use it! It is open whether the appropriate generalization of full model continuity holds here.

On to the actual proof. We show the result when  $\mu = \text{LS}(\mathbf{K})$ . Once this is done, if  $\mu > \text{LS}(\mathbf{K})$  we can apply the " $\mu = \text{LS}(\mathbf{K})$ " case to the class  $\mathbf{K}^{\mu\text{-sat}}$  (by Fact 12.2.7.(2) it is an almost fully good AEC and  $\text{LS}(\mathbf{K}^{\mu\text{-sat}}) = \mu$ ).

We now want to apply [**Jar16**, Theorem 7.8]. The conclusion there is that for any model  $M^0, M^1 \in \mathbf{K}_{\mu^+}, M^0 \leq_{\mathbf{K}_{\mu^+}}^{\mathrm{NF}} M^1$  if and only if  $M^0 \leq_{\mathbf{K}} M^1$ , where  $M^0 \leq_{\mathbf{K}_{\mu^+}}^{\mathrm{NF}} M^1$  is defined to hold if and only if there exists increasing continuous resolutions of  $M^0$  and  $M^1$  as here. Let us check that the hypotheses of [**Jar16**, Theorem 7.8] are satisfied. First, amalgamation in LS( $\mathbf{K}$ )<sup>+</sup> and LS( $\mathbf{K}$ )-tameness hold (by definition of an almost fully good AEC and Fact 12.2.4). Second, [**Jar16**, Hypothesis 6.5] holds:  $\mathbf{K}$  is categorical in LS( $\mathbf{K}$ ), has a semi-good LS( $\mathbf{K}$ )-frame (this is weaker than the existence of an almost fully good independence relation, in fact the frame will be good), satisfies the conjugation property (by [**She09a**, III.1.21] which tells us that conjugation holds in any good LS( $\mathbf{K}$ )-frame categorical in LS( $\mathbf{K}$ )), and has the existence property for uniqueness triples by Hypothesis 12.3.1.(3) and Remark 12.2.10. Therefore the hypotheses of Jarden's theorem are satisfied so its conclusion holds.

We can now generalize the proof of [She09a, Claim III.4.3] to limit cardinals. Roughly, it tells us that every nonalgebraic type over a saturated model has a resolution into domination triples.

<sup>&</sup>lt;sup>4</sup>And hence if  $\mu > LS(\mathbf{K})$  are saturated (Fact 12.2.7.(1)).

LEMMA 12.3.4. Let  $\lambda > \mathrm{LS}(\mathbf{K})$  and let  $\delta := \mathrm{cf} \lambda$ . Let  $M^0 \in \mathbf{K}_{\lambda}$  be saturated and let  $p \in gS(M^0)$  be nonalgebraic. Then there exists a saturated  $M^1 \in \mathbf{K}_{\lambda}$ , an element  $a \in |M^1|$ , and increasing continuous resolutions  $\langle M_i^{\ell} : i \leq \delta \rangle$  of  $M^{\ell}$ ,  $\ell = 0, 1$  such that for all  $i < \delta$ :

- (1)  $p = \operatorname{gtp}(a/M^0; M^1).$
- (2)  $a \in |M_0^1|$ .
- (3) p does not fork over  $M_0^0$ .
- (4) For  $\ell = 0, 1, M_i^{\ell} \in \mathbf{K}_{[\mathrm{LS}(\mathbf{K}),\lambda)}$  and  $M_{i+1}^{\ell}$  is limit over  $M_i^{\ell}$ .
- (5)  $(a, M_i^0, M_i^1)$  is a domination triple.

**PROOF.** For  $\ell = 0, 1$ , we first choose by induction  $\langle N_i^{\ell} : i \leq \lambda \rangle$  increasing continuous and an element a that will satisfy some weaker requirements. In the end, we will rename the  $N_i^{\ell}$ 's to get the desired  $M_i^{\ell}$ 's and  $M^1$ . We require that for all  $i < \lambda$ :

- (i)  $N_0^0 \leq_{\mathbf{K}} M^0$  and p does not fork over  $N_0^0$ .
- (ii)  $a \in |N_0^1|$  and  $p \upharpoonright N_0^0 = \text{gtp}(a/N_0^0; N_0^1)$ . (iii) For  $\ell = 0, 1, N_i^{\ell} \in \mathbf{K}_{|i|+\text{LS}(\mathbf{K})}$  and  $N_i^0 \leq_{\mathbf{K}} N_i^1$ .
- (iv)  $gtp(a/N_i^0; N_i^1)$  does not fork over  $N_0^0$ .
- (v) If *i* is odd, and  $\ell = 0, 1$ , then  $N_{i+1}^{\ell}$  is limit over  $N_i^{\ell}$ .

(vi) If *i* is even and  $(a, N_i^0, N_i^1)$  is not a domination triple, then  $N_i^1 \downarrow_{N_{i+1}^0}^{N_{i+1}^*} N_{i+1}^0$ .

This is possible. First pick  $N_0^0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  such that  $N_0^0 \leq_{\mathbf{K}} M^0$  and p does not fork over  $N_0^0$ . This is possible by local character. Now pick  $N_0^1 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  such that  $N_0^0 \leq_{\mathbf{K}} N_0^1$  and there is  $a \in |N_0^1|$  with  $gtp(a/N_0^0; N_0^1) = p \upharpoonright N_0^0$ . This takes care of the case i = 0. For *i* limit, take unions. Now assume that i = j + 1 is a successor. We consider several cases:

• If j is even and  $(a, N_j^0, N_j^1)$  is not a domination triple, then there must exist witnesses  $N_{j+1}^0, N_{j+1}^1 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})+|j|}$  such that  $N_j^0 \leq_{\mathbf{K}} N_{j+1}^0, N_{j+1}^0 \leq_{\mathbf{K}} N_{j+1}^0, N_{j+1}^0 \leq_{\mathbf{K}} N_{j+1}^0, N_{j+1}^0 \leq_{\mathbf{K}} N_{j+1}^0, N_{j+1}^0 \leq_{\mathbf{K}} N_{j+1}^0, N_{j+1}^0, N_{j+1}^0 \leq_{\mathbf{K}} N_{j+1}^0, N_{j+1}^0 \leq_{\mathbf{$ 

conditions (we know that  $gtp(a/N_j^0; N_j^1)$  does not fork over  $N_0^0$ , so by transitivity also  $gtp(a/N_{j+1}^0; N_{j+1}^1)$  does not fork over  $N_0^0$ ). • If j is even and  $(a, N_j^0, N_j^1)$  is a domination triple, take  $N_{j+1}^{\ell} := N_j^{\ell}$ , for

- $\ell = 0, 1.$
- If j is odd, pick  $N_i^0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})+|j|}$  limit over  $N_j^0$  and  $N_i^1$  limit over  $N_i^0$  and  $N_i^1$  so that  $gtp(a/N_i^0; N_i^1)$  does not fork over  $N_0^0$ . This is possible by the extension property for types of length one.

This is enough. By the odd stages of the construction, and basic properties of universality, for all  $i < \lambda, \ell = 0, 1, N_{i+2}^{\ell}$  is universal over  $N_i^{\ell}$ . Thus for  $\ell = 0, 1$  and  $i \leq \lambda$  a limit ordinal,  $N_i^{\ell}$  is limit. In particular, by Fact 12.2.7.(1),  $N_{\lambda}^{\ell}$  is saturated. By uniqueness of saturated models,  $N^0_{\lambda} \cong_{N^0_0} M^0$ . By uniqueness of the nonforking extension, without loss of generality  $N_{\lambda}^0 = \check{M}^0$ . Now let C be the set of limit  $i < \lambda$ such that  $(a, N_i^0, N_i^1)$  is a domination triple. We claim that C is a club:

• C is closed by Lemma 12.3.2.

• *C* is unbounded: given  $\alpha < \lambda$ , let  $\mu := |\alpha| + \text{LS}(\mathbf{K})$ . Let  $E_{\mu}$  be the set of  $i < \mu^+$  such that *i* is limit and  $N_{\mu^+}^0 \downarrow_{N_i^0}^{N_{\mu^+}^1} N_i^1$ . By Fact 12.3.3,  $E_{\mu}$  is a club. The even stages of the construction imply that for  $i \in E_{\mu}$ ,  $(a, N_i^0, N_i^1)$  is

a domination triple. In other words,  $E_{\mu} \subseteq C$ . Now pick  $\beta \in E_{\mu} \setminus (\alpha, N_i, N_i)$  is we have that  $\alpha < \beta$  and  $\beta \in E_{\mu} \subseteq C$ . This completes the proof that C is unbounded.

Let  $\langle \alpha_i : i < \delta \rangle$  (recall that  $\delta = \operatorname{cf} \lambda$ ) be a cofinal strictly increasing continuous sequence of elements of C. For  $i < \delta$ ,  $\ell = 0, 1$ , let  $M_i^{\ell} := N_{\alpha_i}^{\ell}$  and let  $M^1 := M_{\lambda}^1$ (note that  $M^1$  is saturated by what has been observed above). This works: Clauses (1), (2), (3) are straightforward to check using the monotonicity and uniqueness properties of forking. Clause (5) holds by definition of C. As for (4), we have observed above that for  $\ell = 0, 1$ , for all  $i < \lambda$ ,  $N_{i+2}^{\ell}$  is universal over  $N_i^{\ell}$ . Hence for all limit ordinals  $i < j < \lambda$ ,  $N_j^{\ell}$  is limit over  $N_i^{\ell}$ . In particular because C contains only limit ordinals, for all  $i < \delta$ ,  $N_{\alpha_{i+1}}^{\ell}$  is limit over  $N_{\alpha_i}^{\ell}$ , as desired.

In [She09a, Claim III.4.9], Shelah observes that triples as in the conclusion of Lemma 12.3.4 are prime triples. For the convenience of the reader, we include the proof here. We will use the following fact which follows from the uniqueness property of forking and some renaming.

FACT 12.3.5 (Lemma 6.12.6). For  $\ell < 2$ , i < 4, let  $M_i^{\ell} \in \mathbf{K}$  be such that for  $i = 1, 2, M_0^{\ell} \leq_{\mathbf{K}} M_i^{\ell} \leq_{\mathbf{K}} M_3^{\ell}$ .

If  $M_1^{\ell} \underset{M_0^{\ell}}{\stackrel{M_3^{\ell}}{\rightarrow}} M_2^{\ell}$  for  $\ell < 2$ ,  $f_i : M_i^1 \cong M_i^2$  for i = 0, 1, 2, and  $f_0 \subseteq f_1, f_0 \subseteq f_2$ , then

 $f_1 \cup f_2$  can be extended to  $f_3: M_3^1 \to M_4^2$ , for some  $M_4^2$  with  $M_3^2 \leq_{\mathbf{K}} M_4^2$ .

We can now give a proof of Theorem 12.0.23 from the abstract. For the convenience of the reader we restate Hypothesis 12.3.1 here.

THEOREM 12.3.6. Let  $\mathbf{K}$  be an almost fully good AEC that is categorical in  $LS(\mathbf{K})$  and has the  $LS(\mathbf{K})$ -existence property for domination triples.

For any  $\lambda > \mathrm{LS}(\mathbf{K})$ ,  $\mathbf{K}_{\lambda}^{\lambda-\mathrm{sat}}$  has primes (see Definition 12.2.13). That is, for any saturated  $M \in \mathbf{K}_{\lambda}$  and any nonalgebraic  $p \in \mathrm{gS}(M)$ , there exists a triple (a, M, N) such that  $M \leq_{\mathbf{K}} N$ ,  $N \in \mathbf{K}_{\lambda}$  is saturated,  $p = \mathrm{gtp}(a/M; N)$ , and whenever  $p = \mathrm{gtp}(b/M; N')$  with  $N' \in \mathbf{K}_{\lambda}$  saturated, there exists  $f : N \xrightarrow{M} N'$ such that f(a) = b.

PROOF. Let  $M \in \mathbf{K}_{\lambda}$  be saturated and let  $p \in \mathrm{gS}(M)$  be nonalgebraic. We must find a triple (a, M, N) such that  $M \leq_{\mathbf{K}} N, N \in \mathbf{K}_{\lambda}$  is saturated,  $p = \mathrm{gtp}(a/M; N)$ , and (a, M, N) is a prime triple among the saturated models of size  $\lambda$ .

Set  $M^0 := M$  and let  $\delta := \operatorname{cf} \lambda$ . Let  $M^1$ , a,  $\langle M_i^{\ell} : i \leq \delta \rangle$  be as described by the statement of Lemma 12.3.4. Recall (this is key) that  $||M_i^{\ell}|| < \lambda$  for any  $i < \delta$ . We show that  $(a, M^0, M^1)$  is as desired. By assumption,  $M^0 \leq_{\mathbf{K}} M^1$ ,  $p = \operatorname{gtp}(a/M^0; M^1)$ , and  $M^1 \in \mathbf{K}_{\lambda}$  is saturated. It remains to show that  $(a, M^0, M^1)$ is a prime triple in  $\mathbf{K}_{\lambda}^{\lambda-\operatorname{sat}}$ . Let  $M' \in \mathbf{K}_{\lambda}^{\lambda-\operatorname{sat}}$ ,  $a' \in |M'|$  be given such that  $\operatorname{gtp}(a'/M^0; M') = \operatorname{gtp}(a/M^0; M^1)$ . We want to build  $f : M^1 \xrightarrow[M^0]{} M'$  so that f(a) = a'. We build by induction an increasing continuous chain of embeddings  $\langle f_i : i \leq \delta \rangle$ so that for all  $i \leq \delta$ :

(1)  $f_i: M_i^1 \xrightarrow{M_i^0} M'.$ (2)  $f_i(a) = a'.$ 

This is enough since then  $f := f_{\delta}$  is as required. This is possible: for i = 0, we use that M' is saturated, hence realizes  $p \upharpoonright M_0^0$ , so there exists  $f_0 : M_0^1 \xrightarrow{M_0^0} M'$  witnessing it, i.e.  $f_0(a) = a'$ . At limits, we take unions. For i = j + 1 successor, let  $\mu := \|M_j^1\| + \|M_i^0\|$ . Pick  $N_j \leq_{\mathbf{K}} M'$  with  $N_j \in \mathbf{K}_{\mu}$  and  $N_j$  containing both  $f_j[M_j^1]$  and  $M_i^0$ .

By assumption, p does not fork over  $M_0^0$  and by assumption  $p = \text{gtp}(a'/M^0; M')$ , so by monotonicity of forking,  $a' \underset{M_j^0}{\downarrow} M_{j+1}^0$ . We know that  $(a, M_j^0, M_j^1)$  is a domination triple, hence applying  $f_j$  and using invariance,  $(a', M_j^0, f_j[M_j^1])$  is a domination triple. Therefore  $f_j[M_j^1] \underset{M_j^0}{\downarrow} M_{j+1}^0$ . By a similar argument, we also have

 $M_j^1 \stackrel{M_{j+1}^1}{\bigcup} M_{j+1}^0$ . By Fact 12.3.5, the map  $f_j \cup \operatorname{id}_{M_{j+1}^0}$  can be extended to a **K**-embedding  $a: M_{j+1}^{\alpha} \to N'_i$  for some  $N'_i \geq_{\mathbf{K}} N_i$  of size  $\mu$ . Since  $\mu < \lambda$  and M' is

embedding  $g: M_{j+1}^{\alpha} \to N_{j}'$  for some  $N_{j}' \geq_{\mathbf{K}} N_{j}$  of size  $\mu$ . Since  $\mu < \lambda$  and M' is saturated, there exists  $h: N_{j}' \xrightarrow[N_{j}]{N_{j}} M'$ . Let  $f_{j+1} := h \circ g$ .

# CHAPTER 13

# $\mu$ -Abstract elementary classes and other generalizations

This chapter is based on [**BGL**<sup>+</sup>16] and is joint work with Will Boney, Rami Grossberg, Michael Lieberman, and Jiří Rosický. We thank the referee for questions that helped us clarify some aspects of this chapter.

# Abstract

We introduce  $\mu$ -Abstract Elementary Classes ( $\mu$ -AECs) as a broad framework for model theory that includes complete boolean algebras and metric spaces, and begin to develop their classification theory. Moreover, we note that  $\mu$ -AECs correspond precisely to accessible categories in which all morphisms are monomorphisms, and begin the process of reconciling these divergent perspectives: for example, the preliminary classification-theoretic results for  $\mu$ -AECs transfer directly to accessible categories with monomorphisms.

# 13.1. Introduction

In this chapter, we offer a broad framework for model theory,  $\mu$ -abstract elementary classes, and connect them with existing frameworks, namely abstract elementary classes and, from the realm of categorical model theory, accessible categories (see [**MP89**], [**AR94**]) and  $\mu$ -concrete abstract elementary classes (see [**LR**]).

All of the above frameworks have developed in response to the need to analyze the model theory of nonelementary classes of mathematical structures; that is, classes in which either the structures themselves or the relevant embeddings between them cannot be adequately described in (finitary) first order logic. This project was well underway by the 50's and 60's, which saw fruitful investigations into infinitary logics and into logics with additional quantifiers (see [**Dic75**] and [**BFB85**] for summaries). Indeed, Shelah [**She00**, p. 41] recounts that Keisler and Morley advised him in 1969 that this direction was the future of model theory and that first-order had been mostly explored. The subsequent explosion in stability theory and its applications suggest otherwise, naturally, but the nonelementary context has nonetheless developed into an essential complement to the more classical picture.

On the model-theoretic side, Shelah was the leading figure, publishing work on excellent classes ([She83a] and [She83b]) and classes with expanded quantifiers [She75b], and, of greatest interest here, shifting to a formula-free context through the introduction of *abstract elementary classes* (or *AECs*) in [She87a]. The latter are a purely semantic axiomatic framework for abstract model theory that encompasses first order logic as well as infinitary logics incorporating additional quantifiers and infinite conjuncts and disjuncts, not to mention certain algebraically natural

examples without an obvious syntactic presentation—see [**BET07**]. It is important to note, though, that AECs still lack the generality to encompass the logic  $L_{\omega_1,\omega_1}$  or complete metric structures.

Even these examples are captured by *accessible categories*, a parallel, but significantly more general, notion developed simultaneously among category theorists, first appearing in [MP89] and receiving comprehensive treatments both in [MP89] and [AR94]. An accessible category is, very roughly speaking, an abstract category (hence, in particular, not a category of structures in a fixed signature) that is closed under sufficiently directed colimits, and satisfies a kind of weak Löwenheim-Skolem property: any object in the category can be obtained as a highly directed colimit of objects of small size, the latter notion being purely diagrammatic and internal to the category in question. In particular, an accessible category may not be closed under arbitrary directed colimits, although these are almost indispensable in model-theoretic constructions: the additional assumption of closure under directed colimits was first made in [Ros97]—that paper also experimented with the weaker assumption of directed bounds, an idea that recurs in Section 13.6 below.

Subsequent work (see [**BR12**], [**Lie11a**], and [**LR16**]) has resulted in a precise characterization of AECs as concrete accessible categories with added structure, namely as pairs ( $\mathcal{K}, U$ ), where

- $\mathcal{K}$  is an accessible category with all morphisms monomorphisms and all directed colimits, and
- U : K → Set (with Set the category of sets and functions) is a faithful ("underlying set") functor satisfying certain additional axioms.

Details can be found in Section 3 of [LR16]. Of particular importance is the extent to which U preserves directed colimits; that is, the extent to which directed colimits are concrete. If we assume that U preserves arbitrary directed colimits, we obtain a category equivalent to an AEC. If we make the weaker assumption that U merely preserves colimits of  $\mu$ -directed, rather than directed, diagrams, we arrive at the notion of a  $\mu$ -concrete AEC (see [LR]). Note that, although directed colimits may not be preserved by U (that is, they may not be "Set-like"), they still exist in the category  $\mathcal{K}$ —metric AECs, whose structures are built over complete metric structures, are a crucial example of this phenomenon. One might ask, though, what would happen if we weaken this still further: what can we say if we drop the assumption that  $\mathcal{K}$  is closed under directed colimits, and merely assume that the colimits that exist in  $\mathcal{K}$  and are "Set-like" are those that are  $\mu$ -directed for some  $\mu$ ?

Here we introduce a framework, called  $\mu$ -abstract elementary classes, that represents a model-theoretic approximation of that generalized notion, and which, most importantly, encompasses all of the examples considered in this introduction, including classes of models in infinitary logics  $L_{\kappa,\mu}$ , AECs, and  $\mu$ -concrete AECs. This is not done just for the sake of generalization but in order to be able to deal with specific classes of structures that allow functions with infinite arity (like  $\sigma$ -complete Boolean algebras or formal power series). Moreover, such classes also occur naturally in the development of the classification theory for AECs, as can be seen by their use in Chapters 6 and 7 (there the class studied is the  $\mu$ -AEC of  $\mu$ -saturated models of an AEC).

#### 13.2. PRELIMINARIES

We define  $\mu$ -AECs in Section 13.2. We then show that the examples discussed above fit into this framework. We establish an analog of Shelah's Presentation Theorem for  $\mu$ -AECs in Section 13.3. In Section 13.4 we show that  $\mu$ -AECs are, in fact, extraordinarily general: up to equivalence of categories, the  $\mu$ -AECs are precisely the accessible categories whose morphisms are monomorphisms. Although this presents certain obstacles—it follows immediately that a general  $\mu$ -AEC will not admit Ehrenfeucht-Mostowski constructions—there is a great deal that can be done on the  $\mu$ -AEC side of this equivalence. In section 13.5, we show assuming the existence of large cardinals that  $\mu$ -AECs satisfy tameness, an important locality property in the study of AECs. In Section 13.6, with the additional assumption of directed bounds, we begin to develop the classification theory of  $\mu$ -AECs. Note that the results of Sections 13.5 and 13.6 transfer immediately to accessible categories with monomorphisms.

# 13.2. Preliminaries

We now introduce the notion of a  $\mu$ -abstract elementary class, or  $\mu$ -AEC. As with ordinary AECs, we give a semantic/axiomatic definition for a class of structures and a notion of strong substructure.

DEFINITION 13.2.1. Fix an infinite cardinal  $\mu$ .

A  $\mu$ -ary language L consists of a set of function symbols  $\langle F_i : i \in I_F \rangle$  and relations  $\langle R_j : j \in J_R \rangle$  (here,  $I_F$ ,  $J_R$  are index sets) so that each symbol has an arity, denoted  $n(F_i)$  or  $n(R_j)$ , where n is an ordinal valued function  $n : \{R_j, F_i \mid i \in I_F, j \in J_R\} \to \mu$ .

Given a  $\mu$ -ary language L, an L-structure M is  $\langle |M|, F_i^M, R_j^M \rangle_{i \in L_F, j \in L_R}$  where |M| is a set, called the universe of M;  $F_i^M : {}^{n(F_i)}|M| \to |M|$  is a function of arity  $n(F_i)$ ; and  $R_j^M \subset {}^{n(R_j)}|M|$  is a relation of arity  $n(R_j)$ .

We say that  $\langle \mathbf{K}, \leq_{\mathbf{K}} \rangle$  is a  $\mu$ -abstract class provided

- (1) **K** is a class of *L*-structure for a fixed  $\mu$ -ary language  $L := L(\mathbf{K})$ .
- (2)  $\langle \mathbf{K}, \leq_{\mathbf{K}} \rangle$  is a partially pre-ordered class (that is,  $\leq_{\mathbf{K}}$  is reflexive and transitive) such that  $M \leq_{\mathbf{K}} N$  implies that M is an *L*-submodel of *N*.
- (3)  $\langle \mathbf{K}, \leq_{\mathbf{K}} \rangle$  respects *L*-isomorphisms; that is, if  $f : N \to N'$  is an *L*-isomorphism and  $N \in \mathbf{K}$ , then  $N' \in \mathbf{K}$  and if we also have  $M \in \mathbf{K}$  with  $M \leq_{\mathbf{K}} N$ , then  $f(M) \in \mathbf{K}$  and  $f(M) \leq_{\mathbf{K}} N'$ ;

We often do not make the distinction between **K** and  $(\mathbf{K}, \leq_{\mathbf{K}})$ .

An *L*-homomorphism is called a substructure embedding if it is injective and reflects all relations. Both inclusions of a substructure and isomorphisms are substructure embeddings. Conversely, if  $h: M \to N$  is a substructure embedding then M is isomorphic to the substructure h(M) of N. The category of all *L*-structures and substructure embeddings is denoted by  $\mathbf{Emb}(L)$ . Then an abstract class is the same as a subcategory  $\mathbf{K}$  of  $\mathbf{Emb}(L)$  which is

- (1) Replete, i.e., closed under isomorphic objects.
- (2) Iso-full, i.e., containing isomorphisms between K-objects.

Let  $\langle I, \leq \rangle$  be a partially ordered set. We say that I is  $\mu$ -directed, where  $\mu$  is a regular cardinal, provided that for every  $J \subseteq I$  with card  $(J) < \mu$ , there exists  $r \in I$  such that  $r \geq s$  for all  $s \in J$ . Thus  $\aleph_0$ -directed is the usual notion of directed set. Let  $\langle \mathbf{K}, \leq_{\mathbf{K}} \rangle$  be an abstract class. A family  $\{M_i \mid i \in I\} \subseteq \mathbf{K}$  is called a  $\mu$ -directed system provided I is a  $\mu$ -directed set and i < j implies  $M_i \leq_{\mathbf{K}} M_j$ . This is the same as a  $\mu$ -directed diagram in  $\mathbf{K}$ .

DEFINITION 13.2.2. Suppose  $\langle \mathbf{K}, \leq_{\mathbf{K}} \rangle$  is a  $\mu$ -abstract class, with  $\mu$  a regular cardinal. We say that  $\langle \mathbf{K}, \leq_{\mathbf{K}} \rangle$  is a  $\mu$ -abstract elementary class if the following properties hold:

- (1) (Coherence) if  $M_0, M_1, M_2 \in \mathbf{K}$  with  $M_0 \leq_{\mathbf{K}} M_2, M_1 \leq_{\mathbf{K}} M_2$ , and  $M_0 \subseteq M_1$ , then  $M_0 \leq_{\mathbf{K}} M_1$ ;
- (2) (Tarski-Vaught chain axioms) If  $\{M_i \in \mathbf{K} : i \in I\}$  is a  $\mu$ -directed system, then:
  - (a)  $\bigcup_{i \in I} M_i \in \mathbf{K}$  and, for all  $j \in I$ , we have  $M_j \leq_{\mathbf{K}} \bigcup_{i \in I} M_i$ ; and
  - (b) if there is some  $N \in \mathbf{K}$  so that, for all  $i \in I$ , we have  $M_i \leq_{\mathbf{K}} N$ , then we also have  $\bigcup_{i \in I} M_i \leq_{\mathbf{K}} N$ .
- (3) (Löwenheim-Skolem-Tarski number axiom) There exists a cardinal  $\lambda = \lambda^{<\mu} \ge \operatorname{card}(\operatorname{L}(\mathbf{K})) + \mu$  such that for any  $M \in \mathbf{K}$  and  $A \subseteq |M|$ , there is some  $N \le_{\mathbf{K}} M$  such that  $A \subseteq |N|$  and  $\operatorname{card}(N) \le \operatorname{card}(A)^{<\mu} + \lambda$ . LS(**K**) is the minimal cardinal  $\lambda$  with this property.<sup>1</sup>

Note that this definition mimics the definition of an AEC. We highlight the key differences:

Remark 13.2.3.

- (1) Functions and relations are permitted to have infinite arity.
- (2) The Löwenheim-Skolem-Tarski axiom only guarantees the existence of submodels of certain cardinalities, subject to favorable cardinal arithmetic.
- (3) Closure under unions of  $\leq_{\mathbf{K}}$ -increasing chains does not hold unconditionally: the directed systems must in fact be  $\mu$ -directed.
- (4) The Tarski-Vaught axioms describe  $\mu$ -directed systems rather than chains and say that  $\mathcal{K}$  is closed under  $\mu$ -directed colimits in  $\mathbf{Emb}(L)$ . One could have only required that every chain of models indexed by an ordinal of cofinality at least  $\mu$  has a least upper bound. When  $\mu = \aleph_0$ , this is wellknown to give an equivalent definition (see e.g. [AR94] 1.7, though the central idea of the proof dates back to Iwamura's Lemma, [Iwa44]). In general, though, this is significantly weaker (see [AR94] Exercise 1.c). Concretely, proving the presentation theorem becomes problematic if one opts instead for the chain definition.
- (5) Replacing the Tarski-Vaught axioms by **K** being closed under  $\mu$ -directed colimits in **Emb**(L) makes sense also when **K** is a  $\lambda$ -abstract class for  $\lambda > \mu$ .
- (6) As a notational remark, we use card () to denote the size of sets and (universes) of models. This breaks with convention, but is to avoid | · | being used to denote both universe and cardinality, leading to the notation ||M|| for the cardinality of the universe of a model.

As promised in the introduction,  $\mu$ -AECs subsume many previously studied model theoretic frameworks:

(1) All AECs are  $\aleph_0$ -AECs with the same Löwenheim-Skolem number. This follows directly from the definition. See [**Gro02**] for examples of classes

<sup>&</sup>lt;sup>1</sup>Note that  $LS(\mathbf{K})$  really depends on  $\mu$  but  $\mu$  will always be clear from context.

of structures that are AECs; in particular, AECs subsume classical firstorder model theory.

- (2) Given an AEC **K** with amalgamation (such as models of a first order theory), and a regular cardinal  $\mu > \text{LS}(\mathbf{K})$ , the class of  $\mu$ -saturated<sup>2</sup> models of **K** is a  $\mu$ -AEC with Löwenheim-Skolem number  $\text{LS}(\mathbf{K})^{<\mu}$ . If **K** is also tame and stable (or superstable), then the results of Boney and Vasey (Chapter 7) show that this is true even for certain cardinals below the saturation cardinal  $\mu$ .
- (3) Let  $\lambda \geq \mu$  be cardinals with  $\mu$  regular. Let  $L_A$  be a fragment of  $L_{\lambda,\mu}$  (recall that a fragment is a collection of formulas closed under sub formulas and first order connectives), and let T be a theory in that fragment. Then  $\mathbf{K} = (\text{Mod } T, \leq_{L_A})$  is a  $\mu$ -AEC with  $\text{LS}(\mathbf{K}) = (\text{card}(L_A) + \text{card}(T))^{<\mu}$ , where  $M \leq_{L_A} N$  if and only if for all  $\phi(\bar{x}) \in L_A$  and  $\mathbf{m} \in |M|$  of matching arity (which might be infinite), we have that  $M \models \phi[\bar{a}]$  if and only if  $N \models \phi[\bar{a}]$ .
- (4) Complete metric spaces form an  $\aleph_1$ -AECs. This follows from the above item because metric spaces are axiomatizable in first order and completeness is axiomatized by the  $L_{\omega_1,\omega_1}$  sentence

$$\forall \langle x_n : n < \omega \rangle [(\wedge_{\epsilon \in \mathbb{Q}^+} \vee_{N < \omega} \wedge_{N < n < m < \omega} d(x_n, x_m) < \epsilon) \implies \exists y (\wedge_{\epsilon \in \mathbb{Q}^+} \vee_{N < \omega} \wedge_{N < n < \omega} d(x_n, y) < \epsilon)]$$

Although this does not capture the [0, 1]-value nature of many treatments of the model theory of metric structures, such as **[BYBHU08]**, this can be incorporated in one of two ways. One could add the real numbers as a second sort, interpret relations as functions between the sorts, and axiomatize all of the continuity properties. A less direct approach is taken in [**Bona**], where a complete structure is approximated by a dense subset describable in  $L_{\omega_1,\omega}$ .

- (5) Along the lines of complete metric spaces,  $\mu$ -complete boolean algebras are  $\mu$ -AECs because  $\mu$ -completeness can be written as a  $L_{\mu,\mu}$ -sentence.
- (6) Any  $\mu$ -concrete AEC (or  $\mu$ -CAEC), in the sense of [**LR**], is a  $\mu$ -AEC.
- (7) Any  $\mu$ -ary functorial expansion of a  $\mu$ -AEC is naturally a  $\mu$ -AEC. See Section 2.1 immediately below.
- (8) Generalizing L(Q), consider classes axiomatized by  $L_{\lambda,\mu}(Q^{\chi})$ , where  $Q^{\chi}$  is the quantifier "there exist at least  $\chi$ " (the standard L(Q) is  $L_{\omega,\omega}(Q^{\aleph_1})$  in this notation). As in (3), let T be a theory in  $L_{\lambda,\mu}(Q^{\chi})$  and  $L_A$  be a fragment of this logic containing T. Since  $Q^{\chi}$  is  $L_{\chi,\chi}$  expressible, we already have  $\mathbf{K}_0 := (\text{Mod } T, \leq_{L_A})$  is a  $(\mu + \chi)$ -AEC with  $\text{LS}(\mathbf{K}_0) = (\text{card } (L_A) + \chi)^{<(\mu+\chi)}$ . For a stronger result, if we set  $M \leq_{L_A}^* N$  by

$$M \leq_{L_A} N$$
 and if  $Q^{\chi} x \phi(x, \bar{y}) \in L_A$  with  $M \models \neg Q^{\chi} x \phi(x, \mathbf{m})$ , then  $\phi(M, \bar{a}) = \phi(N, \bar{a})$ 

then  $\mathbf{K}_1 := (\operatorname{Mod} T, \leq_{L_A}^*)$  is a  $\mu$ -AEC with  $\operatorname{LS}(\mathbf{K}_1) = (\operatorname{card} (L_A) + \chi)^{<\mu}$ . Moreover, if  $\chi = \chi_0^+$  and  $L_A$  only contains *negative* instances of  $Q^{\chi}$ , then  $\operatorname{LS}(\mathbf{K}_1) = (\operatorname{card} (L_A) + \chi_0)^{<\mu}$ .

We now briefly discuss the interplay between certain  $\mu\text{-}\mathrm{AECs}$  and functorial expansions.

<sup>&</sup>lt;sup>2</sup>In the sense of Galois types.

**13.2.1. Functorial expansions and infinite summation.** Recall from Definition 2.3.1:

DEFINITION 13.2.4. Let **K** be a  $\mu$ -AEC with  $L = L(\mathbf{K})$  and let  $\widehat{L}$  be a  $\lambda$ -ary expansion of L with  $\lambda \geq \mu$ . A  $\lambda$ -ary  $\widehat{L}$ -functorial expansion of **K** is a class  $\widehat{K}$  of  $\widehat{L}$ -structures satisfying:

- (1) The map  $\widehat{M} \mapsto \widehat{M} \upharpoonright L$  is a bijection from  $\widehat{K}$  onto **K**. For  $M \in \mathbf{K}$ , we write  $\widehat{M}$  for the unique element of  $\widehat{K}$  whose reduct is M.
- (2) Invariance: If  $f: M \cong N$ , then  $f: \widehat{M} \cong \widehat{N}$ .
- (3) Monotonicity: If  $M \leq_{\mathbf{K}} N$ , then  $\widehat{M} \subseteq \widehat{N}$ .

We order  $\widehat{K}$  by  $\widehat{M} \leq_{\widehat{K}} \widehat{N}$  if and only if  $M \leq_{\mathbf{K}} N$ .

FACT 13.2.5 (Proposition 2.3.7). Let **K** be a  $\mu$ -AEC and let  $\hat{K}$  be a  $\mu$ -ary functorial expansion of **K**. Then  $(\hat{K}, \leq_{\hat{K}})$  is a  $\mu$ -AEC with  $LS(\hat{K}) = LS(\mathbf{K})$ .

REMARK 13.2.6. A word of warning: if **K** is an AEC and  $\hat{K}$  is a functorial expansion of **K**, then **K** and  $\hat{K}$  are isomorphic (as categories). In particular, any directed system in  $\hat{K}$  has a colimit. However,  $\hat{K}$  may *not* be an AEC if  $L(\hat{K})$  is not finitary: the colimit of a directed system in  $\hat{K}$  may *not* be the union: relations may need to contain more elements. However, if we change the definition of AEC to allow languages of infinite arity (see Remark 13.2.3.(5)), then  $\hat{K}$  will be an AEC in that new sense, i.e. an "infinitary" AEC.

REMARK 13.2.7. Let **K** be a  $\mu$ -AEC and consider a  $\widehat{L}$ -functorial expansion  $\widehat{K}$  of **K**. Then any function and relation symbols from  $\widehat{L}$  are *interpretable* in **K** in the sense of [**Ros81**] (this idea goes back to [**Law63**]). This means that function symbols of arity  $\alpha$  are natural transformations  $\varphi : U^{\alpha} \to U$  where  $U : \mathbf{K} \to \mathbf{Set}$  is the forgetful functor (given as the domain restriction of the forgetful functor  $\mathbf{Emb}(L) \to \mathbf{Set}$  assigning underlying sets to *L*-structures) and  $U^{\alpha}$  is the functor  $\mathbf{Set}(\alpha, U(-)) : \mathbf{K} \to \mathbf{Set}$ . Similarly, relation symbols of arity  $\alpha$  are subfunctors R of  $U^{\alpha}$ .

If  $\hat{L}$  is  $\mu$ -ary then subfunctors R preserve  $\mu$ -directed colimits. Since  $\mathbf{K}$  is an  $\mathrm{LS}(\mathbf{K})^+$ -accessible category (see 13.4.3), both  $\varphi$  and R are determined by their restrictions to the full subcategory  $\mathbf{K}_{\mathrm{LS}(\mathbf{K})^+}$  of  $\mathbf{K}$  consisting of  $\mathrm{LS}(\mathbf{K})^+$ -presentable objects. Since there is only a set of such objects, there is a largest  $\mu$ -ary functorial expansion where  $\hat{L}$  consists of all symbols for natural transformations and subfunctors as above. For  $\mu = \aleph_0$ , this is contained in [LR16, Remark 3.5].

The main example in Chapter 2 is Galois Morleyization (Definition 2.3.3). However there are many other examples including the original motivation for defining  $\mu$ -AECs: infinite sums in boolean algebras. The point is that even though the language of boolean algebras with a sum operator is infinitary, we really need only to work in an appropriate class in a finitary language that we functorially expand as needed. This shows in a precise sense that the infinite sum operator is already implicit in the (finitary) structure of boolean algebras themselves.

DEFINITION 13.2.8. Fix infinite cardinals  $\lambda \geq \mu$ . Let  $\Phi$  be a set of formulas in  $L_{\lambda,\mu}$ . Let  $(\mathbf{K}, \leq_{\mathbf{K}})$  be an abstract class. Define  $\mathbf{K}_{\Phi} := (\mathbf{K}, \leq_{\mathbf{K}_{\Phi}})$  by  $M \leq_{\mathbf{K}_{\Phi}} N$  if and only if  $M \leq_{\mathbf{K}} N$  and  $M \preceq_{\Phi} N$ .

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LEMMA 13.2.9. Let  $\lambda \geq \mu$ ,  $\Phi$  be a set of formulas in  $L_{\lambda,\mu}$ . Let  $(\mathbf{K}, \leq_{\mathbf{K}})$  be a  $\mu_0$ -AEC with  $\mu_0 \leq \mu$ . Then:

- (1)  $K_{\Phi}$  is a  $\mu$ -AEC.
- (2) If all the formulas in  $\Phi$  have fewer than  $\mu_0$ -many quantifiers, then  $\mathbf{K}_{\Phi}$  satisfies the first Tarski-Vaught chain axiom of  $\mu_0$ -AECs.

PROOF. The first part is straightforward. The second is proven by induction on the quantifier-depth of the formulas in  $\Phi$ .

EXAMPLE 13.2.10. Let T be a completion of the first-order theory of boolean algebras and let  $\mathbf{K} := (\operatorname{Mod}(T), \preceq)$ . Let  $\Phi$  consist of the  $L_{\omega_1,\omega_1}$  formula  $\phi(\bar{x}, y)$ saying that y is a least upper bound of  $\bar{x}$  (here  $\ell(\bar{x}) = \omega$ ). Then  $\phi$  has only one universal quantifier so by the Lemma,  $\mathbf{K}_{\Phi}$  satisfies the first Tarski-Vaught chain axiom of AECs. Of course,  $\mathbf{K}_{\Phi}$  is also an  $\aleph_1$ -AEC. Now expand each  $M \in \mathbf{K}$  to  $\widehat{M}$  by defining  $R_{\Sigma}^M(\bar{a}, b)$  to hold if and only if b is a least upper bound of  $\bar{a}$  (with  $\ell(\bar{a}) = \omega$ ). Let  $\widehat{K}_{\Phi} := {\widehat{M} \mid M \in \mathbf{K}}$ . Then one can check that  $\widehat{K}_{\Phi}$  is a functorial expansion of  $\mathbf{K}_{\Phi}$ .

Basic definitions and concepts for AECs, such as amalgamation or Galois types (see [Bal88] or [Gro] for details), can be easily transferred to  $\mu$ -AECs. In the following sections, we begin the process of translating essential theorems from AECs to  $\mu$ -AECs.

#### 13.3. Presentation Theorem

We now turn to the Presentation Theorem for  $\mu$ -AECs. This theorem has its conceptual roots in Chang's Presentation Theorem [**Cha68**], which shows that  $L_{\lambda,\omega}$  can be captured in a larger finitary language by omitting a set of types. A more immediate predecessor is Shelah's Presentation Theorem, which reaches the same conclusion for an arbitrary AEC. Unfortunately, while Chang's Presentation Theorem gives some insight into the original class, Shelah's theorem does not. However, the presentation is still a useful tool for some arguments and provides a syntactic characterization of what are otherwise purely semantic objects.

DEFINITION 13.3.1. Let  $L \subset L_1$  be  $\mu$ -ary languages,  $T_1$  an  $(L_1)_{\mu,\mu}$  theory, and  $\Gamma$  be a set of  $\mu$ -ary  $(L_1)_{\mu,\mu}$ -types. Here we define a  $\mu$ -ary  $(L_1)_{\mu,\mu}$ -type as a set of  $(L_1)_{\mu,\mu}$  formulas in the same free variables  $\bar{x}$ , where  $\bar{x}$  has arity less than  $\mu$ . We define

EC<sup>μ</sup>(T<sub>1</sub>, Γ) = {M : Man L<sub>1</sub>-structure, M ⊨ T<sub>1</sub>, Momits each type in Γ}
PC<sup>μ</sup>(T<sub>1</sub>, Γ, L) = {M ↾ L : M ∈ EC<sup>μ</sup>(T<sub>1</sub>, Γ)}

THEOREM 13.3.2. Let **K** be a  $\mu$ -AEC with  $LS(\mathbf{K}) = \chi$ . Then we can find some  $L_1 \supset L(\mathbf{K})$ , a  $(L_1)_{\mu,\mu}$ -theory  $T_1$  of size  $\chi$ , and a set  $\Gamma$  of  $\mu$ -ary  $(L_1)_{\mu,\mu}$ -types with card  $(\Gamma) \leq 2^{\chi}$  so that  $\mathbf{K} = PC^{\mu}(T_1, \Gamma, L(\mathbf{K}))$ .

Although we don't state them here, the traditional moreover clauses (see e.g. the statement of [**Bal88**, Theorem 4.15]) apply as well.

PROOF. We adapt the standard proofs; see, for instance, [Bal88, Theorem 4.15]. Set  $\chi := \text{LS}(\mathbf{K})$ . We introduce "Skolem functions"  $L_1 := L(\mathbf{K}) \cup \{F_i^{\alpha}(\bar{x}) : i < \chi, \ell(\bar{x}) = \alpha < \mu\}$  and make very minimal demands by setting

$$T_1 := \{ \exists x(x=x) \} \cup \{ \forall \bar{x} F_i^\alpha(\bar{x}) = x_i : i < \alpha < \mu \}$$

For any  $M_1 \models T_1$  and  $\bar{a} \in |M_1|$ , we define  $N_{\bar{a}}^{M_1}$  to be the minimal  $L_1$ -substructure of  $M_1$  that contains  $\bar{a}$ . We can code the information about  $N_{\bar{a}}^{M_1}$  into  $\bar{a}$ 's quantifier-free type:

 $p_{\bar{a}}^{M_1} := \{ \phi(\bar{x}) : \phi(\bar{x}) \in (L_1)_{\mu,\mu} \text{ is quantifier-free and } M_1 \vDash \phi(\bar{a}) \}$ 

Given tuples  $\bar{a} \in M_1$  and  $\bar{b} \in M_2$  of the same length, we have that  $p_{\bar{a}}^{M_1} = p_{\bar{b}}^{M_2}$ if and only if the map taking  $\bar{a}$  to  $\bar{b}$  induces an isomorphism  $N_{\bar{a}}^{M_1} \cong N_{\bar{b}}^{M_2}$ . Since we have this tight connection between types and structures, we precisely want to exclude types that give rise to structures not coming from **K**. Thus, we set

$$\begin{split} \Gamma^{M_1} &:= & \bigcup \{ p_{\bar{a}}^{M_1} : \exists \bar{b} \subset \bar{a} \text{ such that } (N_{\bar{b}}^{M_1}) \upharpoonright L(\mathbf{K}) \not\leq_{\mathbf{K}} (N_{\bar{a}}^{M_1}) \upharpoonright L(\mathbf{K}) \} \\ \Gamma &:= & \bigcup_{M_1 \vDash T_1} \Gamma^{M_1} \end{split}$$

Note that the  $\bar{b}$  in the first line might be  $\bar{a}$ , in which case the condition becomes  $N_{\bar{a}}^{M_1} \upharpoonright L(\mathbf{K}) \notin \mathbf{K}$ . By counting the number of  $(L_1)_{\mu,\mu}$ -types, we have that card  $(\Gamma) \leq 2^{\chi}$ . Now all we have left to show is the following claim.

Claim:  $\mathbf{K} = PC(T_1, \Gamma, L(\mathbf{K}))$ 

First, let  $M_1 \in \text{EC}(T_1, \Gamma)$ . Given  $\bar{a} \in {}^{<\mu}|M_1|$ , we know that  $\bar{a} \models p_{\bar{a}}^{M_1}$  so  $p_{\bar{a}}^{M_1} \notin \Gamma$ . Thus,  $\{N_{\bar{a}}^{M_1} \upharpoonright \text{L}(\mathbf{K}) : \bar{a} \in {}^{<\mu}|M_1|\}$  is a  $\mu$ -directed system from  $\mathbf{K}_{\leq \chi}$  with union  $M_1 \upharpoonright \text{L}(\mathbf{K})$ , so  $M_1 \upharpoonright \text{L}(\mathbf{K}) \in \mathbf{K}$ .

Second, let  $M \in \mathbf{K}$ . We need to define an expansion  $M_1 \in \mathrm{EC}(T_1, \Gamma)$ . We can build a directed system  $\{M_{\bar{a}} \in \mathbf{K}_{\chi} : \bar{a} \in {}^{<\mu}|M|\}$ . Since each  $M_{\bar{a}}$  has size  $\chi$ , we can define the  $F_i^{\alpha}$  by enumerating  $|M_{\bar{a}}| = \{F_i^{\ell(\bar{a})}(\bar{a}) : i < \chi\}$  with the condition that  $F_i^{\ell(\bar{a})}(\bar{a}) = a_i$  for  $i < \ell(\bar{a})$ . This precisely defines the expansion  $M_1 := \langle M, F_i^{\alpha} \rangle_{i < \chi, \alpha < \mu}$ . It is easy to see  $M_1 \models T_1$ . We also have  $N_{\bar{a}}^{M_1} \upharpoonright L(\mathbf{K}) = M_{\bar{a}}$ , so  $M_1$  omits  $\Gamma$  because  $\{M_{\bar{a}} : \bar{a} \in {}^{<\mu}|M|\}$  is a  $\mu$ -directed system from  $\mathbf{K}$ . So  $M \in PC^{\mu}(T_1, \Gamma, L)$ .

REMARK 13.3.3. A consequence of the presentation theorem for AECs is that an AEC **K** with a model of size  $\beth_{(2^{\text{LS}(\mathbf{K})})^+}$  has arbitrarily large models (see e.g. [**Bal88**, Corollary 4.26]). The lack of Hanf numbers for  $L_{\mu,\mu}$  means that we cannot use this to get similar results for  $\mu$ -AECs. Thus the following question is still open: Can we compute a bound for the Hanf number  $H(\lambda, \mu)$ , where any  $\mu$ -AEC **K** with  $\text{LS}(\mathbf{K}) \leq \lambda$  that has a model larger than  $H(\lambda, \mu)$  has arbitrarily large models?

#### 13.4. $\mu$ -AECs and accessible categories

Accessible categories were introduced in [**MP89**] as categories closely connected with categories of models of  $L_{\kappa,\lambda}$  theories. Roughly speaking, an accessible category is one that is closed under certain directed colimits, and whose objects can be built via certain directed colimits of a set of small objects. To be precise, we say that a category  $\mathcal{K}$  is  $\lambda$ -accessible,  $\lambda$  a regular cardinal, if it closed under  $\lambda$ -directed colimits (i.e. colimits indexed by a  $\lambda$ -directed poset) and contains, up to isomorphism, a set  $\mathcal{A}$  of  $\lambda$ -presentable objects such that each object of  $\mathcal{K}$  is a  $\lambda$ -directed colimit of objects from  $\mathcal{A}$ . Here  $\lambda$ -presentability functions as a notion of size that makes sense in a general, i.e. non-concrete, category: we say an object M is  $\lambda$ -presentable if its hom-functor  $\mathcal{K}(M, -) : \mathcal{K} \to \mathbf{Set}$  preserves  $\lambda$ -directed colimits. Put another way, M is  $\lambda$ -presentable if for any morphism  $f : M \to N$  with N a  $\lambda$ -directed colimit  $\langle \phi_{\alpha} : N_{\alpha} \to N \rangle$ , f factors essentially uniquely through one of the  $N_{\alpha}$ , i.e.  $f = \phi_{\alpha} f_{\alpha}$  for some  $f_{\alpha} : M \to N_{\alpha}$ .

For each regular cardinal  $\kappa$ , an accessible category  $\mathcal{K}$  contains, up to isomorphism, only a set of  $\kappa$ -presentable objects. Any object M of a  $\lambda$ -accessible category is  $\kappa$ -presentable for some regular cardinal  $\kappa$ . Given an object M, the smallest cardinal  $\kappa$  such that M is  $\kappa$ -presentable is called the *presentability rank* of M. If the presentability rank of M is a successor cardinal  $\kappa = ||M||^+$  then ||M|| is called the *internal size* of M (this always happens if  $\mathcal{K}$  has directed colimits or under GCH, see [**BR12**] 4.2 or 2.3.5). This notion of size internal to a particular category more closely resembles a notion of dimension—in the category **Met** of complete metric spaces with isometric embeddings, for example, the internal size of an object is precisely its density character—and, even in case the category is concrete, may not correspond to the cardinality of underlying sets. This distinction will resurface most clearly in the discussion at the beginning of Section 13.6 below.

We consider the category-theoretic structure of  $\mu$ -AECs. As we will see, for any uncountable cardinal  $\mu$ , any  $\mu$ -AEC with Löwenheim-Skolem-Tarski number  $\lambda$ is a  $\lambda^+$ -accessible category whose morphisms are monomorphisms, and that (perhaps more surprisingly) any  $\mu$ -accessible category whose morphisms are monomorphisms is equivalent to a  $\mu$ -AEC with Löwenheim-Skolem-Tarski number  $\lambda = \max(\mu, \nu)^{<\mu}$ , where  $\nu$ , discussed in detail below, is the number of morphisms between  $\mu$ -presentable objects.

It is of no small interest that a general  $\mu$ -accessible category also satisfies a Löwenheim-Skolem-Tarski axiom of sorts, governed by the sharp inequality relation,  $\leq^3$ . As we will see, this notion (see [**MP89**]) matches up perfectly with the behavior of  $\mu$ -AECs conditioned by axiom 13.2.2(3).

We wish to show that  $\mu$ -AECs and accessible categories are equivalent. For the easy direction—that every  $\mu$ -AEC is accessible—we simply follow the argument for the corresponding fact for AECs in Section 4 of [Lie11a].

LEMMA 13.4.1. Let  $\mathcal{K}$  be a  $\mu$ -AEC with Löwenheim-Skolem-Tarski number  $\lambda$ . Any  $M \in \mathcal{K}$  can be expressed as a  $\lambda^+$ -directed union of its  $\leq_{\mathbf{K}}$ -substructures of size at most  $\lambda$ .

PROOF. Consider the diagram consisting of all  $\leq_{\mathbf{K}}$ -substructures of M of size at most  $\lambda$  and with arrows the  $\leq_{\mathbf{K}}$ -inclusions. To check that this diagram is  $\lambda^+$ directed, we must show that any collection of fewer than  $\lambda^+$  many such submodels have a common extension also belonging to the diagram. Let  $\{M_{\alpha} \mid \alpha < \nu\}, \nu < \lambda^+$ , be such a collection. Since  $\lambda^+$  is regular,  $\sup\{|M_{\alpha}| \mid \alpha < \nu\} < \lambda^+$ , whence

$$\operatorname{card}(\bigcup_{\alpha < \nu} M_{\alpha}) \le \nu \cdot \sup\{\operatorname{card}(M_{\alpha}) \mid \alpha < \nu\} \le \nu \cdot \lambda = \lambda$$

This set will be contained in some  $M' \leq_{\mathbf{K}} M$  with  $\operatorname{card}(M') \leq \lambda^{<\mu} + \lambda = \lambda + \lambda = \lambda$ , by the Löwenheim Skolem-Tarski axiom. For each  $\alpha < \nu$ ,  $M_{\alpha} \leq_{\mathbf{K}} M$  and  $M_{\alpha} \subseteq M'$ . Since  $M' \leq_{\mathbf{K}} M$ , coherence implies that  $M_{\alpha} \leq_{\mathbf{K}} M'$ . So we are done.  $\Box$ 

LEMMA 13.4.2. Let  $\mathcal{K}$  be a  $\mu$ -AEC with Löwenheim-Skolem-Tarski number  $\lambda$ . A model M is  $\lambda^+$ -presentable in  $\mathcal{K}$  if and only if  $\operatorname{card}(M) \leq \lambda$ .

<sup>&</sup>lt;sup>3</sup>The sharp inequality was introduced by Makkai and Pare [**MP89**, Section 2.3] and is defined by  $\kappa \leq \kappa'$  if and only if every  $\kappa$ -accessible category is also a  $\kappa'$ -accessible category, among other equivalent conditions.

PROOF. See the proof of Lemma 4.3 in [Lie11a].

Taken together, these lemmas imply that any  $\mu$ -AEC with Löwenheim-Skolem-Tarski number  $\lambda$  contains a set of  $\lambda^+$ -presentable objects, namely  $\mathcal{K}_{<\lambda^+}$ , and that any model can be built as a  $\lambda^+$ -directed colimit of such objects. As the Tarski-Vaught axioms ensure closure under  $\mu$ -directed colimits and  $\lambda \geq \mu$ , it follows that  $\mathcal{K}$  is closed under  $\lambda^+$ -directed colimits. Thus we have:

THEOREM 13.4.3. Let  $\mathcal{K}$  be a  $\mu$ -AEC with Löwenheim-Skolem-Tarski number  $\lambda$ . Then  $\mathcal{K}$  is a  $\lambda^+$ -accessible category.

REMARK 13.4.4. Theorem 13.4.3 is valid for any  $\lambda$  from 13.2.2(3) and not only for the minimal one. Moreover, we only need that  $\lambda$  satisfies the Löwenheim-Skolem-Tarski property for card  $(A) \leq \lambda$ . In this case, we will say that  $\lambda$  is a *weak Löwenheim-Skolem-Tarski number*.

We now aim to prove that any accessible category whose morphisms are monomorphisms is a  $\mu$ -AEC for some  $\mu$ . In fact, there are two cases delineated below, concrete and abstract. In Theorem 13.4.5 we consider the concrete case:  $\mathcal{K}$  is taken to be a  $\kappa$ -accessible category of L-structures and L-embeddings for some  $\mu$ -ary language L where  $\mu + \operatorname{card}(L) \leq \kappa$ . In particular, we insist that  $\mathcal{K}$  sits nicely in  $\operatorname{\mathbf{Emb}}(L)$ , the category of all L-structures and substructure embeddings—we may assume L is relational. In Theorem 13.4.10, we consider abstract accessible categories, with no prescribed signature or underlying sets.

THEOREM 13.4.5. Let L be a  $\mu$ -ary signature and  $\mathcal{K}$  be an iso-full, replete and coherent  $\kappa$ -accessible subcategory of  $\mathbf{Emb}(L)$  where  $\mu + \operatorname{card}(L) \leq \kappa$ . If  $\mathcal{K}$  is closed under  $\mu$ -directed colimits in  $\mathbf{Emb}(L)$  and the embedding  $\mathcal{K} \to \mathbf{Emb}(L)$  preserves  $\kappa$ -presentable objects then  $\mathcal{K}$  is a  $\mu$ -AEC with  $\operatorname{LS}(\mathbf{K}) \leq \lambda = \kappa^{<\mu}$ .

PROOF. We verify that  $\mathcal{K}$  satisfies the axioms in Definitions 13.2.1 and 13.2.2. Given such a category, we define the relation  $\leq_{\mathbf{K}}$  as we must: for  $M, N \in \mathcal{K}$ ,  $M \leq_{\mathbf{K}} N$  if and only if  $M \subseteq N$  and the inclusion is a morphism in  $\mathcal{K}$ . Axiom 13.2.1(1) follows immediately from this definition. Axiom 13.2.1(2) follows from the assumption that the inclusion E is replete and iso-full, while 13.2.2(1) follows from the assumption that the aforementioned inclusion is a coherent functor. 13.2.2(2) is easily verified: given a  $\mu$ -directed system  $\langle M_i | i \in I \rangle$  in  $\mathbf{K}$ , the colimit lies in  $\mathcal{K}$  (by  $\mu$ -accessibility), and since the inclusion E preserves  $\mu$ -directed colimits, it will be precisely the union of the system. So  $\mathcal{K}$  is closed under  $\mu$ -directed unions. The other clauses of 13.2.2(2) are clear as well.

Axiom 13.2.2(3), the Löwenheim-Skolem-Tarski Property, poses more of a challenge. To begin, we recall that in  $\mathbf{Emb}(L)$ , an object is  $\kappa^+$ -presentable for  $\kappa = \kappa^{<\mu} \ge \mu + \operatorname{card}(L)$  precisely if its underlying set is of cardinality at most  $\kappa$ .

Recall that we intend to show that  $\lambda = \kappa^{<\mu}$  satisfies 13.2.2(3). Let  $M \in \mathcal{K}$  and  $A \subseteq |M|$  with  $|A| = \alpha > \lambda$ . We begin by showing that  $\mathcal{K}$  is  $(\alpha^{<\mu})^+$ -accessible. This is an consequence of [**LR**, 4.10] because  $\kappa \leq \lambda < (\alpha^{<\mu})^+$  and  $\mu \leq (\alpha^{<\mu})^+$ . The sharp inequality is a consequence of Example 2.13(4) in [**AR94**]: for any cardinals  $\beta < (\alpha^{<\mu})^+$  and  $\gamma < \mu$ ,

$$\beta^{\gamma} \le (\alpha^{<\mu})^{\gamma} = \alpha^{<\mu} < (\alpha^{<\mu})^+.$$

Since  $\mathcal{K}$  is  $(\alpha^{<\mu})^+$ -accessible, we can express M as an  $(\alpha^{<\mu})^+$ -directed colimit of  $(\alpha^{<\mu})^+$ -presentable objects in  $\mathcal{K}$ , say  $\langle M_i \to M | i \in I \rangle$ —indeed, we may assume

without loss that this is a  $(\alpha^{<\mu})^+$ -directed system of inclusions. Following [**LR**, 4.7],  $E : \mathcal{K} \to \mathbf{Emb}(L)$  preserves  $(\alpha^{<\mu})^+$ -presentable objects—hence the  $M_i$  are also  $(\alpha^{<\mu})^+$ -presentable in  $\mathbf{Emb}(L)$ , and thus of cardinality at most  $\alpha^{<\mu} = \operatorname{card}(A)^{<\mu}$ , by the remark in the previous paragraph. For each  $a \in A$ , choose  $M_{i_a}$  with  $a \in |M_{i_a}|$ . The set of all such  $M_{i_a}$  is of size at most  $\alpha < (\alpha^{<\mu})^+$  and we have chosen the colimit to be  $(\alpha^{<\mu})^+$ -directed, so there is some  $M' = M_j, j \in I$ , with  $M_{i_a} \leq_{\mathbf{K}} M'$  for all  $a \in A$ . Hence  $A \subseteq |M'|, M' \leq_{\mathbf{K}} M$ , and

$$\operatorname{card}(M') \le \alpha^{\le \mu} \le \alpha^{\le \mu} + \lambda = \operatorname{card}(A)^{\le \mu} + \lambda.$$

We now consider the case card  $(A) \leq \lambda$ . Hence

$$\operatorname{card}\left(A\right)^{<\mu} < \lambda^{<\mu} = \lambda$$

So the cardinal bound in the Löwenheim-Skolem-Tarski Property defaults to  $\lambda$ . Since  $\mu \leq \lambda^+$  (by [**AR94**] 2.13(4) again) and  $\kappa \leq \lambda^+$ , [**LR**, 4.7,4.10] imply that  $\mathcal{K}$  is  $\lambda^+$ -accessible and the functor  $E : \mathcal{K} \to \mathbf{Emb}(L)$  preserves  $\lambda^+$ -presentable objects. Thus we may use the same argument as above to find  $M' \leq_{\mathbf{K}} M$  of size  $\lambda$  containing A.

REMARK 13.4.6. Following 13.4.4 and 13.4.5, any  $\mu$ -abstract class from 13.2.2 with (3) weakened to the existence of a weak Löwenheim-Skolem-Tarski number  $\lambda$  is a  $\mu$ -AEC with  $LS(\mathcal{K}) \leq (\lambda^+)^{<\mu}$ .

Assuming Vopěnka's principle, the weak Löwenheim-Skolem-Tarski number axiom is satisfied by any full subcategory  $\mathcal{K}$  of  $\mathbf{Emb}(L)$ . This follows from [**AR95**] and is related to the unpublished theorem of Stavi (see [**MV11**]).

To summarize, we have so far shown that any reasonably embedded  $\kappa$ -accessible subcategory of a category of structures  $\mathbf{Emb}(L)$  is a  $\mu$ -AEC. We wish to go further, however: given any  $\mu$ -accessible category whose morphisms are monomorphisms, we claim that it is equivalent—as an abstract category—to a  $\mu$ -AEC, in a sense that we now recall.

DEFINITION 13.4.7. We say that categories C and D are *equivalent* if the following equivalent conditions (see [Lan98] V.4.1) hold:

- (1) There is a functor  $F : \mathcal{C} \to \mathcal{D}$  that is
  - full: For any  $C_1, C_2$  in  $\mathcal{C}$ , the map  $f \mapsto F(f)$  is a surjection from  $\operatorname{Hom}_{\mathcal{C}}(C_1, C_2)$  to  $\operatorname{Hom}_{\mathcal{D}}(FC_1, FC_2)$ .
  - faithful: For any  $C_1, C_2$  in  $\mathcal{C}$ , the map  $f \mapsto F(f)$  is an injection from  $\operatorname{Hom}_{\mathcal{C}}(C_1, C_2)$  to  $\operatorname{Hom}_{\mathcal{D}}(FC_1, FC_2)$ .
  - essentially surjective: Any object D in  $\mathcal{D}$  is isomorphic to F(C) for some C in  $\mathcal{C}$ .
- (2) There are functors  $F : \mathcal{C} \to \mathcal{D}$  and  $G : \mathcal{D} \to \mathcal{C}$  such that the compositions FG and GF are naturally isomorphic to the identity functors on  $\mathcal{D}$  and  $\mathcal{C}$ , respectively.

One might insist that the compositions in condition (2) are in fact equal to the identity functors, but this notion (*isomorphism of categories*) is typically too strong to be of interest—equivalence of categories as described above is sufficient to ensure that a pair of categories exhibit precisely the same properties. In particular, if  $F : \mathcal{C} \to \mathcal{D}$  gives an equivalence of categories, it preserves and reflects internal sizes and gives a bijection between the isomorphism classes in  $\mathcal{C}$  and those in  $\mathcal{D}$ ; thus questions of, e.g., categoricity have identical answers in  $\mathcal{C}$  and  $\mathcal{D}$ . We proceed by constructing, for a general  $\mu$ -accessible category  $\mathcal{K}$  whose morphisms are monomorphisms, a full, faithful, essentially surjective functor from  $\mathcal{K}$  to  $\mathcal{K}'$ , where  $\mathcal{K}'$  is a  $\mu$ -AEC. We begin by realizing a  $\mu$ -accessible category as a category of structures.

LEMMA 13.4.8. Let  $\mathcal{K}$  be a  $\mu$ -accessible category whose morphisms are monomorphisms. There is a unary many-sorted signature L such that  $\mathcal{K}$  is fully embedded to an equational variety in  $\mathbf{Emb}(L)$ .

Moreover, this full embedding preserves  $\mu$ -directed colimits.

PROOF. Let  $\mathcal{A}$  be the full subcategory of  $\mu$ -presentable objects in  $\mathcal{K}$  (technically, we want  $\mathcal{A}$  to be skeletal, which makes it small). Consider the canonical embedding

$$E: \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{op}}$$

that takes each  $K \in \mathcal{K}$  to the contravariant functor  $\operatorname{Hom}_{\mathcal{K}}(-,K)|_{\mathcal{A}^{op}}$ , and each  $\mathcal{K}$ -morphism  $f: K \to K'$  to the natural transformation  $E(f): \operatorname{Hom}_{\mathcal{K}}(-, K) \to$  $\operatorname{Hom}_{\mathcal{K}}(-,K)$  given by postcomposition with f. We note that, by Proposition 2.8 in [AR94], this functor is fully faithful and preserves  $\mu$ -directed colimits. In fact, we may identify the image  $\mathcal{L}_1$  of  $\mathcal{K}$  in **Set**^{\mathcal{A}^{op}} with an equational variety. Let L be a signature with sorts  $\{S_A | A \in \mathcal{A}\}$ , and with unary function symbols for each morphism in  $\mathcal{A}$ , i.e. a function symbol  $\overline{f}$  of arity  $S_A \to S_B$  for each  $\mathcal{A}$ -map  $f: B \to A$ , subject to certain equations: whenever  $h = f \circ g$  in  $\mathcal{A}$ , we insist that  $h = \bar{g}\bar{f}$ . Concretely, the identification is given by a functor  $F : \mathcal{L}_1 \to \mathbf{Emb}(L)$ that takes each functor  $E(K) = \operatorname{Hom}_{\mathcal{K}}(-, K)$  to the structure FE(K) with sorts  $S_A^{FE(K)} = \operatorname{Hom}_{\mathcal{K}}(A, K)$  and with each  $\overline{f}: S_A \to S_B$  interpreted as the function  $f^{FE(K)}$ : Hom<sub> $\mathcal{K}$ </sub> $(A, K) \to$  Hom<sub> $\mathcal{K}$ </sub>(B, K) given by precomposition with f. Any morphism  $g: K \to K'$  in  $\mathcal{K}$  is first sent to the natural transformation E(g):  $\operatorname{Hom}_{\mathcal{K}}(-,K) \to \operatorname{Hom}_{\mathcal{K}}(-,K')$  then sent, via F, to  $FE(g): FE(K) \to FE(K')$ , which is given sortwise by postcomposition with g, i.e. for any  $A \in \mathcal{A}$  and  $f \in$  $S_A^{FE(K)} = \operatorname{Hom}_{\mathcal{K}}(A, K), g(f) = g \circ f.$  Clearly, morphisms are injective in the image of  $\mathcal{K}$  under FE, as they come from monomorphisms in  $\mathcal{K}$ , and they trivially reflect relations, meaning that in fact  $F : \mathcal{L}_1 \to \mathbf{Emb}(L)$ .

Let  $\mathcal{L}_2$  denote the image of  $\mathcal{K}$  in  $\mathbf{Emb}(L)$  under FE. So we have exhibited  $\mathcal{K}$  as a full subcategory of  $\mathbf{Emb}(L)$  closed under  $\mu$ -directed colimits, where  $\Sigma$  is a finitary language. As a result, the induced relation  $\leq_{\mathbf{K}}$  is simply  $\subseteq$ , and iso-fullness and repleteness of the embedding are trivial. There is only one more wrinkle that we need to consider: the presentability rank of structures in the image of  $\mathcal{K}$  in  $\mathbf{Emb}(L)$  need not correspond to the cardinality of the union of their sorts—that is, if U denotes the forgetful functor  $\mathbf{Emb}(L) \to \mathbf{Set}$ , a  $\mu$ -presentable object FE(K) need not have  $|UFE(K)| < \mu$ —so the argument in Theorem 13.4.5 cannot simply be repeated here. Still, U can only do so much damage:

LEMMA 13.4.9. The functor  $U : \mathcal{L}_2 \to \mathbf{Set}$  sends  $\mu$ -presentable objects to  $\nu^+$ -presentable objects, where  $\nu = \operatorname{card} (\operatorname{Mor}(\mathcal{A}))$ .

THEOREM 13.4.10. Let  $\mathcal{K}$  be a  $\mu$ -accessible category with all morphisms mono. Then  $\mathcal{K}$  is equivalent to a  $\mu$ -AEC with Löwenheim-Skolem-Tarski number  $\lambda = \max(\mu, \nu)^{<\mu}$ .

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PROOF. Consider  $X \subseteq FE(K)$ , and let  $\alpha = \operatorname{card}(X)$ . Since  $\mathcal{L}_2$  is  $\mu$ -accessible, it is  $(\alpha^{<\mu})^+$ -accessible (provided  $\alpha \geq \mu$ ); see the proof of 13.4.5. Thus there is an  $(\alpha^{<\mu})^+$ -presentable  $\mathcal{L}_2$ -subobject  $M_X$  of FE(K) with  $X \subseteq M_X$ . By Theorem 2.3.11 in [**MP89**],  $M_X$  can be expressed as an  $(\alpha^{<\mu})^+$ -small  $\mu$ -directed colimit of  $\mu$ -presentables in  $\mathcal{L}_2$ , meaning that  $U(M_X)$  is an  $(\alpha^{<\mu})^+$ -small  $\mu$ -directed colimit of sets of size less or equal than  $\nu$ . This is of cardinality less or equal than  $\alpha^{<\mu} + \max(\mu, \nu)$ . This suggests  $\max(\mu, \nu)$  might serve as our Löwenheim-Skolem-Tarski number, but we must fulfill the requirement that  $\lambda^{<\mu} = \mu$ . So, take  $\lambda = \max(\mu, \nu)^{<\mu}$ .

The  $\mu$ -AEC from 13.4.10 is a full subcategory of  $\mathbf{Emb}(L)$  where L is a finitary language. Although this equivalence destroys both the ambient language and the underlying sets, and thus moves beyond the methods usually entertained in model theory, it allows us to transfer intuition and concepts between the two contexts.

The equivalence allows us to generate the notion of a Löwenheim-Skolem number in an accessible concrete category, where concreteness is necessary to form the question.

PROPOSITION 13.4.11. Let  $(\mathbf{K}, U)$  be a  $\mu$ -accessible concrete category with all maps monomorphisms such that U preserves  $\mu$ -directed colimits. Then if  $M \in \mathbf{K}$  and  $X \subset UM$ , there is a subobject  $M_0 \in \mathbf{K}$  of M such that  $X \subset UM_0$  and  $M_0$  is  $(\operatorname{card}(X)^{<\mu})^+$ -presentable.

Note that we have proved that every  $\mu$ -accessible category with all maps monomorphism has such a concrete functor: by Theorem 13.4.10, there is a (full and faithful) equivalence  $F : \mathbf{K} \to \mathbf{K}'$ , where  $\mathbf{K}'$  is some  $\mu$ -AEC. The universe functor  $U : \mathbf{K}' \to \mathbf{Set}$  if faithful and preserves  $\mu$ -directed colimits, so  $FU : \mathbf{K} \to \mathbf{K}'$  does as well.

PROOF. Let  $M \in \mathbf{K}$ , where  $\mathbf{K}$  is  $\mu$ -accessible and concrete with monomorphisms. Let  $X \subset UM$ . We want to find  $M_0 \leq M$  with  $X \subset UM_0$  that is  $(\operatorname{card}(X)^{<\mu})^+$ -presentable. By accessibility, we can write M as a  $\mu$ -directed colimit  $\langle M^i, f_{j,i} | j < i \in I \rangle$  where  $M^i \in \mathbf{K}$  is  $\mu$ -presentable, I is  $\mu$ -directed, and  $f_{i,\infty}$  are the colimit maps.

Because U preserves  $\mu$ -directed colimits, there is  $I_0 \subset I$  of size  $\leq \operatorname{card}(X)$ such that, for every  $x \in X$ , there is some  $i_x \in I_0$  such that  $x \in Uf_{i_x,\infty}M^i$ . Close this to a  $\mu$ -directed subset  $I_1 \subset I$  of size  $\leq \operatorname{card}(X)^{<\mu}$  and let  $(M^*, f_{i,*})$  be the colimit of  $\{M^i, f_{j,i} \mid j < i \in I_1\}$ . Since this system also embeds into M, there is a canonical map  $f^* : M^* \to M$ . Set  $M_0 = f^*M^*$ . Then  $M_0$  is a subobject of Mand  $X \subset UM_0$ , so we just need to show  $M_0$  is  $(\operatorname{card}(X)^{<\mu})^+$ -presentable. This follows from [**AR94**] 1.16: since  $\mu \leq (\operatorname{card}(X)^{<\mu})^+$ , each  $M^i$  is  $(\operatorname{card}(X)^{<\mu})^+$ presentable. Since  $\operatorname{card}(I_1) \leq \operatorname{card}(X)^{<\mu}$ ,  $M_0$  is  $(\operatorname{card}(X)^{<\mu})^+$ -presentable by the cited result.

Although the previous theorem doesn't use any model theoretic properties directly, it is inspired by standard proofs of the downward Löwenheim-Skolem theorem and seems not to have been known previously.

Going the other direction, knowledge about accessible categories allows us to show that  $\mu$ -AECs do not, in general, admit Ehrenfeucht-Mostowski constructions. In particular, not every  $\mu$ -AEC  $\mathcal{K}$  admits a faithful functor  $E : \text{Lin} \to \mathcal{K}$ :

EXAMPLE 13.4.12. Let  $\mathcal{K}$  be the category of well-ordered sets and order-preserving injections. By [**AR94**] 2.3(8),  $\mathcal{K}$  is  $\omega_1$ -accessible, and clearly all of its morphisms are monomorphisms. By Theorem 13.4.10, it is therefore equivalent to an  $\omega_1$ -AEC. As  $\mathcal{K}$  is isomorphism rigid—that is, it contains no nonidentity isomorphisms—it cannot admit a faithful functor from **Lin**, which is far from isomorphism rigid.

Ehrenfeucht-Mostowski constructions are a very powerful tool in the study of AECs (see for example [She99]). This suggests that  $\mu$ -AECs may be too general to support a robust classification theory. In particular, the lack Ehrenfeucht-Mostowski models, in turn, means that there is no analogue of the Hanf number that has proven to be very useful in the study of AECs.

A possible substitute to the notion of Hanf number is that of *LS*-accessibility, which was introduced by [**BR12**]. Rather than looking at the cardinality of the models, they asked about the internal size, as computed in the category. The shift stems from the following: it is clear that there are  $\aleph_1$ -AECs that don't have models in arbitrarily large cardinalities: looking at complete (non-discrete) metric spaces or [**BR12**, Example 4.8], there can be no models in cardinalities satisfying  $\lambda < \lambda^{\omega}$ . However, the internal size based on presentability rank mentioned above gives that, e. g., complete metric spaces have models of all sizes. Thus, an accessible category is called LS-accessible iff there is a threshold such that there are object of every size above that threshold. Beke and Rosicky [**BR12**] ask if every large accessible category is LS-accessible. This question is still open and a positive answer (even restricting to accessible categories where all maps are mono) would aid the analysis of  $\mu$ -AECs (see the discussion at the start of Section 13.6).

Still, under the additional assumption of upper bounds for increasing chains of structures—directed bounds, in the language of [**Ros97**], or the  $\delta$ -chain extension property, defined below—we can rule out Example 13.4.12, and begin to develop a genuine classification theory.

#### 13.5. Tameness and large cardinals

In [Bon14b], it was shown by Boney that, assuming the existence of large cardinals, every AEC satisfies the important locality property know as tameness. Tameness was isolated (from an argument of Shelah [She99]) by Grossberg and VanDieren in [GV06b], and was used to prove an upward categoricity transfer from a successor cardinal in [GV06c, GV06a]. Tame AECs have since been a very productive area of study. For example, they admit a well-behaved notion of independence (see Chapters 4 and 6) and many definitions of superstability can be shown to be equivalent in the tame context (Chapter 9).

In this section, we generalize Boney's theorem to  $\mu$ -AECs (in a sense, this also partially generalizes the recent [**BZ**] which proved an analogous result for metric AECs, but for a stronger, metric specialization of tameness). We start by recalling the definition of tameness (and its generalization: full tameness and shortness) to this context. This generalization already appears in Definition 2.2.23.

DEFINITION 13.5.1 (Definitions 3.1 and 3.3 in [Bon14b]). Let **K** be an abstract class and let  $\kappa$  be an infinite cardinal.

(1) **K** is  $(< \kappa)$ -tame if for any distinct  $p, q \in gS(M)$ , there exists  $A \subseteq card(M)$  such that  $|A| < \kappa$  and  $p \upharpoonright A \neq q \upharpoonright A$ .

 $<sup>^{4}</sup>$ We use here Galois types over sets as defined in Definition 2.2.17.

- (2) **K** is fully  $(< \kappa)$ -tame and short if for any distinct  $p, q \in gS^{\alpha}(M)$ , there exists  $I \subseteq \alpha$  and  $A \subseteq |M|$  such that card  $(I) + \text{card}(A) < \kappa$  and  $p^{I} \upharpoonright A \neq q^{I} \upharpoonright A$ .
- (3) We say **K** is *tame* if it is  $(< \kappa)$ -tame for some  $\kappa$ , similarly for fully tame and short.

Instead of strongly compact cardinals, we will (as in [**BU**] and [**BTR17**]) use *almost* strongly compact cardinals:

DEFINITION 13.5.2. An uncountable limit cardinal  $\kappa$  is almost strongly compact if for every  $\mu < \kappa$ , every  $\kappa$ -complete filter extends to a  $\mu$ -complete ultrafilter.

Note that the outline here follows the original model theoretic arguments of [Bon14b]. The category theoretic arguments of [LR16] and [BTR17] can also be used.

A minor variation of the proof of Loś's theorem for  $L_{\kappa,\kappa}$  (see [Dic75, Theorem 3.3.1]) gives:

FACT 13.5.3. Let  $\kappa$  be an almost strongly compact cardinal. Let  $\mu < \kappa$ , let  $(M_i)_{i \in I}$  be *L*-structures, and let *U* be a  $\mu^+$ -complete ultrafilter on *I*. Then for any formula  $\phi \in L_{\mu,\mu}$ ,  $\prod M_i \setminus U \models \phi[[f]_U]$  if and only if  $M_i \models \phi[f(i)]$  for *U*-almost all  $i \in I$ .

Using the presentation theorem, we obtain Los's theorem for  $\mu$ -AECs:

LEMMA 13.5.4. Let **K** be a  $\mu$ -AEC. Let  $(M_i)_{i \in I}$  be models in **K** and let U be a  $(2^{\text{LS}(\mathbf{K})})^+$ -complete ultrafilter on I. Then  $\prod M_i \setminus U \in \mathbf{K}$ .

PROOF SKETCH. Let  $\mu := (2^{\text{LS}(\mathbf{K})})^+$ . By the presentation theorem (Theorem 13.3.2), there exists a language  $L' \supseteq L(\mathbf{K})$  and a sentence  $\phi \in L'_{\mu,\mu}$  such that  $\mathbf{K} = \text{Mod}(\phi) \upharpoonright L = \mathbf{K}$ . Now use Fact 13.5.3 together with the proof of [Bon14b, Theorem 4.3].

All the moreover clauses of [**Bon14b**, Theorem 4.3] are also obtained, thus by the same proof as [**Bon14b**, Theorem 4.5], we get:

THEOREM 13.5.5. Let **K** be a  $\mu$ -AEC and let  $\kappa > \text{LS}(\mathbf{K})$  be almost strongly compact. Then **K** is fully ( $< \kappa$ )-tame and short.

In particular, if there is a proper class of almost strongly compact cardinals, every  $\mu$ -AEC is fully tame and short. Using the recent converse for the special case of AECs due to Boney and Unger [**BU**], we obtain also a converse in  $\mu$ -AECs:

THEOREM 13.5.6. The following are equivalent:

- (1) For every  $\mu$ , every  $\mu$ -AEC is fully tame and short.
- (2) Every AEC is tame.
- (3) There exists a proper class of almost strongly compact cardinals.

PROOF. (1) implies (2) is because AECs are  $\aleph_0$ -AECs. (2) implies (3) is [**BU**] and (3) implies (1) is Theorem 13.5.5.

#### 13.6. On categorical $\mu$ -AECs

Here we show that some non-trivial theorems of classification theory for AECs transfer to  $\mu$ -AECs and, by extension, accessible categories with monomorphisms. Most of the classification theory for AECs has been driven by Shelah's categoricity conjecture<sup>5</sup>. For an abstract class **K**, we write  $I(\lambda, \mathbf{K})$  for the number of pairwise non-isomorphic models of **K** of cardinality  $\lambda$ . An abstract class **K** is said to be *categorical in*  $\lambda$  if  $I(\lambda, \mathbf{K}) = 1$ . Inspired by Morley's categoricity theorem, Shelah conjectured:

CONJECTURE 13.6.1. If an AEC is categorical in a high-enough cardinal, then it is categorical on a tail of cardinals.

Naturally, one can ask the same question for both  $\mu$ -AECs and accessible categories, where, following [**Ros97**], we say an accessible category is *categorical in*  $\lambda$ if it contains exactly one object of internal size  $\lambda$  (up to isomorphism). By shifting the question to these more general frameworks, of course, we make it more difficult to arrive at a positive answer. If the answer is negative, on the other hand, counterexamples should be more readily available in our contexts: if indeed the answer is negative, this would give us a bound on the level of generality at which the categoricity conjecture can hold.

QUESTION 13.6.2. If a large accessible category (whose morphisms are monomorphisms) is categorical in a high-enough cardinal, is it categorical on a tail of cardinals?

A negative answer to the question of Beke and Rosicky from Section 13.4—an example of an large accessible category  $\mathcal{K}$  with arbitrarily large gaps in internal sizes—would yield a negative answer to Question 13.6.2: as noted in [**BR12**] 6.3, it suffices to take the coproduct  $\mathcal{K} \coprod \mathbf{Set}$ . This adds exactly one isomorphism class to each size, resulting in a category that is (internally) categorical in arbitrarily high cardinals—the gaps of  $\mathcal{K}$ —but also fails to be (internally) categorical in arbitrarily large accessible categories whose morphisms are monomorphisms. [**BR12**] and [**LR16**] contain sufficient conditions for LS-accessibility: in particular, it is enough to add the assumption of the existence of arbitrary directed colimits (see [**LR16**, 2.7]).

For  $\mu$ -AECs, the natural formulation is in terms not of the internal size, but of the cardinality of underlying sets. Some adjustments have to be made, as a  $\mu$ -AEC need not have a model of cardinality  $\lambda$  when  $\lambda^{<\mu} > \lambda$ , and thus eventual categoricity would fail more or less trivially.

QUESTION 13.6.3. If a  $\mu$ -AEC is categorical in a high-enough cardinal  $\lambda$  with  $\lambda = \lambda^{<\mu}$ , is it categorical in all sufficiently high  $\lambda'$  such that  $\lambda' = (\lambda')^{<\mu}$ .

For  $\mu = \omega$ , this question reduces to 13.6.1.

REMARK 13.6.4. We will show that a positive answer to Question 13.6.2, the internal version, implies, at the very least, a positive answer to Question 13.6.1. Let  $\mathcal{K}$  be an AEC in a language L. Then  $\mathcal{K}$  is an accessible category and, following **[BR12]** 4.1, 4.3 and 3.6, there is a regular cardinal  $\kappa$  such that  $\mathcal{K}$  is  $\kappa$ -accessible and E preserves sizes  $\lambda \geq \kappa$ . We can assume that, in **Emb**(L), they coincide

<sup>&</sup>lt;sup>5</sup>For more references and history, see the introduction of Shelah's book [She09a]

with cardinalities of underlying sets. Thus, any  $K_1, K_2$  with sufficiently large and distinct  $\operatorname{card}(EK_1), \operatorname{card}(EK_2)$  have distinct sizes  $\operatorname{card}(K_1), \operatorname{card}(K_2)$  and thus  $K_1$  and  $K_2$  are not isomorphic.

At present we do not know whether a positive answer to 13.6.2 implies a positive answer to 13.6.3.

Of course 13.6.2 is currently out of reach, as is 13.6.3. We are not sure about the truth value of either one: it is plausible that there are counterexamples. A possible starting point for 13.6.3 would be to use Theorem 13.5.5 to try to generalize [**Bon14b**] to  $\mu$ -AECs categorical in an appropriate successor above a strongly compact (see also [**Shed**], which proves some model-theoretic results for classes of models of  $L_{\kappa,\kappa}$  with  $\kappa$  a strongly compact cardinal).

We show here that some facts which follow from categoricity in AECs also follow from categoricity in  $\mu$ -AECs. As in [**Ros97**], which considers categoricity in accessible categories with directed bounds (and, ultimately, directed colimits), we have to add the following hypothesis:

DEFINITION 13.6.5. Let  $\delta$  be an ordinal. An abstract class **K** has the  $\delta$ -chain extension property if for every chain  $\langle M_i : i < \delta \rangle$ , there exists  $M_{\delta} \in \mathbf{K}$  such that  $M_i \leq_{\mathbf{K}} M_{\delta}$  for all  $i < \delta$ . We say that **K** has the chain extension property if it has the  $\delta$ -chain extension property for every limit ordinal  $\delta$ .

REMARK 13.6.6. If **K** is a  $\mu$ -AEC, then **K** has the chain extension property if and only if **K** has the  $\delta$ -chain extension property for every limit  $\delta < \mu$ .

REMARK 13.6.7.  $\mu$ -CAECs have the chain extension property (recall the item (6) from the list of examples). Moreover, any  $\mu$ -AEC naturally derived from<sup>6</sup> an AEC (such as the class of  $\mu$ -saturated models of an AEC) will have the chain extension property.

We adapt Shelah's [She09a, Theorem IV.1.12.(1)] to  $\mu$ -AECs:

THEOREM 13.6.8. Let **K** be a  $\mu$ -AEC. Let  $\lambda \geq \text{LS}(\mathbf{K})$  be such that  $\lambda = \lambda^{<\mu}$ and  $\mathbf{K}_{\lambda}$  has the  $\delta$ -chain extension property for all limit  $\delta < \lambda^+$ . Assume **K** is categorical in  $\lambda$ . Let  $M, N \in \mathbf{K}_{\geq \lambda}$ . If  $M \leq_{\mathbf{K}} N$ , then  $M \preceq_{L_{\infty,\mu}} N$ .

Notice that the cardinal arithmetic  $(\lambda^{<\mu} = \lambda)$  is a crucial simplifying assumption in the AEC version that Shelah later worked to remove (see [She09a, Section IV.2]). It appears naturally here in the context of a  $\mu$ -AEC, but note that the chain extension might guarantee the existence of models of intermediate sizes (i.e. in  $\chi < \chi^{<\mu}$ ).

PROOF OF THEOREM 13.6.8. We first assume that  $M, N \in \mathbf{K}_{\lambda}$  and  $M \leq_{\mathbf{K}} N$ . Let  $\phi(\bar{y})$  be an  $L_{\infty,\mu}$ -formula with  $\ell(\bar{y}) = \alpha < \mu$  and let  $\bar{a} \in {}^{\alpha}|M|$ . We show that  $M \models \phi[\bar{a}]$  if and only if  $N \models \phi[\bar{a}]$  by induction on the complexity of  $\phi$ . If  $\phi$  is atomic, this holds because  $M \subseteq N$ . If  $\phi$  is a boolean combination of formulas of lower complexity, this is easy to check too. So assume that  $\phi(\bar{y}) = \exists \bar{x} \psi(\bar{x}, \bar{y})$ . If  $M \models \phi[\bar{a}]$ , then using induction we directly get that  $N \models \phi[\bar{a}]$ . Now assume  $N \models \phi[\bar{a}]$ , and let  $\bar{b} \in {}^{<\mu}|N|$  be such that  $N \models \psi[\bar{b}, \bar{a}]$ .

We build an increasing chain  $\langle M_i : i < \lambda^+ \rangle$  and  $\langle f_i, g_i : i < \lambda^+ \rangle$  such that for all  $i < \lambda^+$ :

 $<sup>^{6}</sup>$ This can be made precise using the notion of a *skeleton*, see Definition 6.5.3.

- (1)  $M_i \in \mathbf{K}_{\lambda}$
- (2) If cf  $i \ge \mu$ , then  $M_i = \bigcup_{j \le i} M_j$ .
- (3)  $f_i: M \cong M_i$ .
- (4)  $g_i: N \cong M_{i+1}$ .
- (5)  $f_i \subseteq g_i$ .

This is possible. If i = 0, let  $M_0 := M$ . For any *i*, given  $M_i$ , use categoricity to pick  $f_i: M \cong M_i$  and extend it to  $g_i: N \cong M_{i+1}$ . If i is limit and cf  $i \ge \mu$ , take unions. If cf  $i < \mu$ , use the chain extension property to find  $M_i \in \mathbf{K}_{\lambda}$  such that  $M_i \leq_{\mathbf{K}} M'_i$  for all j < i.

This is enough. For each  $i < \lambda^+$ , let  $\alpha(i)$  be the least  $\alpha < \lambda^+$  such that  $\operatorname{ran}(f_i(\bar{a})) \subseteq |M_{\alpha}|$ . Let  $S := \{i < \lambda^+ \mid \mathrm{cf} \ i \geq \mu\}$ . Note that S is a stationary subset of  $\lambda^+$  and the map  $i \mapsto \alpha(i)$  is regressive on S. By Fodor's lemma, there exists a stationary  $S_0 \subseteq S$  and  $\alpha_0 < \lambda^+$  such that for any  $i \in S_0$ ,  $\alpha(i) = \alpha_0$ , i.e.  $\operatorname{ran}(f(\bar{a}_i)) \subseteq |M_{\alpha_0}|$ . Now  $\operatorname{card}({}^{<\mu}|M_{\alpha_0}|) = \lambda^{<\mu} = \lambda$  and  $|S_0| = \lambda^+$  so by the pigeonhole principle there exists i < j in  $S_0$  such that  $f_i(\bar{a}) = f_i(\bar{a})$ . Now, since  $N \models \psi[b, \bar{a}]$ , we must have  $M_{i+1} \models \psi[g_i(b), g_i(\bar{a})]$ . By the induction hypothesis,  $M_j \models \psi[g_i(\bar{b}), g_i(\bar{a})]$ . Thus  $M_j \models \phi[g_i(\bar{a})]$ . Since  $f_i \subseteq g_i, g_i(\bar{a}) = f_i(\bar{a})$  so  $M_j \models \phi[f_i(\bar{a})]$ . Since  $f_i(\bar{a}) = f_j(\bar{a})$ , we have that  $M_j \models \phi[f_j(\bar{a})]$ . Applying  $f_j^{-1}$  to this equation, we obtain  $M \models \phi[\bar{a}]$ , as desired.

This proves the result in case  $M, N \in \mathbf{K}_{\lambda}$ . If  $M, N \in \mathbf{K}_{>\lambda}$  and  $M \leq_{\mathbf{K}} N$ , then, as before, we can find a  $\mu$ -directed system  $\langle N_{\bar{a}} \in \mathbf{K}_{\lambda} : \bar{a} \in {}^{<\mu}N \rangle$  with colimit N such that  $\bar{a} \in |N_{\bar{a}}|$  and, if  $\bar{a} \in {}^{<\mu}M$ , then  $N_{\bar{a}} \leq_{\mathbf{K}} M$ .

As before we prove by induction on  $\phi \in L_{\infty,\mu}$  that  $M \models \phi[\bar{a}]$  if and only if  $N \models \phi[\bar{a}]$ . The interesting case is when  $\phi = \exists \bar{x} \psi(\bar{x}, \bar{y})$  and the left to right direction is straightforward, so assume  $N \models \phi[\bar{a}]$ , i. e., there exists  $\bar{b} \in {}^{<\mu}|N|$  such that  $N \models \psi[\bar{b}, \bar{a}]$ . By the previous part,  $N_{\bar{a}} \preceq_{L_{\infty,\mu}} N_{\bar{a}\bar{b}}$ . So there is  $\bar{b}' \in N_{\bar{a}}$  such that  $N_{\bar{a}} \vDash \psi[\bar{b}', \bar{a}]$ . Since  $N_{\bar{a}} \leq_{\mathbf{K}} M$ , by induction, we have  $M \vDash \phi[\bar{a}]$ . 

Another result that can be adapted is Shelah's famous combinatorial argument that amalgamation follows from categoricity in two successive cardinals [She87a, Theorem 3.5]. We start with some simple definitions and lemmas:

DEFINITION 13.6.9. Let  $\mu \leq \lambda$  be regular cardinals.  $C \subseteq \lambda$  is a  $\mu$ -club if it is unbounded and whenever  $\langle \alpha_i : i < \delta \rangle$  is increasing in C with  $\mu \leq \operatorname{cf} \delta < \lambda$ , then  $\sup_{i<\delta}\alpha_i\in C.$ 

REMARK 13.6.10. So  $\aleph_0$ -club is the usual notion of club.

LEMMA 13.6.11. Let  $\mu$  be a regular cardinal. Assume **K** is a  $\mu$ -AEC and  $\lambda \geq \mathrm{LS}(\mathbf{K})$ . Let  $\langle M_i^{\ell} : i < \lambda^+ \rangle$ ,  $\ell = 1, 2$ , be increasing in  $\mathbf{K}_{\lambda}$  such that for all  $\begin{array}{l} i < \lambda^+ \text{ with cf } i \geq \mu, \, M_i^\ell = \bigcup_{j < i} M_j^\ell.\\ \text{ If } f : \bigcup_{i < \lambda^+} M_i^1 \cong \bigcup_{i < \lambda^+} M_i^2, \, \text{then the set } \{i < \lambda^+ \mid f \upharpoonright M_i^1 : M_i^1 \cong M_i^2\} \text{ is a} \end{array}$ 

 $\mu$ -club.

PROOF. Let  $C := \{i < \lambda^+ \mid f \upharpoonright M_i^1 : M_i^1 \cong M_i^2\}$ . By cardinality considerations, for each  $i < \lambda^+$ , there is  $j_i < \lambda^+$  such that  $|f(M_i^1)| \subseteq |M_{j_i}^2|$  (by coherence this implies  $f(M_i^1) \leq_{\mathbf{K}} M_{i_i}^2$ . Let  $\delta$  have cofinality  $\mu$  such that for all  $i < \delta$ ,  $j_i < \delta$ . Then by continuity  $f(M^1_{\delta}) \leq_{\mathbf{K}} M^2_{\delta}$ . Let  $C_0$  be the set of all such  $\delta$ . It is easy to check that  $C_0$  is a  $\mu$ -club. Similarly, let  $C_1$  be the set of all  $\delta$  such that  $f^{-1}(M_{\delta}^2) \leq_{\mathbf{K}} M_{\delta}^1$ .  $C_1$  is also a  $\mu$ -club and it is easy to check that  $C = C_0 \cap C_1$ , and the intersection of two  $\mu$ -clubs is a  $\mu$ -club, so the result follows. 

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THEOREM 13.6.12. Let  $\mu$  be a regular cardinal. Assume **K** is a  $\mu$ -AEC,  $\lambda = \lambda^{<\mu} \geq \text{LS}(\mathbf{K}), I(\lambda, \mathbf{K}) = 1 \leq I(\lambda^+, \mathbf{K}) < 2^{\lambda^+}$ . If:

(1)  $\mathbf{K}_{\lambda}$  has the extension property for  $\delta$ -chains (see above) for every  $\delta < \lambda^+$ .

(2)  $\lambda = \lambda^{\mu}$  and  $2^{\lambda} = \lambda^{+}$ .

Then  ${\bf K}$  has  $\lambda\text{-amalgamation}.$ 

PROOF. Assume not. By failure of amalgamation and some renaming, we have:

(\*) If  $M_1, M_2 \in \mathbf{K}_{\lambda}$  and  $f : M_1 \cong M_2$ , there are  $M'_l \in \mathbf{K}_{\lambda}$ ,  $\ell = 1, 2$ , with  $M_l \leq_{\mathbf{K}} M'_l$ , card  $(|M'_l| - |M_l|) = \lambda$ , such that there is no  $N \in \mathbf{K}_{\lambda}$  and  $g_l : M'_l \to N$  commuting with f.

In particular (taking  $M_1 = M_2$  and f the identity function), the model of size  $\lambda$  is not maximal. By Gregory's theorem (see [**Jec03**, Theorem 23.2]), the combinatorial principle  $\Diamond_{E_{\mu}}$  holds, where  $E_{\mu} := \{i < \lambda^+ \mid \text{cf } i \geq \mu\}$ . With some coding, one can see that  $\Diamond_{E_{\mu}}$  is equivalent to:

(\*\*) There are  $\{\eta_{\alpha}, \nu_{\alpha} : \alpha \to | \alpha < \lambda^+\}, \{g_{\alpha} : \alpha \to \alpha \mid \alpha < \lambda^+\}$  such that for all  $\eta, \nu : \lambda^+ \to 2, g : \lambda^+ \to \lambda^+$ , the set  $\{\alpha \in E_{\mu} \mid \eta_{\alpha} = \eta \upharpoonright \alpha, \nu_{\alpha} = \nu \upharpoonright \alpha, g_{\alpha} = g \upharpoonright \alpha\}$  is stationary.

We build a strictly increasing tree  $\{M_{\eta} \mid \eta \in {}^{\leq \lambda^+}2\}$  such that:

- (1)  $|M_{\eta}| \subseteq \lambda^+$ ,  $\operatorname{card}(M_{\eta}) = \lambda$ ,  $\ell(\eta) \in |M_{\eta \frown \ell}|$  for all  $\eta \in {}^{<\lambda^+}2$  and  $\ell < 2$ .
- (2) If  $\eta \in {}^{\leq \lambda^+}2$  and cf  $\ell(\eta) \geq \mu$ , then  $M_\eta = \bigcup_{j < \ell(\eta)} M_{\eta \upharpoonright j}$ .
- (3) If  $|M_{\eta_{\delta}}| = \delta$ ,  $\eta_{\delta} \neq \nu_{\delta}$ , and  $g_{\delta} : M_{\eta_{\delta}} \cong M_{\nu_{\delta}}$  is an isomorphism, for any  $\ell, \ell' < 2$  and any  $\nu \supseteq \nu_{\delta} \frown \ell', g_{\delta}$  cannot be extended to an embedding of  $M_{\eta_{\delta} \frown \ell}$  into  $M_{\nu}$ .

<u>This is enough</u>. We claim that for any  $\eta \neq \nu \in {}^{\lambda^+}2$ ,  $M_\eta \not\cong M_\nu$ . Indeed, assume  $f: M_\eta \to M_\nu$  is an isomorphism. For  $i < \lambda^+$ , let  $f_i := f \upharpoonright M_{\eta \upharpoonright i}$  and let  $C := \{i < \lambda^+ \mid f_i : M_{\eta \upharpoonright i} \cong M_{\nu \upharpoonright i}\}$ . By Lemma 13.6.11, C is a  $\mu$ -club. Also  $\{i < \lambda^+ \mid |M_{\eta \upharpoonright i}| = i\}$  is a club so without loss of generality is contained in C. Now the stationary set described by (\*\*) intersects C in unboundedly many places (as it only has points of cofinality  $\mu$ ), hence there is  $\delta < \lambda^+$  such that  $\eta \upharpoonright \delta \neq \nu \upharpoonright \delta$ ,  $\eta_{\delta} = \eta \upharpoonright \delta, \nu_{\delta} = \nu \upharpoonright \delta, g_{\delta} = f \upharpoonright \delta, \delta = |M_{\eta_{\delta}}| = |M_{\nu_{\delta}}|$ , and  $g_{\delta} : M_{\eta_{\delta}} \cong M_{\nu_{\delta}}$ . But fextends  $g_{\delta}$  and restricts to an embedding of  $M_{\eta \sim \eta(\delta)}$  into  $M_{\nu \upharpoonright \gamma}$ , for some  $\gamma < \lambda^+$ with  $\gamma > \delta$  sufficiently large. This contradicts (3).

<u>This is possible</u>. Take any  $M_{<>} \in \mathbf{K}$  with  $|M_{\langle\rangle}| = \lambda$  for the base case, take unions at limits of cofinality at least  $\mu$ , and use the extension property for chains (and some renaming) at limits of cofinality less than  $\mu$ .

Now if one wants to define  $M_{\eta \sim l}$  for  $\eta \in {}^{\delta}2$  (assuming by induction that  $M_{\nu}$  for all  $\nu \in {}^{\leq \delta}2$  have been defined) take any two strict extensions, unless  $|M_{\eta}| = \delta$ ,  $\eta_{\delta} \neq \nu_{\delta}, g_{\delta} : M_{\eta_{\delta}} \cong M_{\nu_{\delta}}$  is an isomorphism, and either  $\eta = \eta_{\delta}$ , or  $\eta = \nu_{\delta}$ . We show what to do when  $\eta = \eta_{\delta}$ . The other case is symmetric. Let  $M'_{\eta_{\delta}}, M'_{\nu_{\delta}}$  be as described by (\*) and let  $M_{\eta_{\delta} \sim l}, M_{\nu_{\delta} \sim l}$  be their appropriate renaming to satisfy (1). Now (\*) tells us that (3) is satisfied.

REMARK 13.6.13. Of course, the set-theoretic hypotheses of Theorem 13.6.12 can be weakened. For example, it is enough to require  $\lambda = \lambda^{<\mu}$  and  $\Diamond_{S_{\mu}}$  or even (by Shelah's more complicated proof) a suitable instance of the weak diamond. It is not clear, however, that it follows from just  $2^{\lambda} < 2^{\lambda^+}$ .

# CHAPTER 14

# Downward categoricity from a successor inside a good frame: part I: the main theorem

This chapter and the next one are based on [Vas17a]. I thank John Baldwin and Monica VanDieren for helpful feedback on an earlier draft of this paper. I also thank Will Boney for a conversation on Shelah's omitting type theorem (see Section 15.3). Finally, I thank the referee for comments that helped improve the presentation of this paper.

# Abstract

In the setting of abstract elementary classes (AECs) with amalgamation, Shelah has proven a downward categoricity transfer from categoricity in a successor and Grossberg and VanDieren have established an upward transfer assuming in addition a locality property for Galois types that they called tameness.

We further investigate categoricity transfers in tame AECs. We use orthogonality calculus to prove a downward transfer from categoricity in a successor in AECs that have a good frame (a forking-like notion for types of singletons) on an interval of cardinals:

THEOREM 14.0.14. Let **K** be an AEC and let  $LS(\mathbf{K}) \leq \lambda < \theta$  be cardinals. If **K** has a type-full good  $[\lambda, \theta]$ -frame and **K** is categorical in both  $\lambda$  and  $\theta^+$ , then **K** is categorical in all  $\mu \in [\lambda, \theta]$ .

We deduce improvements on the threshold of several categoricity transfers that do not mention frames. For example, the threshold in Shelah's transfer can be improved from  $\exists_{(2^{LS}(\mathbf{K}))^+}$  to  $\exists_{(2^{LS}(\mathbf{K}))^+}$  assuming that the AEC is  $LS(\mathbf{K})$ -tame. The successor hypothesis can also be removed from Shelah's result by assuming in addition either that the AEC has primes over sets of the form  $M \cup \{a\}$  or (using an unpublished claim of Shelah) that the weak generalized continuum hypothesis holds.

# 14.1. Introduction

14.1.1. Motivation and history. In his two volume book [She09a, She09b] on classification theory for abstract elementary classes (AECs), Shelah introduces the notion of a *good*  $\lambda$ -*frame* [She09a, II.2.1]. Roughly, a good  $\lambda$ -frame is a local notion of independence for types of length one over models of size  $\lambda$ . The independence notion satisfies basic properties of forking in a superstable first-order theory. Good frames are the central concept of the book. In Chapter II and III, Shelah discusses the following three questions regarding frames:

QUESTION 14.1.1.

- (1) Given an AEC **K**, when does there exist a good  $\lambda$ -frame  $\mathfrak{s}$  whose underlying AEC  $\mathbf{K}_{\mathfrak{s}}$  is  $\mathbf{K}_{\lambda}$  (or some subclass of saturated models in  $\mathbf{K}_{\lambda}$ )?
- (2) Given a good  $\lambda$ -frame, under what conditions can it be extended to a good  $\lambda^+$ -frame?
- (3) Once one has a good frame, how can one prove categoricity transfers?

Shelah's answers (see for example II.3.7, III.1, and III.2 in [She09a]) involve a mix of set-theoretic hypotheses (such as the weak generalized continuum hypothesis:  $2^{\theta} < 2^{\theta^+}$  for all cardinals  $\theta$ ) and strong local model-theoretic hypotheses (such as few models in  $\lambda^{++}$ ). While Shelah's approach is very powerful (for example in [She09a, Chapter IV], Shelah proves the eventual categoricity conjecture in AECs with amalgamation assuming some set-theoretic hypotheses, see more below), most of his results do not hold in ZFC.

An alternate approach is to make global model-theoretic assumptions. In [**GV06b**], Grossberg and VanDieren introduced tameness, a locality property which says that Galois types are determined by their small restrictions. In [**Bon14a**], Boney showed that in an AEC which is  $\lambda$ -tame for types of length two and has amalgamation, a good  $\lambda$ -frame can be extended to all models of size at least  $\lambda$  (we call the resulting object a good ( $\geq \lambda$ )-frame, and similarly define good [ $\lambda$ ,  $\theta$ ]-frame for  $\theta > \lambda$  a cardinal). In Chapter 5, tameness for types of length two was improved to tameness for types of length one. In particular, the answer to Question 14.1.1.(2) is always positive in tame AECs with amalgamation. As for existence (Question 14.1.1.(1)), we showed in Chapter 4 how to build good frames in tame AECs with amalgamation assuming categoricity in a cardinal of high-enough cofinality. Further improvements were made in Chapters 6, 7, and 10. This gives answers to Questions 14.1.1.(1),(2) in tame AECs with amalgamation:

FACT 14.1.2. Let **K** be an AEC with amalgamation and let  $\lambda \geq LS(\mathbf{K})$  be such that **K** is  $\lambda$ -tame.

- (1) (Corollary 5.6.9) If there is a good  $\lambda$ -frame  $\mathfrak{s}$  with  $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\lambda}$ , then  $\mathfrak{s}$  can be extended to a good ( $\geq \lambda$ )-frame (with underlying class  $\mathbf{K}$ ).
- (2) (Corollary 10.6.14) If **K** has no maximal models and is categorical in some  $\mu > \lambda$ , then there is a type-full good  $\lambda^+$ -frame with underlying class the Galois saturated models of **K** of size  $\lambda^+$ .

14.1.2. Categoricity in good frames. In this chapter, we study Question 14.1.1.(3) in the global setting: assuming the existence of a good frame together with some global model-theoretic properties, what can we say about the categoricity spectrum? From the two results above, it is natural to assume that we are already working inside a type-full good ( $\geq \lambda$ )-frame (this implies properties such as  $\lambda$ -tameness and amalgamation). It is then known how to transfer categoricity with the additional assumption that the class has primes over sets of the form  $M \cup \{a\}$ . This has been used to prove Shelah's eventual categoricity conjecture for universal classes, see Chapters 8, 16.

DEFINITION 14.1.3 (III.3.2 in [She09a]). An AEC K has primes if for any nonalgebraic Galois type  $p \in gS(M)$  there exists a triple (a, M, N) such that p = gtp(a/M; N) and for every  $N' \in \mathbf{K}$ ,  $a' \in |N'|$ , such that p = gtp(a'/M; N'), there exists  $f: N \xrightarrow{M} N'$  with f(a) = a'.

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FACT 14.1.4 (Theorem 11.2.8). Assume that there is a type-full good  $[\lambda, \theta)$ -frame on the AEC **K**. Assume that **K** has primes and is categorical in  $\lambda$ . If **K** is categorical in *some*  $\mu \in (\lambda, \theta]$ , then **K** is categorical in *all*  $\mu' \in (\lambda, \theta]$ .

What if we do not assume existence of primes? The main result of this chapter is a downward categoricity transfer for global good frames categorical in a successor:

THEOREM 14.6.14. Assume that there is a type-full good  $[\lambda, \theta]$ -frame on the AEC **K**. Assume that **K** is categorical in  $\lambda$ . If **K** is categorical in  $\theta^+$ , then **K** is categorical in all  $\mu \in [\lambda, \theta]$ .

The proof of Theorem 14.6.14 develops orthogonality calculus in this setup (versions of some of our results on orthogonality have been independently derived by Villaveces and Zambrano [VZ14]). We were heavily inspired from Shelah's development of orthogonality calculus in successful good  $\lambda$ -frames [She09a, Section III.6], and use it to define a notion of unidimensionality similar to what is defined in [She09a, Section III.2]. We show unidimensionality in  $\lambda$  is equivalent to categoricity in  $\lambda^+$  and use orthogonality calculus to transfer unidimensionality across cardinals. While we work in a more global setup than Shelah's, we do *not* assume that the good frames we work with are successful [She09a, Definition III.1.1], so we do *not* assume that the forking relation is defined for types of models (it is only defined for types of elements). To get around this difficulty, we use the theory of *independent sequences* introduced by Shelah for good  $\lambda$ -frames in [She09a, Section III.5] and developed in Chapter 5 for global good frames.

14.1.3. Hypotheses of the main theorem. Let us discuss the hypotheses of Theorem 14.6.14. We are assuming that the good frame is *type-full*: the basic types are all the nonalgebraic types. This is a natural assumption to make if we are only interested in tame AECs: by Fact 14.1.2, type-full good frames exist under natural conditions there. Moreover by Remark 9.5.6, if a tame AECs has a good frame, then it has a type-full one (possibly with a different class of models). We do not know if the type-full assumption is necessary; our argument uses it when dealing with minimal types (we do not know in general whether minimal types are basic; if this is the case for the frame we are working with then it is not necessary to assume that it is type-full).

What about categoricity in  $\lambda$ ? This is assumed in order to have some starting degree of saturation (namely all the models of size  $\lambda$  are limit models, see Definition 14.2.2). We do not see it as a strong assumption: in applications, we will take the AEC to be a class of  $\lambda$ -saturated models, where this automatically holds. Still, we do not know if it is necessary.

Another natural question is whether one really needs to assume the existence of a global good frame at all. The Hart-Shelah example [**HS90**, **BK09**] shows that it is not true that any AEC **K** categorical in  $LS(\mathbf{K})$  and in a successor  $\lambda > LS(\mathbf{K})$  is categorical everywhere (even if **K** has amalgamation). One strengthening of Theorem 14.6.14 assumes only that we have a good frame for saturated models. Section 15.7 states this precisely and outlines a proof. Consequently, most of the result stated above hold assuming only weak tameness instead of tameness: that is, only types over saturated models are required to be determined by their small restrictions. In Section 15.8 we mention in which results of the chapter tameness can be replaced by weak tameness.

14.1.4. Application: lowering the bounds in Shelah's transfer. In the second part of this chapter, we give several applications of Theorem 14.6.14 to Shelah's eventual categoricity conjecture, the central test problem in the classification theory of non-elementary classes (see the introduction of Chapter 8 for a history):

CONJECTURE 14.1.5 (Conjecture N.4.2 in [She09a]). An AEC that is categorical in a high-enough cardinal is categorical on a tail of cardinals.

For an AEC **K** we will call Shelah's categoricity conjecture for **K** the statement that if **K** is categorical in some  $\lambda \geq \beth_{(2^{\text{LS}(\mathbf{K})})^+}$ , then **K** is categorical in all  $\lambda' \geq \beth_{(2^{\text{LS}(\mathbf{K})})^+}$  (that is, we explicitly require the "high-enough" threshold to be the first Hanf number).

Shelah [She99] has proven a *downward* categoricity transfer from a successor in AECs with amalgamation where the threshold is the second Hanf number. Complementing it, Grossberg and VanDieren have established an *upward* transfer assuming tameness:

FACT 14.1.6 ([**GV06c**, **GV06a**]). Let **K** be a LS(**K**)-tame AEC with amalgamation and arbitrarily large models. Let  $\lambda > \text{LS}(\mathbf{K})^+$  be a successor cardinal. If **K** is categorical in  $\lambda$ , then **K** is categorical in all  $\lambda' \geq \lambda$ .

Grossberg and VanDieren concluded that Shelah's *eventual* categoricity conjecture from a successor holds in tame AECs with amalgamation. Baldwin [**Bal09**, Problem D.1.(5)] has asked whether the threshold in Shelah's downward transfer can be lowered to the first Hanf number. The answer is not known, but we show here that tameness is the only obstacle: assuming  $LS(\mathbf{K})$ -tameness, the threshold becomes the first Hanf number, and so using Fact 14.1.6, we obtain Shelah's categoricity conjecture from a successor in tame AECs with amalgamation:

COROLLARY 15.4.6. Let **K** be a LS(**K**)-tame AEC with amalgamation and arbitrarily large models. If **K** is categorical in *some successor*  $\lambda > \text{LS}(\mathbf{K})^+$ , then **K** is categorical in *all*  $\lambda' \geq \min(\lambda, \beth_{(2^{\text{LS}(\mathbf{K})})^+})$ .

This can be seen as a generalization (see [**Bon14b**]) of the corresponding result of Makkai and Shelah [**MS90**] for classes of models of an  $\mathbb{L}_{\kappa,\omega}$ -theory,  $\kappa$  a strongly compact cardinal. It is a central open question whether tameness follows from categoricity in AECs with amalgamation (see [**GV06a**, Conjecture 1.5]).

We can use Theorem 14.6.14 to give alternate proofs of Shelah's downward transfer [She99] (see Corollary 15.8.6) and for the Grossberg-VanDieren upward transfer (see Corollary 15.10.6). We also prove a local categoricity transfer that does not mention frames:

COROLLARY 15.4.3. Let **K** be a LS(**K**)-tame AEC with amalgamation and arbitrarily large models. Let<sup>1</sup> LS(**K**)  $< \lambda_0 < \lambda$ . If  $\lambda$  is a successor cardinal and **K** is categorical in both  $\lambda_0$  and  $\lambda$ , then **K** is categorical in all  $\lambda' \in [\lambda_0, \lambda]$ .

REMARK 14.1.7. We believe that the methods of [She99] are not sufficient to prove Corollary 15.4.3 (indeed, Shelah uses models of set theory to prove the transfer

<sup>&</sup>lt;sup>1</sup>On the case  $\lambda_0 = LS(\mathbf{K})$ , see Remark 15.4.4.

#### 14.1. INTRODUCTION

of "no Vaughtian pairs", and hence uses that the starting categoricity cardinal is above the Hanf number, see  $(*)_9$  in the proof of [**She99**, Theorem II.2.7], or [**Bal09**, Theorem 14.12]). However we noticed after posting a first draft of this chapter that they *are* enough to improve the threshold cardinal of [**She99**] from  $\Box_{(2^{LS}(\mathbf{K}))^+}$  to

 $\beth_{(2^{LS}(\mathbf{K}))^+}$ . We sketch the details in Section 15.3.

14.1.5. Application: categoricity in a limit, using primes. Beyond categoricity in a successor, we can appeal to Theorem 14.6.14 to give improvements on the threshold of our previous categoricity transfer in tame AECs with amalgamation and primes (Chapter 11): there the threshold was the second Hanf number (see Fact 15.4.8) and here we show that the first Hanf number suffices:

COROLLARY 15.4.9.. Let **K** be a LS(**K**)-tame AEC with amalgamation and arbitrarily large models. Assume that **K** has primes. If **K** is categorical in *some*  $\lambda > \text{LS}(\mathbf{K})$ , then **K** is categorical in *all*  $\lambda' \ge \min(\lambda, \beth_{(2^{\text{LS}(\mathbf{K})})^+})$ .

REMARK 14.1.8. Compared to Fact 14.1.6 and Corollary 15.4.6, the case  $\lambda = LS(\mathbf{K})^+$  is allowed. We can also allow  $\lambda = LS(\mathbf{K})^+$  if we assume the weak generalized continuum hypothesis instead of the existence of primes, see Corollary 15.5.9.

14.1.6. Categoricity in a limit, using WGCH. Finally, a natural question is how to deal with categoricity in a limit cardinal *without* assuming the existence of prime models. In [She09a, Theorem IV.7.12], Shelah claims assuming the weak generalized continuum hypothesis that if **K** is an AEC with amalgamation<sup>2</sup>, then<sup>3</sup> categoricity in some  $\lambda \geq \beth_{(2^{\aleph_{LS}(\mathbf{K})^+})^+}$  implies categoricity in all  $\lambda' \geq \beth_{(2^{\aleph_{LS}(\mathbf{K})^+})^+}$ . Shelah's argument relies on an unpublished claim (whose proof should appear in [Sheb]), as well as PCF theory and long constructions of linear orders from [She09a, Sections IV.5,IV.6]. We have not fully verified it. In Chapter 6, we gave a way to work around the use of PCF theory and the construction of linear orders (though still using Shelah's unpublished claim) by using the locality assumption of full tameness and shortness (a stronger assumption than tameness introduced by Will Boney in his Ph.D. thesis, see [Bon14b, Definition 3.3]).

In Section 15.5, we give an exposition of Shelah's proof that does not use PCF or the construction of linear orders. This uses a recent result of VanDieren and the author (Corollary 10.7.4), showing that a model at a high-enough categoricity cardinal must have some degree of saturation (regardless of the cofinality of the cardinal). We deduce (using the aforementioned unpublished claim of Shelah) that Shelah's eventual categoricity conjecture is consistent assuming the existence of a proper class of *measurable* cardinals (this was implicit in [She09a, Chapter IV] but we give the details). Furthermore we give an explicit upper bound on the categoricity threshold (see Theorem 15.5.14). This partially answers [She00, Question 6.14.(1)].

 $<sup>^{2}</sup>$ Shelah only assumes some instances of amalgamation and no maximal models at specific cardinals, see the discussion in Section 15.5.

<sup>&</sup>lt;sup>3</sup>Shelah gives a stronger, erroneous statement (it contradicts Morley's categoricity theorem) but this is what his proof gives.

Using Theorem 14.6.14, we also give an improvement on the categoricity threshold of  $\beth_{(2^{\aleph_{LS}(\mathbf{K})+})^+}$  if the AEC is tame:

COROLLARY 15.5.9. Assume the weak generalized continuum hypothesis and an unpublished claim of Shelah (Claim 15.5.2). Let **K** be a LS(**K**)-tame AEC with amalgamation and arbitrarily large models. If **K** is categorical in *some*  $\lambda > \text{LS}(\mathbf{K})$ , then **K** is categorical in *all*  $\lambda' \geq \min(\lambda, \beth_{(2^{\text{LS}(\mathbf{K})})^+})$ .

Moreover, we give two ZFC consequences of a lemma in Shelah's proof (which obtains weak tameness from categoricity in certain cardinals below the Hanf number): an improvement on the Hanf number for constructing good frames (Theorem 15.2.5) and a nontrivial restriction on the categoricity spectrum *below* the Hanf number of an AEC with amalgamation and no maximal models (Theorem 15.2.6).

For clarity, we emphasize once again that Corollary 15.5.9 is due to Shelah when the threshold is  $\beth_{(2^{\aleph_{\text{LS}(\mathbf{K})^+}})^+}$  (and then tameness is not needed). The main contribution of Section 15.5 is a clear outline of Shelah's proof that avoids several of his harder arguments.

In conclusion, the aim of the second part of this chapter is to clarify the status of Shelah's eventual categoricity conjecture by simplifying existing proofs and improving several thresholds. We give a table summarizing the known results on the conjecture in Section 15.6.

14.1.7. Notes and acknowledgments. After the initial circulation of this chapter (in October 2015), we showed (Chapter 17) that if **K** is an AEC with amalgamation and no maximal models that is categorical in  $\lambda > \text{LS}(\mathbf{K})$  then the model of categoricity  $\lambda$  is always saturated. Thus several of the threshold cardinals in Sections 15.2, 15.3, or 15.5 can be improved. In particular in the last two entry of the first column of Table 1 in Section 15.6, the cardinal  $\Box_{\Box_{H_1}}$  can be replaced by  $\Box_{H_1}$ .

The background required to read this chapter is a solid knowledge of AECs (at minimum Baldwin's book [Bal09]) together with some familiarity with good frames (e.g. the first four sections of [She09a, Chapter II]). As mentioned before, the chapter has two parts: The first gives a proof of the main Theorem (Theorem 14.6.14), and the second gives applications. If one is willing to take Theorem 14.6.14 as a black box, the second part can be read independently from the first part. The first part relies on Chapter 5 and the second relies on several other chapters (e.g. Chapters 4, 11, and 10), as well as on parts of [She09a, Chapter IV] (we only use results for which Shelah gives a full proof). We have tried to state all background facts as black boxes that can be used with little understanding of the underlying machinery.

We warn the reader: at the beginning of most sections, we state a *global* hypothesis which applies to any result stated in the section.

#### 14.2. Background

We assume that the reader is familiar with the definition of an AEC and notions such as amalgamation, joint embedding, Galois types, and Ehrenfeucht-Mostowski models (see for example [**Bal09**]). The notation we use is standard and is described in details at the beginning of Chapter 2. For example, we write  $gtp(\bar{b}/M; N)$  for
the Galois type of  $\bar{b}$  over M, as computed in N. Everywhere in this chapter and unless mentioned otherwise, we are working inside a fixed AEC **K**.

14.2.1. Good frames. In [She09a, Definition II.2.1]<sup>4</sup>, Shelah introduces good frames, a local notion of independence for AECs. This is the central concept of his book and has seen several other applications, such as a proof of Shelah's eventual categoricity conjecture for universal classes (Chapters 8 and 16). A good  $\lambda$ -frame is a triple  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  where:

- (1) **K** is a nonempty AEC which has amalgamation in  $\lambda$ , joint embedding in  $\lambda$ , no maximal models in  $\lambda$ , and is stable in  $\lambda$ .
- (2) For each  $M \in \mathbf{K}_{\lambda}$ ,  $\mathrm{gS}^{\mathrm{bs}}(M)$  (called the set of *basic types* over M) is a set of nonalgebraic Galois types over M satisfying (among others) the *density property*: if  $M <_{\mathbf{K}} N$  are in  $\mathbf{K}_{\lambda}$ , there exists  $a \in |N| \setminus |M|$  such that  $\mathrm{gtp}(a/M; N) \in \mathrm{gS}^{\mathrm{bs}}(M)$ .
- (3)  $\downarrow$  is an (abstract) independence relation on types of length one over models in  $\mathbf{K}_{\lambda}$  satisfying the basic properties of first-order forking in a superstable theory: invariance, monotonicity, extension, uniqueness, transitivity, local character, and symmetry (see [She09a, Definition II.2.1]).

As in [She09a, Definition II.6.35], we say that a good  $\lambda$ -frame  $\mathfrak{s}$  is type-full if for each  $M \in \mathbf{K}_{\lambda}$ , gS<sup>bs</sup>(M) consists of all the nonalgebraic types over M. We focus on type-full good frames in this chapter and hence just write  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \downarrow)$ . For notational simplicity, we extend forking to algebraic types by specifying that algebraic types do not fork over their domain. Given a type-full good  $\mu$ -frame  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \downarrow)$  and  $M_0 \leq_{\mathbf{K}} M$  both in  $\mathbf{K}_{\lambda}$ , we say that a nonalgebraic type  $p \in \mathrm{gS}(M)$ does not  $\mathfrak{s}$ -fork over  $M_0$  if it does not fork over  $M_0$  according to the abstract independence relation  $\downarrow$  of  $\mathfrak{s}$ . When  $\mathfrak{s}$  is clear from context, we omit it and just say that p does not fork over  $M_0$ . We write  $\mathbf{K}_{\mathfrak{s}}$  for the underlying class (containing only models of size  $\lambda$ ) of  $\mathfrak{s}$ . We say that a good  $\lambda$ -frame  $\mathfrak{s}$  is on  $\mathbf{K}_{\lambda}$  if  $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\lambda}$ . We might also just say that  $\mathfrak{s}$  is on  $\mathbf{K}$ .

We more generally look at frames where the forking relation works over larger models. For  $\mathcal{F} = [\lambda, \theta)$  an interval with  $\theta \geq \lambda$  a cardinal or  $\infty$ , we define a *type-full* good  $\mathcal{F}$ -frame similarly to a type-full good  $\lambda$ -frame but require forking to be defined over models in  $\mathbf{K}_{\mathcal{F}}$  (similarly, the good properties hold of the class  $\mathbf{K}_{\mathcal{F}}$ , e.g.  $\mathbf{K}$  is stable in every  $\mu \in \mathcal{F}$ ). See Definition 4.2.19 for more details. For a type-full good  $\mathcal{F}$ -frame  $\mathfrak{s} = (\mathbf{K}_{\mathcal{F}}, \downarrow)$  and  $\mathbf{K}'$  a subclass of  $\mathbf{K}_{\mathcal{F}}$ , we define the restriction  $\mathfrak{s} \upharpoonright \mathbf{K}'$  of  $\mathfrak{s}$  to  $\mathbf{K}'$  in the natural way (see Notation 6.3.17).

At one point in the chapter (Section 14.3) we will look at (not necessarily good) frames defined over types longer than one element. This was first defined in Definition 5.3.1 but we use Definition 6.3.1. We require in addition that it satisfies the base monotonicity property and that the underlying class is an AEC.

DEFINITION 14.2.1. A type-full pre- $(\leq \lambda, \lambda)$ -frame is a pair  $\mathfrak{t} := (K_{\lambda}, \downarrow)$ , where **K** is an AEC with amalgamation in  $\lambda$  and  $\downarrow$  is a relation on types of length at most  $\lambda$  over models in  $\mathbf{K}_{\lambda}$  satisfying invariance and monotonicity (including base monotonicity, see Definition 6.3.12.(4)).

<sup>&</sup>lt;sup>4</sup>The definition here is simpler and more general than the original: We will *not* use Shelah's axiom (B) requiring the existence of a superlimit model of size  $\lambda$ . Several papers (e.g. **[JS13**]) define good frames without this assumption.

For a type-full good  $\lambda$ -frame  $\mathfrak{s}$ , we say that  $\mathfrak{t}$  extends  $\mathfrak{s}$  if they have the same underlying class and forking in  $\mathfrak{s}$  and  $\mathfrak{t}$  coincide.

14.2.2. Saturated and limit models. We will make heavy use of limit models (see [GVV16] for history and motivation). Here we give a global definition, where we permit the limit model and the base to have different sizes.

DEFINITION 14.2.2. Let  $M_0 \leq_{\mathbf{K}} M$  be models in  $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$ . M is limit over  $M_0$  if there exists a limit ordinal  $\delta$  and a strictly increasing continuous sequence  $\langle N_i : i \leq \delta \rangle$  such that:

- (1)  $N_0 = M_0$ .
- (2)  $N_{\delta} = M$ .
- (3) For all  $i < \delta$ ,  $N_{i+1}$  is universal over  $N_i$  (that is, for any  $N \in \mathbf{K}_{||N_i||}$  with  $N_0 \leq_{\mathbf{K}} N$ , there exists  $f : N \xrightarrow[N_i]{N_i} N_{i+1}$ ).

We say that M is *limit* if it is limit over some  $M' \leq_{\mathbf{K}} M$ .

DEFINITION 14.2.3. Assume that K has amalgamation.

- (1) For  $\lambda > LS(\mathbf{K})$ ,  $\mathbf{K}^{\lambda-\text{sat}}$  is the class of  $\lambda$ -saturated (in the sense of Galois types) models in  $\mathbf{K}_{>\lambda}$ . We order it with the strong substructure relation inherited from **K**.
- (2) We also define  $\mathbf{K}^{\mathrm{LS}(\mathbf{K})\text{-sat}}$  to be the class of models  $M \in \mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$  such that for all  $A \subseteq |M|$  with  $|A| \leq \mathrm{LS}(\mathbf{K})$ , there exists a limit model  $M_0 \leq_{\mathbf{K}} M$ with  $M_0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  and  $A \subseteq |M_0|$ . We order  $\mathbf{K}^{\mathrm{LS}(\mathbf{K})\text{-sat}}$  with the strong substructure relation inherited from  $\mathbf{K}$ .

REMARK 14.2.4. If **K** has amalgamation and is stable in  $LS(\mathbf{K})$ , then  $\mathbf{K}_{LS(\mathbf{K})}^{LS(\mathbf{K})-\text{sat}}$ is the class of limit models in  $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}$ .

We will repeatedly use the uniqueness of limit models inside a good frame, first proven by Shelah in [She09a, Claim II.4.8] (see also [Bon14a, Theorem 9.2]).

FACT 14.2.5. Let  $\mathfrak{s}$  be a type-full good  $\lambda$ -frame.

Let  $M_0, M_1, M_2 \in \mathbf{K}_{\mathfrak{s}}$ .

- (1) If  $M_1$  and  $M_2$  are limit models, then  $M_1 \cong M_2$ .
- (2) If in addition  $M_1$  and  $M_2$  are both limit over  $M_0$ , then  $M_1 \cong_{M_0} M_2$ .

For global frames, we can combine this with a result of VanDieren [Van16b] to obtain that limit models are saturated and closed under unions:

FACT 14.2.6. Let  $\mathfrak{s}$  be a type-full good  $[\lambda, \theta)$ -frame on the AEC K. Let  $\mu \in$  $[\lambda, \theta).$ 

- M ∈ K<sup>μ-sat</sup><sub>μ</sub> if and only if M is limit.
   If μ > λ, then K<sup>μ-sat</sup><sub>[μ,θ]</sub> is the initial segment of an AEC with Löwenheim-Skolem-Tarski number  $\mu$ .

Proof.

- (1) This is trivial if  $\lambda = LS(\mathbf{K})$ , so assume that  $\lambda > LS(\mathbf{K})$ . If M is limit, then by uniqueness of limit models, M is saturated. Conversely if M is saturated, then it must be unique, hence isomorphic to a limit model.
- (2) By uniqueness of limit models (Fact 14.2.5) and [Van16b, Theorem 1]. Note that there a condition called  $\mu$ -superstability (see Definition 15.1.1)

rather than the existence of a good  $\mu$ -frame is used. However the existence of a type-full good  $\mu$ -frame implies  $\mu$ -superstability (see Fact 15.1.2).

We will use Facts 14.2.5 and 14.2.6 freely.

#### 14.3. Domination and uniqueness triples

In this section, we assume:

HYPOTHESIS 14.3.1.  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \downarrow)$  is a type-full good  $\lambda$ -frame.

REMARK 14.3.2. The results of this section can be adapted to non-type-full good frames, but we assume type-fullness anyway for notational convenience.

Our aim (for the next sections) is to develop some orthogonality calculus as in [She09a, Section III.6]. There Shelah works in a good  $\lambda$ -frame that is *successful* (see [She09a, Definition III.1.1]). Note that by [She09a, Claim III.9.6] such a good frame can be extended to a type-full one. Thus the framework of this section is more general (see Chapter 18 for an example of a non-successful type-full frame).

One of the main components of the definition of successful is the existence property for uniqueness triples (see [She09a, Definition III.1.1], [JS13, Definition 4.1.(5)], or here Definition 15.11.6). It was shown in Lemma 6.11.7 that this property is equivalent to a version of domination assuming the existence of a global independence relation. Using an argument of Makkai and Shelah [MS90, Proposition 4.22], one can see (Lemma 6.11.12) that this version of domination satisfies a natural existence property. We give a slight improvement on this result here by working only locally in  $\lambda$  (i.e. using limit models rather than saturated ones).

Crucial in this section is the uniqueness of limit models (Fact 14.2.5). A consequence is the following conjugation property [She09a, Claim 1.21]. It is stated there for M, N superlimit but the proof goes through if M and N are limit models.

FACT 14.3.3 (The conjugation property). If  $M \leq_{\mathbf{K}} N$  are limit models in  $\mathbf{K}_{\lambda}$  and  $p, q \in \mathrm{gS}(N)$  do not fork over M, then there exists  $f : N \cong M$  so that  $f(p) = p \upharpoonright M$  and  $f(q) = q \upharpoonright M$ .

The next definition is modeled on Definition 6.11.5:

DEFINITION 14.3.4. Let  $\mathfrak{t}$  be a pre- $(\leq \lambda, \lambda)$ -frame extending  $\mathfrak{s}$  (see Definition 14.2.1, we write  $\downarrow$  for the nonforking relation of both  $\mathfrak{s}$  and  $\mathfrak{t}$ ). The triple (a, M, N) is a domination triple for  $\mathfrak{t}$  if  $M, N \in K_{\lambda}, M \leq_{\mathbf{K}} N, a \in |N| \setminus |M|$  and for any  $N' \geq_{\mathbf{K}} N$  and  $M' \leq_{\mathbf{K}} N$  with  $M \leq_{\mathbf{K}} M'$  and  $M', N' \in K_{\lambda}$ , if  $a \downarrow_{M} M'$ , then  $N \downarrow_{M} M'$ .

REMARK 14.3.5. In this chapter, we will take  $\mathfrak{t}$  to be 1-forking (Definition 14.4.2), so the reader who wants a concrete example may substitute it for  $\mathfrak{t}$  throughout this section.

Domination triples are related to Shelah's uniqueness triples (see [She09a, Definition III.1.1] or [JS13, Definition 4.1.(5)]) by the following result (this will not be used outside of this section):

FACT 14.3.6 (Lemma 6.11.7). Let  $\mathfrak{t}$  be a pre- $(\leq \lambda, \lambda)$ -frame extending  $\mathfrak{s}$ . If  $\mathfrak{t}$  has the uniqueness property, then any domination triple for  $\mathfrak{t}$  is a uniqueness triple for  $\mathfrak{s}$ .

We now want to show the existence property for domination triples: For any type  $p \in gS(M)$ , there exists a domination triple (a, M, N) with p = gtp(a/M; N). We manage to do it when M is a limit model. The proof is a local version of Lemma 6.11.12 (which adapted [**MS90**, Proposition 4.22]). We will consider the following local character properties that t may have:

DEFINITION 14.3.7. Let  $\mathfrak{t}$  be a pre- $(\leq \lambda, \lambda)$ -frame extending  $\mathfrak{s}$ . Let  $\kappa \geq 2$  be a cardinal.

- (1) We say that  $\mathfrak{t}$  satisfies local character for  $(\langle \kappa \rangle)$ -length types over  $(\lambda, \lambda^+)$ limits if whenever  $\langle M_i : i < \lambda^+ \rangle$  is increasing in  $\mathbf{K}_{\lambda}$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \lambda^+$  and  $p \in \mathrm{gS}^{<\kappa}(\bigcup_{i < \lambda^+} M_i)$ , there exists  $i < \lambda^+$  such that for any  $j \in [i, \lambda^+)$ ,  $p \upharpoonright M_i$  does not fork over  $M_i$ .
- (2) We say that  $\mathfrak{t}$  reflects down if whenever  $\langle M_i : i < \lambda^+ \rangle$ ,  $\langle N_i : i < \lambda^+ \rangle$  are increasing continuous in  $\mathbf{K}_{\lambda}$  so that for all  $i < \lambda^+$ ,  $M_i \leq_{\mathbf{K}} N_i$ ,  $M_{i+1}$  is universal over  $M_i$ , and  $N_{i+1}$  is universal over  $N_i$ , then there exists  $i < \lambda^+$ such that  $N_i \bigcup_{M_i}^{N_{i+1}} M_{i+1}$ .
- (3) (Definition 6.3.12) For  $\kappa \geq 2$ , we say that  $\mathfrak{t}$  has the left  $(<\kappa)$ -witness property if for any three models  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{\lambda}$  and any  $A \subseteq |N|$ ,  $A \downarrow_{M_0} M$  holds if and only if  $A_0 \downarrow_{M_0} M$  for all  $A_0 \subseteq A$  with  $|A_0| < \kappa$ .

Note that the witness property implies some amount of local character:

LEMMA 14.3.8. Let  $\kappa \geq 2$ . Let  $\mathfrak{t}$  be a pre- $(\leq \lambda, \lambda)$ -frame extending  $\mathfrak{s}$ . If  $\lambda = \lambda^{<\kappa}$ ,  $\mathfrak{t}$  satisfies local character for  $(< \kappa)$ -length types over  $(\lambda, \lambda^+)$ -limits, and  $\mathfrak{t}$  has the  $(< \kappa)$ -witness property, then  $\mathfrak{t}$  satisfies local character for  $(< \lambda^+)$ -length types over  $(\lambda, \lambda^+)$ -limits.

PROOF. Let  $\langle M_i : i < \lambda^+ \rangle$  be increasing in  $\mathbf{K}_{\lambda}$ , with  $M_{i+1}$  universal over  $M_i$  for all  $i < \lambda^+$ . Write  $M_{\lambda^+} := \bigcup_{i < \lambda^+} M_i$ . Let  $p \in \mathrm{gS}^{\alpha}(M_{\lambda^+})$  with  $\alpha < \lambda^+$ . Say  $p = \mathrm{gtp}(\bar{a}/M_{\lambda^+}; N)$ . For each  $I \subseteq \alpha$  with  $|I| < \kappa$ , we will write  $p^I$  for  $\mathrm{gtp}(\bar{a} \upharpoonright I/M_{\lambda^+}; N)$ .

Directly from the local character assumption on  $\mathfrak{t}$ , we have that for each  $I \subseteq \alpha$  with  $|I| < \kappa$ , there exists  $i_I < \lambda^+$  so that  $p^I \upharpoonright M_j$  does not fork over  $M_{i_I}$  for all  $j \ge i_I$ .

Now let  $i := \sup_{I \subseteq \alpha, |I| < \kappa} i_k$ . Since  $\lambda = \lambda^{<\kappa}$ ,  $i < \lambda^+$ . Using the witness property, we get that  $p \upharpoonright M_j$  does not fork over  $M_i$  for all  $j \ge i$ , as desired.  $\Box$ 

Moreover local character together with the witness property imply that  $\mathfrak t$  reflects down:

LEMMA 14.3.9. Let t be a pre- $(\leq \lambda, \lambda)$ -frame extending  $\mathfrak{s}$ . If t has local character for  $(< \lambda^+)$ -length types over  $(\lambda, \lambda^+)$ -limits and has the left  $(< \kappa)$ -witness property for some regular  $\kappa \leq \lambda$ , then t reflects down.

PROOF. Fix  $\langle M_i : i < \lambda^+ \rangle$ ,  $\langle N_i : i < \lambda^+ \rangle$  as in the definition of reflecting down. By local character for  $\mathfrak{t}$ , for each  $i < \lambda^+$ , there exists  $j_i < \lambda^+$  such that  $N_i \bigcup_{M_{j_i}}^{N_j} M_j$ 

for all  $j \ge j_i$ . Let  $i^* < \lambda^+$  be such that cf  $i^* = \kappa$  and  $j_i < i^*$  for all  $i < i^*$ . Then it is easy to check using the left  $(< \kappa)$ -witness property that  $N_{i^*} \underset{M_{i^*}}{\overset{N_j}{\longrightarrow}} M_j$  for all  $j \ge i^*$ , which is as needed.

We have arrived to the existence property for domination triples. For the convenience of the reader, we restate Hypothesis 14.3.1

THEOREM 14.3.10. Let  $\mathfrak{s}$  be a type-full good  $\lambda$ -frame on **K** and let  $\mathfrak{t}$  be a pre- $(\leq \lambda, \lambda)$ -frame extending  $\mathfrak{s}$ . Assume that  $\mathfrak{t}$  reflects down.

Let  $M \in K_{\lambda}$  be a limit model. For each nonalgebraic  $p \in gS(M)$ , there exists a domination triple (a, M, N) for t such that p = gtp(a/M; N).

PROOF. Assume not.

<u>Claim</u>. For any limit  $M' \in \mathbf{K}_{\lambda}$  with  $M' \geq_{\mathbf{K}} M$ , if  $q \in \mathrm{gS}(M')$  is the nonforking extension of p, then there is no domination triple (b, M', N') such that  $q = \mathrm{gtp}(b/M'; N')$ .

<u>Proof of claim</u>. By the conjugation property (Fact 14.3.3), there exists  $f: M' \cong M$  such that f(q) = p. Now use that domination triples are invariant under isomorphisms.  $\dagger_{\text{Claim}}$ 

We construct a,  $\langle M_i : i < \lambda^+ \rangle$ ,  $\langle N_i : i < \lambda^+ \rangle$  increasing continuous such that for all  $i < \lambda^+$ :

- (1)  $M_0 = M$ .
- (2)  $M_i \leq_{\mathbf{K}} N_i$  are both in  $\mathbf{K}_{\lambda}$ .
- (3)  $M_{i+1}$  is limit over  $M_i$  and  $N_{i+1}$  is limit over  $N_i$ .

(4)  $gtp(a/M_i; N_i)$  is the nonforking extension of p. In particular,  $a \bigcup_{M_0}^{N_i} M_i$ .

(5)  $N_i \overset{N_{i+1}}{\underset{M_i}{\not \downarrow}} M_{i+1}.$ 

This is enough, since then we get a contradiction to t reflecting down. This is possible: If i = 0, let  $N_0 \in \mathbf{K}_{\lambda}$  and  $a \in |N_0|$  be such that  $p = \operatorname{gtp}(a/M; N_0)$ . At limits, take unions. Now assume everything up to i has been constructed. By the claim,  $(a, M_i, N_i)$  cannot be a domination triple. This means there exists  $M'_i \geq_{\mathbf{K}} M_i$  and  $N'_i \geq_{\mathbf{K}} N_i$  all in  $\mathbf{K}_{\lambda}$  such that  $a \stackrel{N'_i}{\underset{M_i}{\longrightarrow}} M'_i$  but  $N_i \stackrel{N'_i}{\underset{M_i}{\longrightarrow}} M'_i$ . By the extension property of forking, pick  $M_{i+1} \in \mathbf{K}_{\lambda}$  limit over  $M_i$  containing  $M'_i$  and  $N_{i+1} \geq_{\mathbf{K}} N'_i$  such that  $N_{i+1}$  is limit over  $N_i$  and  $a \stackrel{N'_{i+1}}{\underset{M_i}{\longrightarrow}} M_{i+1}$ .  $\Box$ 

The next corollary will not be used in the rest of this chapter. It improves on Theorem 6.11.13 by working exclusively in  $\lambda$  (so there is no need to assume the existence of a good frame below  $\lambda$ ).

COROLLARY 14.3.11. Assume that **K** is categorical in  $\lambda$ . If there exists a pre-( $\leq \lambda, \lambda$ )-frame  $\mathfrak{t}$  extending  $\mathfrak{s}$ , reflecting down, and satisfying uniqueness, then  $\mathfrak{s}$  has the existence property for uniqueness triples (i.e. it is weakly successful, see [**She09a**, Definition III.1.1] or Definition 15.11.6).

PROOF. By Theorem 14.3.10,  $\mathfrak{t}$  has the existence property for domination triples. By Fact 14.3.6, any domination triple is a uniqueness triple.

QUESTION 14.3.12. Is the converse true? Namely if **K** is categorical in  $\lambda$  and  $\mathfrak{s}$  is a weakly successful good  $\lambda$ -frame on **K**, does there exist a pre- $(\leq \lambda, \lambda)$ -frame t that extends  $\mathfrak{s}$ , satisfies uniqueness, and reflects down?

It is known (see [She09a, Section II.6] and Section 6.12) that weakly successful good  $\lambda$ -frame can be extended to pre- $(\leq \lambda, \lambda)$ -frame with several good properties, including uniqueness, but it is not clear that the pre-frame reflects down.

We finish with a slight improvement on the construction of a weakly successful good frame from Section 8.6 (which improved the threshold cardinals of Chapter 6). Since the result is not needed for the rest of the chapter, we only sketch the proof and quote freely. At this point, we drop Hypothesis 14.3.1.

COROLLARY 14.3.13. Let **K** be an AEC and let  $\lambda > LS(\mathbf{K})$ . Let  $\kappa \leq LS(\mathbf{K})$  be an infinite cardinal. Assume that  $LS(\mathbf{K}) = LS(\mathbf{K})^{<\kappa}$  and  $\lambda = \lambda^{<\kappa}$ . Assume that **K** is  $LS(\mathbf{K})$ -tame for all types of length less than  $\kappa$  over saturated models of size  $\lambda$ , and **K** is  $(<\kappa)$ -type short for all types of length at most  $\lambda$  over saturated models of size  $\lambda$  (see [**Bon14b**, Definition 3.3] and Definition 2.2.23).

If  $\mathbf{K}_{[\mathrm{LS}(\mathbf{K}),\lambda]}$  has amalgamation,  $\mathbf{K}$  is stable in  $\mathrm{LS}(\mathbf{K})$ , and  $\mathfrak{s}$  is a good  $\lambda$ -frame on  $\mathbf{K}_{\lambda}^{\lambda-\mathrm{sat}}$ , then  $\mathfrak{s}$  is weakly successful.

PROOF SKETCH. We use Corollary 14.3.11 with the pre- $(\leq \lambda, \lambda)$ -frame t on  $\mathbf{K}_{\lambda}$ where t-forking is defined as follows: For  $M_0 \leq_{\mathbf{K}} M$  both in  $\mathbf{K}_{\lambda}$ ,  $p \in \mathrm{gS}^{\leq \lambda}(M)$  does not t-fork over  $M_0$  if for every  $I \subseteq \ell(p)$  with  $|I| < \kappa$ , there exists  $M'_0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  with  $M'_0 \leq_{\mathbf{K}} M_0$  such that  $p^I$  does not  $\mathrm{LS}(\mathbf{K})$ -split over  $M'_0$ . We want to show that t extends  $\mathfrak{s}$ , t has uniqueness, and t reflects down.

Following the proof of Lemma 8.6.14, we get that  $\mathfrak{t}$  extends  $\mathfrak{s}$ , has local character for  $(<\lambda^+)$ -length types over  $(\lambda,\lambda^+)$ -limits, and has uniqueness. Also,  $\mathfrak{t}$  clearly has the left  $(<\kappa^+)$ -witness property and by assumption  $\kappa^+ \leq \mathrm{LS}(\mathbf{K})^+ \leq \lambda$ . Thus by Lemma 14.3.9 (where  $\kappa$  there is  $\kappa^+$  here),  $\mathfrak{t}$  reflects down, as desired.

# 14.4. Local orthogonality

HYPOTHESIS 14.4.1.  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \downarrow)$  is a type-full good  $\lambda$ -frame.

The next definition is what we will use for the t of the previous section. One can see it as a replacement for a notion of forking for types over models, when such a notion is not available. It already plays a role in [MS90] (see Lemma 4.17 there) and Definition 7.2.10). A similar notion is called "smooth independence" in [VZ14].

DEFINITION 14.4.2. For  $M \in \mathbf{K}_{\lambda}$  and  $p \in \mathrm{gS}^{<\infty}(M)$ , we say that p does not 1-s-fork over  $M_0$  if  $M_0 \leq_{\mathbf{K}} M$  and for  $I \subseteq \ell(p)$  with |I| = 1, we have that  $p^I$  does not s-fork over  $M_0$ . We see 1-forking as inducing a pre- $(\leq \lambda, \lambda)$ -frame (Definition 14.2.1).

NOTATION 14.4.3. We write  $[A]^1 \stackrel{N}{\underset{M_0}{\downarrow}} M$  if for some (any) enumeration  $\bar{a}$  of A,

 $gtp(\bar{a}/M; N)$  does not 1-s-fork over  $M_0$ . That is,  $a \bigcup_{M_0}^N M$  for all  $a \in A$ .

Remark 14.4.4 (Disjointness). Because nonforking extensions of nonalgebraic types are nonalgebraic, if  $[A]^1 \underset{M_0}{\overset{N}{\downarrow}} M$ , then  $|M| \cap A \subseteq |M_0|$ .

DEFINITION 14.4.5. (a, M, N) is a weak domination triple in  $\mathfrak{s}$  if it is a domination triple (Definition 14.3.4) for the pre-frame induced by 1-forking. That is,  $M, N \in K_{\lambda}, M \leq_{\mathbf{K}} N, a \in |N| \setminus |M|$  and for any  $N' \geq_{\mathbf{K}} N$  and  $M' \leq_{\mathbf{K}} N$  with  $M \leq_{\mathbf{K}} M'$  and  $M', N' \in K_{\lambda}$ , if  $a \downarrow_{M}^{N'} M'$ , then  $[N]^{1} \downarrow_{M}^{N'} M'$ .

From the results of the previous section, we deduce the existence property for weak domination triples:

LEMMA 14.4.6. 1-forking reflects down (see Definition 14.3.7).

**PROOF.** It is clear that 1-forking has the left (< 2)-witness property (Definition 14.3.7) so by Lemmas 14.3.8 and 14.3.9, it is enough to show that 1-forking has local character for (< 2)-length types over  $(\lambda, \lambda^+)$ -limits. This follows from the local character property of  $\mathfrak{s}$  (see [She09a, Claim II.2.11]). In details:

<u>Claim</u>. Let  $\langle M_i : i < \lambda^+ \rangle$  be increasing continuous in  $\mathbf{K}_{\lambda}$ . Let  $p \in \mathrm{gS}(\bigcup_{i < \lambda^+})$ . There exists  $i < \lambda^+$  so that  $p \upharpoonright M_j$  does not fork over  $M_i$  for all  $j \ge i$ .

<u>Proof of claim</u>. By local character (in  $\mathfrak{s}$ ), for each limit  $j < \lambda^+$ , there exists  $\gamma_j < j$  so that  $p \upharpoonright M_j$  does not fork over  $M_{\gamma_j}$ . By Fodor's lemma, there exists  $i < \lambda^+$  such that for unboundedly many  $j < \lambda^+$ ,  $\gamma_j = i$ . By monotonicity of forking, i is as desired.  $\square$ 

THEOREM 14.4.7. Let  $M \in K_{\lambda}$  be a limit model. For each nonalgebraic  $p \in$ gS(M), there exists a weak domination triple (a, M, N) such that p = gtp(a/M; N).

PROOF. By Lemma 14.4.6 and Theorem 14.3.10.

We now give a definition of orthogonality in terms of independent sequences.

DEFINITION 14.4.8 (Independent sequence, III.5.2 in [She09a]). Let  $\alpha$  be an ordinal.

- (1)  $\langle a_i : i < \alpha \rangle \frown \langle M_i : i \le \alpha \rangle$  is said to be *independent in* (M, M', N) when: (a)  $(M_i)_{i < \alpha}$  is increasing continuous in  $\mathbf{K}_{\lambda}$ .
  - (b)  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M_0$  and  $M, M' \in \mathbf{K}_{\lambda}$ . (c)  $M_{\alpha} \leq_{\mathbf{K}} N$  and  $N \in \mathbf{K}_{\lambda}$ .

(d) For every  $i < \alpha$ ,  $a_i \downarrow_M^{M_{i+1}} M_i$ .  $\langle a_i : i < \alpha \rangle \frown \langle M_i : i \le \alpha \rangle$  is said to be *independent over* M when it is independent in  $(M, M_0, M_\alpha)$ .

(2)  $\bar{a} := \langle a_i : i < \alpha \rangle$  is said to be independent in  $(M, M_0, N)$  when  $M \leq_{\mathbf{K}}$  $M_0 \leq_{\mathbf{K}} N, \bar{a} \in {}^{\alpha}|N|$ , and for some  $\langle M_i : i \leq \alpha \rangle$  and a model  $N^+$  such that  $M_{\alpha} \leq_{\mathbf{K}} N^+, N \leq_{\mathbf{K}} N^+, \text{ and } \langle a_i : i < \alpha \rangle \frown \langle M_i : i \leq \alpha \rangle$  is independent over M. When  $M = M_0$ , we omit it and just say that  $\bar{a}$  is independent in (M, N).

REMARK 14.4.9. We will use the definition above when  $\alpha = 2$ . In this case, we have that  $\langle ab \rangle$  is independent in (M, N) if and only if  $a \underset{M}{\overset{N}{\downarrow}} b$  (technically, the right hand side of the  $\downarrow$  relation must be a model but we can remedy this by extending the nonforking relation in the natural way, as in the definition of the minimal closure in Definition 3.3.4).

DEFINITION 14.4.10. Let  $M \in \mathbf{K}_{\lambda}$  and let  $p, q \in \mathrm{gS}(M)$  be nonalgebraic. We say that p is weakly orthogonal to q and write  $p \perp_{wk} q$  (or just  $p \perp_{wk} q$  if  $\mathfrak{s}$  is clear from context) if for all  $N \in \mathbf{K}_{\lambda}$  with  $N \geq_{\mathbf{K}} M$  and all  $a, b \in |N|$  such that  $\mathrm{gtp}(a/M; N) = p$  and  $\mathrm{gtp}(b/M; N) = q$ ,  $\langle ab \rangle$  is independent in (M, N).

We say that p is orthogonal to q (written  $p \perp q$ , or just  $p \perp q$  if  $\mathfrak{s}$  is clear from context) if for every  $N \in \mathbf{K}_{\lambda}$  with  $N \geq_{\mathbf{K}} M$ ,  $p' \perp_{wk} q'$ , where p', q' are the nonforking extensions to N of p and q respectively.

REMARK 14.4.11. Definition 14.4.10 is equivalent to Shelah's ([**She09a**, Definition III.6.2]), see [**She09a**, Claim III.6.4.(2)] assuming that  $\mathfrak{s}$  is *successful*. By a similar proof (and assuming that  $\mathbf{K}_{\mathfrak{s}}$  has primes), it is also equivalent to the definition in terms of primes in Definition 11.2.2.

We will use the following consequence of symmetry:

FACT 14.4.12 (Theorem 4.2 in [**JS12**]). For any  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  all in  $\mathbf{K}_{\lambda}$ , if  $a, b \in |N| \setminus |M_0|$ , then  $\langle ab \rangle$  is independent in  $(M_0, M, N)$  if and only if  $\langle ba \rangle$  is independent in  $(M_0, M, N)$ .

LEMMA 14.4.13. Let  $M \in \mathbf{K}_{\lambda}$ . Let  $p, q \in \mathrm{gS}(M)$  be nonalgebraic.

- (1) If M is limit, then  $p \perp q$  if and only if  $p \perp q$ .
- (2)  $p \perp_{wk} q$  if and only if  $q \perp_{wk} p$ .
- (3) If  $p \perp_{wk} q$ , then whenever (a, M, N) is a weak domination triple representing q, p is omitted in N. In particular, if M is limit, there exists  $N \in \mathbf{K}_{\lambda}$  with  $M <_{\mathbf{K}} N$  so that N realizes q but p is omitted in N.

Proof.

- (1) By the conjugation property (Fact 14.3.3). See the proof of Lemma 11.2.6.
- (2) By Fact 14.4.12.
- (3) Let  $N' \in \mathbf{K}_{\lambda}$  be such that  $N \leq_{\mathbf{K}} N'$  and let  $b \in |N'|$  realize p. We have that  $\langle ab \rangle$  is independent in (M, N'). Therefore there exists  $M', N'' \in \mathbf{K}_{\lambda}$

so that 
$$N \leq_{\mathbf{K}} N''$$
,  $M \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} N''$ ,  $b \in |M'|$ , and  $a \downarrow_{M} M'$ . By

domination,  $[N]^1 \stackrel{N}{\underset{M}{\longrightarrow}} M'$ , so by disjointness (Remark 14.4.4),  $b \notin |N|$ . The last sentence follows from the existence property for weak domination triple (Theorem 14.4.7).

# 14.5. Unidimensionality

HYPOTHESIS 14.5.1.  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \downarrow)$  is a type-full good  $\lambda$ -frame on **K** and **K** is categorical in  $\lambda$ .

In this section we give a definition of unidimensionality similar to the ones in [She90, Definition V.2.2] or [She09a, Section III.2]. We show that  $\mathfrak{s}$  is unidimensional if and only if **K** is categorical in  $\lambda^+$  (this uses categoricity in  $\lambda$ ). In the next

section, we will show how to transfer unidimensionality across cardinals, hence getting the promised categoricity transfer. In [She09a, Section III.2], Shelah gives several different definitions of unidimensionality and also shows (see [She09a, III.2.3, III.2.9]) that the so-called "weak-unidimensionality" is equivalent to categoricity in  $\lambda^+$  (hence our definition is equivalent to Shelah's weak unidimensionality) but it is unclear how to transfer weak-unidimensionality across cardinals without assuming that the frame is successful.

Note that the hypothesis of categoricity in  $\lambda$  implies that the model of size  $\lambda$  is limit, hence weak orthogonality and orthogonality coincide, see Lemma 14.4.13.

Rather than defining what it means to be unidimensional, we find it clearer to define what it means to *not* be unidimensional:

DEFINITION 14.5.2.  $\mathfrak{s}$  is unidimensional if the following is false: for every  $M \in \mathbf{K}_{\lambda}$  and every nonalgebraic  $p \in \mathrm{gS}(M)$ , there exists  $M' \in \mathbf{K}_{\lambda}$  with  $M' \geq_{\mathbf{K}} M$  and nonalgebraic  $p', q \in \mathrm{gS}(M')$  so that p' extends p and  $p' \perp q$ .

We first give an equivalent definition using minimal types:

DEFINITION 14.5.3. For  $M \in \mathbf{K}_{\lambda}$ , a type  $p \in \mathrm{gS}(M)$  is minimal if for every  $M' \in \mathbf{K}_{\lambda}$  with  $M \leq_{\mathbf{K}} M'$ , p has a unique nonalgebraic extension to  $\mathrm{gS}(M')$ .

REMARK 14.5.4. Since we are working inside a good frame, any nonalgebraic type will have at least one nonalgebraic extension (the nonforking one). The non-trivial part of the definition is its uniqueness.

REMARK 14.5.5. If  $M \leq_{\mathbf{K}} N$  are both in  $\mathbf{K}_{\lambda}$  and  $p \in \mathrm{gS}(N)$  is nonalgebraic such that  $p \upharpoonright M$  is minimal, then p does not fork over M (because the nonforking extension of  $p \upharpoonright M$  has to be p).

By the proof of  $(*)_5$  in [She99, Theorem II.2.7]:

FACT 14.5.6 (Density of minimal types). For any  $M \in \mathbf{K}_{\lambda}$  and nonalgebraic  $p \in \mathrm{gS}(M)$ , there exists  $M' \in \mathbf{K}_{\lambda}$  and  $p' \in \mathbf{K}_{\lambda}$  such that  $M \leq_{\mathbf{K}} M'$ , p' extends p, and p' is minimal.

LEMMA 14.5.7. The following are equivalent:

- (1)  $\mathfrak{s}$  is not unidimensional.
- (2) For every  $M \in \mathbf{K}_{\lambda}$  and every minimal  $p \in \mathrm{gS}(M)$ , there exists  $M' \in \mathbf{K}_{\lambda}$  with  $M \leq_{\mathbf{K}} M'$  and  $p', q \in \mathrm{gS}(M')$  nonalgebraic so that p' extends p and  $p' \perp q$ .
- (3) For every  $M \in \mathbf{K}_{\lambda}$  and every minimal  $p \in gS(M)$ , there exists a nonalgebraic  $q \in gS(M)$  with  $p \perp q$ .

PROOF. (1) implies (2) because (2) is a special case of (1). Conversely, (2) implies (1): given  $M \in \mathbf{K}_{\lambda}$  and  $p \in \mathrm{gS}(M)$ , first use density of minimal types to extend p to a minimal  $p' \in \mathrm{gS}(M')$  (so  $M' \in \mathbf{K}_{\lambda}$   $M \leq_{\mathbf{K}} M'$ ). Then apply (2).

Also, if (3) holds, then (2) holds with M = M'. Conversely, assume that (2) holds. Let  $p \in gS(M)$  be minimal and let p', q, M' witness (2), i.e.  $p', q \in gS(M')$ , p' extends p and  $p' \perp q$ . By Remark 14.5.5, p' does not fork over M. By the conjugation property (Fact 14.3.3), there exists  $f : M' \cong M$  so that f(p') = p. Thus  $p \perp f(q)$ , hence (2) holds.

We use the characterization to show that unidimensionality implies categoricity in  $\lambda^+$ . This is similar to [**MS90**, Proposition 4.25] but the proof is slightly more involved since our definition of unidimensionality is weaker. We start with a version of density of minimal types inside a fixed model. We will use the following fact, whose proof is a straightforward direct limit argument:

FACT 14.5.8 (Claim 0.32.(1) in [She01a]). Let  $\langle M_i : i \leq \omega \rangle$  be an increasing continuous chain in  $\mathbf{K}_{\lambda}$  and for each  $i < \omega$ , let  $p_i \in \mathrm{gS}(M_i)$  be such that j < i implies  $p_i \upharpoonright M_j = p_j$ . Then there exists  $p \in \mathrm{gS}(M_\omega)$  so that  $p \upharpoonright M_i = p_i$  for all  $i < \omega$ .

LEMMA 14.5.9. Let  $M_0 \leq_{\mathbf{K}} M$  with  $M_0 \in \mathbf{K}_{\lambda}$  and  $M \in \mathbf{K}_{>\lambda}$ . Let  $p \in \mathrm{gS}(M_0)$ . Then there exists  $M_1 \in \mathbf{K}_{\lambda}$  with  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$  and  $q \in \mathrm{gS}(M_1)$  so that q extends p and for all  $M' \in \mathbf{K}_{\lambda}$  with  $M_1 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M$ , any extension of q to  $\mathrm{gS}(M')$  does not fork over  $M_1$ .

PROOF. Suppose not. Build  $\langle N_i : i < \omega \rangle$  increasing in  $\mathbf{K}_{\lambda}$  and  $\langle q_i : i < \omega \rangle$  such that for all  $i < \omega$ :

(1)  $N_0 = M_0, q_0 = p.$ 

(2)  $N_i \leq_{\mathbf{K}} M$ .

(3)  $q_i \in gS(N_i)$  and  $q_{i+1}$  extends  $q_i$ .

(4)  $q_{i+1}$  forks over  $N_i$ .

This is possible since we assumed that the lemma failed. This is enough: let  $N_{\omega} := \bigcup_{i < \omega} N_i$ . Let  $q \in gS(N_{\omega})$  extend each  $q_i$  (exists by Fact 14.5.8). By local character, there exists  $i < \omega$  such that q does not fork over  $N_i$ , so  $q \upharpoonright N_{i+1} = q_{i+1}$  does not fork over  $N_i$ , contradiction.

LEMMA 14.5.10. If  $\mathfrak{s}$  is unidimensional, then **K** is categorical in  $\lambda^+$ .

PROOF. Assume that **K** is *not* categorical in  $\lambda^+$ . We show that (2) of Lemma 14.5.7 holds so  $\mathfrak{s}$  is not unidimensional. Let  $M_0 \in \mathbf{K}_{\lambda}$  and let  $p \in \mathrm{gS}(M_0)$  be minimal. We consider two cases:

<u>Case 1</u>. There exists  $M \in \mathbf{K}_{\lambda^+}$ ,  $M_1 \in \mathbf{K}_{\lambda}$  with  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$  and an extension  $p' \in gS(M_1)$  of p so that p' is omitted in M.

Let  $c \in |M| \setminus |M_1|$ . Fix  $M' \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{\lambda}$  containing c so that  $M_1 \leq_{\mathbf{K}} M'$  and let  $q := \operatorname{gtp}(c/M_1; M')$ . We claim that  $q \perp_{w^k} p'$  (and so by Lemma 14.4.13,  $p' \perp q$ , as needed). Let  $N \in \mathbf{K}_{\lambda}$  be such that  $N \geq_{\mathbf{K}} M_1$  and let  $a, b \in |N|$  be such that  $p' = \operatorname{gtp}(b/M_1; N), q = \operatorname{gtp}(a/M_1; N)$ . We want to see that  $\langle ba \rangle$  is independent in  $(M_1, N)$ . We have that  $\operatorname{gtp}(a/M_1; N) = \operatorname{gtp}(c/M_1; M')$ , so let  $N' \in \mathbf{K}_{\lambda}$  with  $M' \leq_{\mathbf{K}} N'$  and  $f : N \xrightarrow[M_1]{M_1} N'$  witness it, i.e. f(a) = c. Let b' := f(b). We have that  $\operatorname{gtp}(b'/M'; N')$  extends p', and  $b' \notin |M'|$  since p' is omitted in M, hence by minimality  $\operatorname{gtp}(b'/M'; N')$  does not fork over  $M_1$ . In particular,  $\langle cb' \rangle$  is independent in  $(M_1, N')$ . By invariance and monotonicity,  $\langle ba \rangle$  is independent in  $(M_1, N)$ .

<u>Case 2</u>. Not Case 1: For every  $M \in \mathbf{K}_{\lambda^+}$ , every  $M_1 \in \mathbf{K}_{\lambda}$  with  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$ , every extension  $p' \in \mathrm{gS}(M_1)$  of p is realized in M.

By categoricity in  $\lambda$  and non-categoricity in  $\lambda^+$ , we can find  $M \in \mathbf{K}_{\lambda^+}$  with  $M_0 \leq_{\mathbf{K}} M$  and  $q_0 \in \mathrm{gS}(M_0)$  omitted in M. Let  $M_1 \in \mathbf{K}_{\lambda}$ ,  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$  and  $q \in \mathrm{gS}(M_1)$  extend  $q_0$  so that any extension of q to a model  $M' \leq_{\mathbf{K}} M$  in  $\mathbf{K}_{\lambda}$  does not fork over  $M_1$  (this exists by Lemma 14.5.9). Let  $p' \in \mathrm{gS}(M_1)$  be a nonalgebraic extension of p. By assumption, p is realized by some  $c \in |M|$ . Now by the same argument as above (reversing the roles of p' and q),  $p' \perp_{wk} q$ , hence  $p' \perp q$ , as desired.

REMARK 14.5.11. In fact, the second case cannot happen. Otherwise, we could use the conjugation property to show that **K** has no  $(p, \lambda)$ -Vaughtian pair (in the sense of [**GV06c**, Definition 3.1]) and apply [**GV06c**, Theorem 4.1] to get that **K** is categorical in  $\lambda^+$  in that case. Since the proof of case 2 is shorter, we prefer to let it stand.

For the converse of Lemma 14.5.10, we will use:

FACT 14.5.12 (Theorem 6.1 in [**GV06a**]). Assume that **K** is categorical in  $\lambda^+$ . Then there exists  $M \in \mathbf{K}_{\lambda}$  and a minimal type  $p \in gS(M)$  which is realized in every  $N \in \mathbf{K}_{\lambda}$  with  $M <_{\mathbf{K}} N$ .

REMARK 14.5.13. The proof of Fact 14.5.12 uses categoricity in  $\lambda$  in a strong way (it uses that the union of an increasing chain of limit models is limit).

LEMMA 14.5.14. If **K** is categorical  $\lambda^+$ , then  $\mathfrak{s}$  is unidimensional.

PROOF. By Fact 14.5.12, there exists  $M \in \mathbf{K}_{\lambda}$  and a minimal  $p \in gS(M)$  so that p is realized in every  $N >_{\mathbf{K}} M$ . Now assume for a contradiction that  $\mathbf{K}$  is *not* unidimensional. Then by Lemma 14.5.7, there exists a nonalgebraic  $q \in gS(M)$  such that  $p \perp q$ . By Lemma 14.4.13.(3) (note that M is limit by categoricity in  $\lambda$ ), there exists  $N \in \mathbf{K}_{\lambda}$  with  $N >_{\mathbf{K}} M$  so that p is omitted in N, a contradiction to the choice of p.

We have arrived to the main result of this section. For the convenience of the reader, we repeat Hypothesis 14.5.1.

THEOREM 14.5.15. Let  $\mathfrak{s}$  be a type-full good  $\lambda$ -frame on **K**. Assume that **K** is categorical in  $\lambda$ . Then  $\mathfrak{s}$  is unidimensional if and only if **K** is categorical in  $\lambda^+$ .

PROOF. By Lemmas 14.5.10 and 14.5.14.

# 14.6. Global orthogonality

Hypothesis 14.6.1.

- (1)  $\mathbf{K}$  is an AEC.
- (2)  $\theta > \mathrm{LS}(\mathbf{K})$  is a cardinal or  $\infty$ . We set  $\mathcal{F} := [\mathrm{LS}(\mathbf{K}), \theta)$ .
- (3)  $\mathfrak{s} = (\mathbf{K}_{\mathcal{F}}, \downarrow)$  is a type-full good  $\mathcal{F}$ -frame.

We start developing the theory of orthogonality and unidimensionality in a more global context (with no real loss, the reader can think of  $\theta = \infty$  as being the main case). The main problem is to show that for M sufficiently saturated, if  $p, q \in gS(M)$  do not fork over  $M_0$ , then  $p \perp q$  if and only if  $p \upharpoonright M_0 \perp q \upharpoonright M_0$ . This can be done with the conjugation property in case  $||M_0|| = ||M||$  but in general one needs to use more tools from the study of independent sequences. We will use Fact 14.2.6 without further mention. We will also use a few facts about independent sequences:

FACT 14.6.2 (Corollary 5.6.10). Independent sequences of length two satisfy the axioms of a good  $\mathcal{F}$ -frame. For example:

- (1) Monotonicity: If  $\langle ab \rangle$  is independent in  $(M_0, M, N)$  and  $M_0 \leq_{\mathbf{K}} M'_0 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} N'$ , then  $\langle ab \rangle$  is independent in  $(M'_0, M', N')$ .
- (2) Continuity: If  $\langle M_i : i \leq \delta \rangle$  is increasing continuous,  $M_{\delta} \leq_{\mathbf{K}} N$ , and  $\langle ab \rangle$  is independent in  $(M_0, M_i, N)$  for all  $i < \delta$ , then  $\langle ab \rangle$  is independent in  $(M_0, M_{\delta}, N)$ .

REMARK 14.6.3. Inside which frame do we work in when we say that  $\langle ab \rangle$  is independent, the global frame  $\mathfrak{s}$  or its restriction to a single cardinal? By monotonicity, the answer does not matter, i.e. the independent sequences are the same either way. Similarly, if  $\lambda > \mathrm{LS}(\mathbf{K})$  and  $M_0, M, N \in \mathbf{K}_{[\lambda,\theta)}^{\lambda-\mathrm{sat}}$ , then  $\langle ab \rangle$  is independent in  $(M_0, M, N)$  with respect to  $\mathfrak{s}$  if and only if it is independent in  $(M_0, N)$  with respect to  $\mathfrak{s} \upharpoonright \mathbf{K}_{[\lambda,\theta)}^{\lambda-\mathrm{sat}}$  (i.e. we can require the models witnessing the independence to be saturated). This is a simple consequence of the extension property.

We now define global orthogonality.

DEFINITION 14.6.4. Let  $M \in \mathbf{K}_{\mathcal{F}}$ . For  $p, q \in \mathrm{gS}(M)$  nonalgebraic, we write  $p \perp q$  for  $p \perp q$ , and  $p \perp q$  for  $p \perp q$ .

Note that a priori we need not have that if  $p \perp q$  and p', q' are nonforking extensions of p and q to big models, then  $p' \perp q'$ . This will be proven first (Lemma 14.6.7).

LEMMA 14.6.5. Let  $\delta$  be a limit ordinal. Let  $\langle M_i : i \leq \delta \rangle$  be increasing continuous in  $\mathbf{K}_{\mathcal{F}}$ . Let  $p, q \in \mathrm{gS}(M_{\delta})$  be nonalgebraic and assume that  $p \upharpoonright M_i \perp_{\mathrm{wk}} q \upharpoonright M_i$  for all  $i < \delta$ . Then  $p \perp q$ .

**PROOF.** By the continuity property of independent sequences (Fact 14.6.2).  $\Box$ 

The difference between the next lemma and the previous one is the use of  $\bot$  instead of  $\underset{_{\rm wk}}{\bot}.$ 

LEMMA 14.6.6. Let  $\delta$  be a limit ordinal. Let  $\langle M_i : i \leq \delta \rangle$  be increasing continuous in  $\mathbf{K}_{\mathcal{F}}$ . Let  $p, q \in \mathrm{gS}(M_{\delta})$  be nonalgebraic and assume that  $p \upharpoonright M_i \perp q \upharpoonright M_i$  for all  $i < \delta$ . Then  $p \perp q$ .

PROOF. By local character, there exists  $i < \delta$  so that both p and q do not fork over  $M_i$ . Without loss of generality, i = 0. Let  $\lambda := ||M_{\delta}||$ . If there exists  $i < \delta$ so that  $\lambda = ||M_i||$ , then the result follows from the definition of orthogonality. So assume that  $||M_i|| < \lambda$  for all  $i < \delta$ . Let  $M' \in \mathbf{K}_{\lambda}$  be such that  $M_{\delta} \leq_{\mathbf{K}} M'$  and let p', q' be the nonforking extensions to M' of p, q respectively. We want to see that  $p' \perp_{wk} q'$ . Let  $\langle M'_i : i \leq \delta \rangle$  be an increasing continuous resolution of M' such that  $M_i \leq_{\mathbf{K}} M'_i$  and  $||M'_i|| = ||M_i||$  for all  $i < \delta$ . We know that  $p' \upharpoonright M'_i$  does not fork over  $M_0$ , hence over  $M_i$  and similarly  $q' \upharpoonright M'_i$  does not fork over  $M_i$ . Therefore by definition of orthogonality,  $p' \upharpoonright M'_i \perp_{wk} q' \upharpoonright M'_i$ . By Lemma 14.6.5,  $p' \perp_{wk} q'$ .

LEMMA 14.6.7. Let  $M_0 \leq_{\mathbf{K}} M$  be both in  $\mathbf{K}_{\mathcal{F}}$ . Let  $p, q \in gS(M)$  be nonalgebraic so that both do not fork over  $M_0$ . If  $p \upharpoonright M_0 \perp q \upharpoonright M_0$ , then  $p \perp q$ .

PROOF. Let  $\delta := \operatorname{cf} \|M\|$ . Build  $\langle N_i : i \leq \delta \rangle$  increasing continuous such that  $N_0 = M_0, N_{\delta} = M$ , and  $p \upharpoonright N_i \perp q \upharpoonright N_i$  for all  $i \leq \delta$ . This is easy: at successor steps, we require  $\|N_i\| = \|N_{i+1}\|$  and use the definition of orthogonality. At limit steps, we use Lemma 14.6.6. Then  $p \upharpoonright N_{\delta} \perp q \upharpoonright N_{\delta}$ , but  $N_{\delta} = M$  so  $p \perp q$ .  $\Box$ 

QUESTION 14.6.8. Is the converse true? That is if  $M_0 \leq_{\mathbf{K}} M$  are in  $\mathbf{K}_{\mathcal{F}}$ ,  $p, q \in \mathrm{gS}(M)$  do not fork over  $M_0$  and  $p \perp q$ , do we have that  $p \upharpoonright M_0 \perp q \upharpoonright M_0$ ?

An answer to this question would be useful in order to transfer unidimensionality up in a more conceptual way than below. With a very mild additional hypothesis, we give a positive answer in Theorem 15.10.4 but this is not needed for the rest of the chapter.

We now go back to studying unidimensionality. We give a global definition:

DEFINITION 14.6.9. For  $\lambda \in \mathcal{F}$ , we say that  $\mathfrak{s}$  is  $\lambda$ -unidimensional if the following is *false*: for every limit  $M \in \mathbf{K}_{\lambda}$  and every nonalgebraic  $p \in \mathrm{gS}(M)$ , there exists a limit  $M' \in \mathbf{K}_{\lambda}$  with  $M \leq_{\mathbf{K}} M'$  and  $p', q \in \mathrm{gS}(M')$  so that p' extends p and  $p' \perp q$ .

REMARK 14.6.10. When  $\lambda > LS(\mathbf{K})$ ,  $\mathfrak{s}$  is  $\lambda$ -unidimensional if and only if  $\mathfrak{s} \upharpoonright \mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  is unidimensional (see Definition 14.5.2). If  $\mathbf{K}$  is categorical in  $LS(\mathbf{K})$ , this also holds when  $\lambda = LS(\mathbf{K})$  (if  $\mathbf{K}$  is not categorical in  $LS(\mathbf{K})$ , we do not know that  $\mathbf{K}^{LS(\mathbf{K})-\text{sat}}$  is an AEC).

Our next goal is to prove (assuming categoricity in  $LS(\mathbf{K})$ ) that  $\lambda$ -unidimensionality is equivalent to  $\mu$ -unidimensionality for every  $\lambda, \mu \in \mathcal{F}$ . We will use another characterization of  $\lambda$ -unidimensionality when  $\lambda > LS(\mathbf{K})$ . In that case, it is enough to check failure of unidimensionality with a single minimal type.

LEMMA 14.6.11. Let  $\lambda > LS(\mathbf{K})$  be in  $\mathcal{F}$ . The following are equivalent:

- (1)  $\mathfrak{s}$  is not  $\lambda$ -unidimensional.
- (2) There exists a saturated  $M \in \mathbf{K}_{\lambda}$  and nonalgebraic types  $p, q \in \mathrm{gS}(M)$  such that p is minimal and  $p \perp q$ .

Proof.

- (1) implies (2): Assume that  $\mathfrak{s}$  is not  $\lambda$ -unidimensional. Let  $M \in \mathbf{K}_{\lambda}^{\lambda-\text{sat}}$ and let  $p \in \mathrm{gS}(M)$  be minimal (exists by density of minimal types and uniqueness of saturated models). By Lemma 14.5.7, there exists  $q \in \mathrm{gS}(M)$  so that  $p \perp q$ , as desired.
- (2) implies (1): Let  $M \in \mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  and let  $p, q \in \mathrm{gS}(M)$  be nonalgebraic so that p is minimal and  $p \perp q$ . We show that  $\mathbf{K}^{\lambda-\text{sat}}$  is not categorical in  $\lambda^+$ , which is enough by Theorem 14.5.15. Fix  $N \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  with  $N \leq_{\mathbf{K}} M$  so that p does not fork over N. Build a strictly increasing continuous chain  $\langle M_i : i \leq \lambda^+ \rangle$  such that for all  $i < \lambda^+$ :
  - (1)  $M_i \in \mathbf{K}_{\lambda}^{\lambda-\mathrm{sat}}$ .
  - (2)  $M_0 = M$ .
  - (3) p is omitted in  $M_i$ .

This is enough, since then p is omitted in  $M_{\lambda^+}$  so  $M_{\lambda^+} \in \mathbf{K}_{\lambda^+}$  cannot be saturated. This is possible: at limits we take unions and for i = 0we set  $M_0 := M$ . Now let i = j + 1 be given. Let  $p' \in \mathrm{gS}(M_j)$  be the nonforking extension of p. By uniqueness of saturated models, there exists  $f : M_j \cong_N M_0$ . By uniqueness of nonforking extension, f(p') = p. By Lemma 14.4.13.(3), there exists  $M' \geq_{\mathbf{K}} M_0$  in  $\mathbf{K}_{\lambda}^{\lambda-\mathrm{sat}}$  so that p is omitted in M'. Let  $M_{j+1} := f^{-1}[M']$ . Then p' is omitted in  $M_{j+1}$ . Since p is minimal, p is omitted in  $|M_{j+1}| \setminus |M_j|$ , and hence by induction in  $M_{j+1}$ .

An issue in transferring unidimensionality up is that we do not have a converse to Lemma 14.6.7 (see Question 14.6.8), so we will "cheat" and use the following transfer which follows from the proof of [**GV06a**, Theorem 6.3] (recall that we are assuming Hypothesis 14.6.1).

FACT 14.6.12. If **K** is categorical in  $LS(\mathbf{K})$  and  $LS(\mathbf{K})^+$ , then **K** is categorical in all  $\mu \in [LS(\mathbf{K}), \theta]$ .

For the convenience of the reader, we have repeated Hypothesis 14.6.1 in the statement of the next two theorems.

THEOREM 14.6.13. Let  $\theta > LS(\mathbf{K})$  be a cardinal or  $\infty$ . and let  $\mathcal{F} := [LS(\mathbf{K}), \theta)$ . Let  $\mathfrak{s}$  be a type-full good  $\mathcal{F}$ -frame on  $\mathbf{K}_{\mathcal{F}}$ .

Assume that **K** is categorical in LS(**K**). Let  $\lambda$  and  $\mu$  both be in  $\mathcal{F}$ . Then  $\mathfrak{s}$  is  $\lambda$ -unidimensional if and only if  $\mathfrak{s}$  is  $\mu$ -unidimensional.

PROOF. Without loss of generality,  $\mu < \lambda$ . We first show that if  $\mathfrak{s}$  is not  $\mu$ unidimensional, then  $\mathfrak{s}$  is not  $\lambda$ -unidimensional. Assume that  $\mathfrak{s}$  is not  $\mu$ -unidimensional. Let  $M_0 \in \mathbf{K}_{\mu}^{\mu\text{-sat}}$  and let  $p \in \mathrm{gS}(M_0)$  be minimal (exists by density of minimal types). By definition (and the proof of Lemma 14.5.7), there exists  $q \in \mathrm{gS}(M_0)$ so that  $p \perp q$ . Now let  $M \in \mathbf{K}_{\lambda}^{\lambda\text{-sat}}$  be such that  $M_0 \leq_{\mathbf{K}} M$ . Let p', q' be the nonforking extensions to M of p and q respectively. By Lemma 14.6.7,  $p' \perp q'$ . By Lemma 14.6.11,  $\mathfrak{s}$  is not  $\lambda$ -unidimensional.

Conversely, assume that  $\mathfrak{s}$  is  $\mu$ -unidimensional. By the first part,  $\mathfrak{s}$  is LS(**K**)unidimensional. By Theorem 14.5.15, **K** is categorical in LS(**K**)<sup>+</sup>. By Fact 14.6.12, **K**, and hence  $\mathbf{K}^{\lambda\text{-sat}}$ , is categorical in  $\lambda^+$ . By Theorem 14.5.15 again,  $\mathfrak{s}$  is  $\lambda$ unidimensional.

We obtain the promised categoricity transfer. Note that it suffices to assume that  $\mathbf{K}^{\lambda\text{-sat}}$  (instead of  $\mathbf{K}$ ) is categorical in  $\lambda^+$ .

THEOREM 14.6.14. Let  $\theta > LS(\mathbf{K})$  be a cardinal or  $\infty$ . and let  $\mathcal{F} := [LS(\mathbf{K}), \theta)$ . Let  $\mathfrak{s}$  be a type-full good  $\mathcal{F}$ -frame on  $\mathbf{K}_{\mathcal{F}}$ .

Assume that **K** is categorical in LS(**K**) and let  $\lambda \in \mathcal{F}$ . If  $\mathbf{K}^{\lambda-\text{sat}}$  is categorical in  $\lambda^+$ , then **K** is categorical in every  $\mu \in [\text{LS}(\mathbf{K}), \theta]$ .

PROOF. Assume that  $\mathbf{K}^{\lambda\text{-sat}}$  is categorical in  $\lambda^+$ . We prove by induction on  $\mu \in [\mathrm{LS}(\mathbf{K}), \theta]$  that  $\mathbf{K}$  is categorical in  $\mu$ . By assumption,  $\mathbf{K}$  is categorical in  $\mathrm{LS}(\mathbf{K})$ . Now let  $\mu \in (\mathrm{LS}(\mathbf{K}), \theta]$  and assume that  $\mathbf{K}$  is categorical in every  $\mu_0 \in [\mathrm{LS}(\mathbf{K}), \mu)$ . If  $\mu$  is limit, then it is easy to see that every model of size  $\mu$  must be saturated, hence  $\mathbf{K}$  is categorical in  $\mu$ . Now assume that  $\mu$  is a successor, say  $\mu = \mu_0^+$  for  $\mu_0 \in \mathcal{F}$ . By assumption,  $\mathbf{K}^{\lambda\text{-sat}}$  is categorical in  $\lambda^+$ . By Theorem 14.5.15,  $\mathfrak{s}$  is  $\lambda$ -unidimensional. By Theorem 14.6.13,  $\mathfrak{s}$  is  $\mu_0$ -unidimensional. By Theorem 14.5.15,  $\mathbf{K}^{\mu_0\text{-sat}}$  is categorical in  $\mu_0^+$ . By the induction hypothesis,  $\mathbf{K}$  is categorical in  $\mu_0$ , hence  $\mathbf{K}^{\mu_0\text{-sat}} = \mathbf{K}_{\geq \mu_0}$ , so  $\mathbf{K}$  is categorical in  $\mu_0^+ = \mu$ , as desired.  $\Box$ 

# CHAPTER 15

# Downward categoricity from a successor inside a good frame: part II: applications

This chapter and the preceeding one are based on [Vas17a].

# Abstract

We present applications of Theorem 14.0.14 from the preceeding chapter.

# 15.1. Background

The definition of superstability below is already implicit in [SV99] and several variants were studied in, e.g. [Van06, GVV16], Chapters 6 7, 9, 10. We will use the statement from Definition 6.10.1.

DEFINITION 15.1.1. **K** is  $\mu$ -superstable (or superstable in  $\mu$ ) if:

- (1)  $\mu \geq \text{LS}(\mathbf{K})$ .
- (2)  $\mathbf{K}_{\mu}$  is nonempty, has amalgamation, joint embedding, and no maximal models.
- (3) **K** is stable in  $\mu$ , and:
- (4)  $\mu$ -splitting in **K** satisfies the following locality property: for every limit ordinal  $\delta < \mu^+$  and every increasing continuous sequence  $\langle M_i : i \leq \delta \rangle$  in  $\mathbf{K}_{\mu}$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \delta$ , if  $p \in \mathrm{gS}(M_{\delta})$ , then there exists  $i < \delta$  so that p does not  $\mu$ -split over  $M_i$ .

We will use the following without comments. See Fact 10.4.8.

FACT 15.1.2. If  $\mathfrak{s}$  is a type-full good  $\lambda$ -frame on **K**, then **K** is  $\lambda$ -superstable.

In the setup of this chapter, superstability follows from categoricity. If (as will be the case in most of this chapter) the AEC is categorical in a successor, this is due to Shelah and appears as [She99, Lemma 6.3]. The heart of the proof in the general case appears as [SV99, Theorem 2.2.1] and a full axiomatic proof appears in Chapter 20.

FACT 15.1.3 (The Shelah-Villaveces theorem). If **K** has amalgamation, no maximal models, and is categorical in a  $\lambda > LS(\mathbf{K})$ , then **K** is  $LS(\mathbf{K})$ -superstable.

Together with superstability, a powerful tool is the symmetry property for splitting, first isolated by VanDieren [Van16a]:

DEFINITION 15.1.4. Let  $\mu \geq \text{LS}(\mathbf{K})$  and assume that  $\mathbf{K}$  has amalgamation in  $\mu$ .  $\mathbf{K}$  exhibits symmetry for  $\mu$ -splitting (or  $\mu$ -symmetry for short) if whenever models  $M, M_0, N \in \mathbf{K}_{\mu}$  and elements a and b satisfy the conditions 1-4 below, then there exists  $M^b$  a limit model over  $M_0$ , containing b, so that  $\text{gtp}(a/M^b)$  does not  $\mu$ -split over N.

- (1) M is universal over  $M_0$  and  $M_0$  is a limit model over N.
- (2)  $a \in M \setminus M_0$ .
- (3)  $gtp(a/M_0)$  is non-algebraic and does not  $\mu$ -split over N.
- (4) gtp(b/M) is non-algebraic and does not  $\mu$ -split over  $M_0$ .

When the class is tame, symmetry follows from superstability (Corollary 10.6.9) and superstability transfers upward (Proposition 6.10.10) hence they both hold everywhere:

FACT 15.1.5. If **K** has amalgamation, is  $LS(\mathbf{K})$ -tame, and is  $LS(\mathbf{K})$ -superstable, then **K** is superstable and has symmetry in every  $\mu \ge LS(\mathbf{K})$ .

One consequence of the symmetry property is given by the following more precise statement of Fact 14.2.6 (see Lemma 10.2.20 and [Van16b, Theorem 1]):

FACT 15.1.6. Assume that **K** has amalgamation. Let  $\chi > \text{LS}(\mathbf{K})$ . If for every  $\mu \in [\text{LS}(\mathbf{K}), \chi)$ , **K** is superstable in  $\mu$  and  $\mu^+$  and has symmetry in  $\mu^+$ , then  $\mathbf{K}^{\chi\text{-sat}}$  is an AEC with  $\text{LS}(\mathbf{K}^{\chi\text{-sat}}) = \chi$ .

We will use the following consequences of categoricity in a suitable cardinal.

FACT 15.1.7. Assume that **K** has amalgamation and no maximal models. Let  $\lambda > LS(\mathbf{K})$  be such that **K** is categorical in  $\lambda$ .

- (1) (Corollary 10.7.2) If  $\lambda \ge H_1$  or the model of size  $\lambda$  is  $LS(\mathbf{K})^+$ -saturated (e.g. if  $cf \lambda > LS(\mathbf{K})$ ), then **K** has  $LS(\mathbf{K})$ -symmetry.
- (2) (Corollary 10.7.4) If  $\lambda \ge h(\mathrm{LS}(\mathbf{K})^+)$ , then the model of size  $\lambda$  is  $\mathrm{LS}(\mathbf{K})^+$ -saturated.

As a special case, we obtain the following result that is already stated in [She99, Claim I.6.7].

COROLLARY 15.1.8. Assume that **K** has amalgamation and no maximal models. Let  $\lambda > \mu > \text{LS}(\mathbf{K})$  be such that **K** is categorical in  $\lambda$ . If the model of size  $\lambda$  is  $\mu^+$ -saturated, then  $\mathbf{K}^{\mu\text{-sat}}$  is an AEC with  $\text{LS}(\mathbf{K}^{\mu\text{-sat}}) = \mu$ .

PROOF. By Fact 15.1.3, **K** is superstable in every  $\chi \in [LS(\mathbf{K}), \lambda)$ . By Fact 15.1.7, **K** has symmetry in every  $\chi \in [LS(\mathbf{K}), \mu]$ . Now apply Fact 15.1.6.

The following fact tells us that we can often assume without loss of generality that a categorical AECs with amalgamation also has no maximal models. The proof is folklore (see e.g. Proposition 6.10.13).

FACT 15.1.9. Assume that **K** has amalgamation. Let  $\lambda \geq \text{LS}(\mathbf{K})$  be such that **K** has joint embedding in  $\lambda$ . Then there exists  $\chi < \beth_{(2^{\text{LS}(\mathbf{K})})^+}$  and an AEC  $\mathbf{K}^*$  such that:

- (1)  $\mathbf{K}^* \subseteq \mathbf{K}$  and  $\mathbf{K}^*$  has the same strong substructure relation as  $\mathbf{K}$ .
- (2)  $LS(\mathbf{K}^*) = LS(\mathbf{K}).$
- (3)  $\mathbf{K}^*$  has amalgamation, joint embedding, and no maximal models.
- (4)  $\mathbf{K}_{\geq\min(\lambda,\chi)} = \mathbf{K}^*_{\geq\min(\lambda,\chi)}.$

Let us also recall the definition of tameness (first isolated in [**GV06b**]) and weak tameness (already implicit in [**She99**]). We use the notation from [**Bal09**, Definition 11.6]

DEFINITION 15.1.10 (Tameness). Let  $\chi, \mu$  be cardinals with  $\mathrm{LS}(\mathbf{K}) \leq \chi \leq \mu$ . Assume that  $\mathbf{K}_{[\chi,\mu]}$  has amalgamation. **K** is  $(\chi,\mu)$ -tame if for any  $M \in \mathbf{K}_{\mu}$ , any  $p, q \in \mathrm{gS}(M)$ , if  $p \neq q$ , there exists  $M_0 \in \mathbf{K}_{\chi}$  with  $M_0 \leq_{\mathbf{K}} M$  and  $p \upharpoonright M_0 \neq q \upharpoonright M_0$ . For  $\theta \geq \mu$ , **K** is  $(\chi, < \theta)$ -tame if it is  $(\chi, \mu)$ -tame for every  $\mu \in [\chi, \theta)$ .  $(\chi, \leq \theta)$ -tame means  $(\chi, < \theta^+)$ -weakly tame. Finally, **K** is  $\chi$ -tame if it is  $(\chi, \mu)$ -tame.

Let us also define **K** is  $(\chi, \mu)$ -weakly tame to mean that for any saturated  $M \in \mathbf{K}_{\mu}$ , any  $p, q \in \mathrm{gS}(M)$ , if  $p \neq q$ , there exists  $M_0 \in \mathbf{K}_{\chi}$  with  $M_0 \leq_{\mathbf{K}} M$  and  $p \upharpoonright M_0 \neq q \upharpoonright M_0$ . Define variations such as  $(\chi, < \mu)$ -weakly tame as above.

REMARK 15.1.11. Tameness says that type over *any* models are determined by their small restrictions. Weak tameness says that only types over *saturated* models have this property. While there is no known example of an AEC that is weakly tame but not tame, it is known that weak tameness follows from categoricity in a suitable cardinal (but the corresponding result for non-weak tameness is open, see **[GV06a**, Conjecture 1.5]), see Section 15.2.

It was noticed in Chapter 4 (and further improvements in Section 6.10 or Corollary 10.6.14) that tameness can be combined with superstability to build a good frame. This can also be done using only weak tameness:

FACT 15.1.12 (Theorem 10.6.4). Let  $\lambda > LS(\mathbf{K})$ . Assume that **K** is superstable in every  $\mu \in [LS(\mathbf{K}), \lambda]$  and has  $\lambda$ -symmetry.

If **K** is  $(LS(\mathbf{K}), \lambda)$ -weakly tame, then there exists a type-full good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  (so in particular,  $\mathbf{K}_{\lambda}^{\lambda-\text{sat}}$  is the initial segment of an AEC).

Once we have a good  $\lambda$ -frame, we can enlarge it so that the forking relation works over larger models.

FACT 15.1.13 (Corollary 5.6.9). Let  $\theta > \lambda \ge LS(\mathbf{K})$ . Let  $\mathcal{F} := [\lambda, \theta)$ . Assume that  $\mathbf{K}_{\mathcal{F}}$  has amalgamation. Let  $\mathfrak{s}$  be a type-full good  $\lambda$ -frame on  $\mathbf{K}_{\lambda}$ . If  $\mathbf{K}$  is  $(\lambda, < \theta)$ -tame, then there exists a type-full good  $\mathcal{F}$ -frame  $\mathfrak{s}'$  extending  $\mathfrak{s}: \mathfrak{s}' \upharpoonright \mathbf{K}_{\lambda} = \mathfrak{s}$ .

Assuming only weak tameness, we can show that if  $\mathfrak{s}$  is a (type-full) good  $\mu$ -frame and  $\mathfrak{s}'$  is a good  $\lambda$ -frame with  $\mu < \lambda$  and the underlying class of  $\mathfrak{s}'$  is the saturated models in the underlying class of  $\mathfrak{s}$ , then forking in  $\mathfrak{s}'$  can be described in terms of forking in  $\mathfrak{s}$ . This is proven as Theorem 15.7.5 and is used to replace tameness by weak tameness in the main theorem.

#### 15.2. Weak tameness from categoricity

We quote a result of Shelah from [She09a, Chapter IV] on deriving weak tameness from categoricity and combine it with the corresponding results in Chapter 10. We derive a small improvement on some of the Hanf numbers, positively answering a question of Baldwin [Bal09, Question 11.16] (see also [Bal09, Remark 14.15]) which asked whether it was possible to obtain  $\chi$ -weak tameness for some  $\chi < H_1$ rather than ( $< H_1$ )-weak tameness. We give two applications in AECs with amalgamation and no maximal models that are categorical in a high-enough cardinal that is still *below* the Hanf number: the construction of a good frame and a non-trivial restriction on the categoricity spectrum.

The following appears as [She99, Main Claim II.2.3] (a simplified and improved argument is in [Bal09, Theorem 11.15]):

FACT 15.2.1. Assume that **K** has amalgamation. Let  $\lambda > \mu \ge H_1$ . Assume that **K** is categorical in  $\lambda$ , and the model of cardinality  $\lambda$  is  $\mu^+$ -saturated. Then there exists  $\chi < H_1$  such that **K** is  $(\chi, \mu)$ -weakly tame.

As opposed to Fact 15.2.1, the following also applies when the categoricity cardinal is below the Hanf number.

- FACT 15.2.2 (Claim IV.7.2 in [She09a]). Let  $\mu > LS(\mathbf{K})$ . If:
- (1)  $\mathbf{K}_{<\mu}$  has amalgamation.
- (2) cf  $\mu > LS(\mathbf{K})$ .
- (3)  $\Phi$  is a proper for linear orders, and if  $\theta \in (LS(\mathbf{K}), \mu)$ , I is a  $\theta$ -wide<sup>1</sup> linear order, then  $EM_{\tau(\mathbf{K})}(I, \Phi)$  is  $\theta$ -saturated.

Then there exists  $\chi \in (LS(\mathbf{K}), \mu)$  such that **K** is  $(\chi, < \mu)$ -weakly tame.

Condition (3) in Fact 15.2.2 can be derived from categoricity if the model in the categoricity cardinal is sufficiently saturated. This is implicit in [She99] and appears as [Bal09, Lemma 10.11].

FACT 15.2.3. If **K** has amalgamation and no maximal models,  $\mu > \text{LS}(\mathbf{K})$ , **K** is categorical in a  $\lambda \ge \mu$ , so that the model of size  $\lambda$  is  $\mu$ -saturated, then for every  $\Phi$  proper for linear orders, if  $\theta \in (\text{LS}(\mathbf{K}), \mu)$  and I is a  $\theta$ -wide linear order, we have that  $\text{EM}_{\tau(\mathbf{K})}(I, \Phi)$  is  $\theta$ -saturated.

Combining these facts, we obtain the following result. Note that the second part is a slight improvement on Fact 15.2.1, as the model of size  $\lambda$  is allowed to be  $H_1$ saturated. Moreover the amount of weak tameness  $\chi$  can be chosen independently of  $\mu$ :

THEOREM 15.2.4. Assume that **K** has amalgamation. Let  $\lambda > LS(\mathbf{K})$  be such that **K** is categorical in  $\lambda$ .

- (1) Let  $\mu$  be a limit cardinal such that  $\operatorname{cf} \mu > \operatorname{LS}(\mathbf{K})$ . If  $\mathbf{K}$  has no maximal models and the model of size  $\lambda$  is  $\mu$ -saturated, then there exists  $\chi < \mu$  such that  $\mathbf{K}$  is  $(\chi, < \mu)$ -weakly tame.
- (2) If the model of size  $\lambda$  is  $H_1$ -saturated, then there exists  $\chi < H_1$  such that whenever  $\mu \ge H_1$  is so that the model of size  $\lambda$  is  $\mu$ -saturated, we have that **K** is  $(\chi, < \mu)$ -weakly tame.

### Proof.

- (1) By Fact 15.2.2 (using Fact 15.2.3 to see that (3) is satisfied).
- (2) Without loss of generality (Fact 15.1.9), **K** has no maximal models. By the first part (with  $\mu$  there standing for  $H_1$  here), there exists  $\chi < H_1$ such that **K** is  $(\chi, < H_1)$ -weakly tame. Now assume that the model of size  $\lambda$  is  $\mu$ -saturated, for  $\mu > H_1$ . Let  $\mu' \in [H_1, \mu)$ . We show that **K** is  $(\chi, \mu')$ -weakly tame. By Fact 15.2.1 (with  $\mu$  there standing for  $\mu'$  here), there exists  $\chi' < H_1$  such that **K** is  $(\chi', \mu')$ -weakly tame. In particular, **K** is  $(H_1, \mu')$ -weakly tame. Now by Corollary 15.1.8,  $\mathbf{K}^{H_1$ -sat is an AEC with  $\mathrm{LS}(\mathbf{K}^{H_1\text{-sat}}) = H_1$ . Thus we can combine  $(\chi, H_1)$ -weak and  $(H_1, \mu')$ -weak tameness to get  $(\chi, \mu)$ -weak tameness, as desired.

<sup>&</sup>lt;sup>1</sup>A linear order is  $\theta$ -wide if for every  $\theta_0 < \theta$ , I contains an increasing sequence of length  $\theta_0^+$ , see [**She09a**, Definition IV.0.14.(1)].

We give two applications of (the first part of) Theorem 15.2.4. First, we obtain an improvement on the Hanf number for the construction of a good frame in Corollary 10.7.9 ( $\mu$  below can be less than  $H_1$ , e.g.  $\mu = \aleph_{\text{LS}(\mathbf{K})^+}$ ).

THEOREM 15.2.5. Assume that **K** has amalgamation and no maximal models. Let  $\mu$  be a limit cardinal such that  $\operatorname{cf} \mu > \operatorname{LS}(\mathbf{K})$  and assume that **K** is categorical in a  $\lambda \geq \mu$ . If the model of size  $\lambda$  is  $\mu$ -saturated, then there exists  $\chi < \mu$  such that for all  $\mu_0 \in [\chi, \mu)$ , there is a good  $\mu_0$ -frame on  $\mathbf{K}^{\mu_0\text{-sat}}$ .

PROOF. By Fact 15.1.7, **K** has  $\mu_0$ -symmetry for every  $\mu_0 < \chi$ . By Fact 15.1.3, **K** is also superstable in every  $\mu_0 \in [LS(K), \lambda)$ . Now by Theorem 15.2.4, there exists  $\chi < \mu$  so that **K** is  $(\chi, < \mu)$ -weakly tame. We finish by applying Fact 15.1.12.  $\Box$ 

Second, we can study the categoricity spectrum *below* the Hanf number of an AEC with amalgamation and no maximal models. While it is known that the categoricity spectrum in such AECs must be a closed set (see the proof of  $[\mathbf{GV06c}, \mathbf{Corollary } 4.3]$ ), we show (in ZFC) that there are other restrictions:

THEOREM 15.2.6. Assume that **K** has amalgamation and arbitrarily large models. Let  $\mu$  be a limit cardinal such that cf  $\mu > \text{LS}(\mathbf{K})$ . If **K** is categorical in unboundedly many successor cardinals below  $\mu$ , then there exists  $\mu_0 < \mu$  such that **K** is categorical in every  $\lambda \in [\mu_0, \mu]$ .

In particular (setting  $\mu := \aleph_{\mathrm{LS}(\mathbf{K})^+}$ ), if **K** is categorical in  $\aleph_{\alpha+1}$  for unboundedly many  $\alpha < \mathrm{LS}(\mathbf{K})^+$ , then there exists  $\alpha_0 < \mathrm{LS}(\mathbf{K})^+$  such that **K** is categorical in  $\aleph_\beta$  for every  $\beta \in [\alpha_0, \mathrm{LS}(\mathbf{K})^+]$ .

Before starting the proof, we make a remark:

REMARK 15.2.7. Fact 14.1.6 generalizes to AECs that are only  $LS(\mathbf{K})$ -weakly tame, or just  $(LS(\mathbf{K}), < \theta)$ -weakly tame (in the second case, we can only conclude categoricity up to and including  $\theta$ ). This is implicit in [**GV06c**, **GV06a**] and stated explicitly in Chapter 13 of [**Bal09**].

PROOF OF THEOREM 15.2.6. Without loss of generality (Fact 15.1.9), **K** has no maximal models. By amalgamation, every model of size  $\mu$  is saturated. In particular **K** is categorical in  $\mu$ . By Theorem 15.2.4 (with  $\lambda, \mu$  there standing for  $\mu, \mu$  here), there exists  $\mu'_0 < \mu$  such that **K** is  $(\mu'_0, < \mu)$ -weakly tame. By making  $\mu'_0$ bigger if necessary, we can assume without loss of generality that  $\mu'_0 > \text{LS}(\mathbf{K})$  and **K** is categorical in  $\mu_0 := (\chi'_0)^+$ . By the upward categoricity transfer of Grossberg and VanDieren (Fact 14.1.6, keeping in mind Remark 15.2.7), **K** is categorical in every  $\lambda \in [\mu_0, \mu]$ .

## 15.3. Shelah's omitting type theorem

In this section, we give a nonlocal proof of the improvement of the bounds in [She99] from  $\beth_{H_1}$  to  $H_1$  using the methods of [She99]. We will present a more powerful local proof in the next sections. We also give several partial categoricity transfers in AECs with amalgamation, including Theorem 15.3.8 which says that in a tame AEC with amalgamation, categoricity in *some* cardinal (above the tameness cardinal) implies categoricity in a proper class of cardinals. The main tool is a powerful generalization of Morley's omitting type theorem (Fact 15.3.3), an early form of which appears in [MS90].

All throughout, we assume:

HYPOTHESIS 15.3.1. K is an AEC with amalgamation.

As a motivation, we first state Morley's omitting type theorem for AECs [She99, II.1.10]. We state a slightly stronger conclusion (replacing  $H_1$  by some  $\chi < H_1$ ) that is implicit e.g. in [She99] but to the best of our knowledge, a proof of this stronger result has not appeared in print before. We include a proof (similar to the proof of [BG, Theorem 5.4], though there is an additional step involved) for the convenience of the reader.

FACT 15.3.2 (Morley's omitting type theorem for AECs). Let  $\lambda > LS(\mathbf{K})$ . If every model in  $\mathbf{K}_{\lambda}$  is  $LS(\mathbf{K})^+$ -saturated, then there exists  $\chi < H_1$  such that every model in  $\mathbf{K}_{\geq\chi}$  is  $LS(\mathbf{K})^+$ -saturated.

PROOF SKETCH. Without loss of generality (Fact 15.1.9), **K** has no maximal models. Suppose the conclusion fails. Then for every  $\chi \in [LS(\mathbf{K}), H_1)$ , there exists  $M_{\chi} \in \mathbf{K}_{\chi}$  which is not  $LS(\mathbf{K})^+$ -saturated. Pick witnesses  $M_{0,\chi} \leq_{\mathbf{K}} M_{\chi}$ and  $p_{\chi} \in gS(M_{0,\chi})$  such that  $||M_{0,\chi}|| = LS(\mathbf{K})$  and  $M_{\chi}$  omits  $p_{\chi}$ . Now there are only  $2^{LS(K)}$  isomorphism types of Galois types over models of size  $LS(\mathbf{K})$ , and cf  $H_1 = (2^{LS(\mathbf{K})})^+ > 2^{LS(\mathbf{K})}$ , so there exists  $N \in \mathbf{K}_{LS(\mathbf{K})}$ ,  $p \in gS(M)$ , and an unbounded  $S \subseteq [LS(\mathbf{K}), H_1)$  such that for all  $\chi \in S$ ,  $p_{\chi}$  is isomorphic to p (in the natural sense). Look at the AEC  $\mathbf{K}_{\neg p}$  of all the models of  $\mathbf{K}$  omitting p, with constants added for N (see e.g. the definition of  $\mathbf{K}^+$  in the proof of [**BG**, Theorem 5.4]). For each  $\chi \in S$ , an appropriate expansion of a copy of  $M_{\chi}$  is in  $\mathbf{K}_{\neg p}$ .  $\mathbf{K}_{\neg p}$ has Löwenheim-Skolem-Taski number  $LS(\mathbf{K})$ , so by Shelah's presentation theorem and Morley's omitting type theorem (for first-order theories),  $\mathbf{K}_{\neg p}$  has arbitrarily large models, contradicting the assumptions on  $\lambda$ .

A generalization of Fact 15.3.2 is what we call Shelah's omitting type theorem. The statement appears (in a more general form) in [**She99**, Lemma II.1.6], but the full proof (for models of an  $\mathbb{L}_{\kappa,\omega}$  theory,  $\kappa$  a strongly compact cardinal) can already be found in [**MS90**, Proposition 3.3] (see also Will Boney's note [**Bonb**]). We state a simplified version:

FACT 15.3.3 (Shelah's omitting type theorem). Let  $M_0 \leq_{\mathbf{K}} M$  both be in  $\mathbf{K}_{\geq \mathrm{LS}(\mathbf{K})}$  and let  $p \in \mathrm{gS}(M_0)$ . Assume that M omits  $p/E_{\mathrm{LS}(\mathbf{K})}$ . That is, for every  $a \in |M|$ , there is  $M'_0 \leq_{\mathbf{K}} M_0$  with  $||M'_0|| = \mathrm{LS}(\mathbf{K})$  such that  $\mathrm{gtp}(a/M'_0; M) \neq p \upharpoonright M'_0$ .

If  $\beth_{(2^{\text{LS}(\mathbf{K})})^+}(||M_0||) \leq ||M||$ , then there is a non-LS(**K**)<sup>+</sup>-saturated model in every cardinal.

Note that taking  $||M_0|| = LS(\mathbf{K})$ , we recover Morley's omitting type theorem for AECs. Note also that when  $\mathbf{K}$  is  $LS(\mathbf{K})$ -tame, M above omits  $p/E_{LS(\mathbf{K})}$  if and only if M omits p. The following two direct consequences are implicit in [She99].

LEMMA 15.3.4. Let  $LS(\mathbf{K}) < \lambda$  and let  $LS(\mathbf{K}) \leq \chi < \mu$ . Assume that:

- (1)  $\beth_{(2^{LS(\mathbf{K})})^+}(\chi) \le \mu.$
- (2) **K** is  $(LS(\mathbf{K}), \leq \chi)$ -weakly tame.
- (3) For every  $\chi_0 \in [\mathrm{LS}(\mathbf{K})^+, \chi], \mathbf{K}^{\chi_0 \text{-sat}}$  is an AEC with  $\mathrm{LS}(\mathbf{K}^{\chi_0 \text{-sat}}) = \chi_0$ .

If every model in  $\mathbf{K}_{\lambda}$  is  $\mathrm{LS}(\mathbf{K})^+$ -saturated, then every model in  $\mathbf{K}_{\mu}$  is  $\chi^+$ -saturated.

PROOF. Assume that every model in  $\mathbf{K}_{\lambda}$  is  $\mathrm{LS}(\mathbf{K})^+$ -saturated. We prove that for all  $\chi_0 \in [\mathrm{LS}(\mathbf{K}), \chi]$ , every model in  $\mathbf{K}_{\mu}$  is  $\chi_0^+$ -saturated. We proceed by induction on  $\chi_0$ . If  $\chi_0 = \mathrm{LS}(\mathbf{K})$ , take  $M \in \mathbf{K}_{\mu}$  and  $M_0 \leq_{\mathbf{K}} M$  of size  $\chi_0$ . If there is a type over  $M_0$  omitted in  $\mathbf{K}_{\mu}$ , then by Fact 15.3.3, there is a non-LS( $\mathbf{K}$ )<sup>+</sup>-saturated model of size  $\lambda$ , a contradiction. Therefore every model in  $\mathbf{K}_{\mu}$  is LS( $\mathbf{K}$ )<sup>+</sup>-saturated.

Now let  $\chi_0 > \mathrm{LS}(\mathbf{K})$  and assume inductively that every model in  $\mathbf{K}_{\mu}$  is  $\chi_0$ -saturated. Let  $M \in \mathbf{K}_{\mu}$  and let  $M_0 \leq_{\mathbf{K}} M$  have size  $\chi_0$ . Since  $\mathrm{LS}(\mathbf{K}^{\chi_0-\mathrm{sat}}) = \chi_0$  and M is  $\chi_0$ -saturated, we take enlarge  $M_0$  if necessary to assume without loss of generality that  $M_0$  is  $\chi_0$ -saturated. Let  $p \in \mathrm{gS}(M_0)$ . Then by the weak tameness hypothesis  $p/E_{\mathrm{LS}(\mathbf{K})} = p$ . So if M omits p, then by Fact 15.3.3,  $\mathbf{K}_{\lambda}$  has a non- $\mathrm{LS}(\mathbf{K})^+$ -saturated model, a contradiction, so M realizes p, as needed.

REMARK 15.3.5. We only use above that  $LS(\mathbf{K}^{\chi_0\text{-sat}}) = \chi_0$ , not that  $\mathbf{K}^{\chi_0\text{-sat}}$ is an AEC. That is, we only use that for  $M \in \mathbf{K}^{\chi_0\text{-sat}}$  and  $A \subseteq |M|$ , there exists a  $\chi_0$ -saturated  $M_0 \leq_{\mathbf{K}} M$  such that  $||M_0|| \leq |A| + \chi_0$ .

LEMMA 15.3.6. Let  $LS(\mathbf{K}) < \mu < \lambda$ . Assume that:

- (1) **K** is categorical in  $\lambda$ .
- (2)  $\mu = \beth_{\delta}$ , for some limit ordinal  $\delta$  divisible by  $(2^{\text{LS}(\mathbf{K})})^+$ .
- (3) **K** is  $(LS(\mathbf{K}), < \mu)$ -weakly tame.

If the model of size  $\lambda$  is  $\mu$ -saturated, then **K** is categorical in  $\mu$ .

PROOF. Without loss of generality (by Fact 15.1.9), **K** has joint embedding and no maximal models. Thus it is enough to show that every model of size  $\mu$  is saturated. Let  $\chi \in [\text{LS}(\mathbf{K}), \mu)$ . We have to check that the hypotheses of Lemma 15.3.4 hold. The only problematic part is to see that  $\mathbf{K}^{\chi\text{-sat}}$  is an AEC with  $\text{LS}(\mathbf{K}^{\chi\text{-sat}}) = \chi$  (when  $\chi > \text{LS}(\mathbf{K})$ ). But this holds by Corollary 15.1.8.

We obtain the following downward transfer result that slightly improves on Corollary 10.7.7:

COROLLARY 15.3.7. Let  $LS(\mathbf{K}) < \mu < \lambda$  be such that:

(1) **K** is categorical in  $\lambda$ .

(2)  $\mu = \beth_{\delta}$ , for some limit ordinal  $\delta$  divisible by  $H_1$ .

If the model of size  $\lambda$  is  $\mu$ -saturated (e.g. if  $cf \lambda \geq \mu$  or by Fact 15.1.7 if  $\lambda \geq \sup_{\mu_0 < \mu} h(\mu_0^+)$ ), then **K** is categorical in  $\mu$ .

PROOF. By Theorem 15.2.4, there exists  $\chi < H_1$  such that **K** is  $(\chi, < \mu)$ -weakly tame. Now apply Lemma 15.3.6 to  $\mathbf{K}_{>\chi}$ .

When tameness holds instead of weak tameness, we obtain the following generalization of the second main result of [**MS90**]:

THEOREM 15.3.8. If **K** is LS(**K**)-tame, has arbitrarily large models, and is categorical in a  $\lambda > \text{LS}(\mathbf{K})$ , then **K** is categorical in all cardinals of the form  $\beth_{\delta}$ , where  $(2^{\text{LS}(\mathbf{K})})^+$  divides  $\delta$ .

PROOF. Without loss of generality (Fact 15.1.9), **K** has no maximal models. Let  $\delta$  be a limit ordinal divisible by  $(2^{\text{LS}(\mathbf{K})})^+$ . Let  $\mu := \beth_{\delta}$ . We prove that every model in  $\mathbf{K}_{\mu}$  is saturated. Observe first that for every  $\chi \geq \text{LS}(\mathbf{K})^+$ ,  $\mathbf{K}^{\chi\text{-sat}}$  is an AEC with  $\text{LS}(\mathbf{K}^{\chi\text{-sat}}) = \chi$ . This follows from Facts 15.1.3, 15.1.5, and 15.1.6. In particular, the model of size  $\lambda$  is  $\text{LS}(\mathbf{K})^+$ -saturated. Therefore for each  $\chi < \mu$ , Lemma 15.3.4 tells us that every model in  $\mathbf{K}_{\mu}$  is  $\chi^+$ -saturated. Thus every model in  $\mathbf{K}_{\mu}$  is saturated.

REMARK 15.3.9. We could rely on fewer background facts by directly using Fact 15.3.3. In that case, all that one needs to show is that the model of size  $\lambda$  is  $LS(\mathbf{K})^+$ -saturated. This holds by combining Fact 15.1.3 (telling us that  $\mathbf{K}$  is  $LS(\mathbf{K})$ -superstable) and Theorem 4.5.6 (saying that  $\mathbf{K}$  is stable in  $\lambda$ ).

We can derive the desired improvements on the bounds of [She99].

COROLLARY 15.3.10. Let **K** be an LS(**K**)-tame AEC with amalgamation. If **K** is categorical in some successor  $\lambda \geq H_1$ , then **K** is categorical in all  $\lambda' \geq H_1$ .

PROOF. By Theorem 15.3.8, **K** is categorical in  $H_1$ . Now proceed as in the proof of the main result of **[She99**], see e.g. **[Bal09**, Theorem 14.14].

Note that this method does not allow us to go lower than the Hanf number, even if we know for example that  $\mathbf{K}$  is categorical below it (Shelah's argument for transferring Vaughtian pairs is not local enough). See Corollary 15.4.6 for another proof.

### 15.4. Categoricity at a successor or with primes

We apply Theorem 14.6.14 to categorical tame AECs with amalgamation when the categoricity cardinal is a successor or the AEC has primes (recall Definition 14.1.3). All throughout, we assume:

HYPOTHESIS 15.4.1. K is an AEC with amalgamation.

LEMMA 15.4.2 (Main lemma). Assume that **K** has no maximal models and is  $LS(\mathbf{K})$ -tame. Let  $\lambda > LS(\mathbf{K})^+$  be a successor cardinal. If **K** is categorical in *some* successor  $\lambda > LS(\mathbf{K})^+$ , then  $\mathbf{K}^{LS(\mathbf{K})^+}$ -sat is categorical in all  $\lambda' \ge LS(\mathbf{K})^+$ .

PROOF. By Fact 15.1.3, **K** is LS(**K**)-superstable. By Fact 15.1.5, **K** is superstable and has symmetry in every  $\mu \geq \text{LS}(\mathbf{K})$ . By Facts 15.1.12 and 15.1.13, there exists a type-full good ( $\geq \text{LS}(\mathbf{K})^+$ )-frame  $\mathfrak{s}$  with underlying class  $\mathbf{K}_{\mathfrak{s}} := \mathbf{K}^{\text{LS}(\mathbf{K})^+\text{-sat}}$ . Moreover,  $\mathbf{K}_{\mathfrak{s}}$  is categorical in  $\text{LS}(\mathbf{K})^+$ . Thus we can apply Theorem 14.6.14 where  $\mathbf{K}, \text{LS}(\mathbf{K}), \theta$  there stand for  $\mathbf{K}^{\text{LS}(\mathbf{K})^+\text{-sat}}, \text{LS}(\mathbf{K})^+, \infty$  here.  $\Box$ 

COROLLARY 15.4.3. Assume that **K** has arbitrarily large models and is  $LS(\mathbf{K})$ -tame. Let  $LS(\mathbf{K}) < \lambda_0 < \lambda$ . If  $\lambda$  is a successor and **K** is categorical in  $\lambda_0$  and  $\lambda$ , then **K** is categorical in all  $\lambda' \geq \lambda_0$ .

PROOF. By Fact 15.1.9, we can assume without loss of generality that **K** has no maximal models. By Lemma 15.4.2,  $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$  is categorical in all  $\lambda' \geq \mathrm{LS}(\mathbf{K})^+$ . Moreover by the proof of Lemma 15.4.2, **K** is stable in every  $\mu \in [\mathrm{LS}(\mathbf{K})^+, \lambda)$ , hence the model of size  $\lambda_0$  is saturated. Therefore  $\mathbf{K}_{\geq\lambda_0}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}} = \mathbf{K}_{\geq\lambda_0}$ , and the result follows.

REMARK 15.4.4. We can allow  $\lambda_0 = LS(\mathbf{K})$  but then the proof is more complicated: we do not know how to build a good  $LS(\mathbf{K})$ -frame so have to work with  $LS(\mathbf{K})$ -splitting.

REMARK 15.4.5. The case  $\lambda' \geq \lambda$  in Lemma 15.4.2 and Corollary 15.4.3 is Fact 14.1.6. The contribution of this chapter is the case  $\lambda' < \lambda$ .

We deduce another proof of Corollary 15.3.10. We prove a slightly stronger result:

COROLLARY 15.4.6. Assume that **K** is  $LS(\mathbf{K})$ -tame and has arbitrarily large models. If **K** is categorical in *some successor*  $\lambda > LS(\mathbf{K})^+$ , then there exists  $\chi < H_1$  such that **K** is categorical in all  $\lambda' \ge \min(\chi, \lambda)$ .

PROOF. By Lemma 15.4.2,  $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$  is categorical in all  $\mu \geq \mathrm{LS}(\mathbf{K})^+$ . Since  $\lambda$  is regular, the model of size  $\lambda$  is saturated hence  $\mathrm{LS}(\mathbf{K})^+\text{-saturated}$ , so every model in  $\mathbf{K}_{\geq\lambda}$  is  $\mathrm{LS}(\mathbf{K})^+\text{-saturated}$ . By Fact 15.3.2, there exists  $\chi < H_1$  such that every model in  $\mathbf{K}_{\geq\chi}$  is  $\mathrm{LS}(\mathbf{K})^+\text{-saturated}$ . Thus every model in  $\mathbf{K}_{\geq\min(\chi,\lambda)}$  is  $\mathrm{LS}(\mathbf{K})^+\text{-saturated}$ , that is:

$$\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-}\mathrm{sat}}_{\geq\min(\chi,\lambda)}=\mathbf{K}_{\geq\min(\chi,\lambda)}$$

The result follows.

REMARK 15.4.7. An alternate proof (which just deduces categoricity in all  $\lambda' \geq \min(\lambda, H_1)$ ) goes as follows. By Theorem 15.3.8, **K** is categorical in  $H_1$ . By Fact 14.1.6, **K** is categorical in all  $\lambda' \geq \lambda$ . By Corollary 15.4.3 (where  $\lambda_0, \lambda_1$  there stand for  $\min(\lambda, H_1), \max(\lambda^+, H_1^+)$  here), we get that **K** is categorical in every  $\lambda' \geq \min(\lambda, H_1)$ .

We can similarly deduce several consequences on tame AECs with primes. One of the main results of Chapter 11 was (the point compared to Shelah's downward categoricity transfer [She99] is that  $\lambda$  need not be a successor):

FACT 15.4.8 (Theorem 11.3.8). Assume that **K** has no maximal models, is  $H_2$ -tame, and  $\mathbf{K}_{\geq H_2}$  has primes. If **K** is categorical in some  $\lambda > H_2$ , then it is categorical in all  $\lambda' \geq H_2$ .

We show that we can obtain categoricity in more cardinals provided that  $\mathbf{K}$  has more tameness:

COROLLARY 15.4.9. Assume that **K** is  $LS(\mathbf{K})$ -tame and has arbitrarily large models. Assume also that **K** has primes (or just that  $\mathbf{K}_{\geq\mu}$  has prime, for some  $\mu$ ). If **K** is categorical in *some*  $\lambda > LS(\mathbf{K})$ , then **K** is categorical in *all*  $\lambda' \geq \min(\lambda, H_1)$ .

PROOF. Without loss of generality (Fact 15.1.9), **K** has no maximal models. By Theorem 15.3.8, **K** is categorical in a proper class of cardinals. By Fact 15.4.8 (applied to  $\mathbf{K}_{\geq\mu}$ , where  $\mu$  is such that  $\mathbf{K}_{\geq\mu}$  has primes), **K** is categorical in a successor cardinal. By Corollary 15.4.6, **K** is categorical in all  $\lambda' \geq H_1$ . By Corollary 15.4.3 (with  $\lambda_0, \lambda$  there standing for  $\lambda, H_1^+$  here), **K** is categorical also in all  $\lambda' \in [\lambda, H_1^+]$ .

REMARK 15.4.10. Similarly to Corollary 15.4.6, we get that there is a  $\chi < H_1$  such that **K** is categorical in all  $\lambda' \geq \chi$ .

Specializing to universal classes, we can improve some of the Hanf number bounds in Chapter 8, obtaining in particular the full categoricity conjecture (i.e. the Hanf number is  $H_1$ ) assuming amalgamation.

COROLLARY 15.4.11. Let **K** be a universal class with amalgamation and arbitrarily large models. If **K** is categorical in *some*  $\lambda > \text{LS}(\mathbf{K})$ , then **K** is categorical in *all*  $\lambda' \geq \min(\lambda, H_1)$ .

PROOF. By a result of Boney [Bonc] (a full proof appears as Theorem 8.3.6), **K** is  $LS(\mathbf{K})$ -tame. Also (Remark 8.5.3), **K** has primes. Thus we can apply Corollary 15.4.9.

#### 15.5. Categoricity in a limit without primes

In this section, we give an exposition of Shelah's proof of the eventual categoricity conjecture in AECs with amalgamation [She09a, Theorem IV.7.12] (assuming the weak generalized continuum hypothesis), see Corollary 15.5.12. We improve the threshold cardinal to  $H_1$  assuming tameness (Corollary 15.5.9), and moreover give alternate proofs for several of the hard steps in Shelah's argument.

Most of the results of this section will use the weak generalized continuum hypothesis. We adopt the following notation:

NOTATION 15.5.1. For a cardinal  $\lambda$ , WGCH( $\lambda$ ) is the statement " $2^{\lambda} < 2^{\lambda^+}$ ". More generally, for S a class of cardinals, WGCH(S) is the statement "WGCH( $\lambda$ ) for all  $\lambda \in S$ ". WGCH will stand for WGCH(Card), where Card is the class of all cardinals.

We assume familiarity with the definitions of a weakly successful, successful, and  $\omega$ -successful good  $\lambda$ -frame (see [She09a, Definition III.1.1], and on weakly successful see Definition 15.11.6). We use the notation from [JS13]. We will say a good  $\lambda$ -frame is successful<sup>+</sup> if it is successful and  $\leq_{\mathbf{K}_{\lambda^+}}^{\mathrm{NF}}$  is just  $\leq_{\mathbf{K}_{\lambda^+}}$  on the saturated models in  $\mathbf{K}_{\lambda^+}$ , see [JS13, Definition 6.1.4]. We say a good  $\lambda$ -frame is  $\omega$ -successful<sup>+</sup> if it is  $\omega$ -successful and successful<sup>+</sup>.

We first state the unpublished Claim of Shelah mentioned in the introduction. This stems from [She09a, Discussion III.12.40]. A proof should appear in [Sheb].

CLAIM 15.5.2. Let  $\mathfrak{s}$  be an  $\omega$ -successful<sup>+</sup> good  $\lambda$ -frame on  $\mathbf{K}$ . Assume WGCH( $[\lambda, \lambda^{+\omega})$ ). If  $\mathbf{K}^{\lambda^{+\omega}\text{-sat}}$  is categorical in some  $\mu > \lambda^{+\omega}$ , then  $\mathbf{K}^{\lambda^{+\omega}\text{-sat}}$  is categorical in all  $\mu' > \lambda^{+\omega}$ . Moreover, for any  $\mu' > \lambda$ ,  $\mathbf{K}^{\mu'\text{-sat}}$  has amalgamation in  $\mu'$ .

Next, we discuss how to obtain an  $\omega$ -successful<sup>+</sup> good frame from a good frame. The proof of the following fact is contained in the proof of [**She09a**, Theorem IV.7.12] (see  $\odot_4$  there). We give a full proof in Section 15.11. Note that as opposed to the results in [**She09a**, Section II.5], we do *not* assume that **K** has few models in  $\lambda^{+2}$ .

FACT 15.5.3. Assume WGCH( $\lambda$ ). If  $\mathfrak{s}$  is a good  $\lambda$ -frame on  $\mathbf{K}$ ,  $\mathbf{K}$  is categorical in  $\lambda$ , has amalgamation in  $\lambda^+$  and is stable in  $\lambda^+$ , then  $\mathfrak{s}$  is weakly successful.

We can obtain the stability hypothesis and successfulness using weak tameness:

FACT 15.5.4 (Theorem 4.5 in [**BKV06**]). Let  $\lambda \geq \text{LS}(\mathbf{K})$ , be such that **K** has amalgamation in  $\lambda$  and  $\lambda^+$ , and **K** is stable in  $\lambda$ . If **K** is  $(\lambda, \lambda^+)$ -weakly tame, then **K** is stable in  $\lambda^+$ .

FACT 15.5.5 (Corollary 7.19 in [**Jar16**]). If  $\mathfrak{s}$  is a weakly successful good  $\lambda$ -frame on **K**, **K** is categorical<sup>2</sup> in  $\lambda$ , **K** has amalgamation in  $\lambda^+$ , and **K** is  $(\lambda, \lambda^+)$ -weakly tame, then  $\mathfrak{s}$  is successful<sup>+</sup>.

<sup>&</sup>lt;sup>2</sup>Note that by [She09a, Claim III.1.21], this implies the conjugation property, so Hypothesis 6.5 in [Jar16] is satisfied.

REMARK 15.5.6. Although we will not need it, the converse (i.e. obtaining weak tameness from being successful<sup>+</sup>) is also true, see [**JS13**, Theorem 7.1.13.(b)].

COROLLARY 15.5.7. Assume WGCH( $\lambda$ ). If  $\mathfrak{s}$  is a good  $\lambda$ -frame on  $\mathbf{K}$ ,  $\mathbf{K}$  is categorical in  $\lambda$ , has amalgamation in  $\lambda^+$ , and is  $(\lambda, \lambda^+)$ -weakly tame, then  $\mathfrak{s}$  is successful<sup>+</sup>.

PROOF. By Fact 15.5.4, **K** is stable in  $\lambda^+$ . By Fact 15.5.3,  $\mathfrak{s}$  is weakly successful. By Fact 15.5.5,  $\mathfrak{s}$  is successful<sup>+</sup>.

Using Fact 15.1.12 to build the good frame, we obtain:

LEMMA 15.5.8. Assume that **K** has amalgamation and no maximal models. Assume WGCH([LS(**K**), LS(**K**)<sup>+ $\omega$ </sup>)) and Claim 15.5.2. If **K** is categorical in a  $\lambda > \text{LS}(\mathbf{K})^{+\omega}$  and:

(1) The model of size  $\lambda$  is  $LS(\mathbf{K})^{++}$ -saturated.

(2) **K** is  $(LS(\mathbf{K}), < LS(\mathbf{K})^{+\omega})$ -weakly tame.

Then  $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^{+\omega}\text{-sat}}$  is categorical in all  $\lambda' > \mathrm{LS}(\mathbf{K})^{+\omega}$ . In particular, **K** is categorical in all  $\lambda' \ge \min(\lambda, \sup_{n \le \omega} h(\mathrm{LS}(\mathbf{K})^{+n}))$ .

PROOF. By Fact 15.1.7, **K** has symmetry in  $LS(\mathbf{K})$  and  $LS(\mathbf{K})^+$ . By Fact 15.1.3, **K** is superstable in every  $\chi \in [LS(\mathbf{K}), \lambda)$ . By Fact 15.1.12, there is a good  $LS(\mathbf{K})^+$ -frame  $\mathfrak{s}$  on  $\mathbf{K}^{LS(\mathbf{K})^+$ -sat}. By repeated applications of Corollary 15.5.7,  $\mathfrak{s}$  is  $\omega$ -successful<sup>+</sup>. By Claim 15.5.2,  $\mathbf{K}^{LS(\mathbf{K})^{+\omega}$ -sat is categorical in all  $\lambda' > LS(\mathbf{K})^{+\omega}$ , and hence by Fact 15.3.2, **K** is categorical in every  $\lambda' \geq \sup_{n < \omega} h(LS(\mathbf{K})^{+\omega})$ . In particular by Fact 15.1.3, **K** is stable in  $\lambda$ , so the model of size  $\lambda$  is saturated (hence  $LS(\mathbf{K})^{+\omega}$ -saturated), and so **K** must be categorical in all  $\lambda' \geq \min(\lambda, \sup_{n < \omega} h(LS(\mathbf{K})^{+\omega}))$ .

COROLLARY 15.5.9. Assume WGCH([LS(**K**), LS(**K**)<sup>+ $\omega$ </sup>)) and Claim 15.5.2. Assume that **K** has amalgamation, arbitrarily large models, and is LS(**K**)-tame. If **K** is categorical in a  $\lambda >$  LS(**K**), then there exists  $\chi < H_1$  such that **K** is categorical in all  $\lambda' \ge \min(\lambda, \chi)$ .

PROOF. By Fact 15.1.9, without loss of generality **K** has no maximal models. By Theorem 15.3.8 we can assume without loss of generality that  $\lambda \geq H_1$  (then we can use Corollary 15.4.3 to transfer categoricity downward). By Fact 15.1.3, **K** is LS(**K**)-superstable. By<sup>3</sup> Fact 15.1.5, **K** is stable in  $\lambda$ , hence the model of size  $\lambda$  is saturated. By Lemma 15.5.8, **K** is categorical in all  $\lambda' \geq h(\text{LS}(\mathbf{K})^{+\omega})$ . In particular, **K** is categorical in  $(h(\text{LS}(\mathbf{K})^{+\omega}))^+$ . By Corollary 15.4.6, there exists  $\chi < H_1$  such that **K** is categorical in all  $\lambda' \geq \chi$ .

Without tameness, we can obtain the hypotheses of Lemma 15.5.8 from categoricity in a high-enough cardinal. More precisely, to obtain enough weak tameness, we use Theorem 15.2.4. To obtain the first condition in Lemma 15.5.8 (i.e. that the model in the categoricity cardinal is sufficiently saturated), we will use<sup>4</sup> Fact 15.1.7.

<sup>&</sup>lt;sup>3</sup>In our case, we only use stability in  $\lambda$ , which follows by Theorem 4.5.6.

<sup>&</sup>lt;sup>4</sup>Shelah claims [**She09a**, Claim IV.7.8] a slightly different result using PCF theory and the existence of certain linear orders: under amalgamation and no maximal models, for  $\mu > LS(\mathbf{K})$ , categoricity in some  $\lambda$  so that  $\lambda \geq \aleph_{\mu+4} + 2^{2^{\mu}}$  implies that the model of size  $\lambda$  is  $\mu$ -saturated.

This allows us to give a proof of [She09a, Theorem IV.7.12]. Note that, while we give a slightly different proof to attempt to convince doubters, the result is due to Shelah. In fact, Shelah assumes amalgamation more locally but we haven't fully verified his general proof. As explained in the introduction, we avoid relying on PCF theory or on Shelah's construction of certain linear orders in [She09a, Sections IV.5, IV.6].

FACT 15.5.10. Assume Claim 15.5.2. Assume that **K** has a malgamation. Let  $\lambda$  and  $\mu$  be cardinals such that:

- (1) **K** is categorical in  $\lambda$ .
- (2)  $\mu$  is a limit cardinal with cf  $\mu > LS(\mathbf{K})$ .
- (3) For all  $\chi < \mu$ ,  $h(\chi) < \lambda$ .
- (4) For unboundedly many  $\chi < \mu$ , WGCH( $[\chi, \chi^{+\omega})$ ).

Then there exists  $\mu_* < \mu$  such that **K** is categorical in all  $\lambda' \ge h(\mu_*)$ .

PROOF. By Fact 15.1.9, without loss of generality **K** has no maximal models. By Fact 15.1.7 (used with **K** there standing for  $\mathbf{K}_{\geq \chi^+}$  here, for each  $\chi < \mu$ ), the model of size  $\lambda$  is  $\mu$ -saturated. By Theorem 15.2.4, there exists  $\chi < \mu$  so that **K** is  $(\chi, < \mu)$ -weakly tame. Increasing  $\chi$  if necessary, assume without loss of generality that WGCH( $(\chi, \chi^{+\omega})$ ) holds. Now apply Lemma 15.5.8 with **K** there standing for  $\mathbf{K}_{\geq \chi}$  here. We get that **K** is categorical in all  $\lambda' \geq h(\chi^{+\omega})$ , so we obtain the desired conclusion with  $\mu_* := \chi^{+\omega}$ .

REMARK 15.5.11. The proof of Fact 15.5.10 given above goes through assuming only that **K** has amalgamation below the categoricity cardinal  $\lambda$  (using the moreover part of Claim 15.5.2 to check uniqueness of saturated models).

COROLLARY 15.5.12. Assume Claim 15.5.2 and WGCH. If **K** has amalgamation and is categorical in some  $\lambda \geq h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$ , then **K** is categorical in all  $\lambda' \geq h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$ .

PROOF. Set 
$$\mu := \aleph_{\mathrm{LS}(\mathbf{K})^+}$$
 in Fact 15.5.10.

We can also state a version using large cardinals instead of amalgamation. This is implicit in Shelah's work (see the remark after [She09a, Theorem IV.7.12]), but to the best of our knowledge, the details have not appeared in print before. We will use the following fact, which follows from [SK96, She01b]. Note that while the results there are stated when **K** is the class of models of an  $\mathbb{L}_{\kappa,\omega}$ -theory, Boney observed that the proofs go through just as well in an AEC **K** with  $\kappa > \mathrm{LS}(\mathbf{K})$ , see the discussion around [Bon14b, Theorem 7.6].

FACT 15.5.13. Let **K** be an AEC and let  $\kappa > \text{LS}(\mathbf{K})$  be a measurable cardinal. Let  $\lambda \ge h(\kappa)$  be such that **K** is categorical in  $\lambda$ . Then:

- (1) **[SK96]**  $\mathbf{K}_{[\kappa,\lambda)}$  has amalgamation and no maximal models.
- (2) [She01b, Claim 1.16] The model of size  $\lambda$  is saturated.
- (3) [She01b, Corollary 3.7] K is  $(\kappa, < \lambda)$ -tame.
- (4) [She01b, Theorem 3.16] If λ is a successor cardinal, then K is categorical in all λ' ≥ h(κ).

COROLLARY 15.5.14. Assume Claim 15.5.2 and WGCH. Let  $\kappa > LS(\mathbf{K})$  be a measurable cardinal. If **K** is categorical in *some*  $\lambda \ge h(\kappa)$ , then **K** is categorical in *all*<sup>5</sup>  $\lambda' \ge h(\kappa)$ 

<sup>&</sup>lt;sup>5</sup>The proof gives that there exists  $\chi < h(\kappa)$  such that **K** is categorical in all  $\lambda' \ge \chi$ .

#### 15.6. SUMMARY

PROOF. By Fact 15.5.13.(1),  $\mathbf{K}_{[\kappa,\lambda)}$  has amalgamation (and no maximal models, by taking ultrapowers). Note that by Remark 15.5.11 we do not need amalgamation in  $\mathbf{K}_{\geq\lambda}$ . By Fact 15.5.13.(2, the model of size  $\lambda$  is saturated. Let  $\mu := \aleph_{\kappa^+}$ . By Theorem 15.2.4, there exists  $\chi < \mu$  so that  $\mathbf{K}$  is  $(\chi, < \mu)$ -weakly tame. By Lemma 15.5.8 (with  $\mathbf{K}$  there standing for  $\mathbf{K}_{\geq\chi}$  here),  $\mathbf{K}$  is categorical in all  $\lambda' \geq h(\mu)$ . In particular,  $\mathbf{K}$  is categorical in  $(h(\mu))^+$ . By Fact 15.5.13.(4) (or by Corollary 15.4.6, since by Fact 15.5.13.(3)  $\mathbf{K}$  has enough tameness),  $\mathbf{K}$  is also categorical in all  $\lambda' \in [h(\kappa), h(\mu))$ .

The same proof gives:

COROLLARY 15.5.15. Assume Claim 15.5.2 and WGCH. Let  $\kappa$  be a measurable cardinal and let T be a theory in  $\mathbb{L}_{\kappa,\omega}$ . If T is categorical in some  $\lambda \geq h(|T| + \kappa)$ , then T is categorical in all  $\lambda' \geq h(|T| + \kappa)$ .

### 15.6. Summary

Table 1 summarizes several known approximations of Shelah's eventual categoricity conjecture, for a fixed AEC **K**. The topmost line and leftmost column contain properties that are either model-theoretic, set-theoretic, or about the categoricity cardinal. The intersection of a line and a column gives a known categoricity transfer for a class having these properties. "AP" stands for "**K** has the amalgamation property", "Primes" is short for "**K** has primes" (Definition 14.1.3), "s.c." is short for "strongly compact", and  $(*)_{\mathbf{K}}$  is the statement " $\mathrm{LS}(\mathbf{K}) = \kappa$ , **K** has amalgamation, and **K** is  $\mathrm{LS}(\mathbf{K})$ -tame".

Each transfer is described by its type, a comma, and a threshold  $\mu$ . A "Full" type means that categoricity in some  $\lambda \geq \mu$  implies categoricity in all  $\lambda' \geq \mu$ . A "Down" type means that we only know a downward transfer: categoricity in some  $\lambda \geq \mu$  implies categoricity in all  $\lambda' \in [\mu, \lambda]$  (in this case, we can still do an argument similar to the existence of Hanf numbers [Han60] to deduce Shelah's eventual categoricity conjecture, see [Bal09, Conclusion 15.13]). A "Partial" type means that we only know that categoricity in some  $\lambda \geq \mu$  implies categoricity in some  $\lambda'$  with  $\lambda' \neq \lambda$  (we do not require that  $\lambda' \geq \mu$ ). When reading the line beginning with "categ. in a successor", one should assume that the starting categoricity cardinal  $\lambda$  is a successor.

For example, the first entry says that if an AEC **K** satisfies the amalgamation property and WGCH together with Claim 15.5.2 hold, then categoricity in some  $\lambda \ge h(\aleph_{LS(\mathbf{K})^+})$  implies categoricity in all  $\lambda' \ge h(\aleph_{LS(\mathbf{K})^+})$ .

Note that in the first column, it is enough to assume amalgamation below the categoricity cardinal (see Remark 15.5.11). So by Fact 15.5.13.(1) and because strongly compact cardinals are measurable, we can see the properties in the topmost row as being arranged in increasing order of strength. Moreover, the existence of a strongly compact cardinal implies (\*): amalgamation follows from the methods of [**MS90**, Proposition 1.13] and tameness from the main theorem of [**Bon14b**].

FACT 15.6.1. Let **K** be an AEC and let  $\kappa > LS(\mathbf{K})$  be a strongly compact cardinal. Let  $\lambda \ge h(\kappa)$ . If **K** is categorical in  $\lambda$ , then  $(*)_{\mathbf{K}>\kappa}$  holds.

REMARK 15.6.2. An analog of  $(*)_{\mathbf{K}}$  in the case  $\kappa$  is measurable would be given by conclusions (1)-(3) in Fact 15.5.13.

	AP	$\kappa > \mathrm{LS}(\mathbf{K})$ measurable	$\kappa > \mathrm{LS}(\mathbf{K}) \text{ s.c. or } (*)_{\mathbf{K}}$
WGCH and 15.5.2	Full, $h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$	Full, $h(\kappa)$	Full, $h(\kappa)$
Categ. in a successor	Down, $\beth_{H_1}$	Down, $h(\kappa)$	Full, $h(\kappa)$
Primes	Partial, $\beth_{\beth_{H_1}}$	Down, $h(\kappa)$	Full, $h(\kappa)$
No extra hypothesis	Partial, $\beth_{\beth_{H_1}}$	Partial, $h(\kappa)^+$	Partial, $h(\kappa)$

TABLE 1. Some approximations to Shelah's categoricity conjecture. Properties in the top row are consequences of large cardinal axioms while properties in the first column do not follow (or are not known to follow) from large cardinals. Each entry gives a type of transfer (full, down, or partial) as well as a cardinal threshold. See the beginning of this section for more information on how to read the table.

The results in the first row are Corollary 15.5.12 and Corollary 15.5.14 (for the strongly compact case, recall that the properties in the topmost row are in increasing order of strength). The first result in the second row is the downward transfer of [She99] (see also Corollary 15.8.6 for an alternate proof). The second is Fact 15.5.13.(4). The third is given by Corollary 15.4.3 (recalling Fact 15.6.1).

The last two results in the first column are given by Corollary 15.3.7 (categoricity above  $\beth_{\beth_{H_1}}$  implies categoricity in  $\beth_{H_1}$ ). As for the last column, the third result is by Corollary 15.4.9, and the fourth is by Theorem 15.3.8. Very similar proofs (using Fact 15.5.13 to deduce the needed amount of amalgamation and tameness) give the corresponding results in the second column.

# 15.7. Shrinking good frames

We state a generalization of Theorem 14.6.14 to frames that are only defined over classes of saturated models (Shelah studies these frames in more details in **[Sheb**]). This allows us to replace the assumption of tameness by only weak tameness in several results (see Section 15.8).

We start by giving a precise definition of these frames (we call them shrinking frames for reasons that will soon become apparent).

DEFINITION 15.7.1 (Shrinking frame). Let  $\lambda$  be an infinite cardinal and let  $\theta > \lambda$  be a cardinal or  $\infty$ . Let  $\mathcal{F} := [\lambda, \theta)$  and let **K** be an AEC.

We say that  $\langle \mathfrak{s}_{\mu} : \mu \in \mathcal{F} \rangle$  is a *shrinking type-full good*  $\mathcal{F}$ -frame on  $\mathbf{K}_{\mathcal{F}}$  (or on **K**) if:

- (1) **K** is  $(\lambda, < \theta)$ -weakly tame.
- (2)  $\mathfrak{s}_{\lambda}$  has underlying class  $\mathbf{K}_{\lambda}$ .
- (3) For each  $\mu \in \mathcal{F}$ ,  $\mathfrak{s}_{\mu}$  is a type-full good  $\mu$ -frame with  $\mathbf{K}_{\mathfrak{s}_{\mu}} = \mathbf{K}_{\mu}^{\mu-\text{sat}}$ . In particular, **K** is categorical in  $\lambda$ .

The reason for the name *shrinking* is that if  $\mu < \mu'$  are in  $\mathcal{F}$ , then the AEC generated by  $\mathbf{K}_{\mathfrak{s}_{\mu'}}$  is  $\mathbf{K}^{\mu'\text{-sat}}$ , but the underlying class  $\mathbf{K}_{\mathfrak{s}_{\mu}}$  is only  $\mathbf{K}^{\mu\text{-sat}}$  which could be a proper subclass of  $\mathbf{K}^{\mu'\text{-sat}}$  (if  $\mathbf{K}$  is not categorical in  $\mu'$ ). Note that that a type-full good  $[\lambda, \theta)$ -frame (which is categorical in  $\lambda$ ) induces a shrinking frame in a natural way.

PROPOSITION 15.7.2. If  $\mathfrak{s}$  is a type-full good  $[\lambda, \theta)$ -frame and  $\mathbf{K}_{\mathfrak{s}}$  is categorical in  $\lambda$ , then  $\langle \mathfrak{s} \upharpoonright \mathbf{K}^{\mu\text{-sat}} : \mu \in [\lambda, \theta) \rangle$  is a shrinking type-full good  $[\lambda, \theta)$ -frame.

PROOF. Straightforward, recalling Fact 14.2.6.

Shrinking frames can be built using Fact 15.1.12:

THEOREM 15.7.3. Let **K** be an AEC. Let  $\lambda > \text{LS}(\mathbf{K})$ . Assume that for every  $\mu \in [\text{LS}(\mathbf{K}), \lambda)$ , **K** is  $\mu$ -superstable and has  $\mu$ -symmetry.

If **K** is  $(LS(\mathbf{K}), < \lambda)$ -weakly tame, then there exists a shrinking type-full good  $[LS(\mathbf{K})^+, \lambda)$ -frame on **K**.

PROOF. Let  $\mathcal{F} := [\mathrm{LS}(\mathbf{K})^+, \lambda)$ . By Fact 15.1.12, for each  $\mu \in \mathcal{F}$ , there exists a type-full good  $\mu$ -frame on  $\mathbf{K}^{\mu-\mathrm{sat}}_{\mu}$ . The result follows.

We now study how forking in two different cardinals interact in a shrinking frame. The following notion is key:

DEFINITION 15.7.4. Let **K** be an AEC. Let  $\mathrm{LS}(\mathbf{K}) \leq \lambda < \mu$ . Let  $\mathfrak{s}_{\lambda}$  be a type-full good  $\lambda$ -frame on  $\mathbf{K}_{\lambda}^{\lambda-\mathrm{sat}}$  and  $\mathfrak{s}_{\mu}$  be a type-full good  $\mu$ -frame on  $\mathbf{K}_{\mu}^{\mu-\mathrm{sat}}$ . We say that  $\mathfrak{s}_{\lambda}$  and  $\mathfrak{s}_{\mu}$  are *compatible* if for any  $M \leq_{\mathbf{K}} N$  in  $\mathbf{K}_{\mathfrak{s}_{\mu}}$ , and  $p \in \mathrm{gS}(N)$ , p does not  $\mathfrak{s}_{\mu}$ -fork over M if and only if there exists  $M_0 \leq_{\mathbf{K}} M$  with  $M_0 \in \mathbf{K}_{\mathfrak{s}_{\lambda}}$  so that  $p \upharpoonright N_0$  does not  $\mathfrak{s}_{\lambda}$ -fork over  $M_0$  for every  $N_0 \in \mathbf{K}_{\mathfrak{s}_{\lambda}}$  with  $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$ .

Intuitively, compatibility says that that forking in  $\mathfrak{s}_{\mu}$  can be computed using forking in  $\mathfrak{s}_{\lambda}$ . In fact, it can be described in a canonical way (i.e. using Shelah's description of the extended frame, see [She09a, Section II.2]). The following result is key:

THEOREM 15.7.5. Let  $\langle \mathfrak{s}_{\mu} : \mu \in \mathcal{F} \rangle$  be a shrinking type-full good  $\mathcal{F}$ -frame on the AEC **K**. Let  $\lambda < \mu$  be in  $\mathcal{F}$ . Then  $\mathfrak{s}_{\lambda}$  and  $\mathfrak{s}_{\mu}$  are compatible.

For the proof, we will use the following result which gives an explicit description of forking in any categorical good frame:

FACT 15.7.6 (The canonicity theorem, see Lemma 6.9.6). Let  $\mathfrak{s}$  be a type-full good  $\lambda$ -frame with underlying class  $\mathbf{K}_{\lambda}$ . If  $M \leq_{\mathbf{K}} N$  are limit models in  $\mathbf{K}_{\lambda}$ , then for any  $p \in \mathrm{gS}(N)$ , p does not  $\mathfrak{s}$ -fork over M if and only if there exists  $M' \in \mathbf{K}_{\lambda}$  such that M is limit over M' and p does not  $\lambda$ -split over M'.

PROOF OF THEOREM 15.7.5. Note that by uniqueness of limit models, every model in  $\mathbf{K}_{\mathfrak{s}_{\mu}}$  is limit.

For  $M, N \in \mathbf{K}_{\mathfrak{s}_{\mu}}$  with  $M \leq_{\mathbf{K}} N$ , let us say that  $p \in \mathrm{gS}(N)$  does not  $(\geq \mathfrak{s}_{\lambda})$ fork over M if it satisfies the condition in Definition 15.7.4, namely there exists  $M_0 \leq_{\mathbf{K}} M$  with  $M_0 \in \mathbf{K}_{\mathfrak{s}_{\lambda}}$  so that  $p \upharpoonright N_0$  does not  $\mathfrak{s}_{\lambda}$ -fork over  $M_0$  for every  $N_0 \in \mathbf{K}_{\mathfrak{s}_{\lambda}}$  with  $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$ . Let us say that p does not  $\mu$ -fork over M if it satisfies the description of the canonicity theorem, namely there exists  $M' \in \mathbf{K}_{\mathfrak{s}_{\mu}}$ such that M is limit over M' and p does not  $\mu$ -split over M'. Notice that by the canonicity theorem (Fact 15.7.6), p does not  $\mathfrak{s}_{\mu}$ -fork over M if and only if p does not  $\mu$ -fork over M. Thus it is enough to show that p does not  $(\geq \mathfrak{s}_{\lambda})$ -fork over Mif and only if p does not  $\mu$ -fork over M. We first show one direction:

<u>Claim</u>. Let  $M \leq_{\mathbf{K}} N$  both be in  $\mathbf{K}_{\mathfrak{s}_{\mu}}$  and let  $p \in \mathrm{gS}(N)$ . If p does not  $(\geq \mathfrak{s}_{\lambda})$ -fork over M, then p does not  $\mu$ -fork over M.

Proof of Claim. We know that M is limit, so let  $\langle M_i : i < \delta \rangle$  witness it, i.e.  $\delta$  is limit, for all  $i < \delta$ ,  $M_i \in \mathbf{K}_{\mathfrak{s}_{\mu}}$ ,  $M_{i+1}$  is universal over  $M_i$ , and  $\bigcup_{i < \delta} M_i = M$ . By [She09a, Claim II.2.11.(5)], there exists  $i < \delta$  such that  $p \upharpoonright M$  does not  $(\geq \mathfrak{s}_{\lambda})$ -fork over  $M_i$ . By [She09a, Claim II.2.11.(4)], p does not  $(\geq \mathfrak{s}_{\lambda})$ -fork over  $M_i$ . By weak tameness and the uniqueness property of  $\mathfrak{s}$ ,  $(\geq \mathfrak{s}_{\lambda})$ -forking has the uniqueness property (see the proof of [Bon14a, Theorem 3.2]). By Lemma 3.4.2,  $(\geq \mathfrak{s}_{\lambda})$ -nonforking must be extended by  $\mu$ -nonsplitting, so p does not  $\mu$ -split over  $M_i$ . Therefore p does not  $\mu$ -fork over M, as desired.  $\dagger_{\text{Claim}}$ .

Now as observed above,  $(\geq \mathfrak{s}_{\lambda})$ -forking has the uniqueness property. Also,  $\mu$ -forking has the extension property (as  $\mathfrak{s}_{\mu}$ -forking has it). The claim tells us that  $\mu$ -nonforking extends  $(\geq \mathfrak{s}_{\lambda})$ -forking and hence by Lemma 3.4.1, they are the same.

Thus we can define a global notion of forking inside the frame:

DEFINITION 15.7.7. Assume that  $\langle \mathfrak{s}_{\mu} : \mu \in \mathcal{F} \rangle$  is a shrinking type-full good  $[\lambda, \theta)$ -frame. Let  $\mu \leq \mu'$  be in  $\mathcal{F}$  and let  $M \leq_{\mathbf{K}} N$  be such that  $M \in \mathbf{K}_{\mathfrak{s}_{\mu}}$  and  $M' \in \mathbf{K}_{\mathfrak{s}_{\mu'}}$ . Let  $p \in \mathrm{gS}(N)$ . We say that p does not fork over M if there exists  $M_0 \leq_{\mathbf{K}} M$  so that  $M_0 \in \mathbf{K}_{\mathfrak{s}_{\lambda}}$  and for every  $N_0 \in \mathbf{K}_{\mathfrak{s}_{\lambda}}$  with  $M_0 \leq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$ ,  $p \upharpoonright N_0$  does not  $\mathfrak{s}_{\lambda}$ -fork over  $M_0$ .

THEOREM 15.7.8. Assume that  $\langle \mathfrak{s}_{\mu} : \mu \in \mathcal{F} \rangle$  is a shrinking type-full good  $[\lambda, \theta)$ -frame. Then forking (as defined in Definition 15.7.7) has the usual properties: invariance, monotonicity, extension, uniqueness, transitivity, local character, and symmetry.

PROOF SKETCH. Invariance, monotonicity, transitivity, and local character are straightforward. Symmetry is also straightforward (once we have it when the domain and the base have the same size, it is a simple use of monotonicity). Uniqueness is by weak tameness, and extension is as in Proposition 10.5.1.  $\Box$ 

We can now state a generalization of Theorem 14.6.14 and sketch a proof:

THEOREM 15.7.9. Let  $\langle \mathbf{s}_{\mu} : \mu \in [\lambda, \theta) \rangle$  be a shrinking type-full good  $[\lambda, \theta)$ -frame on **K**. Let  $\mu \in [\lambda, \theta)$ . If  $\mathbf{K}^{\mu\text{-sat}}$  is categorical in  $\mu^+$ , then **K** is categorical in every  $\mu \in [\lambda, \theta]$ .

PROOF SKETCH. First note that in the upward transfer of Grossberg and VanDieren (Fact 14.6.12), it is implicit that tameness can be weakened to weak tameness (Remark 15.2.7). The rest of the proof of Theorem 14.6.14 (the downward part) is as before: we use Theorem 15.7.8 and make sure that anytime a resolution is taken, all the components are saturated.  $\Box$ 

#### 15.8. More on weak tameness

We use Theorem 15.7.9 to replace tameness by weak tameness in some of the results of the second part of this chapter. Everywhere in this section, we assume:

HYPOTHESIS 15.8.1. K is an AEC with amalgamation.

First, we state a stronger version of the main lemma (Lemma 15.4.2):

LEMMA 15.8.2. Assume that **K** has no maximal models. Let  $\theta \ge \lambda > LS(\mathbf{K})^+$  be such that **K** is  $(LS(\mathbf{K}), < \theta)$ -weakly tame. Assume that  $\lambda$  is a successor cardinal. If **K** is categorical in  $\lambda$ , then  $\mathbf{K}^{LS(\mathbf{K})^+}$ -sat is categorical in all  $\mu \in [LS(\mathbf{K})^+, \theta]$ .

PROOF. By the upward transfer of Grossberg and VanDieren (Fact 14.1.6), **K** (and therefore  $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$ ) is categorical in every  $\lambda' \in [\lambda, \theta]$ . It remains to show the downward part. By Fact 15.1.3, **K** is superstable in every  $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$ . Since  $\lambda$  is a successor, the model of size  $\lambda$  is saturated. By Fact 15.1.7, **K** has symmetry in every  $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$ . By Theorem 15.7.3, there is a shrinking type-full good  $[\mathrm{LS}(\mathbf{K})^+, \lambda)$ -frame on **K**. By Theorem 15.7.9,  $\mathbf{K}^{\mathrm{LS}(\mathbf{K})^+\text{-sat}}$  is categorical in all  $\lambda' \in [\mathrm{LS}(\mathbf{K})^+, \lambda]$ .

We can improve on Corollary 15.4.3:

COROLLARY 15.8.3. Assume that **K** has arbitrarily large models. Let  $LS(\mathbf{K}) < \lambda_0 < \lambda$ . Assume that **K** is  $(LS(\mathbf{K}), < \lambda)$ -weakly tame. If  $\lambda$  is a successor cardinal and **K** is categorical in  $\lambda_0$  and  $\lambda$ , then **K** is categorical in all  $\lambda' \in (\lambda_0, \lambda)$ .

PROOF. As in the proof of Corollary 15.4.3, using Lemma 15.8.2 (with  $\lambda, \theta$  there standing for  $\lambda, \lambda$  here).

Corollary 15.4.6 can similarly be generalized:

COROLLARY 15.8.4. Let  $\lambda \geq H_1$  be a successor cardinal and assume that **K** is  $(LS(\mathbf{K}), < \lambda)$ -weakly tame. If **K** is categorical in  $\lambda$ , then there exists  $\chi < H_1$  such that **K** is categorical in all  $\lambda' \in [\chi, \lambda)$ .

PROOF. As in the proof of Corollary 15.4.6, using Lemma 15.8.2 (with  $\lambda, \theta$  there standing for  $\lambda, \lambda$  here).

REMARK 15.8.5. It is unclear how to generalize the results using primes: the proof of Fact 15.4.8 uses tameness (for all models) heavily, and we do not know how to generalize it to weakly tame AECs.

We can use Corollary 15.8.4 to give an alternate proof to the main theorem of [She99].

COROLLARY 15.8.6. If **K** is categorical in some successor  $\lambda \geq \beth_{H_1}$ , then there exists  $\mu < \beth_{H_1}$  such that **K** is categorical in all  $\lambda' \in [\mu, \lambda)$ .

PROOF. Without loss of generality (Fact 15.1.9), **K** has no maximal models. By Fact 15.1.3, **K** is stable below  $\lambda$ , so the model of size  $\lambda$  is saturated. By Theorem 15.2.4, there exists  $\chi < H_1$  such that **K** is  $(\chi, < \lambda)$ -weakly tame. By Corollary 15.8.4 (applied to  $\mathbf{K}_{\geq \chi}$ ), there exists  $\mu < h(\chi) < \beth_{H_1}$  such that **K** is categorical in all  $\lambda' \in [\mu, \lambda)$ .

Generalizing Corollary 15.5.9 is harder. The problem is how to ensure that the model in the categoricity cardinal has enough saturation. We give a consistency result in case  $\lambda \geq H_1$ .

COROLLARY 15.8.7. Assume that  $2^{\text{LS}(\mathbf{K})} = 2^{\text{LS}(\mathbf{K})^+}$ , WGCH([LS( $\mathbf{K}$ )<sup>+</sup>, LS( $\mathbf{K}$ )<sup>+ $\omega$ </sup>)), and Claim 15.5.2 holds. Assume that  $\mathbf{K}$  is (LS( $\mathbf{K}$ ),  $< H_1$ )-weakly tame. If  $\mathbf{K}$  is categorical in *some*  $\lambda \ge H_1$ , then there exists  $\chi < H_1$  such that  $\mathbf{K}$  is categorical in *all*  $\lambda' \ge \chi$ .

PROOF. Without loss of generality (Fact 15.1.9), **K** has no maximal models. By Fact 15.1.3, **K** is superstable in every  $\mu \in [\text{LS}(\mathbf{K}), \lambda)$ . Since  $2^{\text{LS}(\mathbf{K})} = 2^{\text{LS}(\mathbf{K})^+}$ ,  $H_1 = h(\text{LS}(\mathbf{K})^+)$ , so by Fact 15.1.7, **K** has symmetry in  $\text{LS}(\mathbf{K})^+$ . By Fact 15.1.6,  $\mathbf{K}^{\text{LS}(\mathbf{K})^+\text{-sat}}$  is an AEC with  $\text{LS}(\mathbf{K}^{\text{LS}(\mathbf{K})^+\text{-sat}}) = \text{LS}(\mathbf{K})^+$ . In particular, the model of size  $\lambda$  is  $\mathrm{LS}(\mathbf{K})^+$ -saturated. By Fact 15.1.12, there exists a type-full good  $\mathrm{LS}(\mathbf{K})^+$ frame  $\mathfrak{s}$  on  $\mathbf{K}_{\mathrm{LS}(\mathbf{K})^+}^{\mathrm{LS}(\mathbf{K})^+}$ . By iterating Corollary 15.5.7,  $\mathfrak{s}$  is  $\omega$ -successful<sup>+</sup>. As in
the proof of Corollary 15.5.9, we get that  $\mathbf{K}$  is categorical on a tail of cardinals.
By Theorem 15.2.4,  $\mathbf{K}$  is  $\chi$ -weakly tame, so combining this with the hypothesis of
( $\mathrm{LS}(\mathbf{K}), < H_1$ )-tameness,  $\mathbf{K}$  is  $\mathrm{LS}(\mathbf{K})$ -weakly tame. Now apply Corollary 15.8.4.

### 15.9. Superstability for long types

We generalize Definition 15.1.1 to types of more than one element and use it to prove an extension property for 1-forking (recall Definition 14.4.2). This is used to give a converse to Lemma 14.6.7 in the next section (but is not needed for the main body of this chapter). Everywhere below,  $\mathbf{K}$  is an AEC.

DEFINITION 15.9.1. Let  $\alpha \leq \omega$  be a cardinal. **K** is  $(\langle \alpha, \mu \rangle)$ -superstable (or  $(\langle \alpha \rangle)$ -superstable in  $\mu$ ) if it satisfies Definition 15.9.1 except that in addition in condition (4) there we allow  $p \in gS^{\langle \alpha}(M_{\delta})$  rather than just  $p \in gS(M_{\delta})$  (that is, p need not have length one).  $(\leq \alpha, \mu)$ -superstable means  $(\langle \alpha^+, \mu)$ -superstable. When  $\alpha = 2$ , we omit it (that is,  $\mu$ -superstable means  $(\langle 2, \mu \rangle)$ -superstable which is the same as  $(\leq 1, \mu)$ -superstable).

While not formally equivalent (although we do not know of any examples separating the two),  $\mu$ -superstability and ( $< \omega, \mu$ )-superstability are very close. For example, the proof of Fact 15.1.3 also gives:

FACT 15.9.2. Let  $\mu \geq \text{LS}(\mathbf{K})$ . If **K** has amalgamation, no maximal models, and is categorical in a  $\lambda > \mu$ , then **K** is  $(< \omega, \mu)$ -superstable.

Even without categoricity, we can obtain eventual ( $< \omega$ )-superstability from just ( $\leq 1$ )-superstability and tameness. This uses another equivalent definition of superstability: solvability:

THEOREM 15.9.3. Assume **K** has amalgamation, no maximal models, and is  $LS(\mathbf{K})$ -tame. If **K** is  $LS(\mathbf{K})$ -superstable, then there exists  $\mu_0 < H_1$  such that **K** is  $(<\omega)$ -superstable in every  $\mu \ge \mu_0$ .

PROOF SKETCH. By Theorem 9.4.9, there exists  $\mu_0 < H_1$  such that **K** is  $(\mu_0, \mu)$ -solvable for every  $\mu \ge \mu_0$ . This means [**She09a**, Definition IV.1.4.(1)] that for every  $\mu \ge \mu_0$ , there exists an EM Blueprint  $\Phi$  so that  $\text{EM}_{\tau(\mathbf{K})}(I, \Phi)$  is a superlimit in **K** for every linear order I of size  $\mu$ . Intuitively, it gives a weak version of categoricity in  $\mu$ . As observed in Chapter 20, this weak version is enough for the proof of the Shelah-Villaveces theorem to go through, hence by Fact 15.9.2, **K** is  $(<\omega)$ -superstable in  $\mu$  for every  $\mu \ge \mu_0$ .

REMARK 15.9.4. If **K** has amalgamation, is LS(**K**)-tame for types of length less than  $\omega$ , and is ( $\langle \omega, \text{LS}(\mathbf{K})$ )-superstable, then (by the proof of Proposition 6.10.10) **K** is ( $\langle \omega \rangle$ -superstable in every  $\mu \geq \text{LS}(\mathbf{K})$ . However here we want to stick to using regular tameness (i.e. tameness for types of length one).

To prove the extension property for 1-forking, we will use:

FACT 15.9.5 (Extension property for splitting, Proposition 10.5.1). Let **K** be an AEC,  $\theta > \text{LS}(\mathbf{K})$ . Let  $\alpha \leq \omega$  be a cardinal and assume that **K** is  $(< \alpha)$ -superstable in every  $\mu \in [\text{LS}(\mathbf{K}), \theta)$ . Let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  be in  $\mathbf{K}_{[\text{LS}(\mathbf{K}),\theta)}$ , with M limit over

 $M_0$ . Let  $p \in gS^{<\alpha}(M)$  be such that p does not  $LS(\mathbf{K})$ -split over  $M_0$ . Then there exists an extension  $q \in gS^{<\alpha}(N)$  of p which does not  $LS(\mathbf{K})$ -split over  $M_0$ .

THEOREM 15.9.6. Let  $\theta > \mathrm{LS}(\mathbf{K})$ . Write  $\mathcal{F} := [\mathrm{LS}(\mathbf{K}), \theta)$ . Let  $\mathfrak{s}$  be a type-full good  $\mathcal{F}$ -frame with underlying class  $\mathbf{K}_{\mathcal{F}}$ . Let  $\alpha \leq \omega$  be a cardinal and assume that  $\mathbf{K}$  is  $(< \alpha, \mu)$ -superstable for every  $\mu \in \mathcal{F}$ . Let  $M \leq_{\mathbf{K}} N$  be in  $\mathbf{K}_{\mathcal{F}}$  with M a limit model. Let  $p \in \mathrm{gS}^{<\alpha}(M)$ . Then there exists  $q \in \mathrm{gS}^{<\alpha}(N)$  that extends p so that qdoes not 1- $\mathfrak{s}$ -fork over M (recall Definition 14.4.2).

PROOF. Without loss of generality, N is a limit model. Let  $\mu := ||M||$ . By  $(< \alpha)$ -superstability, there exists  $M_0 \in \mathbf{K}_{\mu}$  such that M is limit over  $M_0$  and p does not  $\mu$ -split over  $M_0$ . By Fact 15.9.5, there exists  $q \in \mathrm{gS}(N)$  extending p so that q does not  $\mu$ -split over  $M_0$ . We claim that q does not 1-s-fork over M. Let  $I \subseteq \ell(p)$  have size one. By monotonicity of splitting,  $q^I$  does not  $\mu$ -split over  $M_0$ . Superstability,  $q^I$  does not  $\mu$ -split over  $M_0$ . By local character, let  $N_0 \leq_{\mathbf{K}} N$  be such that  $M \leq_{\mathbf{K}} N_0$ ,  $N_0 \in \mathbf{K}_{\mu}$ ,  $N_0$  is limit, and  $q^I$  does not s-fork over  $N_0$ . By monotonicity of splitting again,  $q^I \upharpoonright N_0$  does not  $\mu$ -split over  $M_0$ . By the canonicity theorem (Fact 15.7.6) applied to the frame  $\mathfrak{s} \upharpoonright \mathbf{K}_{\mu}$ ,  $q^I \upharpoonright N_0$  does not  $\mathfrak{s}$ -fork over M. By transitivity,  $q^I$  does not  $\mathfrak{s}$ -fork over M, as desired.

# 15.10. More on global orthogonality

Assuming superstability for types of length two, we prove a converse to Lemma 14.6.7, partially answering Question 14.6.8. We then prove a few more facts about global orthogonality and derive an alternative proof of the upward categoricity transfer of Grossberg and VanDieren (Fact 14.1.6). This material is not needed for the main body of this chapter.

Hypothesis 15.10.1.

- (1)  $\mathbf{K}$  is an AEC.
- (2)  $\theta > \mathrm{LS}(\mathbf{K})$  is a cardinal or  $\infty$ . We set  $\mathcal{F} := [\mathrm{LS}(\mathbf{K}), \theta)$ .
- (3)  $\mathfrak{s} = (\mathbf{K}_{\mathcal{F}}, \downarrow)$  is a type-full good  $\mathcal{F}$ -frame.
- (4) **K** is  $(\leq 2)$ -superstable in every  $\mu \in \mathcal{F}$ .

REMARK 15.10.2. Compared to Hypothesis 14.6.1, we have added ( $\leq 2$ )-superstability. Note that this would follow automatically if  $\mathfrak{s}$  was a type-full good frame for types of length two, hence it is a minor addition. It also holds if **K** is categorical above  $\mathcal{F}$  (Fact 15.9.2) or even if it is just tame (Theorem 15.9.3).

LEMMA 15.10.3. Let  $M_0 \leq_{\mathbf{K}} M$  be both in  $\mathbf{K}_{\mathcal{F}}$  with  $M_0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  limit. Let  $p, q \in \mathrm{gS}(M)$  be nonalgebraic so that both do not fork over  $M_0$ . If  $p \perp_{\mathrm{wk}} q$ , then  $p \upharpoonright M_0 \perp_{\mathrm{wk}} q \upharpoonright M_0$ .

PROOF. Assume that  $p \upharpoonright M_0 \oiint_{wk} q \upharpoonright M_0$ . We show that  $p \oiint_{wk} q$ . Fix  $N \in \mathbf{K}_{\mathcal{F}}$ with  $M_0 \leq_{\mathbf{K}} N$  and let  $a, b \in |N|$  realize in  $N p \upharpoonright M_0$  and  $q \upharpoonright M_0$  respectively. Assume that  $\langle ab \rangle$  is not independent in  $(M_0, N)$ . Let  $r := \operatorname{gtp}(ab/M_0; N)$ . By Theorem 15.9.6, there exists  $r' \in \operatorname{gS}^2(M)$  that extends r and so that r' does not 1-s-fork over  $M_0$  (recall Definition 14.4.2). Let  $N' \geq_{\mathbf{K}} M$  and let  $\langle a'b' \rangle$  realize r' in N'. Then  $\operatorname{gtp}(a'/M; N')$  does not fork over  $M_0$  and extends  $p \upharpoonright M_0$ , hence a' must realize p in N'. Similarly, b' realizes q. We claim that  $\langle a'b' \rangle$  is not independent in (M, N'), hence  $p \not \leq q$ . If  $\langle a'b' \rangle$  were independent in (M, N'), there would exist  $N'' \geq_{\mathbf{K}} N'$  and  $M' \leq_{\mathbf{K}} N''$  so that  $M \leq_{\mathbf{K}} M', b \in |M'|$ , and gtp(a'/M'; N'') does not fork over M. By transitivity, gtp(a'/M'; N'') does not fork over  $M_0$ . This shows that  $\langle a'b' \rangle$  is independent in  $(M_0, N'')$ , so since  $gtp(a'b'/M_0; N'') = gtp(ab/M_0; N)$ , we must have that  $\langle ab \rangle$  is independent in  $(M_0, N)$ , a contradiction.  $\square$ 

We obtain:

THEOREM 15.10.4. Let  $M \in \mathbf{K}_{\mathcal{F}}$  and  $p, q \in \mathrm{gS}(M)$ . Then:

- (1) If  $M \in \mathbf{K}_{\mathcal{F}}^{\mathrm{LS}(\mathbf{K})\text{-sat}}$ , then  $p \perp q$  if and only if  $p \perp q$ .
- (2) If M ∈ K<sup>LS(K)-sat</sup><sub>F</sub>, then p ⊥ q if and only if q ⊥ p.
  (3) If M<sub>0</sub> ∈ K<sup>LS(K)-sat</sup><sub>F</sub> is such that M<sub>0</sub> ≤<sub>K</sub> M and both p and q do not fork over M<sub>0</sub>, then p ⊥ q if and only if p ↾ M<sub>0</sub> ⊥ q ↾ M<sub>0</sub>.

Proof.

- (1) If  $p \perp q$ , then  $p \perp_{\text{wk}} q$  by definition. Conversely, assume that  $p \perp_{\text{wk}} q$ . Fix a limit  $M_0 \in \mathbf{K}_{\text{LS}(\mathbf{K})}$  such that  $M_0 \leq_{\mathbf{K}} M$  and both p and q do not fork over  $M_0$ . By Lemma 15.10.3,  $p \upharpoonright M_0 \perp_{wk} q \upharpoonright M_0$ . By Lemma 14.4.13.(1),  $p \upharpoonright M_0 \perp q \upharpoonright M_0$ . By Lemma 14.6.7,  $p \perp q$ .
- (2) A similar proof, using (2) instead of (1) in Lemma 14.4.13.
- (3) By local character and transitivity, we can fix a limit  $M'_0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  such that  $M'_0 \leq_{\mathbf{K}} M_0$  and both p and q do not fork over  $M'_0$ . Now by what has been proven above and Lemmas 14.6.7 and 15.10.3,  $p \perp q$  if and only if  $p \upharpoonright M'_0 \perp q \upharpoonright M'_0$  if and only if  $p \upharpoonright M_0 \perp q \upharpoonright M_0$ .

We can now give another proof of the upward transfer of unidimensionality (the second part of the proof of Theorem 14.6.13). This does not use Fact 14.6.12.

LEMMA 15.10.5. Let  $\mu < \lambda$  be in  $\mathcal{F}$ . If  $\mathfrak{s}$  is  $\mu$ -unidimensional, then  $\mathfrak{s}$  is  $\lambda$ unidimensional.

PROOF. Assume that  $\mathfrak{s}$  is not  $\lambda$ -unidimensional. Let  $M_0 \in \mathbf{K}_{\mu}$  be limit and let  $p_0 \in gS(M_0)$  be minimal. We show that there exists a limit  $M'_0 \in \mathbf{K}_{\mu}, p'_0, q'_0 \in$  $gS(M'_0)$  such that  $p'_0$  extends  $p_0$  and  $p'_0 \perp q'_0$ . This will show that **K** is not  $\mu$ -unidimensional by Lemma 14.5.7. Let  $M \in \mathbf{K}_{\lambda}$  be saturated such that  $M_0 \leq_{\mathbf{K}} M$ and let  $p \in gS(M)$  be the nonforking extension of  $p_0$ . By non- $\lambda$ -unidimensionality (and Lemma 14.5.7), there exists  $q \in gS(M)$  so that  $p \perp q$ . Let  $M'_0 \in \mathbf{K}_{\mu}$  be limit such that  $M_0 \leq_{\mathbf{K}} M'_0 \leq_{\mathbf{K}} M$  and q does not fork over  $M'_0$ . Let  $p'_0 := p \upharpoonright M'_0$ ,  $q'_0 := q \upharpoonright M'_0$ . By Theorem 15.10.4,  $p'_0 \perp q'_0$ , as desired.

We obtain the promised alternate proof to Grossberg-VanDieren. For this corollary, we drop Hypothesis 15.10.1.

COROLLARY 15.10.6. Let K be an AEC with amalgamation and arbitrarily large models. If K is LS(K)-tame and categorical in a successor  $\lambda > LS(K)^+$ , then **K** is categorical in all  $\mu \geq \lambda$ .

PROOF. By Fact 15.1.9, we can assume without loss of generality that K has no maximal models. By Fact 15.1.3, K is LS(K)-superstable. By Fact 15.1.5, K is superstable and has symmetry in every  $\mu \geq LS(\mathbf{K})$ . By Theorem 15.3.8, **K** is categorical in a proper class of cardinals. By Fact 15.9.2, **K** is  $(\langle \omega \rangle)$ -superstable in every  $\mu \geq \text{LS}(\mathbf{K})$ . Now say  $\lambda = \lambda_0^+$ . By Facts 15.1.12 and 15.1.13, there exists a type-full good  $(\geq \lambda_0)$ -frame  $\mathfrak{s}$  with underlying class  $\mathbf{K}_{\geq \text{LS}(\mathbf{K})^+}^{\text{LS}(\mathbf{K})^+-\text{sat}}$ . By Fact 14.2.6, we can restrict the frame further to have underlying class  $\mathbf{K}_{\geq \lambda_0}^{\lambda_0-\text{sat}}$ . By Corollary 14.6.14 (using Lemma 15.10.5 to transfer unidimensionality up),  $\mathbf{K}^{\lambda_0-\text{sat}}$  is categorical in every  $\mu \geq \lambda_0$ . Now  $\mathbf{K}_{\geq \lambda}^{\lambda_0-\text{sat}} = \mathbf{K}_{\geq \lambda}$  (by categoricity in  $\lambda$ ), so the result follows.  $\Box$ 

REMARK 15.10.7. Similarly to what was said in Remark 15.4.4, it is also possible to use this argument to prove that a  $LS(\mathbf{K})$ -tame AEC with amalgamation and arbitrarily large models categorical in  $LS(\mathbf{K})$  and  $LS(\mathbf{K})^+$  is categorical everywhere [**GV06a**, Theorem 6.3]. However we do not know that we have a good frame in  $LS(\mathbf{K})$ , so the proof is more complicated.

REMARK 15.10.8. As opposed to Grossberg and VanDieren's proof, our proof of Corollary 15.10.6 is global: it cannot be turned into an argument for Fact 14.6.12 (e.g. if  $\theta$  there is  $LS(\mathbf{K})^{+\omega}$  we cannot use Theorem 15.3.8). It is also not clear how to replace tameness with weak tameness in the proof.

# 15.11. A proof of Fact 15.5.3

HYPOTHESIS 15.11.1. **K** is an AEC,  $\lambda \geq LS(\mathbf{K})$ .

We give a full proof of Fact 15.5.3. Shelah's proof in  $\odot_4$  of the proof of [**She09a**, Theorem IV.7.12] skips some steps (for example it is not clear how we can make sure there that  $f_{\eta \sim 0}[M_{\ell(\eta)+1}] = f_{\eta \sim 1}[M_{\ell(\eta)+1}]$ ). The proof we give here is similar in spirit to Shelah's, but we use a stronger blackbox (relying on weak diamond) which avoids having to deal with all of Shelah's renaming steps.

We will use the combinatorial principle  $\Theta_{\lambda^+}$  introduced (for  $\lambda = \aleph_0$ ) in [**DS78**].

DEFINITION 15.11.2.  $\Theta_{\lambda^+}$  holds if for every  $\langle f_\eta \in {}^{\lambda^+}\lambda^+ : \eta \in {}^{\lambda^+}2 \rangle$ , there exists  $\eta \in {}^{\lambda^+}2$  such that the set

$$S_{\eta} := \{ \delta < \lambda^+ \mid \exists \nu \in {}^{\lambda^+}2 : f_{\eta} \upharpoonright \delta = f_{\nu} \upharpoonright \delta \land \eta \upharpoonright \delta = \nu \upharpoonright \delta \land \eta(\delta) \neq \nu(\delta) \}$$

is stationary.

REMARK 15.11.3. Instead of requiring that  $f_{\eta} \in \lambda^+ \lambda^+$ , we can assume only that  $f_{\eta}$  is a partial function from  $\lambda^+$  to  $\lambda^+$  (we can always extend  $f_{\eta}$  arbitrarily to an actual function).

FACT 15.11.4 (6.1 in [**DS78**]). If  $2^{\lambda} < 2^{\lambda^+}$ , then  $\Theta_{\lambda^+}$  holds.

In [She01a, Claim 1.4.(2)], Shelah shows assuming the weak diamond that given a tree witnessing failure of amalgamation in  $\lambda$ , there cannot be a universal model of cardinality  $\lambda^+$ . A similar proof gives a two-dimensional version:

LEMMA 15.11.5. Assume  $2^{\lambda} < 2^{\lambda^+}$ . Let  $\langle M_{\eta} : \eta \in {}^{\leq \lambda^+} 2 \rangle$  be a strictly increasing continuous tree with  $M_{\eta} \in \mathbf{K}_{\lambda}$  for all  $\eta \in {}^{<\lambda^+} 2$ . Let  $\langle M_{\alpha} : \alpha \leq \lambda^+ \rangle$  be an increasing continuous chain and let  $\langle f_{\eta} : \eta \in {}^{\leq \lambda^+} 2 \rangle$  be such that for any  $\eta \in {}^{\leq \lambda^+} 2$ ,  $f_{\eta} : M_{\ell(\eta)} \to M_{\eta}$  and for any  $\alpha < \ell(\eta)$ ,  $f_{\eta} \upharpoonright M_{\alpha} = f_{\eta \upharpoonright \alpha}$ .

Assume that there exists  $N \in \mathbf{K}_{\lambda^+}$  with  $M_{\lambda^+} \leq_{\mathbf{K}} N$  such that for all  $\eta \in {}^{\lambda^+}2$ , there exists  $g_\eta : M_\eta \to N$  such that the following diagram commutes:



Then there exists  $\rho \in {}^{<\lambda^+}2$ ,  $\eta, \nu \in {}^{\lambda^+}2$ ,  $\delta < \lambda^+$  such that  $\rho = \eta \upharpoonright \delta = \nu \upharpoonright \delta$ ,  $0 = \eta(\delta) \neq \nu(\delta) = 1$ , and the following diagram commutes:



PROOF. Fix  $N \in \mathbf{K}_{\lambda^+}$ ,  $f_{\eta}$ ,  $g_{\eta}$  as in the statement of the lemma. By renaming everything, we can assume without loss of generality that  $|N| \subseteq \lambda^+$  and  $|M_{\eta}| \subseteq \lambda^+$  for all  $\eta \in \leq^{\lambda^+} 2$ .

By Fact 15.11.4,  $\Theta_{\lambda^+}$  holds. We use it with the sequence  $\langle g_\eta : \eta \in {}^{\lambda^+}2 \rangle$  (see Remark 15.11.3). We obtain  $\eta \in {}^{\lambda^+}2$  such that the set  $S_\eta$  of Definition 15.11.2 is stationary. Let  $C := \{\delta < \lambda^+ \mid |M_{\eta \restriction \delta}| \subseteq \delta\}$ . Clearly, C is club so let  $\delta \in S_\eta \cap C$  and let  $\nu \in {}^{\lambda^+}2$  be as given by the definition of  $S_\eta$  (i.e.  $g_\eta \restriction \delta = f_\nu \restriction \delta, \eta \restriction \delta = \nu \restriction \delta$ , and without loss of generality  $0 = \eta(\delta) \neq \nu(\delta) = 1$ ). Let  $\rho := \eta \restriction \delta = \nu \restriction \delta$ . Then  $\rho, \eta, \nu, \delta$  are as desired. The main point is that since  $|M_\rho| \subseteq \delta$  (by definition of  $\delta$ ),  $g_\eta \restriction M_\rho = g_\nu \restriction M_\rho$ .

The next technical property is of great importance in Chapter II and III of [She09a]. The definition below follows [JS13, Definition 4.1.5] (but as usual, we work only with type-full frames).

Definition 15.11.6.

- (1) For  $M_0 \leq_{\mathbf{K}} M_\ell$  all in  $\mathbf{K}_{\lambda}$ ,  $\ell = 1, 2$ , an amalgam of  $M_1$  and  $M_2$  over  $M_0$  is a triple  $(f_1, f_2, N)$  such that  $N \in \mathbf{K}_{\lambda}$  and  $f_\ell : M_\ell \xrightarrow[M_0]{M_0} N$ .
- (2) Let  $(f_1^x, f_2^x, N^x)$ , x = a, b be amalgams of  $M_1$  and  $M_2$  over  $M_0$ . We say  $(f_1^a, f_2^a, N^a)$  and  $(f_1^b, f_2^b, N^b)$  are equivalent over  $M_0$  if there exists  $N_* \in \mathbf{K}_{\lambda}$  and  $f^x : N^x \to N_*$  such that  $f^b \circ f_1^b = f^a \circ f_1^a$  and  $f^b \circ f_2^b = f^a \circ f_2^a$ , namely, the following commutes:


Note that being "equivalent over  $M_0$ " is an equivalence relation ([**JS13**, Proposition 4.3]).

- (3) Let  $\mathfrak{s}$  be a type-full good  $\lambda$ -frame on **K**.
  - (a) Let  $\mathbf{K}_{\mathfrak{s}}^{3,\mathrm{uq}}$  denote the set of triples (a, M, N) such that  $M \leq_{\mathbf{K}} N$  are in  $\mathbf{K}_{\lambda}$ ,  $a \in |N| \setminus |M|$  and for any  $M_1 \geq_{\mathbf{K}} M$  in  $\mathbf{K}_{\lambda}$ , there exists a unique (up to equivalence over M) amalgam  $(f_1, f_2, N_1)$  of N and  $M_1$  over M such that  $\operatorname{gtp}(f_1(a)/f_2[M_1]; N_1)$  does not fork over M. We call the elements of  $\mathbf{K}_{\mathfrak{s}}^{3,\mathrm{uq}}$  uniqueness triples. When  $\mathfrak{s}$  is clear from context, we just write  $\mathbf{K}^{3,\mathrm{uq}}$ .
  - (b)  $\mathbf{K}_{\mathfrak{s}}^{3,\mathrm{uq}}$  has the existence property if for any  $M \in \mathbf{K}_{\lambda}$  and any nonalgebraic  $p \in \mathrm{gS}(M)$ , one can write  $p = \mathrm{gtp}(a/M; N)$  with  $(a, M, N) \in \mathbf{K}_{\mathfrak{s}}^{3,\mathrm{uq}}$ .
  - (c) We say that  $\mathfrak{s}$  has the existence property for uniqueness triples or  $\mathfrak{s}$  is weakly successful if  $\mathbf{K}_{\mathfrak{s}}^{3,\mathrm{uq}}$  has the existence property.

REMARK 15.11.7. Let  $M_0 \leq_{\mathbf{K}} M_1, M_0 \leq_{\mathbf{K}} M_2$  all be in  $\mathbf{K}_{\lambda}$ .

- (1) If  $(f_1, f_2, N)$  is an amalgam of  $M_1$  and  $M_2$  over  $M_0$ , there exists an equivalent amalgam  $(g_1, g_2, N')$  of  $M_1$  and  $M_2$  over  $M_0$  with  $g_2 = \mathrm{id}_{M_2}$ .
- (2) For x = a, b, assume that  $(f_1^x, f_2^x, N^x)$  are non-equivalent amalgams of  $M_1$  and  $M_2$  over  $M_0$ . We have the following monotonicity properties:
  - (a) For x = a, b, if  $N^x \leq_{\mathbf{K}} N^x_*$ , then  $(f_1^x, f_2^x, N^x_*)$  are non-equivalent amalgams of  $M_1$  and  $M_2$  over  $M_0$ .
  - (b) If  $M_1 \leq_{\mathbf{K}} M'_1$ ,  $M_2 \leq_{\mathbf{K}} M'_2$  (with  $M'_1, M'_2 \in \mathbf{K}_{\lambda}$ ), and for x = a, b, there exists  $g_1^x \supseteq f_1^x, g_2^x \supseteq f_2^x$  such that  $(g_1^x, g_2^x, N^x)$  is an amalgam of  $M'_1$  and  $M'_2$  over  $M_0$ , then  $(g_1^a, g_2^a, N^a)$  and  $(g_1^b, g_2^b, N^b)$  are not equivalent over  $M_0$ .

We are now ready to prove the desired result.

THEOREM 15.11.8 (Shelah). Assume  $2^{\lambda} < 2^{\lambda^+}$ . Let  $\mathfrak{s}$  be a good  $\lambda$ -frame on **K**. If **K** is categorical in  $\lambda$  and for any saturated  $M \in \mathbf{K}_{\lambda^+}$  there exists  $N \in \mathbf{K}_{\lambda^+}$  universal over M, then  $\mathfrak{s}$  is weakly successful.

PROOF. Below, we assume to simplify the notation that  $\mathfrak{s}$  is type-full but the same proof goes through in the general case. Suppose that the conclusion of the theorem fails. Fix  $\langle M_{\alpha} : \alpha < \lambda^+ \rangle$  increasing continuous in  $\mathbf{K}_{\lambda}$  such that  $M_{\alpha+1}$  is limit over  $M_{\alpha}$  for all  $\alpha < \lambda^+$ . Let  $M_{\lambda^+} := \bigcup_{\alpha < \lambda^+} M_{\alpha}$ . Since  $\mathfrak{s}$  is not weakly successful, there exists a nonalgebraic type  $p \in \mathrm{gS}(M_0)$  which cannot be represented by a uniqueness triple. Say  $p = \mathrm{gtp}(a/M_0; M)$ .

We build a strictly increasing continuous tree  $\langle M_{\eta} : \eta \in {}^{\leq \lambda^+} 2 \rangle$  with  $M_{\eta} \in \mathbf{K}_{\lambda}$  for  $\eta \in {}^{<\lambda^+} 2$ , as well as a strictly increasing continuous tree of embeddings  $\langle f_{\eta} : \eta \in {}^{\leq \lambda^+} 2 \rangle$  such that for any  $\eta \in {}^{<\lambda^+} 2$ :

- (1)  $f_{\eta}: M_{\ell(\eta)} \to M_{\eta}.$
- (2)  $M_{<>} = M$  and  $f_{<>} = \mathrm{id}_{M_0}$ .
- (3)  $gtp(a/f_{\eta}[M_{\ell(\eta)}]; M_{\eta})$  does not fork over  $M_0$ .
- (4) There is no  $N \in \mathbf{K}_{\lambda}$  and  $g_{\ell} : M_{\eta \frown l} \to N, \ \ell = 0, 1$ , such that the following diagram commutes:



<u>This is enough</u>: We have that  $M_{\lambda^+}$  is saturated and we know by assumption that there is a universal model N over  $M_{\lambda^+}$  in  $\mathbf{K}_{\lambda^+}$ . In particular, for every  $\eta \in {}^{\lambda^+}2$ , there exists  $g_\eta : M_\eta \to N$  such that  $f_\eta^{-1} \subseteq g_\eta$ . By Lemma 15.11.5, requirement (4) must fail somewhere in the construction, contradiction.

This is possible: The construction is by induction on the length of  $\eta \in {}^{\leq \lambda^+} 2$ . If  $\ell(\overline{\eta}) = 0$ , then (2) specifies what to do and if the length is limit, we take unions (and use the local character and transitivity properties of forking to see that (3) is preserved). Assume now that  $\alpha < \lambda^+$  and that  $\eta \in {}^{\alpha} 2$  is such that  $M_{\eta}$ ,  $f_{\eta}$  have been defined. We want to build  $M_{\eta \sim \ell}$ ,  $f_{\eta \sim \ell}$  for  $\ell = 0, 1$ . Let  $q := \operatorname{gtp}(a/f_{\eta}[M_{\alpha}]; M_{\eta})$ . We know that q is the nonforking extension of p (by (3) and the definition of p), so by the conjugation property (Fact 14.3.3, note that by assumption **K** is categorical in  $\lambda$ ) p and q are conjugates, hence q cannot be represented by a uniqueness triple. Therefore  $(a, f_{\eta}[M_{\alpha}], M_{\eta})$  is not a uniqueness triple. This means that there exists  $M'_{\alpha} \in \mathbf{K}_{\lambda}$  with  $f_{\eta}[M_{\alpha}] \leq M'_{\alpha}$  and two non-equivalent amalgams  $(f_{1}^{\ell}, f_{2}^{\ell}, M_{\eta \sim \ell})$  such that  $\operatorname{gtp}(f_{2}^{\ell}(a)/f_{1}^{\ell}[M'_{\alpha}]; M_{\eta \sim \ell})$  does not fork over  $M_{\eta}$  for  $\ell = 0, 1$ . Without loss of generality (Remark 15.11.7)  $f_{2}^{\ell}$  is the identity for  $\ell = 0, 1$ :



By the monotonicity property of being non-equivalent amalgams (Remark 15.11.7) and the extension property of forking, we can increase  $M'_{\alpha}$ ,  $M^0_{\eta}$ , and  $M^1_{\eta}$  to assume without loss of generality that  $M'_{\alpha}$  is limit over  $f_{\eta}[M_{\alpha}]$ . In particular, there exists  $g: M_{\alpha+1} \cong M'_{\alpha}$  with  $f_{\eta} \subseteq g$ . For  $\ell = 0, 1$ , let  $f_{\eta \sim \ell} := f_1^{\ell} \circ g$ .

PROOF OF FACT 15.5.3. Note that amalgamation and stability in  $\lambda^+$  imply that over every  $M \in \mathbf{K}_{\lambda^+}$  there exists  $N \in \mathbf{K}_{\lambda^+}$  universal over M. Thus the hypotheses of Theorem 15.11.8 hold.

# CHAPTER 16

# Shelah's eventual categoricity conjecture in universal classes: part II

This chapter is based on [Vas17c]. I thank John Baldwin for inviting me to visit UIC in Fall 2015 to present a preliminary version of Chapter 8. The present chapter is an answer to several questions he asked me. This paper was also presented at seminars in Harvard and Rutgers University. I thank the organizers of these seminars for showing interest in my work and inviting me to talk. I thank the participants of these seminars for helpful feedback that helped me refine the presentation and motivation for this paper. I thank the referee for a detailed report that helped me improve the presentation of this paper. Finally, this chapter would not exist without the constant support and encouragements of Samaneh. I would like to dedicate this work to her.

# Abstract

We prove that a universal class categorical in a high-enough cardinal is categorical on a tail of cardinals. As opposed to other results in the literature, we work in ZFC, do not require the categoricity cardinal to be a successor, do not assume amalgamation, and do not use large cardinals. Moreover we give an explicit bound on the "high-enough" threshold:

THEOREM 16.0.9. Let  $\psi$  be a universal  $\mathbb{L}_{\omega_1,\omega}$  sentence (in a countable vocabulary). If  $\psi$  is categorical in *some*  $\lambda \geq \beth_{\beth_{\omega_1}}$ , then  $\psi$  is categorical in *all*  $\lambda' \geq \beth_{\beth_{\omega_1}}$ .

As a byproduct of the proof, we show that a conjecture of Grossberg holds in universal classes:

COROLLARY 16.0.10. Let  $\psi$  be a universal  $\mathbb{L}_{\omega_1,\omega}$  sentence (in a countable vocabulary) that is categorical in some  $\lambda \geq \beth_{\beth_{\omega_1}}$ , then the class of models of  $\psi$  has the amalgamation property for models of size at least  $\beth_{\beth_{\omega_1}}$ .

We also establish generalizations of these two results to uncountable languages. As part of the argument, we develop machinery to transfer model-theoretic properties between two different classes satisfying a compatibility condition (agreeing on any sufficiently large cardinals in which either is categorical). This is used as a bridge between Shelah's milestone study of universal classes (which we use extensively) and a categoricity transfer theorem of the author for abstract elementary classes that have amalgamation, are tame, and have primes over sets of the form  $M \cup \{a\}$ .

# 16.1. Introduction

In 1965, Morley [Mor65] started what is now called stability theory by proving:

FACT 16.1.1. If a countable first-order theory is categorical<sup>1</sup> in *some* uncountable cardinal, then it is categorical in *all* uncountable cardinals.

In 1976, Shelah proposed [She90, Open Problem D.(3a)] the following farreaching generalization:

CONJECTURE 16.1.2 (Shelah's categoricity conjecture for  $\mathbb{L}_{\omega_1,\omega}$ ). Let  $\psi$  be an  $\mathbb{L}_{\omega_1,\omega}$  sentence. If  $\psi$  is categorical in *some* cardinal  $\lambda \geq \beth_{\omega_1}$ , then  $\psi$  is categorical in *all* cardinals  $\lambda' \geq \beth_{\omega_1}$ .

This is now recognized as the central test question in nonelementary model theory. In 1977, Shelah introduced abstract elementary classes (AECs) [She87a], an abstract framework encompassing classes of models of an  $\mathbb{L}_{\lambda^+,\omega}(Q)$  theory and several other examples of interest. Shelah has stated in [She09b, N.4.2] the following version of the conjecture:

CONJECTURE 16.1.3 (Shelah's eventual categoricity conjecture for AECs). If an AEC is categorical in a high-enough cardinal, then it is categorical on a tail of cardinals.

While many pages of approximations exist (see the references given after the statement of the main theorem below) both conjectures are still open.

In this chapter, we prove an approximation of Conjecture 16.1.2 when  $\psi$  is a *universal* (see Definition 16.2.3) sentence  $(\beth_{\omega_1}$  is replaced by  $\beth_{\beth_{\omega_1}}$ , see more below). More generally, we confirm Conjecture 16.1.3 for universal classes: classes of models of a universal  $\mathbb{L}_{\infty,\omega}$  theory, or equivalently classes of models K in a fixed vocabulary  $\tau(K)$  closed under isomorphisms, substructure, and unions of  $\subseteq$ -increasing chains.

MAIN THEOREM 16.7.3. Let K be a universal class. If K is categorical in some  $\lambda \geq \beth_{(2^{|\tau(K)|+\aleph_0})^+}$ , then K is categorical in all  $\lambda' \geq \beth_{(2^{|\tau(K)|+\aleph_0})^+}$ .

Let us compare the main theorem to earlier approximations to Shelah's eventual categoricity conjecture<sup>2</sup>: In a series of papers [**GV06b**, **GV06c**, **GV06a**], Grossberg and VanDieren isolated *tameness*, a locality properties of AECs, and (using earlier work of Shelah [**She99**]) proved Shelah's eventual categoricity conjecture in tame AECs with amalgamation assuming that the starting categoricity cardinal is a successor. Boney [**Bon14b**] later showed (building on work of Makkai-Shelah [**MS90**]) that tameness (as well as amalgamation, if in addition categoricity in a high-enough cardinal is assumed) follows from a large cardinal axiom (a proper class of strongly compact cardinals exists). Therefore the eventual categoricity cardinal is a successor, and a large cardinal axiom holds. In [**She09a**, IV.7.12], Shelah removes the successor hypothesis assuming amalgamation<sup>3</sup> and the generalized continuum hypothesis (GCH)<sup>4</sup>. Shelah's proof is clarified in Section 15.5, but it relies on a claim which Shelah has yet to publish a proof of.

<sup>&</sup>lt;sup>1</sup>We say that a class of structures is *categorical* in a cardinal  $\lambda$  if it has a unique (up to isomorphism) model of size  $\lambda$ . We say that a theory or sentence (in some logic) is categorical in  $\lambda$  if its class of models is.

 $<sup>^{2}</sup>$ We do not present a complete history or an exhaustive list of recent results here. See the introduction of Chapter 8 for the former and [**BVd**] for the latter.

<sup>&</sup>lt;sup>3</sup>By [Bon14b, Theorem 7.6], this can also be replaced by a large cardinal axiom.

<sup>&</sup>lt;sup>4</sup>It is enough to assume the existence of a suitable family of cardinals  $\theta$  such that  $2^{\theta} < 2^{\theta^+}$ .

In any case, all known categoricity transfers (which do not make model-theoretic assumptions on the AEC) rely on the existence of large cardinals together with either GCH or the assumption that the categoricity cardinal is a successor.

In the prequel to this chapter (Chapter 8) we showed that some of these limitations could be overcome in the case of universal classes:

FACT 16.1.4 (Corollary 8.5.28). Let K be a universal class.

- (1) If K is categorical in cardinals of arbitrarily high cofinality, then K is categorical on a tail of cardinals.
- (2) If  $\kappa > |\tau(K)| + \aleph_0$  is a measurable cardinal and K is categorical in some  $\lambda \ge \Box_{\Box_{\Box_{\tau}}}$  then K is categorical in all  $\lambda' \ge \Box_{\Box_{\Box_{\tau}}}$ .

Still, requirements on the categoricity cardinal in the first case and the existence of large cardinals in the second case could not be completely eliminated. These hypotheses were made to prove the amalgamation property, which is known to be the only obstacle:

FACT 16.1.5 (Corollary 15.4.11). Let K be a universal class with amalgamation. If K is categorical in some  $\lambda \geq \beth_{(2^{|\tau(K)|+\aleph_0})^+}$ , then K is categorical in all  $\lambda' \geq \beth_{(2^{|\tau(K)|+\aleph_0})^+}$ .

Note that (see Chapters 8, 11) all the facts stated above hold in a much wider context than universal classes: tame AECs with primes. However for the specific case of universal classes there is a well-developed structure theory [She87b]. This chapter uses it to remove the assumption of amalgamation from Fact 16.1.5 and prove the main theorem. Further, a conjecture of Grossberg [Gro02, Conjecture 2.3] says that any AEC categorical in a high-enough cardinal should have amalgamation on a tail. A byproduct of this chapter is that Grossberg's conjecture holds in universal classes (see the proof of Theorem 16.7.3). Note that the behavior of amalgamation in universal classes is nontrivial: Kolesnikov and Lambie-Hanson have shown [KLH16] that for each  $\alpha < \omega_1$ , there is a universal class in a countable vocabulary that has amalgamation up to  $\beth_{\alpha}$  but fails amalgamation everywhere above  $\beth_{\omega_1}$  (the example is not categorical in any uncountable cardinal).

One might think that Grossberg's conjecture should be established *before* transferring categoricity (in order to be able to assume amalgamation in the transfer), but our proof of Theorem 16.7.3 is more subtle. First we use Shelah's structure theory of universal classes to show that there exists an ordering  $\leq$  (potentially different from substructure) such that  $(K, \leq)$  has amalgamation and other structural properties. We then work inside  $(K, \leq)$  to transfer categoricity (proving Theorem 16.7.3 since its statement does not depend on the ordering of the class). It is only after that we are able to conclude that  $\leq$  is actually substructure (on a tail of cardinals), and hence that Grossberg's conjecture holds in universal classes.

The main difficulty in the argument just outlined is that it is unclear that  $(K, \leq)$  is an AEC (it may fail the smoothness axiom). The hard part of this chapter is proving that it actually *is* an AEC. This is done by working inside a framework for forking-like independence in  $(K, \leq)$  that Shelah calls AxFri<sub>1</sub> and proving new results for that framework, including Theorem 16.5.40 telling us how to copy a chain witnessing the failure of smoothness into an independent tree of models.

It should be noted that these new results (in Section 16.5) are really the only new pieces needed to prove the main theorem. The rest of the chapter is about combining the structure theory of universal classes developed by Shelah [She09b, Chapter V] with known categoricity transfers (Chapters 8, 11, 14, 15). Another contribution of this chapter is Section 16.3 which considers two weak AECs  $\mathbf{K}^1, \mathbf{K}^2$ satisfying a compatibility condition (the isomorphism types of models in a categoricity cardinal is the same). The motivation here is the aforementioned change from  $\mathbf{K}^1 = (K, \subseteq)$  to  $\mathbf{K}^2 = (K, \leq)$ : In general, we may want to study an AEC  $\mathbf{K}^1$ by changing its ordering, giving a new class  $\mathbf{K}^2$  which has certain properties P of  $\mathbf{K}^1$  together with some new properties P' that  $\mathbf{K}^1$  may not have. We may know a theorem telling us that a single class that has both P and P' is well-behaved. Section 16.3 gives tools to generalize the original theorem to the case when we do *not* have a single class (i.e.  $\mathbf{K}^1 = \mathbf{K}^2$ ) but instead have potentially different classes  $\mathbf{K}^1$  and  $\mathbf{K}^2$ .

Note in passing that this chapter does not make Chapter 8 obsolete: the results there hold for a wider context than universal classes, whereas we do not know how to generalize the proof of the main theorem here. Furthermore, we rely heavily here on Chapter 8.

A natural question is why, the threshold in Theorem 16.0.9 is  $\beth_{\beth_{\omega_1}}$  and not  $\beth_{\omega_1}$  as in Conjecture 16.1.2. The  $\beth_{\beth_{\omega_1}}$  comes from the fact that, in the argument outlined in the second paragraph after Fact 16.1.5, the class  $(K, \leq)$  has Löwenheim-Skolem-Tarski number  $\chi$ , for some  $\chi < \beth_{\omega_1}$ . After proving that it is an AEC, we apply known categoricity transfers to this class, hence the final threshold for categoricity is of order  $\beth_{(2\chi)^+} \leq \beth_{\beth_{\omega_1}}$  (a similar phenomenon occurs in [She99], where Shelah proves that the class  $\mathbf{K}$  is  $\chi$ -weakly tame for some  $\chi < \beth_{(2^{\mathrm{LS}(\mathbf{K}))^+}}$  and then obtains a threshold of  $\beth_{(2\chi)^+}$ ). We do not know whether the threshold in Theorem 16.0.9 can be lowered to  $\beth_{\omega_1}$ .

Let us discuss the background required to read this chapter. It is assumed that the reader has a solid knowledge of AECs (including at minimum the material in [**Bal09**]). Still, except for the basic concepts, we have tried to explicitly state all the definitions and facts. Only little understanding of Chapters 8, 11, 14, 15 is required: they are used only as black boxes. While some results in Chapter 8 rely on deep results of Shelah from the first sections of Chapter IV of [**She09a**], we do not use them<sup>5</sup>. At one point (Lemma 16.3.4) we rely on Shelah's construction of a certain linear order [**She09a**, IV.5]. This can also be taken as a black box. Last but not least, we rely on part of Shelah's original study of universal classes [**She87b**] (we quote from the updated version in Chapter V of [**She09b**]). All the results that we use from there have full proofs. We do not rely on any of Shelah's nonstructure results.

## 16.2. Preliminaries

We state definitions and facts that will be used later. All throughout this chapter, we use the letters M, N for models and write |M| for the universe of a model M and ||M|| for the cardinality of its universe. We may abuse notation and write e.g.  $a \in M$  when we really mean  $a \in |M|$ .

Recall the definition of a universal class (for examples, see e.g. Example 8.2.2).

DEFINITION 16.2.1 ([**Tar54, She87b**]). A class of structure K is *universal* if:

 $<sup>{}^{5}</sup>$ The one exception is [She09a, IV.1.12.(2)] (see Fact 16.2.13), but the proof is short and elementary.

- (1) It is a class of  $\tau$ -structures for a fixed vocabulary  $\tau = \tau(K)$ , closed under isomorphisms.
- (2) If  $\langle M_i : i < \delta \rangle$  is  $\subseteq$ -increasing in K, then  $\bigcup_{i < \delta} M_i \in K$ .
- (3) If  $M \in K$  and  $M_0 \subseteq M$ , then  $M_0 \in K$ .

REMARK 16.2.2. Notice the following fundamental property of a universal class K. Given a subset A of  $N \in K$ ,  $cl^N(A)$ , the closure of A under the functions of N (or equivalently  $\bigcap \{N_0 \in K \mid A \subseteq |N|, N_0 \subseteq N\}$ ) is in K.

It is known that universal classes can be characterized syntactically. We will use the following definition.

DEFINITION 16.2.3. A sentence  $\psi$  of  $\mathbb{L}_{\infty,\omega}$  is *universal* if it is of the form  $\forall x_0 \forall x_1 \dots \forall x_n \phi(x_0, x_1, \dots, x_n)$ , where  $\phi$  is a quantifier-free  $\mathbb{L}_{\infty,\omega}$  formula. An  $\mathbb{L}_{\infty,\omega}$ -theory is *universal* if it consists only of universal  $\mathbb{L}_{\infty,\omega}$  formulas.

The following is essentially due to Tarski [**Tar54**]. Only "(2) implies (1)" will be used. Tarski proved the result for  $\mathbb{L}_{\omega,\omega}$ , so we sketch a proof of the  $\mathbb{L}_{\infty,\omega}$  case for the convenience of the reader.

FACT 16.2.4. Let K be a class of structures in a fixed vocabulary  $\tau = \tau(K)$ . Set  $\lambda := |\tau| + \aleph_0$ . The following are equivalent.

- (1) K is a universal class.
- (2) K = Mod(T), for some universal  $\mathbb{L}_{\lambda^+,\omega}$  theory T with  $|T| \leq \lambda$ .

PROOF SKETCH. (2) implies (1) is straightforward. We show (1) implies (2). Note that for any fixed finitely generated  $\tau$ -structure M, the class  $K_{\neg M}$  of  $\tau$ structures that do not contain (as a substructure) a copy of M is axiomatized by a universal  $\mathbb{L}_{\lambda^+,\omega}$ -sentence. Further, there are only  $\lambda$ -many isomorphism types of finitely generated  $\tau$ -structures.

Now for any universal class K in the vocabulary  $\tau$ , let  $\Gamma$  be the class of finitely generated  $\tau$ -structures that are not contained in any member of K. With a directed system argument, one sees that K is exactly the class of  $\tau$ -structures that do not contain a copy of a member of  $\Gamma$ .

REMARK 16.2.5. Fact 16.2.4 shows that K is axiomatized by a single  $\mathbb{L}_{\lambda^+,\omega^-}$ sentence (take the conjunctions of all the formulas in T). However it need not be true that K is axiomatized by a single universal  $\mathbb{L}_{\lambda^+,\omega}$ -sentence: consider the class of directed graphs that do not contain a finite cycle. Confusingly, Malitz [Mal69] calls a sentence universal (we will say it is Malitz-universal) if it has no existential quantifiers and negations are only applied to atomic formulas. Thus the class of directed graphs without finite cycles is axiomatizable by a single Malitz-universal sentence but not by a single universal sentence. Even worse, the class of all finite sets is axiomatizable by a single Malitz-universal sentence but is not a universal class (it is not closed under unions).

Universal classes are abstract elementary classes:

DEFINITION 16.2.6 (Definition 1.2 in [She87a]). An abstract elementary class (AEC for short) is a pair  $\mathbf{K} = (K, \leq_{\mathbf{K}})$ , where:

- (1) K is a class of  $\tau$ -structures, for some fixed vocabulary  $\tau = \tau(\mathbf{K})$ .
- (2)  $\leq_{\mathbf{K}}$  is a partial order (that is, a reflexive and transitive relation) on K.

- (3)  $(K, \leq_{\mathbf{K}})$  respects isomorphisms: If  $M \leq_{\mathbf{K}} N$  are in K and  $f : N \cong N'$ , then  $f[M] \leq_{\mathbf{K}} N'$ . In particular (taking M = N), K is closed under isomorphisms.
- (4) If  $M \leq_{\mathbf{K}} N$ , then  $M \subseteq N$ .
- (5) Coherence: If  $M_0, M_1, M_2 \in K$  satisfy  $M_0 \leq_{\mathbf{K}} M_2, M_1 \leq_{\mathbf{K}} M_2$ , and  $M_0 \subseteq M_1$ , then  $M_0 \leq_{\mathbf{K}} M_1$ ;
- (6) Tarski-Vaught axioms: Suppose  $\delta$  is a limit ordinal and  $\langle M_i \in K : i < \delta \rangle$  is an increasing chain. Then:
  - (a)  $M_{\delta} := \bigcup_{i < \delta} M_i \in K$  and  $M_0 \leq_{\mathbf{K}} M_{\delta}$ .
  - (b) Smoothness: If there is some  $N \in K$  so that for all  $i < \delta$  we have  $M_i \leq_{\mathbf{K}} N$ , then we also have  $M_\delta \leq_{\mathbf{K}} N$ .
- (7) Löwenheim-Skolem-Tarski axiom: There exists a cardinal  $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any  $M \in K$  and  $A \subseteq |M|$ , there is some  $M_0 \leq_{\mathbf{K}} M$  such that  $A \subseteq |M_0|$  and  $||M_0|| \leq |A| + \lambda$ . We write  $\mathrm{LS}(\mathbf{K})$  for the minimal such cardinal.

Remark 16.2.7.

- (1) When we write  $M \leq_{\mathbf{K}} N$ , we implicitly also mean that  $M, N \in K$ .
- (2) We write **K** for the pair  $(K, \leq_{\mathbf{K}})$ , and K (no boldface) for the actual class. However we may abuse notation and write for example  $M \in \mathbf{K}$  instead of  $M \in K$  when there is no danger of confusion. Note that in this chapter we will sometimes work with two AECs  $\mathbf{K}^1$ ,  $\mathbf{K}^2$  that happen to have the same underlying class K but not the same ordering.

Notice that if K is a universal class, then  $\mathbf{K} := (K, \subseteq)$  is an AEC with  $\mathrm{LS}(\mathbf{K}) = |\tau(K)| + \aleph_0$ . Throughout this chapter we will use the following notation:

NOTATION 16.2.8. Let K be a universal class. We think of K as the AEC  $\mathbf{K} := (K, \subseteq)$ , and may write "**K** is a universal class" instead of "K is a universal class".

We will also have to deal with AECs that may not satisfy the smoothness axiom:

DEFINITION 16.2.9 (I.1.2.(2) in [She09a]). A weak AEC is a pair  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  satisfying all the axioms of AECs except perhaps smoothness ((6b) in Definition 16.2.6).

Shelah introduced the following parametrized version of smoothness:

DEFINITION 16.2.10 (V.1.18.(3) in [She09b]). Let **K** be a weak AEC. Let  $\lambda \geq \text{LS}(\mathbf{K})$  and let  $\delta$  be a limit ordinal. We say that **K** is  $(\leq \lambda, \delta)$ -smooth if for any increasing chain  $\langle M_i : i \leq \delta \rangle$  with  $||M_i|| \leq \lambda$  for all  $i < \delta$  and  $||M_\delta|| \leq \lambda + \delta$ , we have that  $\bigcup_{i < \delta} M_i \leq_{\mathbf{K}} M_{\delta}$ .  $(\leq \lambda, \leq \kappa)$ -smooth means  $(\leq \lambda, \delta)$ -smooth for all  $\delta \leq \kappa$ , and similarly for the other variations.

REMARK 16.2.11. Above, we could have allowed  $||M_{\delta}|| > \lambda + \delta$  and gotten an equivalent definition. Indeed, if  $M_i \leq_{\mathbf{K}} M_{\delta}$  for all  $i < \delta$  and we want to see that  $\bigcup_{i < \delta} M_i \leq_{\mathbf{K}} M_{\delta}$ , we can use the Löwenheim-Skolem-Tarski axiom to take  $N \leq_{\mathbf{K}} M_{\delta}$  containing  $\bigcup_{i < \delta} |M_i|$  and having size at most  $\lambda + \delta$ . Then we can use coherence to see that  $M_i \leq_{\mathbf{K}} N$  for all  $i < \delta$ , hence by smoothness,  $\bigcup_{i < \delta} M_i \leq_{\mathbf{K}} N$ and so by transitivity of  $\leq_{\mathbf{K}}, \bigcup_{i < \delta} M_i \leq_{\mathbf{K}} M_{\delta}$ .

We now list a several known facts about AECs that we will use. First, recall that an AEC **K** is determined by its restriction  $\mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  to models of size  $\mathrm{LS}(\mathbf{K})$ . More precisely:

FACT 16.2.12 (II.1.23 in [She09a]). Assume  $\mathbf{K}^1$  and  $\mathbf{K}^2$  are AECs with  $\lambda := \mathrm{LS}(\mathbf{K}^1) = \mathrm{LS}(\mathbf{K}^2)$ . If  $\mathbf{K}^1_{\lambda} = \mathbf{K}^2_{\lambda}$  (so also  $\leq_{\mathbf{K}^1}$  and  $\leq_{\mathbf{K}^2}$  coincide on the models of size  $\lambda$ ), then  $\mathbf{K}^1_{\geq \lambda} = \mathbf{K}^2_{\geq \lambda}$ .

We will use the relationship between the ordering of any AEC and elementary equivalence in a sufficiently powerful infinitary logic:

FACT 16.2.13. Let **K** be an AEC and let  $M, N \in \mathbf{K}$ .

- (1) **[Kue08**, Theorem 7.2.(b)] If  $M \preceq_{\mathbb{L}_{\infty, \mathrm{LS}(\mathbf{K})^+}} N$ , then  $M \leq_{\mathbf{K}} N$ .
- (2) [She09a, IV.1.12.(2)] Let  $\lambda$  be an infinite cardinal such that **K** is categorical in  $\lambda$  and  $\lambda = \lambda^{\text{LS}(\mathbf{K})}$ . If  $M, N \in \mathbf{K}_{\lambda}$  and  $M \leq_{\mathbf{K}} N$ , then  $M \preceq_{\mathbb{L}_{\infty,\text{LS}(\mathbf{K})^+}} N$ .

REMARK 16.2.14. Shelah's proof of Fact 16.2.13.(2) is short and elementary but in [**She09a**, Section IV.2], he attempts to remove the " $\lambda = \lambda^{\text{LS}(\mathbf{K})}$ " restriction. We rely on parts of Shelah's argument to get amalgamation in Chapter 8 (e.g. in the proof of Fact 16.1.4.(1)), but in this chapter we have a different strategy to get amalgamation and hence do not need to rely on the deep results from [**She09a**, Chapter IV].

We will also use that AECs have a Hanf number. Below, we write  $\delta(\lambda)$  for the pinning down ordinal at  $\lambda$ : the first ordinal that is not definable in  $\mathbb{L}_{\lambda^+,\omega}$ . We will also deal with the more general  $\delta(\lambda, \kappa)$  (the least ordinal not definable using a PC<sub> $\lambda,\kappa$ </sub> class, see [**She90**, VII.5.5.1] for a precise definition). Recall the following well-known facts about this ordinal (see e.g. [**She90**, VII.5]):

Fact 16.2.15.

- (1) (Lopez-Escobar)  $\delta(\aleph_0) = \omega_1$ .
- (2) (Morley and C.C. Chang) For any infinite cardinals  $\lambda$  and  $\kappa$ ,  $\delta(\lambda, \kappa) \leq (2^{\lambda})^{+}$ .

DEFINITION 16.2.16. Let  $\mathbf{K}$  be an AEC.

(1) Let  $\lambda(\mathbf{K})$  be the least cardinal  $\lambda \geq \mathrm{LS}(\mathbf{K})$  such that there exists a vocabulary  $\tau_1 \supseteq \tau(\mathbf{K})$ , a first-order  $\tau_1$ -theory  $T_1$ , and a set of  $T_1$ -types  $\Gamma$  such that:

(a)  $\mathbf{K} = PC(T_1, \Gamma, \tau(\mathbf{K})).$ 

- (b) For  $M, N \in \text{EC}(T_1, \Gamma)$ , if  $M \subseteq N$ , then  $M \upharpoonright \tau(\mathbf{K}) \leq_{\mathbf{K}} N \upharpoonright \tau(\mathbf{K})$ .
- (c)  $|T_1| + |\tau_1| \leq \mathrm{LS}(\mathbf{K})$  and  $|\Gamma| \leq \lambda$ .
- (2) Let  $\delta(\mathbf{K}) := \delta(\mathrm{LS}(\mathbf{K}), \lambda(\mathbf{K})).$
- (3) Let  $h(\mathbf{K}) := \beth_{\delta(\mathbf{K})}$ .

REMARK 16.2.17. By Chang's presentation theorem [**Cha68**], if **K** is axiomatized by an  $\mathbb{L}_{\lambda^+,\omega}$  sentence, and the ordering is just substructure (as for universal classes), then  $\lambda(\mathbf{K}) \leq \lambda$ . In particular (see Fact 16.2.4)  $\lambda(\mathbf{K}) = |\tau(\mathbf{K})| + \aleph_0$  for any universal class **K**.

It makes sense to talk of  $\lambda(\mathbf{K})$  because of Shelah's presentation theorem:

FACT 16.2.18 (I.1.9 in [She09a]). For any AEC K, there exists a vocabulary  $\tau_1 \supseteq \tau(\mathbf{K})$ , a first-order  $\tau_1$ -theory  $T_1$ , and a set of  $T_1$ -types  $\Gamma$  such that (1a) and (1b) in Definition 16.2.16 hold and  $|T_1| + |\tau_1| \leq \mathrm{LS}(\mathbf{K})$ . Thus  $\lambda(\mathbf{K}) \leq 2^{\mathrm{LS}(\mathbf{K})}$ .

REMARK 16.2.19. By Facts 16.2.15 and 16.2.18, For any AEC K,  $h(\mathbf{K}) \leq h(\mathrm{LS}(\mathbf{K}))$ .

The reason  $h(\mathbf{K})$  is interesting is because it is a Hanf number for  $\mathbf{K}$  (this follows from Chang's result on the Hanf number of PC classes [Cha68]).

FACT 16.2.20. Let **K** be an AEC. If **K** has a model of size  $h(\mathbf{K})$ , then **K** has arbitrarily large models.

In the rest of this section, we quote categoricity transfer results that we will use. We assume that the reader is familiar with notions such as amalgamation, joint embedding, Galois types, Ehrenfeucht-Mostowski models, and tameness (see for example [**Bal09**]). The notation we use is standard and is described in details at the beginning of Chapter 2 (for Ehrenfeucht-Mostowski models, we use the notation in [**She09a**, IV.0.8]<sup>6</sup>). For example, we write  $gtp(\bar{b}/M; N)$  for the Galois type of  $\bar{b}$ over M, as computed in N. This assumes that we are working inside an AEC **K** that is clear from context. When we want to emphasize **K**, we will write  $gtp_{\mathbf{K}}(\bar{b}/M; N)$ .

The following result is implicit in the proof of [**GV06c**, Corollary 4.3]. For completeness, we sketch a proof.

FACT 16.2.21. If **K** is an AEC with amalgamation and arbitrarily large models, then the categoricity spectrum (i.e. the class of cardinals  $\lambda \geq \text{LS}(\mathbf{K})$  such that **K** is categorical in  $\lambda$ ) is closed. That is, if  $\lambda > \text{LS}(\mathbf{K})$  is a limit cardinal and **K** is categorical in unboundedly many cardinals below  $\lambda$ , then **K** is also categorical in  $\lambda$ .

PROOF. Let  $\lambda > LS(\mathbf{K})$  be a limit cardinal such that  $\mathbf{K}$  is categorical in unboundedly many cardinals below  $\lambda$ . We show that  $\mathbf{K}$  is categorical in  $\lambda$ . We proceed in several steps:

- (1) **K** is (Galois) stable in every  $\mu \in [LS(\mathbf{K}), \lambda)$ . [Why? Pick  $\mu' \in (\mu, \lambda)$  such that **K** is categorical in  $\mu'$ . Since **K** has arbitrarily large models, we can use Ehrenfeucht-Mostowski models and the standard argument of Morley (see e.g. the proof of [**Shea**, Claim I.1.7]) to see that **K** is stable in  $\mu$ .].
- (2) For every categoricity cardinal  $\mu \in (LS(\mathbf{K}), \lambda)$ , the model of size  $\mu$  is (Galois) saturated. [Why? Using stability we can build a  $\mu_0$ -saturated model of size  $\mu$  for every  $\mu_0 \in (LS(\mathbf{K}), \mu)$ , and then use categoricity.]
- (3) Every model of size  $\lambda$  is saturated. [Why? Let  $M \in \mathbf{K}_{\lambda}$ . Let  $N \in \mathbf{K}_{<\lambda}$  be such that  $N \leq_{\mathbf{K}} M$ . Let  $p \in \mathrm{gS}(N)$ . Let  $\mu := \|N\|$  and let  $\mu' \in (\mu, \lambda)$  be a categoricity cardinal. Let  $N' \in \mathbf{K}_{\mu'}$  be such that  $N \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} M$ . By the previous step, N' is saturated, and therefore realizes p. Since  $N' \leq_{\mathbf{K}} M$ , M also realizes p.
- (4) **K** is categorical in  $\lambda$ . [Why? By uniqueness of saturated models.]

To state the next categoricity transfer, we first recall Shelah's notion of an AEC having primes. The intuition is that the AEC has prime models over every set of the form  $M \cup \{a\}$ , for  $M \in \mathbf{K}$ . This is described formally using Galois types.

<sup>&</sup>lt;sup>6</sup>For **K** an AEC, we call  $\Phi$  an *EM blueprint for* **K** if  $\Phi \in \Upsilon_{\mathbf{K}}^{\text{or}}$ .

#### 16.2. PRELIMINARIES

DEFINITION 16.2.22 (III.3.2 in [She09a]). Let K be an AEC.

- (1) (a, M, N) is a prime triple if  $M \leq_{\mathbf{K}} N$ ,  $a \in |N| \setminus |M|$ , and for every  $N' \in \mathbf{K}$ ,  $a' \in |N'|$ , such that gtp(a/M; N) = gtp(a'/M; N'), there exists  $f: N \xrightarrow{M} N'$  with f(a) = a'.
- (2) **K** has primes if for any nonalgebraic Galois type  $p \in gS(M)$  there exists a prime triple (a, M, N) such that p = gtp(a/M; N).

By taking the closure of the relevant set under the functions of an ambient model, we obtain:

FACT 16.2.23 (Remark 8.5.3). Any universal class  $\mathbf{K} = (K, \subseteq)$  has primes.

REMARK 16.2.24. Having primes is a property of the AEC  $\mathbf{K} = (K, \leq_{\mathbf{K}})$ , not just of the class K. Thus even though for any universal class K,  $(K, \subseteq)$  has primes, changing the order may lead to an AEC  $(K, \leq_{\mathbf{K}})$  that may *not* have primes anymore.

The following is a ZFC approximation of Shelah's eventual categoricity conjecture in tame AECs with amalgamation. It combines works of Makkai-Shelah [**MS90**], Shelah [**She99**], Grossberg and VanDieren [**GV06c**, **GV06a**], and the author Chapters 8, 11, 14, 15.

FACT 16.2.25. Let **K** be a LS(**K**)-tame AEC with amalgamation and arbitrarily large models. Let  $\lambda > \text{LS}(\mathbf{K})$  be such that **K** is categorical in  $\lambda$ .

- (1) (Theorem 15.3.8)<sup>7</sup> If  $\delta$  is a limit ordinal that is divisible by  $(2^{\text{LS}(\mathbf{K})})^+$ , then **K** is categorical in  $\beth_{\delta}$ .
- (2) **K** is categorical in all  $\lambda' \geq \min(\lambda, h(\mathrm{LS}(\mathbf{K})))$  when at least one of the following holds:
  - (a) (Corollaries 15.4.3, 15.4.6)<sup>8</sup> There exists a successor cardinal  $\mu > LS(\mathbf{K})^+$  such that **K** is categorical in  $\mu$ .
  - (b) (Corollary 15.4.9)<sup>9</sup> K has primes.

REMARK 16.2.26. In Fact 16.2.25, we do not use that **K** has joint embedding: we can find a sub-AEC  $\mathbf{K}^0$  of **K** that has joint embedding and work within  $\mathbf{K}^0$ . See Definition 16.6.11.

REMARK 16.2.27. If in Fact 16.2.25 we start instead with a  $\chi$ -tame AEC (with  $\chi > \text{LS}(\mathbf{K})$ ), the same conclusions hold for  $\mathbf{K}_{>\chi}$ .

<sup>&</sup>lt;sup>7</sup>The version for classes of models axiomatized by an  $\mathbb{L}_{\kappa,\omega}$  theory,  $\kappa$  strongly compact, appears in [**MS90**]. It generalizes to AECs with amalgamation when the model in the categoricity cardinal is saturated (see [**She99**, Lemma II.1.6] or [**Bal09**, Theorem 14.8]). In the tame case, the model in the categoricity cardinal is always saturated (by the Shelah-Villaveces theorem [**SV99**, Theorem 2.2.1] together with the upward superstability transfer of the author, Proposition 6.10.10). In all these arguments, it seems that the amalgamation property is used in a strong way.

<sup>&</sup>lt;sup>8</sup>The *upward* part of this transfer (i.e. concluding categoricity in all  $\mu' \ge \mu$  is due to Grossberg and VanDieren [**GV06a**]).

<sup>&</sup>lt;sup>9</sup>The main ideas of the transfer with primes appear in Chapters 8 and 11 but there the threshold is higher (around  $\beth_{h(\mathrm{LS}(\mathbf{K}))}$ ). The improved threshold of  $h(\mathrm{LS}(\mathbf{K}))$  can be obtained from Fact 16.2.25.(2a).

#### 16.3. Compatible pairs of AECs

Let K be a universal class. A central result of Shelah [She09b, V.B] is that if K does not have the order property, there is an ordering  $\leq$  such that  $(K, \leq)$ has several structural properties, including amalgamation. The downside is that  $(K, \leq)$  might loose the smoothness axiom, i.e. it may only be a weak AEC. We will give the precise statement of Shelah's result and discuss its implications in the next sections.

Here, we look at the situation abstractly: we consider pairs of weak AECs  $\mathbf{K}^1 = (K^1, \leq_{\mathbf{K}^1})$  and  $\mathbf{K}^2 = (K^2, \leq_{\mathbf{K}^2})$  satisfying a compatibility condition. The case of interest is  $\mathbf{K}^1 = (K, \subseteq)$  and  $\mathbf{K}^2 = (K, \leq)$ .

DEFINITION 16.3.1. For  $\ell = 1, 2$ , let  $\mathbf{K}^{\ell} = (K^{\ell}, \leq_{\mathbf{K}^{\ell}})$  be weak AECs.  $\mathbf{K}^{1}$  and  $\mathbf{K}^{2}$  are *compatible* if:

- (1)  $\tau(\mathbf{K}^1) = \tau(\mathbf{K}^2).$
- (2) For any  $\lambda > LS(\mathbf{K}^1) + LS(\mathbf{K}^2)$ , if either  $\mathbf{K}^1$  or  $\mathbf{K}^2$  is categorical in  $\lambda$ , then  $K^1_{\lambda} = K^2_{\lambda}$ .

We write  $LS(\mathbf{K}^1, \mathbf{K}^2)$  instead of  $LS(\mathbf{K}^1) + LS(\mathbf{K}^2)$ .

REMARK 16.3.2. This definition is really only useful when one of the classes is categorical. Note that in (2), we only ask for  $K_{\lambda}^1 = K_{\lambda}^2$ , i.e. the isomorphism type of the model of size  $\lambda$  must be the same in both classes, but the orderings need not agree.

For the rest of this section, we assume (and will emphasize the compatibility hypothesis again):

HYPOTHESIS 16.3.3.  $\mathbf{K}^1 = (K^1, \leq_{\mathbf{K}^1})$  and  $\mathbf{K}^2 = (K^2, \leq_{\mathbf{K}^2})$  are compatible weak AECs. We set  $\tau := \tau(\mathbf{K}^1) = \tau(\mathbf{K}^2)$ .

Assume that  $\mathbf{K}^1$  is categorical in a  $\lambda > \mathrm{LS}(\mathbf{K}^1, \mathbf{K}^2)$ . What can we say about  $\mathbf{K}^2$ ? If  $\mathbf{K}^1$  is a universal class and  $\mathbf{K}^2$  is as above,  $\mathbf{K}^1$  is an AEC, and one of our ultimate goal is to show that  $\mathbf{K}^2$  is also an AEC. The following result will turn out to be key. Under some assumptions,  $\mathbf{K}^2$  is stable below the categoricity cardinal.

LEMMA 16.3.4. Assume:

- (1)  $\mathbf{K}^1$  is an AEC with arbitrarily large models.
- (2)  $\mathbf{K}^2$  has amalgamation and joint embedding.
- (3)  $\mathbf{K}^1$  and  $\mathbf{K}^2$  are compatible.

Let  $\lambda > \mathrm{LS}(\mathbf{K}^1, \mathbf{K}^2)$ . If  $\mathbf{K}^2$  (and so by compatibility also  $\mathbf{K}^1$ ) is categorical in  $\lambda$ , then  $\mathbf{K}^2$  is  $(<\omega)$ -stable in all  $\mu \in [\mathrm{LS}(\mathbf{K}^1, \mathbf{K}^2), \lambda)$  such that  $\mu^+ < \lambda$ . That is, for any such  $\mu$  and any  $M \in \mathbf{K}^2_{\mu}$ ,  $|\mathrm{gS}^{<\omega}_{\mathbf{K}^2}(M)| \leq \mu$ 

Before starting the proof, a few comments are in order. First note that the case  $\mathbf{K}^1 = \mathbf{K}^2$  is a classical result that can be traced back to Morley [**Mor65**, Theorem 3.7]. It appears explicitly as [**She99**, Claim I.1.7]. The proof uses Ehrenfeucht-Mostowski (EM) models. Here, we have additional difficulties since the EM models are well-behaved really only for  $\mathbf{K}^1$  and not for  $\mathbf{K}^2$  (in fact,  $\mathbf{K}^2$  may be only a weak AEC, so may not have any suitable EM blueprint). More precisely, if  $\Phi$  is an EM blueprint for  $\mathbf{K}^1$  and  $I \subseteq J$  are linear orders, then  $\mathrm{EM}_{\tau}(I, \Phi) \leq_{\mathbf{K}^1} \mathrm{EM}_{\tau}(J, \Phi)$  but possibly  $\mathrm{EM}_{\tau}(I, \Phi) \not\leq_{\mathbf{K}^2} \mathrm{EM}_{\tau}(J, \Phi)$ . Thus a Galois type of  $\mathbf{K}^2$  computed inside  $\mathrm{EM}_{\tau}(I, \Phi)$  may not be the same as one computed in  $\mathrm{EM}_{\tau}(J, \Phi)$ . For this reason,

we want to use only that Galois types are invariant under isomorphisms in the proof, and hence want to use the existence of certain linear orderings with many automorphisms.

Fortunately, Shelah gives a proof of the case  $\mathbf{K}^1 = \mathbf{K}^2$  in [Shea, Claim I.1.7] (the online version of [She99]) that we can imitate. It uses the following fact:

FACT 16.3.5 (IV.5.1.(2) in [She09a]). Let  $\theta < \lambda$  be infinite cardinals with  $\theta$  regular. There exists a linear order I of size  $\lambda$  such that for every  $I_0 \subseteq I$  of size less than  $\theta$ , there is  $J \subseteq I$  such that:

- (1)  $I_0 \subseteq J$ .
- (2)  $||J|| = ||I_0|| + \aleph_0.$
- (3) For any  $\bar{a} \in {}^{<\omega}I$ , there is  $f \in \operatorname{Aut}_{I_0}(I)$  such that  $f(\bar{a}) \in {}^{<\omega}J$ .

PROOF OF LEMMA 16.3.4. Since  $\mathbf{K}^1$  has arbitrarily large models and is an AEC, it has an Ehrenfeucht-Mostowski blueprint  $\Phi$ . Let  $\mu \in [\mathrm{LS}(\mathbf{K}^1, \mathbf{K}^2), \lambda)$  and let  $M \in K^2_{\mu}$ . We want to see that  $|\mathrm{gS}_{\mathbf{K}^2}(M)| \leq \mu$ . Let I be as described by Fact 16.3.5 (where  $\theta$  there stands for  $\mu^+$  here, we are using that  $\mu^+ < \lambda$ ). Suppose for a contradiction that  $|\mathrm{gS}_{\mathbf{K}^2}^{<\omega}(M)| > \mu$ . Then using amalgamation we can find  $N \in K^2$  with  $M \leq_{\mathbf{K}^2} N$  and a sequence  $\langle \bar{a}_i \in {}^{<\omega}|N| : i < \mu^+ \rangle$  such that for  $i < j < \mu^+$ ,  $\mathrm{gtp}_{\mathbf{K}^2}(\bar{a}_i/M; N) \neq \mathrm{gtp}_{\mathbf{K}^2}(\bar{a}_j/M; N)$ .

By joint embedding and categoricity, without loss of generality  $N = \text{EM}_{\tau}(I, \Phi)$ . Now let  $I_0 \subseteq I$  be such that  $|I_0| = \mu$  and  $M \subseteq \text{EM}_{\tau}(I_0, \Phi)$ . Let J be as given by the definition of I and let  $M_1 := \text{EM}_{\tau}(J, \Phi)$ . We have that for each  $i < \mu^+$ , there is a finite linear order  $I_i \subseteq I$  generating  $\bar{a}_i$ , so pick  $f_i \in \text{Aut}_{I_0}(I)$  such that  $f_i[I_i] \subseteq J$ . Let  $\hat{f}_i \in \text{Aut}_{M_1}(N)$  be the automorphism of  $N = \text{EM}_{\tau}(I, \Phi)$  naturally induced by  $f_i$ . Then  $\hat{f}_i(\bar{a}_i) \in |M_1|$ . By the pigeonhole principle, without loss of generality there is  $\bar{b} \in |M_1|$  such that for all  $i < \mu^+$ ,  $\hat{f}_i(\bar{a}_i) = \bar{b}$ . But this means that for  $i < \mu^+$ :

$$\operatorname{gtp}_{\mathbf{K}^{2}}(\bar{a}_{i}/M; N) = \operatorname{gtp}_{\mathbf{K}^{2}}(\widehat{f}_{i}(\bar{a}_{i})/M; N) = \operatorname{gtp}_{\mathbf{K}^{2}}(\bar{b}/M; N)$$
  
So for  $i < j < \mu^{+}$ ,  $\operatorname{gtp}_{\mathbf{K}^{2}}(\bar{a}_{i}/M; N) = \operatorname{gtp}_{\mathbf{K}^{2}}(\bar{a}_{j}/M; N)$ , a contradiction.  $\Box$ 

REMARK 16.3.6. We emphasize that Lemma 16.3.4 establishes stability for all finite types and not just stability for types of length one (in the framework of weak AECs we do not know if the two notions are the same). This slightly stronger statement will be used in the proof of Theorem 16.7.2. There we want to derive a contradiction with Theorem 16.6.16, which only concludes unstability for finite types, not unstability for types of length one.

For the rest of this section, we assume that  $\mathbf{K}^1$  and  $\mathbf{K}^2$  are both AECs and discuss categoricity transfers (generalizing Fact 16.2.25) to this setup. First, we show that categoricity in a suitable cardinal implies that the two classes (and their ordering) are equal on a tail.

LEMMA 16.3.7. Assume  $\mathbf{K}^1$  and  $\mathbf{K}^2$  are compatible AECs. Let  $\lambda$  be an infinite cardinal such that:

- (1)  $\mathbf{K}^1$  is categorical in  $\lambda$ .
- (2)  $\lambda = \lambda^{\mathrm{LS}(\mathbf{K}^1, \mathbf{K}^2)}$ .

Then  $\mathbf{K}_{>\lambda}^1 = \mathbf{K}_{>\lambda}^2$  (so also the orderings are equal).

PROOF. By compatibility,  $K_{\lambda}^1 = K_{\lambda}^2$ . By Fact 16.2.12 (where  $\mathbf{K}^1, \mathbf{K}^2$  there stand for  $\mathbf{K}_{\geq\lambda}^1, \mathbf{K}_{\geq\lambda}^2$  here), it is enough to show that the orderings of  $\mathbf{K}^1$  and  $\mathbf{K}^2$  coincide on  $K_{\lambda}^1$ . So let  $M, N \in K_{\lambda}^1$ . We show that  $M \leq_{\mathbf{K}^1} N$  implies  $M \leq_{\mathbf{K}^2} N$  (the converse is symmetric).

So assume that  $M \leq_{\mathbf{K}^1} N$ . By Fact 16.2.13.(2) (where  $\mathbf{K}$ ,  $\lambda$  there stand for  $\mathbf{K}^1_{\geq \mathrm{LS}(\mathbf{K}^1,\mathbf{K}^2)}$ ,  $\lambda$  here),  $M \preceq_{\mathbb{L}_{\infty,\mathrm{LS}(\mathbf{K}^1,\mathbf{K}^2)^+}} N$ . By Fact 16.2.13.(1) (where  $\mathbf{K}$  there stands for  $\mathbf{K}^2_{\geq \mathrm{LS}(\mathbf{K}^1,\mathbf{K}^2)}$  here),  $M \leq_{\mathbf{K}^2} N$ , as desired.

The next result shows that if one of the classes has amalgamation, we can find a categoricity cardinal satisfying the condition of the previous lemma.

THEOREM 16.3.8. Assume  $\mathbf{K}^1$  and  $\mathbf{K}^2$  are compatible AECs categorical in a proper class of cardinals. If  $\mathbf{K}^1$  has amalgamation, then there exists  $\lambda$  such that  $\mathbf{K}^1_{>\lambda} = \mathbf{K}^2_{>\lambda}$  (so also the orderings are equal).

PROOF. Because  $\mathbf{K}^1$  is categorical in a proper class of cardinals, it has arbitrarily large models, so by Fact 16.2.21,  $\mathbf{K}^1$  is categorical on a closed unbounded class of cardinals. In particular, one can find an infinite cardinal  $\lambda$  such that  $\mathbf{K}^1$  is categorical in  $\lambda$  and  $\lambda = \lambda^{\mathrm{LS}(\mathbf{K}^1, \mathbf{K}^2)}$ . By Lemma 16.3.7,  $\mathbf{K}^1_{>\lambda} = \mathbf{K}^2_{>\lambda}$ .

We end this section with a categoricity transfer. Intuitively, this shows that if we start with an AEC  $\mathbf{K}^1$  with primes, it is enough to change its ordering (getting an AEC  $\mathbf{K}^2$ ) so that  $\mathbf{K}^2$  has amalgamation and is tame (it may lose existence of primes, see Remark 16.2.24). This is especially relevant to universal classes, since they always have primes (Fact 16.2.23). Note that Fact 16.2.25.(2b) is the case  $\mathbf{K}^1 = \mathbf{K}^2$ .

THEOREM 16.3.9. Assume  $\mathbf{K}^1$  and  $\mathbf{K}^2$  are compatible AECs such that:

(1)  $\mathbf{K}^1$  has primes.

(2)  $\mathbf{K}^2$  has amalgamation, arbitrarily large models, and is  $LS(\mathbf{K}^2)$ -tame.

If  $\mathbf{K}^2$  is categorical in a  $\lambda > \mathrm{LS}(\mathbf{K}^2)$ , then  $\mathbf{K}^2$  is categorical in all  $\lambda' \geq \min(\lambda, h(\mathrm{LS}(\mathbf{K}^2)))$ .

PROOF. By Fact 16.2.25.(1),  $\mathbf{K}^2$  is categorical in a proper class of cardinals. By Theorem 16.3.8 (where the role of  $\mathbf{K}^1$  and  $\mathbf{K}^2$  is switched), we can fix a cardinal  $\lambda_0$ such that  $\mathbf{K}_{\geq\lambda_0}^1 = \mathbf{K}_{\geq\lambda_0}^2$ . In particular, their orderings also coincide and so  $\mathbf{K}_{\geq\lambda_0}^2$ has primes. By Fact 16.2.25.(2b),  $\mathbf{K}_{\geq\lambda_0}^2$  is categorical on a tail, and in particular in a successor cardinal. Applying Fact 16.2.25.(2a) to  $\mathbf{K}^2$ , this implies that  $\mathbf{K}^2$  is categorical in all  $\lambda' \geq \min(\lambda, h(\mathrm{LS}(\mathbf{K}^2)))$ , as desired.

# 16.4. Independence in weak AECs

AxFri<sub>1</sub> is an axiomatic framework for independence in weak AECs that Shelah introduces in [She87b]. The main motivation for the axioms is that if K is a universal class that does not have the order property, then there is an ordering  $\leq$  such that  $(K, \leq)$  satisfies AxFri<sub>1</sub> (see Section 16.6). Here, we repeat the definition and state some facts that we will use. We quote from Chapter V of [She09b], an updated version of [She87b].

DEFINITION 16.4.1 (AxFri<sub>1</sub>, V.B in [**She09b**]). ( $\mathbf{K}, \, \downarrow, \, \mathrm{cl}$ ) satisfies<sup>10</sup> AxFri<sub>1</sub> if:

 $<sup>^{10}</sup>$  In order to be consistent with Chapter 8, we write cl rather than Shelah's  $\langle\rangle_{\rm gn}.$ 

- (1)  $\mathbf{K}$  is a weak AEC.
- (2) For each  $N \in \mathbf{K}$ ,  $\mathrm{cl}^N$  is a function from  $\mathcal{P}(|N|)$  to  $\mathcal{P}(|N|)$ . Often,  $\mathrm{cl}^N(A)$ induces a  $\tau(\mathbf{K})$ -substructure M of N. In this case, we identify  $\mathrm{cl}^{N}(A)$ with M. We require cl to satisfy the following axioms: For  $N, N' \in \mathbf{K}$ ,  $A, B \subseteq |N|$ :
  - (a) Invariance: If  $f : N \cong N'$ , then  $\operatorname{cl}^{N'}(f[A]) = f[\operatorname{cl}^N(A)]$ . (b) Monotonicity 1: If  $A \subseteq B$ , then  $\operatorname{cl}^N(A) \subseteq \operatorname{cl}^N(B)$ .

  - (c) Monotonicity 2: If  $N \leq_{\mathbf{K}} N'$ , then  $\operatorname{cl}^{N}(A) = \operatorname{cl}^{N'}(A)$ .
  - (d) Idempotence:  $\operatorname{cl}^{N}(\operatorname{cl}^{N}(A)) = \operatorname{cl}^{N}(A)$ .

(3)  $\downarrow$  is a 4-ary relation on **K**. We write  $M_1 \downarrow_{M_0}^{M_3} M_2$  instead of  $\downarrow (M_0, M_1, M_2, M_3)$ . We require that  $\downarrow$  satisfies the following axioms:

- (a)  $M_1 \downarrow_{M_2}^{M_3} M_2$  implies that for  $\ell = 1, 2, M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$ .
- (b) Invariance: If  $f: M_3 \cong M'_3$  and  $M_1 \stackrel{M_3}{\underset{M_0}{\downarrow}} M_2$ , then  $f[M_1] \stackrel{M'_3}{\underset{f[M_0]}{\downarrow}} f[M_2]$ .
- (c) Monotonicity 1: If  $M_1 \underset{M_0}{\overset{M_3}{\downarrow}} M_2$  and  $M_3 \leq_{\mathbf{K}} M'_3$ , then  $M_1 \underset{M_0}{\overset{M'_3}{\downarrow}} M_2$ . (d) Monotonicity 2: If  $M_1 \underset{M_0}{\overset{M_3}{\downarrow}} M_2$  and  $M_0 \leq_{\mathbf{K}} M'_2 \leq_{\mathbf{K}} M_2$ , then  $M_1 \underset{M_0}{\overset{M_3}{\downarrow}} M'_2$ .
- (e) Base enlargement: If  $M_1 \underset{M_2}{\downarrow} M_2$  and  $M_0 \leq_{\mathbf{K}} M_2' \leq_{\mathbf{K}} M_2$ , then  $M_{2}$

$$\mathrm{cl}^{M_3}(M_2'\cup M_1) \underset{M_2'}{\overset{}{\bigcup}} M_2.$$

- (f) Symmetry: If  $M_1 \underset{M_0}{\overset{M_3}{\downarrow}} M_2$ , then  $M_2 \underset{M_0}{\overset{M_3}{\downarrow}} M_1$ . (g) Existence: If  $M_0 \leq_{\mathbf{K}} M_\ell$ ,  $\ell = 1, 2$ , then there exists  $N \in \mathbf{K}$  and
- $f_{\ell}: M_{\ell} \xrightarrow{M_0} N, \ \ell = 1, 2$ , such that  $f[M_1] \underset{M_0}{\downarrow}^N f[M_2].$
- (h) Uniqueness: If for  $\ell = 1, 2$ ,  $M_1^{\ell} \underset{M_0^{\ell}}{\overset{M_3^{\ell}}{\cup}} M_2^{\ell}$  and fori < 3,  $f_i : M_i^1 \cong M_i^2$ are such that  $f_0 \subseteq f_1$ ,  $f_0 \subseteq f_2$ , then there exists  $N \in \mathbf{K}$  with  $M_3^2 \leq_{\mathbf{K}} N$  and  $h: M_3^1 \to N$  such that  $f_1 \cup f_2 \subseteq h$ . (i) Finite character: If  $\delta$  is a limit ordinal,  $\langle M_{2,i}: i \leq \delta \rangle$  is increasing and
- continuous,  $M_0 \leq_{\mathbf{K}} M_{1,0}$ , and  $M_1 \bigcup_{M_0}^{M_3} M_{2,\delta}$ , then  $\mathrm{cl}^{M_3}(M_1 \cup M_{2,\delta}) =$  $\bigcup_{i<\delta} \mathrm{cl}^{M_3}(M_1\cup M_{2,i}).$

We say that a weak AEC K satisfies  $AxFri_1$  if there exists  $\downarrow$  and cl such that  $(\mathbf{K}, \downarrow, \mathrm{cl})$  satisfies AxFri<sub>1</sub>.

REMARK 16.4.2. The definition we give is slightly different from Shelah's: Shelah does not assume that  $\mathbf{K}$  has a Löwenheim-Skolem-Tarski number. We do not need the extra generality, although there are places (e.g. Section 16.5) where the existence of a Löwenheim-Skolem-Tarski number is not used.

REMARK 16.4.3. There is an example (derived from the class of metric graphs, see [She09b, V.B.1.22]) of a triple  $(\mathbf{K}, \downarrow, cl)$  that satisfies AxFri<sub>1</sub> but where **K** is not an AEC.

REMARK 16.4.4. If a weak AEC K satisfies  $AxFri_1$ , then by the existence property for  $\downarrow$ , K has amalgamation.

In the rest of this section, we assume:

HYPOTHESIS 16.4.5.  $(\mathbf{K}, \downarrow, \mathrm{cl})$  satisfies AxFri<sub>1</sub>.

The following is easy to see from the definition of the closure operator.

FACT 16.4.6. Let  $N \in \mathbf{K}$  and let  $\langle A_i : i \in I \rangle$  be a sequence of subsets of |N|,  $I \neq \emptyset$ . Then:

(1) 
$$\bigcup_{i \in I} \operatorname{cl}^{N}(A_{i}) \subseteq \operatorname{cl}^{N}(\bigcup_{i \in I} A_{i}).$$
  
(2) 
$$\operatorname{cl}^{N}(\bigcup_{i \in I} A_{i}) = \operatorname{cl}^{N}(\bigcup_{i \in I} \operatorname{cl}^{N}(A_{i})).$$

The following are consequences of the axioms and will all be used in the rest of this chapter (as forking calculus tools for Sections 16.5 and 16.6).

# Fact 16.4.7.

(1) [She09b, V.B.1.21.(1)] If  $M_1 \underset{M_0}{\stackrel{M_3}{\downarrow}} M_2$ , then  $cl^{M_3}(M_1 \cup M_2) \leq_{\mathbf{K}} M_3$  and  $M_1 \underset{M_0}{\stackrel{cl^{M_3}(M_1 \cup M_2)}{\downarrow}} M_2.$ 

(2) [She09b, V.C.1.3] Transitivity: If  $M_1 \stackrel{M_3}{\underset{M_0}{\downarrow}} M_2$  and  $M_3 \stackrel{M_5}{\underset{M_2}{\downarrow}} M_4$ , then  $M_1 \stackrel{M_5}{\underset{M_0}{\downarrow}} M_4$ . (3) [She09b, V.C.1.6] Let  $\delta$  be a limit ordinal. Let  $\langle M_i : i \leq \delta \rangle$ ,  $\langle N_i : i \leq \delta \rangle$ 

(3) [She09b, V.C.1.6] Let  $\delta$  be a limit ordinal. Let  $\langle M_i : i \leq \delta \rangle$ ,  $\langle N_i : i \leq \delta \rangle$ be  $\subseteq$ -increasing continuous chains such that for all  $i < j < \delta$ ,  $M_j \downarrow_{M_i} N_i$ .

Then for all  $i \leq \delta$ ,  $M_{\delta} \stackrel{N_{\delta}}{\underset{M_i}{\cup}} N_i$ .

- (4) [She09b, V.C.1.10.(1)] Let  $\delta$  be a limit ordinal. Let  $\langle M_i : i \leq \delta + 1 \rangle$ ,  $\langle N_i^a : i \leq \delta \rangle$ ,  $\langle N_i^b : i \leq \delta \rangle$  be increasing continuous chains such that for all  $i < \delta$ ,  $N_i^a \downarrow_{M_i}^b M_{\delta+1}$  and  $N_i^b = \operatorname{cl}^{N_i^b}(M_{\delta+1} \cup N_i^a)$ . Then  $N_{\delta}^a \downarrow_{M_{\delta}}^b M_{\delta+1}$ . (5) Let  $\delta$  be a limit ordinal. Let  $\langle M_i : i \leq \delta \rangle$ ,  $\langle N_i : i \leq \delta \rangle$  be increasing
- 5) Let  $\delta$  be a limit ordinal. Let  $\langle M_i : i \leq \delta \rangle$ ,  $\langle N_i : i \leq \delta \rangle$  be increasing continuous so that for  $i, j < \delta$ ,  $M_j \underset{M_i}{\cup} N_i$ . Let  $M \in \mathbf{K}$  be such that  $M_i \leq_{\mathbf{K}} M$  for all  $i < \delta$  (but possibly  $M_\delta \not\leq_{\mathbf{K}} M$ ). Then there exists  $N \in \mathbf{K}$  and an embedding  $f : M \xrightarrow{M_\delta} N$  such that for all  $i < \delta$ :
  - (a)  $N_i \leq_{\mathbf{K}} N.$ (b)  $f[M] \underset{M_i}{\downarrow} N_i.$ (c)  $N = \operatorname{cl}^N (f[M] \cup N_\delta).$

PROOF OF (5). This is given by the proof of [She09b, V.C.1.11], but Shelah omits the end of the proof. We give it here. We build  $\langle N_i^a, N_i^b, f_i : i \leq \delta \rangle$  such that:

- (1)  $\langle N_i^x : i \leq \delta \rangle$  is increasing continuous for  $x \in \{a, b\}$ .
- (2) For  $i < \delta$ ,  $M \bigcup_{M_i}^{N_i^b} N_i^a$ .
- (3) For  $i < \delta$ ,  $N_i^b = \operatorname{cl}^{N_i^b}(M \cup N_i^a)$ . (4) For  $i \le \delta$ ,  $f_i : N_i \cong_{M_i} N_i^a$ .

This is possible by the proof of [She09b, V.C.1.11]. Let us see that it is enough. Find  $N \in \mathbf{K}$  and  $f: N^b_{\delta} \cong N$  that extends  $f^{-1}_{\delta}$ . We claim that this works. First observe that  $f \upharpoonright M : M \xrightarrow[M_{\delta}]{N}$  as f fixes  $M_i$  for each  $i < \delta$  and  $M \leq_{\mathbf{K}} N_0^b \leq_{\mathbf{K}} N_{\delta}^b$ . Now:

- (1) For all  $i < \delta$ ,  $N_i \leq_{\mathbf{K}} N$ , since  $N_i^a \leq_{\mathbf{K}} N_i^b \leq_{\mathbf{K}} N_{\delta}^b$  and  $f_i^{-1} : N_i^a \cong N_i$ . (2) For all  $i < \delta$ , we have that  $M \underset{M_i}{\overset{N_i^b}{\downarrow}} N_i^a$  by construction, so applying f to this  $f[N_i^b]$   $f[N_i^b]$  N

we get 
$$f[M] \stackrel{j[N_i]}{\underset{M_i}{\bigcup}} f[N_i^a]$$
, i.e.  $f[M] \stackrel{j[N_i]}{\underset{M_i}{\bigcup}} N_i$ , so  $f[M] \stackrel{N}{\underset{M_i}{\bigcup}} N_i$  by monotonicity.  
(3)  $N = \operatorname{cl}^N(f[M] \cup N_{\delta})$ : Why? Note that by continuity  $N_{\delta}^b = \bigcup_{i < \delta} N_i^b$  and  
the latter is  $\bigcup_{i < \delta} \operatorname{cl}^{N_{\delta}^b}(M \cup N_i^a)$  by construction. Now,  $N_{\delta}^b = \operatorname{cl}^{N_{\delta}^b}(N_{\delta}^b) =$   
 $\operatorname{cl}^{N_{\delta}^b}(\bigcup_{i < \delta} \operatorname{cl}^{N_{\delta}^b}(M \cup N_i^a))$ . By Fact 16.4.6, this is  $\operatorname{cl}^{N_{\delta}^b}(\bigcup_{i < \delta} M \cup N_i^a) =$   
 $\operatorname{cl}^{N_{\delta}^b}(M \cup N_{\delta}^a)$ . We have shown that  $N_{\delta}^b = \operatorname{cl}^{N_{\delta}^b}(M \cup N_{\delta}^a)$ . Applying  $f$  to  
this equation, we obtain  $N = \operatorname{cl}^N(f[M] \cup N_{\delta})$ , as desired.

The next notion is studied explicitly in [She09b, [V.E.1.2] and Definition 3.3.4 (where it is called the minimal closure of  $\downarrow$ ). It is a way to extend  $\downarrow$  to take sets on the left and right hand side.

DEFINITION 16.4.8. We write  $A \underset{M_0}{\overset{M_3}{\longrightarrow}} B$  if  $M_0 \leq_{\mathbf{K}} M_3$ ,  $A \cup B \subseteq |M_3|$ , and there exists  $M'_3 \geq_{\mathbf{K}} M_3$ ,  $M_1 \leq_{\mathbf{K}} M'_3$ , and  $M_2 \leq_{\mathbf{K}} M'_3$  such that  $A \subseteq |M_1|$ ,  $B \subseteq |M_2|$ , and  $M_1 \underset{M_0}{\overset{\widetilde{M}'_3}{\bigcup}} M_2$ . LEMMA 16.4.9.

(1)  $M_1 \stackrel{M_3}{\underset{M_0}{\downarrow}} M_2$  if and only if  $M_1 \stackrel{M_3}{\underset{M_0}{\downarrow}} M_2$  and  $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$  for  $\ell = 1, 2.$ (2) Invariance: if  $A \stackrel{M_3}{\underset{M_0}{\downarrow}} B$  and  $f : M_3 \cong M'_3$ , then  $f[A] \stackrel{M'_3}{\underset{f[M_0]}{\downarrow}} f[B]$ . (3) Monotonicity: if  $A \stackrel{M_3}{\underset{M_0}{\downarrow}} B$  and  $A_0 \subseteq A$ ,  $B_0 \subseteq B$ , and  $M_3 \leq_{\mathbf{K}} M'_3$ , then  $M'_2$  $A_0 \frac{M'_3}{\bigcup} B_0.$ 

**PROOF.** Straight from the definitions.

NOTATION 16.4.10.

- (1) When  $N \in \mathbf{K}$ ,  $M \leq_{\mathbf{K}} N$ ,  $B \subseteq |N|$ , and  $\bar{a} \in {}^{<\infty}|N|$ , we write  $\bar{a} \downarrow_{M} B$  for  $\operatorname{ran}(\bar{a}) \downarrow_{R} B$ .
- ran $(\bar{a}) \overset{N}{\underset{M}{\bigcup}} B$ . (2) For  $p \in gS^{<\infty}(B; N)$  and  $M \leq_{\mathbf{K}} N$ , we say p does not fork over M if whenever  $p = gtp(\bar{a}/B; N)$ , we have that  $\bar{a} \overset{N}{\underset{M}{\bigcup}} B$ . Note that this does not depend on the choices of representatives by Lemma 16.4.9.

The following properties all appear either in Section 3.5.1 or Sections 6.4, 6.12. We will use them without comments.

Fact 16.4.11.

(1) Normality: If 
$$A \underset{M_0}{\overset{M_3}{\longrightarrow}} B$$
, then  $A \cup |M_0| \underset{M_0}{\overset{M_3}{\longrightarrow}} B \cup |M_0|$ .  
(2) Base monotonicity: if  $A \underset{M_0}{\overset{M_3}{\longrightarrow}} B$  and  $M_0 \leq_{\mathbf{K}} M'_0 \leq_{\mathbf{K}} M_3$  is such that  $|M'_0| \subseteq B$ , then  $A \underset{M'_0}{\overset{M_3}{\longrightarrow}} B$ .

(3) Symmetry: If 
$$A \bigcup_{M_0}^{M_3} B$$
, then  $B \bigcup_{M_0}^{M_3} A$ 

- (4) Extension: Let  $M \leq_{\mathbf{K}} N$  and  $B \subseteq C \subseteq |N|$  be given. If  $p \in gS^{<\infty}(B; N)$  does not fork over M, then there exists  $N' \geq_{\mathbf{K}} N$  and  $q \in gS^{<\infty}(C; N')$  extending p and not forking over M.
- (5) Uniqueness: Let  $M \leq_{\mathbf{K}} N$  and let  $|M| \subseteq B \subseteq |N|$ . If  $p, q \in \mathrm{gS}^{<\infty}(B; N)$  do not fork over M and  $p \upharpoonright M = q \upharpoonright M$ , then p = q.

(6) Transitivity: If 
$$A \downarrow_{M_0} M$$
,  $A \downarrow_M B$ , and  $M_0 \leq_{\mathbf{K}} M$ , then  $A \downarrow_M B$ .

The following is a form of local character that  $\downarrow$  may have:

DEFINITION 16.4.12 (V.C.3.7 in [She09b]). We say that  $\downarrow$  is  $\chi$ -based if whenever  $M \leq_{\mathbf{K}} M^*$  and  $A \subseteq |M^*|$  then there are  $N_0$  and  $N_1$  so that  $||N_1|| \leq |A| + \chi$ ,  $N_0 = M \cap N_1, A \subseteq |N_1|$ , and  $N_1 \underset{N_0}{\downarrow} M$ .

Interestingly, if  $\downarrow$  is based then smoothness for small lengths implies smoothness for all lengths.

FACT 16.4.13 (V.D.1.2 in [She09b]). If K is  $(\leq LS(K), \leq LS(K)^+)$ -smooth (recall Definition 16.2.10) and  $\downarrow$  is LS(K)-based, then K is smooth, i.e. it is an AEC.

A consequence of  $\downarrow$  being based is that the class is tame. The argument is folklore and appears already in [**GK**, p. 15].

LEMMA 16.4.14. Assume that  $\downarrow$  is LS(**K**)-based.

(1) Set local character: if  $p \in gS^{<\infty}(M)$ , then there are  $M_0 \leq_{\mathbf{K}} M$  such that  $||M_0|| \leq |\ell(p)| + \mathrm{LS}(\mathbf{K})$  and p does not fork over  $M_0$ .

(2)  $\mathbf{K}$  is  $LS(\mathbf{K})$ -tame.

Proof.

- (1) Straight from the definitions.
- (2) Combine (1) with the uniqueness property.

### 16.5. Enumerated trees and generalized symmetry

HYPOTHESIS 16.5.1.  $(\mathbf{K}, \bigcup, cl)$  satisfies AxFri<sub>1</sub>. Eventually, we will also assume Hypotheses 16.5.8 and 16.5.26

Consider a minimal failure of smoothness: an increasing chain  $\langle M_i : i \leq \delta \rangle$  that is continuous below  $\delta$  but so that  $\bigcup_{i < \delta} M_i \not\leq_{\mathbf{K}} M_{\delta}$ . We would like to copy this chain into a tree indexed by  $\leq^{\delta} \lambda$ . The branches of the tree should be as independent as possible. The main theorem of this section, Theorem 16.5.40, shows that it can be done. We show in Theorem 16.6.16 that the resulting tree of failures witnesses unstability.

The main difficulty in the proof of Theorem 16.5.40 is that we cannot assume smoothness when we construct the tree, so we have difficulties at limits (because, to quote the referee, the tree is "wider than it is high"). We work around this by studying trees enumerated in some order, giving a definition of a closed subset of such tree (Definition 16.5.9) and proving a generalized symmetry theorem for these sets (Theorem 16.5.35). Generalized symmetry says intuitively (as in [She83a, She83b]) that whether a tree is independent does not depend on its enumeration, so closed sets will be as independent of each other as possible. Once generalized symmetry is proven, the construction of the desired tree can be carried out.

This section draws a lot of inspiration from [She09b, V.C.4], where Shelah defines a notion of stable construction which is supposed to accomplish similar goals than here. Shelah even states Theorem 16.5.40 as an exercise [She09b, V.C.4.14]. However, we cannot solve it when smoothness fails. It seems that clause (vi) in [She09b, Definition V.C.4.2] is too restrictive and precisely prevents us from copying a non-smooth chain into a tree.

We start by setting up the notation of this section for trees. The universe of the trees we will use is always an ordinal  $\alpha$ , and we think of  $(\alpha, \leq)$  as giving the order in which the tree is enumerated and  $(\alpha, \leq)$  as being the tree order.

DEFINITION 16.5.2. An enumerated tree is a pair  $(\alpha, \trianglelefteq)$ , where  $\alpha$  is an ordinal and  $\trianglelefteq$  is a partial order on  $\alpha$  such that for all  $i, j < \alpha$ :

- (1)  $0 \leq i$  (i.e. 0 is the root of the tree).
- (2)  $i \leq j$  implies  $i \leq j$  (i.e. if j is above i in the tree, then it is enumerated later).
- (3)  $(\{k < \alpha \mid k \leq i\}, \leq)$  is a well-ordering.

DEFINITION 16.5.3. Let  $(\alpha, \trianglelefteq)$  be an enumerated tree.

- (1) For  $i < \alpha$ , and  $R \in \{\triangleleft, \trianglelefteq\}$ , let  $\operatorname{pred}_R(i) := \{k \le i \mid kRi\}$ . When  $R = \triangleleft$ , we omit the subscript.
- (2) A branch of  $(\alpha, \trianglelefteq)$  is a set  $b \subseteq \alpha$  such that:
  - (a)  $\leq$  linearly orders b.
  - (b)  $i \in b$  implies  $pred(i) \subseteq b$ .

- (3) A branch  $b \subseteq \alpha$  is bounded (in  $(\alpha, \leq)$ ) if either it has a maximum or  $b = \operatorname{pred}(i)$  for some  $i < \alpha$ . It is unbounded otherwise. We say that a set  $u \subseteq \alpha$  is bounded if any branch  $b \subseteq u$  is bounded.
- (4) We say that  $(\alpha, \leq)$  is *continuous* when for any  $i, j < \alpha$ , if pred(i) = pred(j) and pred(i) does not have a maximum, then i = j.
- (5) When  $(\alpha, \trianglelefteq)$  is continuous and  $b \subseteq \alpha$  is a bounded branch, we let:

$$\operatorname{top}(b) := \begin{cases} \max(b) & \text{if } b \text{ has a } \trianglelefteq \operatorname{-maximum} \\ \text{The unique } i < \alpha \text{ such that } b = \operatorname{pred}(i) & \text{otherwise.} \end{cases}$$

(6) When  $u \subseteq \alpha$ , let:

 $B(u) := \{ b \subseteq u \mid b \text{ is a branch and for any branch } b', b \subseteq b' \subseteq u \text{ implies } b' = b \}$ 

be the set of branches in u that are maximal in u.

(7) When  $(\alpha, \trianglelefteq)$  is continuous and  $u \subseteq \alpha$  is a bounded set, we let  $top(u) := \sup_{b \in B(u)} top(b)$  (this will only be used when B(u) is finite, so in that case the supremum is actually a maximum).

LEMMA 16.5.4. If  $u \subseteq v$  and  $b \in B(u)$ , then there is a branch  $b' \in B(v)$  such that  $b \subseteq b'$ . Consequently,  $|B(u)| \leq |B(v)|$ .

PROOF. Straightforward from the definition of B(u). The last sentence is because the map  $b \mapsto b'$  (for some choice of b') is an injection from B(u) to B(v).  $\Box$ 

We now define a tree of structures coming from the class  $\mathbf{K}$ . Note that continuity of chains of models is only required when the chain is smooth (see (5) below).

DEFINITION 16.5.5. A continuous enumerated tree of models is a tuple ( $\langle M_i : i < \alpha \rangle, N, \alpha, \trianglelefteq$ ) satisfying:

- (1)  $(\alpha, \trianglelefteq)$  is a continuous enumerated tree.
- (2)  $N \in \mathbf{K}$ .
- (3) For all  $i < \alpha, M_i \leq_{\mathbf{K}} N$ .
- (4) For all  $i, j < \alpha, i \leq j$  implies  $M_i \leq_{\mathbf{K}} M_j$ .
- (5) For all  $i < \alpha$ , if pred(i) has no maximum and  $\bigcup_{j \leq i} M_j \leq_{\mathbf{K}} N$ , then  $M_i = \bigcup_{j \leq i} M_j$ .

REMARK 16.5.6. By coherence, for all  $i < \alpha$ ,  $\bigcup_{j < i} M_j \leq_{\mathbf{K}} N$  if and only if  $\bigcup_{j < i} M_j \leq_{\mathbf{K}} M_i$ 

REMARK 16.5.7. In Definition 16.5.5, N is just an ambient model. Eventually, we will want to also ensure that it satisfies a minimality condition (see the conclusion of Theorem 16.5.39).

From now on until Lemma 16.5.37, we assume:

HYPOTHESIS 16.5.8.  $\mathcal{T} := (\langle M_i : i < \alpha \rangle, N, \alpha, \trianglelefteq)$  is a continuous enumerated tree of models.

The following is a key definition. Intuitively, a set is closed if it is closed under initial segments and all its branches smoothly embed inside N.

DEFINITION 16.5.9.  $u \subseteq \alpha$  is closed if:

- (1)  $i \in u$  implies  $\operatorname{pred}(i) \subseteq u$ .
- (2)  $b \in B(u) \setminus \{\emptyset\}$  implies  $\bigcup_{i \in b} M_i \leq_{\mathbf{K}} N$ .

Lemma 16.5.10.

- (1) An arbitrary intersection of closed sets is closed.
- (2) A finite union of closed sets is closed.

# Proof.

- (1) Let  $\langle u_i : i < \gamma \rangle$  be closed,  $\gamma > 0$ . Let  $u := \bigcap_{i < \gamma} u_i$ . We show that u is closed. It is easy to check that u satisfies (1) from the definition of a closed set. We check (2). Let  $b \in B(u) \setminus \{\emptyset\}$ . We want to see that  $\bigcup_{j \in b} M_j \leq_{\mathbf{K}} N$ . By Lemma 16.5.4, for each  $i < \gamma$  there exists  $b_i \in B(u_i)$  such that  $b \subseteq b_i$ . Since  $u_i$  is closed, we have that  $\bigcup_{j \in b_i} M_j \leq_{\mathbf{K}} N$ . If there exists  $j < \gamma$  such that  $b = b_j$ , we are done so assume that this is not the case. This implies that b is bounded. Let k := top(b). We know that  $b \subsetneq b_j$  for all  $j < \gamma$ , so by downward closure we must have that  $k \in b_j$  for all  $j < \gamma$ . But then this means that  $k \in u$ , so  $k \in b$ , a contradiction.
- (2) Let u, v be closed. We show that  $u \cup v$  is closed. As before, (1) is straightforward to see. As for (2), let  $b \in B(u \cup v)$ . It is straightforward to see that either  $b \in B(u)$  or  $b \in B(v)$ . In either case we get that  $\bigcup_{i \in b} M_i \leq_{\mathbf{K}} N$ , as desired.

REMARK 16.5.11. Lemma 16.5.10 almost tells us that closed sets induce a topology on  $\alpha$ . While it is easy to check that the empty set is closed,  $\alpha$  itself may not be closed (think of a chain  $\langle M_i : i \leq \delta \rangle$  where  $\bigcup_{i < \delta} M_i \not\leq_{\mathbf{K}} M_{\delta}$ . The tree could consist of  $\langle M_i : i < \delta \rangle$  and  $N = M_{\delta}$ ). However  $\alpha$  will be closed when all the maximal branches of the tree have a maximum (e.g. if  $(\alpha, \trianglelefteq)$  looks like  $\leq \delta \lambda$  for some cardinal  $\lambda \geq 2$  and limit ordinal  $\delta$ ).

The next definition describes the model  $M^u$  generated by a set  $u \subseteq \alpha$ . Typically, u will be closed and in case the tree is sufficiently independent (see Definition 16.5.25),  $M^u$  will be in **K**.

DEFINITION 16.5.12. For  $u \subseteq \alpha$ ,  $M^u := \operatorname{cl}^N(|M_0| \cup \bigcup_{i \in u} |M_i|)$ .

LEMMA 16.5.13. Let  $u, v \subseteq \alpha$ .  $M^{u \cup v} = \operatorname{cl}^N(M^u \cup M^v)$ .

PROOF. By Fact 16.4.6.

LEMMA 16.5.14. If b is a closed and bounded branch, then  $M^b = M_i$ , where i := top(b).

PROOF. If *i* is a maximum of *b* or *b* is empty, this is clear. If not, we know since *b* is closed that  $\bigcup_{j \triangleleft i} M_j = \bigcup_{j \in b} M_j \leq_{\mathbf{K}} N$ . By (5) in Definition 16.5.5,  $M_i = \bigcup_{j \triangleleft i} M_j$ . Note that  $\bigcup_{j \triangleleft i} M_j = \operatorname{cl}^N(\bigcup_{j \triangleleft i} M_j)$  and by Fact 16.4.6, this is equal to  $M^b$ . So  $\bigcup_{j \triangleleft i} M_j = M^b$ , as desired.

The next definition describes when two (typically closed) sets u and v are "as independent as possible", i.e. the model generated by u is independent of the one generated by v over the model generated by  $u \cap v$ . There are two variations depending on whether the ambient model is N or the model generated by  $u \cup v$ .

Generalized symmetry (Theorem 16.5.35) will say that under appropriate conditions, if the tree is independent then any closed sets u and v are as independent as possible.

DEFINITION 16.5.15. Let  $u, v \subseteq \alpha$ .

- (1) We write uv for  $u \cup v$ .
- (2) We write  $u \, {\downarrow} \, v$  if  $M^u \stackrel{M^{uv}}{\underset{M^{u\cap v}}{\downarrow}} M^v$ . (3) We write  $u \stackrel{N}{\downarrow} v$  if  $M^u \stackrel{M^{u \cap v}}{\downarrow} M^v$ .

Note that to make the notation lighter we omit the base and write  $u \downarrow v$  instead of e.g.  $u \perp v$ .

The following will be used without comment.

LEMMA 16.5.16. 
$$u \stackrel{N}{\downarrow} v$$
 if and only if  $[u \downarrow v \text{ and } M^{uv} \leq_{\mathbf{K}} N]$ .

PROOF. If  $u \stackrel{N}{\downarrow} v$ , then by Fact 16.4.7.(1),  $M^{uv} \leq_{\mathbf{K}} N$  and  $u \downarrow v$ . The converse is by the monotonicity 2 property of  $\downarrow$  in Definition 16.4.1. 

If  $u \subseteq v$ , there is an easy way to determine whether  $u \downarrow v$ .

LEMMA 16.5.17. If  $u \subseteq v$ ,  $M^u \leq_{\mathbf{K}} N$ , and  $M^v \leq_{\mathbf{K}} N$ , then  $u \downarrow v$ .

**PROOF.** Straight from the definition.

We now translate the properties of Section 16.4 into properties of the relations  $u \downarrow v$  and  $u \downarrow v$ .

LEMMA 16.5.18. Let  $u, v, w \subseteq \alpha$  be closed.

- (1) Symmetry: If  $u \downarrow v$ , then  $v \downarrow u$ . If  $u \downarrow v$ , then  $v \downarrow u$ .
- (2) Base enlargement: If  $u \stackrel{N}{\downarrow} v, u \cap v \subseteq w \subseteq v$ , and  $M^w \leq_{\mathbf{K}} M^v$ , then  $uw \stackrel{N}{\downarrow} v$ .
- (3) Transitivity: If  $u \downarrow v$ ,  $uv \downarrow w$ ,  $uv \cap w = u$ , and  $M^{v \cap w} \leq_{\mathbf{K}} M^w$ , then  $v \stackrel{N}{\downarrow} w.$

Proof.

- (1) Straightforward from the symmetry axiom.
- (2) Directly from the base enlargement axiom (note that  $uw \cap v = w$ ), see Definition 16.4.1.
- (3) Let  $M_0 := M^{u \cap v}$ ,  $M_1 := M^v$ ,  $M_2 := M^u$ ,  $M_3 := M^{uv}$ ,  $M_4 := M^w$ ,  $M_5 := N$ . We know that  $u \downarrow v$ , so  $M^{uv} \leq_{\mathbf{K}} N$  and  $u \downarrow v$ , hence  $M_1 \downarrow M_2$  holds. We know that  $uv \downarrow w$  (so in particular  $M^{uvw} \leq_{\mathbf{K}} N$ )

and 
$$uv \cap w = u$$
, i.e.  $M^{uv \cap w} = M^u = M_2$ , so  $M_3 \underset{M_2}{\overset{M_5}{\downarrow}} M_4$  holds. Apply-  
ing Fact 16.4.7.(2), we obtain  $M_1 \underset{M_2}{\overset{M_5}{\downarrow}} M_4$ , i.e.  $M^v \underset{M^{u \cap v}}{\overset{N}{\downarrow}} M^w$ . Now since

A key part of the proof of generalized symmetry is a concatenation property telling us when  $uv \downarrow w$  if we know something about u and v separately. We start with the following result:

LEMMA 16.5.19. Let  $u, v, w \subseteq \alpha$  be closed. If:

(1) 
$$u(v \cap w) \stackrel{N}{\downarrow} w.$$
  
(2)  $v \stackrel{N}{\downarrow} uw.$   
(3)  $M^{u(v \cap w)} \leq_{\mathbf{K}} M^{uw}$   
(4)  $M^{uv \cap w} \leq_{\mathbf{K}} M^{uv}.$   
Then  $uv \stackrel{N}{\downarrow} w$ 

Then  $uv \, {\perp} w$ .

**PROOF.** We apply base enlargement with u, v, w in 16.5.18.(2) standing for  $v, uw, u(v \cap w)$  here. The hypotheses hold by (2) and (3). We obtain  $uv \stackrel{N}{\downarrow} uw$ . We want to apply transitivity, where u, v, w in 16.5.18.(3) stand for  $u(v \cap w), w, uv$ here. The conditions there are:

- $u \stackrel{N}{\downarrow} v$ , which translates to  $u(v \cap w) \stackrel{N}{\downarrow} w$  here (holds by (1)).  $uv \stackrel{N}{\downarrow} w$ , which translates to  $uw \stackrel{N}{\downarrow} uv$  here (holds by the paragraph above and symmetry).
- $uv \cap w = u$ , which translates to  $u(v \cap w)w \cap uv = u(v \cap w)$ , i.e.  $uw \cap uv = u(v \cap w)$  $u(v \cap w)$ , which is true.
- $M^{v \cap w} \leq_{\mathbf{K}} M^w$ , which translates to  $M^{w \cap uv} \leq_{\mathbf{K}} M^{uv}$ , which is true by (4).

Therefore the conclusion of transitivity holds. In our case, this means that  $_N$  $w \stackrel{N}{\downarrow} uv$ . By symmetry,  $uv \stackrel{N}{\downarrow} w$ , as desired. 

LEMMA 16.5.20. Let  $u, v, w \subseteq \alpha$  be closed. If:

(1) 
$$u \stackrel{N}{\downarrow} w.$$
  
(2)  $M^{uv \cap w} \leq_{\mathbf{K}} N.$   
(3)  $M^{u(v \cap w)} \leq_{\mathbf{K}} M^{u(v \cap w)}$ 

Then  $u(v \cap w) \perp w$ .

**PROOF.** We use Lemma 16.5.19 with u, v, w there standing for  $w \cap v, u, w$  here. Let us check the hypotheses:

- (1) there translates to  $(w \cap v)(u \cap w) \stackrel{N}{\downarrow} w$  here. So it is enough to see that  $M^w \leq_{\mathbf{K}} N$  and  $M^{uv \cap w} \leq_{\mathbf{K}} N$ . This holds by (1) and (2).
- (2) there translates to  $u \downarrow w$  here, which is (1).

- (3) there translates to  $M^{uv \cap w} \leq_{\mathbf{K}} M^w$  here. This holds by (1), (2), and coherence.
- (4) there translates to  $M^{uv \cap w} \leq_{\mathbf{K}} M^{u(v \cap w)}$  here. This holds by (3).

The hypotheses hold, so we obtain that  $u(v \cap w) \stackrel{N}{\downarrow} w$ , as needed.

Finally, we obtain a usable concatenation property.

LEMMA 16.5.21 (Concatenation). Let  $u, v, w \subseteq \alpha$  be closed. If:

(1)  $u \stackrel{N}{\downarrow} w$ . (1)  $u \downarrow w$ . (2)  $v \downarrow uw$ . (3)  $M^{uv \cap w} \leq_{\mathbf{K}} N$ . (4)  $M^{u(v \cap w)} \leq_{\mathbf{K}} N$ . (5)  $M^{uv} \leq_{\mathbf{K}} N$ . Then  $uv \stackrel{N}{\downarrow} w$ .

**PROOF.** We use Lemma 16.5.19. Let us check the hypotheses:

- (1) says  $u(v \cap w) \downarrow w$ . This holds by Lemma 16.5.20. Note that (1) there holds by (1), (2) there holds by (3), and (3) there holds by (3), (4), and coherence.
- (2) there is (2) here.
- (3) there is given by (1), (4), and coherence.
- (4) there is given by (3), (5), and coherence.

The hypotheses hold, so we obtain that  $uv \stackrel{N}{\downarrow} w$ , as needed.

Another key ingredient of the proof of generalized symmetry is a continuity property that tells us how to deal with increasing chains  $\langle u_i : i < \delta \rangle$  of closed sets. At that point, the following hypothesis will appear in some of the statements (we do *not* assume it globally).

DEFINITION 16.5.22. We say that cl is algebraic if for any  $M, N \in \mathbf{K}$  with  $M \subseteq N$  and any  $A \subseteq |M|$ ,  $\mathrm{cl}^{M}(A) = \mathrm{cl}^{N}(A)$ .

Recall that we are working under Hypothesis 16.5.1, so cl is in particular a fixed operator satisfying Monotonicity 2 (Definition 16.4.1.(2c)). The difference here is that we assume that closure is the same whenever  $M \subseteq N$  (not only under the stronger condition  $M \leq_{\mathbf{K}} N$ ).

Note that if  $cl^{N}(A)$  is the closure of A under the functions of N, then cl is algebraic. This will be the closure operator when we study universal classes, so we do not lose much by assuming it here. In fact, we could have assumed from the beginning that  $cl^{N}(A)$  was the closure of A under the functions of N. For the purpose of proving the main result of this chapter, we would not lose anything.

LEMMA 16.5.23. Assume that cl is algebraic. Let  $\delta$  be a limit ordinal and let  $\langle u_i : i \leq \delta \rangle$  be an increasing continuous chain of closed sets. If for all  $i < \delta$ ,  $M^{u_i}$  is a  $\tau(\mathbf{K})$ -structure, then  $M^{u_{\delta}} = \bigcup_{i < \delta} M^{u_i}$ .

PROOF. Let  $M_{\delta} := \bigcup_{i < \delta} M^{u_i}$ .

First observe that  $M_{\delta} \subseteq N$ , because for all  $i < \delta$ ,  $M^{u_i} \subseteq N$  (as we are assuming it is a  $\tau(\mathbf{K})$ -structure and by definition it must inherit the function symbols from N). Therefore because cl is algebraic,  $\operatorname{cl}^N(M_{\delta}) = \operatorname{cl}^{M_{\delta}}(M_{\delta}) = M_{\delta}$ . But  $\operatorname{cl}^N(M_{\delta}) = \operatorname{cl}^N\left(\bigcup_{i < \delta} \operatorname{cl}^N(M_0 \cup \bigcup_{j \in u_i} M_j)\right)$ . By Fact 16.4.6, this is just  $\operatorname{cl}^N\left(\bigcup_{i < \delta} (M_0 \cup \bigcup_{j \in u_i} M_i)\right) = \operatorname{cl}^N(M_0 \cup \bigcup_{i \in u_{\delta}} M_i) = M^{u_{\delta}}$ . Combining the chains of equalities, we have the result.

LEMMA 16.5.24 (Continuity). Assume that cl is algebraic.

Let  $\delta$  be a limit ordinal and let  $\langle u_i : i \leq \delta \rangle$ ,  $\langle v_i : i \leq \delta \rangle$  be increasing continuous chains of closed sets. If for all  $i, j < \delta$ :

- (1)  $u_i \, {\downarrow} \, v_j$ . (2)  $u_\delta \cap v_i \, {\downarrow} \, v_j$ .
- (3)  $v_{\delta} \cap u_i \perp u_j$ .

Then  $u_{\delta} \perp v_{\delta}$ .

PROOF. <u>Claim 1</u>:  $M^{u_{\delta} \cap v_{\delta}} \leq_{\mathbf{K}} M^{v_{\delta}}$ .

<u>Proof of Claim 1</u>: We use Fact 16.4.7.(3) where  $M_i, N_i$  there stand for  $M^{u_\delta \cap v_i}$ ,  $M^{v_i}$  here. Why is  $\langle M^{u_\delta \cap v_i} : i \leq \delta \rangle \subseteq$ -increasing and continuous? Note that  $M^{u_\delta \cap v_i}$ is a member of **K** for each  $i < \delta$  (by (2)), and the chain is increasing by definition of  $M^{u_\delta \cap v_i}$ . The continuity is because  $\langle v_i : i \leq \delta \rangle$  is itself continuous (use Lemma 16.5.23). Similarly,  $\langle M^{v_i} : i \leq \delta \rangle$  is  $\subseteq$ -increasing continuous. Also, (2) ensures that the independence hypothesis of Fact 16.4.7.(3) is satisfied. Therefore we have in particular that  $M_\delta \leq_{\mathbf{K}} N_\delta$  there. That is,  $M^{u_\delta \cap v_\delta} \leq_{\mathbf{K}} M^{v_\delta}$ .  $\dagger_{\text{Claim}}$ .

<u>Claim 2</u>: For all  $j < \delta$ ,  $u_j \perp v_{\delta}$ .

<u>Proof of Claim 2</u>: Fix  $j < \delta$ . We will show that  $v_{\delta} \downarrow u_j$ . For this, we use Fact 16.4.7.(4) where  $M_i, M_{\delta+1}, N_i^a, N_i^b$  there stand for  $M^{u_j \cap v_{j+i}}, M^{u_j}, M^{v_{j+i}}, M^{u_j v_{j+i}}$  here (so we see  $M_{\delta+1}$  as really the "fixed" part and the  $N_i^a$ 's as the "growing" part). All the hypotheses of Fact 16.4.7.(4) are satisfied. In detail, we have to check that there  $M_{\delta} \leq_{\mathbf{K}} M_{\delta+1}$ , which here translates to  $M^{u_j \cap v_{\delta}} \leq_{\mathbf{K}} M^{u_j}$ , but this holds by (3). Also,  $N_i^b = \operatorname{cl}^{N_i^b}(M_{\delta+1} \cup N_i^a)$  there translates to  $M^{u_j v_{j+i}} = \operatorname{cl}^{M_{u_j}v_{j+i}}(M_{u_j} \cup M^{v_{j+i}})$ . This holds because  $M^{u_j v_{j+i}} \subseteq N$  (by (1),  $M^{u_j v_{j+i}} \in \mathbf{K}$ , and hence by definition it must be a substructure of N), and hence because cl is algebraic,  $\operatorname{cl}^N(A) = \operatorname{cl}^{M^{u_j v_{j+i}}}(A)$  for any set A. The other conditions are checked similarly. Applying Fact 16.4.7.(4), we obtain that  $v_{\delta} \downarrow u_j$ , and hence by symmetry  $u_j \downarrow v_{\delta}$  as desired.  $\dagger_{\operatorname{Claim} 2}$ .

To prove that  $u_{\delta} \downarrow v_{\delta}$ , we use Fact 16.4.7.(4) again where  $M_i, M_{\delta+1}, N_i^a, N_i^b$ there stand for  $M^{u_i \cap v_i}, M^{v_\delta}, M^{u_i}, M^{u_i v_\delta}$  here. We need to know there that  $M_{\delta} \leq_{\mathbf{K}} M_{\delta+1}$ , i.e.  $M^{u_{\delta} \cap v_{\delta}} \leq_{\mathbf{K}} M^{v_{\delta}}$ , but this is given by Claim 1. Further by Claim 2,  $u_i \downarrow v_{\delta}$  for every  $i < \delta$ , so the hypotheses of Fact 16.4.7.(4) hold.  $\Box$ 

With the forking calculus out of the way, we are ready to start proving generalized symmetry. First, we state what it means for a tree to be independent. The intuition is that for any  $i \leq j$ ,  $M_j$  is independent over  $M_i$  of as much as possible that comes before j in the enumeration of the tree. We use a slightly different notation than in e.g. [She83a, She83b] but the notion described is the same.

DEFINITION 16.5.25.  $\mathcal{T}$  is *independent* if for any  $i \leq j < \alpha$ :

$$M_j \bigcup_{M_i}^{\underline{N}} \bigcup_{k \in A_{i,j}} M_k$$

where  $\overline{\downarrow}$  is from Definition 16.4.8 and:

$$A_{i,j} := \{k < j \mid \operatorname{pred}_{\trianglelefteq}(k) \cap \operatorname{pred}_{\trianglelefteq}(j) \subseteq \operatorname{pred}_{\trianglelefteq}(i)\}$$

From now on until Lemma 16.5.37, we assume:

HYPOTHESIS 16.5.26.  $\mathcal{T}$  (from Hypothesis 16.5.8) is independent.

Our aim is to prove Theorem 16.5.35 which gives conditions under which  $u \downarrow v$  for any closed sets u and v. We prove increasingly stronger approximations to this result, each time using the previously proven approximations. First, we prove it when u and v are closed bounded branches.

LEMMA 16.5.27. If a and b are closed bounded branches, then  $a \downarrow b$ .

PROOF. Let  $i := \operatorname{top}(a), j := \operatorname{top}(b)$ . By Lemma 16.5.14,  $M^a = M_i, M^b = M_j$ . By Definition 16.5.5.(3),  $M^a \leq_{\mathbf{K}} N$  and  $M^b \leq_{\mathbf{K}} N$ . Note that  $a \cap b$  is also a closed bounded branch so  $M^{a \cap b} \leq_{\mathbf{K}} N$  also. By coherence,  $M^{a \cap b} \leq_{\mathbf{K}} M^x$  for  $x \in \{a, b\}$ . By symmetry, we can assume without loss of generality that  $j \leq i$ . Furthermore, if i = j then Lemma 16.5.17 gives the result, so assume j < i. Let  $k := \operatorname{top}(a \cap b)$ . By Lemma 16.5.14 again,  $M^{a \cap b} = M_k$ . Now by Definition 16.5.25, we must have that  $M_i \underbrace{\bigcup_{M_k} M_j}_{M_k}$ . By what we have argued, we must actually have  $M_i \underbrace{\bigcup_{M_k} M_j}_{M_k}$ , i.e.  $a \underbrace{\bigcup_{M_k} N}_{M_k}$  be as needed.

Next, we prove it when u is a closed and bounded branch and v is a bounded finite union of closed branches that comes before u in the enumeration of the tree (see Condition (3) below).

LEMMA 16.5.28. If:

(1) a is a closed and bounded branch.

- (2) v is a closed and bounded set with B(v) finite.
- (3)  $\operatorname{top}(a) \ge \operatorname{top}(v).$
- Then  $a \stackrel{N}{\downarrow} v$ .

PROOF. Let n := |B(v)|. We work by induction on n. If n = 1, the result holds by Lemma 16.5.27. Otherwise, say  $B(v) = \{b_0, \ldots, b_{n-1}\}$ , where without loss of generality  $\operatorname{top}(b_0) < \operatorname{top}(b_1) < \ldots < \operatorname{top}(b_{n-1})$ . By the induction hypothesis,  $b_{n-1} \downarrow b_0 \ldots b_{n-2}$ . In particular,  $M^v = M^{b_0 \ldots b_{n-1}} \leq_{\mathbf{K}} N$ . Now using Definition 16.5.25 (or Lemma 16.5.17 if  $\operatorname{top}(a) = \operatorname{top}(v)$ , so  $a \subseteq v$ ), it is easy to check that  $M^a \underset{M^{a \cap v}}{\overset{N}{\longrightarrow}} M^v$ , so the result follows.  $\Box$ 

Next, we can show that  $M^u \leq_{\mathbf{K}} N$  when u is a bounded finite union of closed branches.

LEMMA 16.5.29. If u and v are bounded closed sets with B(u) and B(v) both finite, then:

(1) 
$$M^u \leq_{\mathbf{K}} N.$$

(2)  $u \subseteq v$  implies  $M^u \leq_{\mathbf{K}} M^v$ .

PROOF. The second part follows from the first and coherence. For the first part, let n := |B(u)| and write  $B(u) = \{b_0, \ldots, b_{n-1}\}$  with  $top(b_0) < \ldots < top(b_{n-1})$ . If n = 1, the result follows from Lemma 16.5.27 (where a, b there stand for u, u here) so assume that  $n \ge 2$ . Apply Lemma 16.5.28 where a, v there stand for  $b_{n-1}$ ,  $b_0 \ldots b_{n-2}$  here.

We now use the previous result together with concatenation to show that  $u \perp v$  when u and v are bounded finite union of closed branches.

LEMMA 16.5.30. If u and v are closed bounded sets with B(u) and B(v) both finite, then  $u \stackrel{N}{\downarrow} v$ .

PROOF. Work by induction on |B(u)| + |B(v)|. By symmetry, without loss of generality  $top(u) \ge top(v)$ . Let n := |B(u)|. Write  $B(u) = \{a_0, \ldots, a_{n-1}\}$  with  $top(a_0) < \ldots < top(a_{n-1})$ . If n = 1, the result is given by Lemma 16.5.28, so assume now that  $n \ge 2$ . We use concatenation (Lemma 16.5.21) with u, v, w there standing for  $a_0 \ldots a_{n-2}, a_{n-1}, v$  here. Let us check the hypotheses:

- (1) there translates to  $a_0 \dots a_{n-2} \stackrel{N}{\downarrow} v$  here. This holds by the induction hypothesis.
- (2) there translates to  $a_{n-1} \downarrow^N a_0 \dots a_{n-2} v$  here. This holds by Lemma 16.5.28.
- (3)-(5) there hold by Lemma 16.5.29.

Next, we can use the continuity property to prove generalized symmetry for all closed bounded sets.

The hypotheses hold, so we obtain  $a_0 \ldots a_{n-1} \stackrel{N}{\bigcup} v$ , as desired.

LEMMA 16.5.31. Assume that cl is algebraic. If u and v are closed bounded sets, then  $u \downarrow v$ .

PROOF. Let  $\lambda := |B(u \cup v)|$ . We work by induction on  $\lambda$ . If  $\lambda < \aleph_0$ , then this is taken care of by Lemma 16.5.30. Otherwise, say  $B(u) = \langle a_i : i < \lambda \rangle$  and  $B(v) = \langle b_i : i < \lambda \rangle$  (we allow repetition in the enumerations). For  $i \leq \lambda$ , let  $u_i := \bigcup_{j < i} b_j$  and  $v_i := \bigcup_{j < i} b_j$ . It is easy to check that  $\langle u_i : i \leq \lambda \rangle$ ,  $\langle v_i : i \leq \lambda \rangle$  are increasing continuous resolutions of u and v respectively. Moreover, each member of the chain is a closed bounded set. We apply Lemma 16.5.24 (where  $\delta$  there stands for  $\lambda$  here). Its hypotheses hold by the induction hypothesis. We obtain that  $u_\lambda \downarrow v_\lambda$ , as desired.

When u or v is not bounded, we will make an additional hypothesis which says that branches do not have too many non-smooth points. In the case we are interested in (see Theorem 16.5.40), each branch will have at most one nonsmooth point, so this hypothesis is reasonable. Note again that we *do not* assume this globally, only in some statements.

DEFINITION 16.5.32.  $\mathcal{T}$  is *resolvable* if for any branch  $b \subseteq \alpha$ ,  $\{i \in b \mid \bigcup_{j \triangleleft i} M_j \not\leq_{\mathbf{K}} N\}$  is finite.

DEFINITION 16.5.33. For  $u \subseteq \alpha$ , let  $B'(u) := \{b \in B(u) \mid b \text{ is unbounded}\}.$ 

Assuming that  $\mathcal{T}$  is resolvable, we show that every closed set has a resolution with fewer unbounded branches than the original set. This will allow us to do a proof by induction on |B'(u)|.

LEMMA 16.5.34. Assume that  $\mathcal{T}$  is resolvable.

- (1) Let b be a closed branch. Then there is a limit ordinal  $\delta$  and an increasing continuous sequence of closed bounded branches  $\langle b_i : i \leq \delta \rangle$  such that  $b = b_{\delta}$ .
- (2) Let u be a closed unbounded set. Then there is a limit ordinal  $\delta$  and an increasing continuous sequence  $\langle u_i : i \leq \delta \rangle$  of closed sets such that  $u_{\delta} = u$  and for all  $i < \delta$ ,  $|B'(u_i)| < |B'(u)|$ .

Proof.

- (1) If b is bounded, we can take  $b = b_i$  for all  $i \leq \delta$ , so assume that b is unbounded. Since  $\mathcal{T}$  is resolvable, we know that there exists  $i \in b$  such that for all  $i' \geq i$ ,  $\bigcup_{j \leq i'} M_j \leq_{\mathbf{K}} N$ . In other words,  $\operatorname{pred}(i')$  is closed. So let  $\delta := \operatorname{gtp}(b)$  and write  $b \setminus i = \langle i_j : j < \delta \rangle$ . For  $j < \delta$ , let  $b_j := \operatorname{pred}(i_j)$ .
- (2) Say  $B'(u) = \{b_i : i < \lambda\}$ . Let  $v := u \setminus \bigcup_{i < \lambda} b_i$ . Note that v is closed and bounded. If  $\lambda$  is infinite, we can let  $\delta := \lambda$  and for  $i \leq \delta$ ,  $u_i := v \cup \bigcup_{j < i} b_j$ . So assume that  $\lambda$  is finite. By the first part, for each  $i < \lambda$  there exists a limit ordinal  $\delta_i$  and a resolution  $\langle b_i^j : j < \delta_i \rangle$  of  $b_i$  into closed bounded branches. Let  $\delta := \sum_{i < \lambda} \delta_i$ . Now for  $j < \delta$ , there are unique  $i < \lambda$ and  $k < \delta_i$  such that  $j = \sum_{i_0 < i} \delta_{i_0} + k$ . Set  $u_j := \bigcup_{i_0 < i} b_{i_0} \cup b_i^k$ . It is straightforward to check that this works.

THEOREM 16.5.35 (Generalized symmetry). Assume that  $\mathcal{T}$  is resolvable and cl is algebraic. If u and v are closed sets, then  $u \downarrow v$ .

PROOF. Work by induction on  $\lambda := |B'(u)| + |B'(v)|$ . If  $\lambda = 0$ , this is given by Lemma 16.5.31. If  $\lambda$  is infinite, we can use an argument analogous to the proof of Lemma 16.5.31, so assume that  $\lambda$  is finite and non-zero.

By Lemma 16.5.34, we can find limit ordinals  $\delta_1, \delta_2$  and  $\langle u_i : i \leq \delta_1 \rangle$ ,  $\langle v_i : i \leq \delta_2 \rangle$  that are increasing continuous resolutions of u and v respectively so that each member in the chain is closed, and for all  $i < \delta_1$ ,  $|B'(u_i)| < |B'(u)|$ , and similarly for v.

By symmetry, without loss of generality,  $\delta_1 \leq \delta_2$ . We first use Lemma 16.5.24 with  $\delta$  there standing for  $\delta_1$  here. The hypotheses hold by the induction hypothesis. So we obtain  $u \perp v_{\delta_1}$ . If  $\delta_1 = \delta_2$ , we are done. Otherwise by the induction hypothesis (using that  $\lambda$  is finite) we have that  $u \perp v_i$  for all  $i < \delta_2$ . So we use Lemma 16.5.24 a second time with  $\delta$ ,  $u_i$ ,  $v_i$  there standing for  $\delta_2$ , u,  $v_i$  here. We obtain that  $u \perp v_{\delta_2}$ , as desired.

For the remainder of this section, we focus on building independent trees. We "start from scratch" and drop Hypotheses 16.5.8 and 16.5.26. It will be convenient to have the tree enumerated in a particular order:

DEFINITION 16.5.36. An enumerated tree  $(\alpha, \leq)$  is in pre-order if for any  $i < \alpha$ and any  $b \in B(i)$ , either b = pred(i) or  $b \in B(\alpha)$ .

The idea is that (Lemma 16.5.38) if the tree is in pre-order, then the set  $A_{i,i}$ from Definition 16.5.25 is closed, so we can use generalized symmetry (Theorem 16.5.35) on it. Before proving this, we show that the tree we care about has an enumeration in pre-order. For this, we simply keep building the same branch until it becomes maximal, then start a different branch.

LEMMA 16.5.37. Let  $\delta$  be a limit ordinal and let  $\lambda$  be a cardinal with  $\lambda \geq 2$ . Then there exists an enumeration  $\langle \eta_i : i < \alpha \rangle$  of  $\leq \delta \lambda$  such that defining  $i \leq j$  if and only if  $\eta_i$  is an initial segment of  $\eta_j$ , we have that  $(\alpha, \trianglelefteq)$  is a continuous enumerated tree which is in pre-order.

**PROOF.** Let  $\langle \nu_j : j < \beta \rangle$  be an enumeration (without repetitions) of  $\leq \delta \lambda$  such that if  $\nu_j$  is an initial segment of  $\nu_{j'}$ , then  $j \leq j'$ . We define  $\alpha$  and  $\langle \eta_i : i < \alpha \rangle$  by induction on i such that:

(1)  $(i, \leq)$  is a continuous enumerated tree.

(2) If  $b \in B(i)$ , then either there is  $j \in b$  such that  $\eta_j \in {}^{\delta}\lambda$ , or b = pred(i). There are three cases:

- $\{\eta_j : j < i\} = \{\nu_j : j < \beta\}$ . Then we are done and let  $\alpha := i$ .
- If there is  $b \in B(i)$  such that for some  $j < \beta$ ,  $\bigcup_{k \in b} \eta_k$  is an initial segment of  $\nu_j$  but  $\nu_j \notin \{\eta_k \mid k \in b\}$ , then pick any such b and the least such j, and let  $\eta_i := \nu_j$ .
- Otherwise, let  $j < \beta$  be least such that  $\nu_j \neq \eta_k$  for any k < i. Let  $\eta_i := \nu_j$ .

It is straightforward to see that this works.

We can now prove that  $A_{i,j}$  is closed:

LEMMA 16.5.38. Let  $\mathcal{T} := (\langle M_i : i < \alpha \rangle, N, \alpha, \triangleleft)$  be a continuous enumerated tree of models. If:

- (1)  $(\alpha, \trianglelefteq)$  is in pre-order.
- (2) For any  $b \in B(\alpha)$ , b is bounded.

Then for any  $i \leq j < \alpha$ ,  $A_{i,j} = \{k < j \mid \operatorname{pred}_{\leq}(k) \cap \operatorname{pred}_{\leq}(j) \subseteq \operatorname{pred}_{\leq}(i)\}$  (see Definition 16.5.25) is closed (see Definition 16.5.9)

**PROOF.** Let  $b \in B(A_{i,j})$ . We have to see that  $\bigcup_{k \in b} M_k \leq_{\mathbf{K}} N$ . Now either  $b = \operatorname{pred}_{\leq}(i)$ , in which case  $\bigcup_{k \in b} M_k = M_i \leq_{\mathbf{K}} N$ , or  $b \not\subseteq \operatorname{pred}_{\leq}(j)$ . In this case, it is easy to check that  $b \in B(j)$  (otherwise we could just extend the branch), so since  $(\alpha, \trianglelefteq)$  is in pre-order, either  $b = \operatorname{pred}(j)$  or  $b \in B(\alpha)$ . The first case was dealt with before and in the second case, b is bounded so has a maximum j' (otherwise it would not be in  $B(\alpha)$  and so  $\bigcup_{k \in b} M_k = M_{j'} \leq_{\mathbf{K}} N$ . 

We can now prove that any reasonable tree can be "made independent" (and further, it will generate its ambient model N). This can be seen as a generalization of the existence axiom (see Definition 16.4.1.(3g)). Note that generalized symmetry is used in the proof.

LEMMA 16.5.39. Assume that cl is algebraic and we are given a resolvable continuous enumerated tree of models  $\mathcal{T}^0 := (\langle M_i^0 : i < \alpha \rangle, N^0, \alpha, \leq)$ . If:

- (1)  $(\alpha, \triangleleft)$  is in pre-order.
- (2) For any  $b \in B(\alpha)$ , b is bounded.

Then we can find  $\langle M_i : i < \alpha \rangle$ , N, and  $\langle f_i : i < \alpha \rangle$  such that:

- (1)  $\mathcal{T} := (\langle M_i : i < \alpha \rangle, N, \alpha, \trianglelefteq)$  is a resolvable *independent* continuous enumerated tree.
- (2) For all  $i, j < \alpha, f_i : M_i^0 \cong M_i$  and  $i \leq j$  implies  $f_i \subseteq f_j$ .
- (3)  $N = M^{\alpha} := \operatorname{cl}^{N}(\bigcup_{i < \alpha} M_{i}).$

**PROOF.** We build  $\langle N_i : i < \alpha \rangle$ ,  $\langle M_i : i < \alpha \rangle$ ,  $\langle f_i : i < \alpha \rangle$  such that:

- (1)  $\langle N_i : i \leq \alpha \rangle$  is increasing.
- (2)  $\langle f_i : i < \alpha \rangle$  satisfies (2).
- (3) For all  $i \in (0, \alpha)$ ,  $\mathcal{T}_i := (\langle M_j : j < i \rangle, \bigcup_{j < i} N_j, i, \trianglelefteq)$  is a resolvable independent continuous enumerated tree of models.
- (4) For all  $i < \alpha$ ,  $N_i = \operatorname{cl}^{N_i}(M_0 \cup \bigcup_{j < i} M_j) (= M^i)$ .

This is enough, as we can then take  $N := \bigcup_{i < \alpha} N_i$ . This is possible. When i = 0, set  $N_0 := M_0$ ,  $f_0 := \mathrm{id}_{M_0^0}$ . Now assume that i > 0. Let  $N'_i := \bigcup_{j < i} N_j$ . There are two cases:

• Case 1: pred(i) has a maximum: Let  $j := \max(\operatorname{pred}(i))$ . Use the existence axiom (Definition 16.4.1.(3g)) to find  $f_i$  extending  $f_j$  and  $N_i \geq_{\mathbf{K}} N'_i$  so that  $f_i : M_i^0 \cong M_i, M_i \underset{M_j}{\stackrel{N_i}{\longrightarrow}} N'_i$ , and  $N_i = \operatorname{cl}^{N_i}(M_i \cup N'_i)$ . It is easy to check that this works.

• Case 2: pred(i) does not have a maximum: Let  $M'_i := \bigcup_{j \triangleleft i} M_j$ ,  $(M^0_i)' := \bigcup_{j \triangleleft i} M_j^0$ ,  $f'_i := \bigcup_{j \triangleleft i} f_j$ . Let  $M''_i$ ,  $g : M^0_i \cong M''_i$  be such that g extends  $f'_i$ .

Let  $\delta := \operatorname{gtp}(\operatorname{pred}(i))$ . Note that  $\delta$  is a limit ordinal. Let  $\langle i_j : j < \delta \rangle$ list  $\operatorname{pred}(i)$  in increasing order. For j < i, let  $u_j := A_{i,j}$ , where  $A_{i,j}$  is as in Definition 16.5.25. Note that  $\bigcup_{j < \delta} u_j = i$ . By Lemma 16.5.38,  $u_j$ is closed in  $\mathcal{T}^0$ , hence (taking the image of  $\mathcal{T}^0$  by  $\bigcup_{j < i} f_j$ ) in  $\mathcal{T}_i$ . We use Fact 16.4.7.(5) with  $M_j$ ,  $N_j$ , M there standing for  $M_{i_j}$ ,  $M^{u_{i_j}}$ ,  $M''_i$ here. The hypotheses are satisfied by Theorem 16.5.35 (applied to  $\mathcal{T}_i$ ) and monotonicity. We obtain  $N_i \in \mathbf{K}$  and a map  $f : M''_i \xrightarrow{M'} N_i$  such that for

all 
$$j < \delta$$
:  
(1)  $N_{i_j} \leq_{\mathbf{K}} N_i$ .  
(2)  $f[M_i''] \stackrel{N_i}{\downarrow} M^{u_{i_j}}$ .  
(3)  $N_i = \operatorname{cl}^{N_i}(f[M_i''] \cup M^i)$ .  
Let  $f_i := f \circ g$  and let  $M_i := f[M_i'']$ . This works by the above properties.

A specialization of Lemma 16.5.39 yields the main theorem of this section.

THEOREM 16.5.40 (Tree construction). Assume that cl is algebraic. Let  $\delta$  be a limit ordinal and let  $\lambda \geq 2$  be a cardinal. Let  $\langle M_i : i \leq \delta \rangle$  be an increasing chain (we do *not* need to assume that the models have size  $\lambda$ ).

If  $\langle M_i : i < \delta \rangle$  is continuous but  $\bigcup_{i < \delta} M_i \not\leq_{\mathbf{K}} M_\delta$  (so  $\delta$  is the least failure of smoothness for the chain  $\langle M_i : i < \delta \rangle$ ), then there is  $\langle M_\eta \mid \eta \in {}^{\leq \delta}\lambda \rangle$ ,  $\langle f_\eta \mid \eta \in {}^{\leq \delta}\lambda \rangle$  and  $N \in \mathbf{K}$  such that for all  $\eta, \nu \in {}^{\leq \delta}\lambda$ :

- (1)  $M_{\eta} \leq_{\mathbf{K}} N, f_{\eta} : M_{\ell(\eta)} \cong M_{\eta}.$
- (2) If  $\eta$  is an initial segment of  $\nu$ , then  $M_{\eta} \leq_{\mathbf{K}} M_{\nu}$  and  $f_{\eta} \subseteq f_{\nu}$ .

(3) If  $\eta \neq \nu$  have length  $\delta$  and  $\alpha < \delta$  is least such that  $\eta \upharpoonright (\alpha+1) \neq \nu \upharpoonright (\alpha+1)$ ,

then 
$$M_\eta \bigcup_{M_{\nu \uparrow \alpha}}^N M_{\nu}$$
.

PROOF. By Lemma 16.5.37, we can find an enumeration  $\langle \eta_i : i < \alpha \rangle$  of  $\leq \delta \lambda$ such that  $(\alpha, \trianglelefteq)$  is a continuous enumerated tree in pre-order and  $i \trianglelefteq j < \alpha$  implies that  $\eta_i$  is an initial segment of  $\eta_j$ . For  $i < \alpha$ , let  $M_i^0 := M_{\ell(\eta_i)}$  and let  $N^0 := M_{\delta}$ . Then it is straightforward to check that  $\mathcal{T}^0 := (\langle M_i^0 : i < \alpha \rangle, N^0, \alpha, \trianglelefteq)$  satisfies the hypotheses of Lemma 16.5.39. We obtain  $\langle M_i : i < \alpha \rangle$ , N, and  $\langle f_i : i < \alpha \rangle$  there that correspond to  $\langle M_{\eta_i} : i < \alpha \rangle$ , N, and  $\langle f_{\eta_i} : i < \alpha \rangle$  here. Since the resulting tree is independent, we obtain the independence condition via Lemma 16.5.27.  $\Box$ 

# 16.6. Structure theory of universal classes

In this section, we precisely state a result of Shelah saying that for a universal class K which does not have the order property there is an ordering  $\leq$  so that  $\mathbf{K}^0 := (K, \leq)$  is a weak AEC satisfying AxFri<sub>1</sub> (see Definition 16.6.9). To simplify matters, we partition  $\mathbf{K}^0$  into disjoint classes, each of which has joint embedding, pick an appropriate such class and name it  $\mathbf{K}^*$  (Definition 16.6.11). We then use the tree construction theorem (Theorem 16.5.40) to show that failure of smoothness in  $\mathbf{K}^*$  implies unstability at certain cardinals (see Theorem 16.6.16).

We start by specializing the order property from [She09b, Definition V.A.1.1] (or Definition 2.4.2) to the quantifier-free version for universal classes:

DEFINITION 16.6.1. A universal class **K** has the order property of length  $\chi$ if there exists a quantifier-free first-order formula  $\phi(\bar{x}, \bar{y}, \bar{z})$ , a model  $M \in K$ , a sequence  $\bar{c} \in {}^{\ell(\bar{z})}|M|$ , and sequences  $\langle \bar{a}_i : i < \chi \rangle$ ,  $\langle \bar{b}_i : i < \chi \rangle$  from M (with  $\ell(\bar{a}_i) = \ell(\bar{x}), \ell(\bar{b}_i) = \ell(\bar{y})$  for all  $i < \chi$ ) so that for all  $i, j < \chi$ ,  $M \models \phi[\bar{a}_i; \bar{b}_j; \bar{c}]$  if and only if i < j. We say that **K** has the order property if it has the order property of length  $\chi$  for all cardinals  $\chi$ .

REMARK 16.6.2. In the next section, we will show (Lemma 16.7.1) that categoricity in some  $\lambda > LS(\mathbf{K})$  implies failure of the order property.

The following result is proven (in a more general form) in  $\S2$  of [GS86b].

FACT 16.6.3. Let **K** be a universal class. If **K** does not have the order property, then there exists  $\chi < h(\mathbf{K})$  (recall Definition 16.2.16) such that **K** does not have the order property of length  $\chi$ .

From failure of the order property, Shelah shows that there exists a certain ordering  $\leq^{\chi^+,\mu^+}$  on K such that  $(K,\leq^{\chi^+,\mu^+})$  satisfies AxFri<sub>1</sub> (recall Definition 16.4.1). We now proceed to define this ordering.

DEFINITION 16.6.4 (Averages, V.A.2 in [She09b]). Let **K** be a universal class. Let  $M \in \mathbf{K}$ , let I be an index set, and let  $\mathbf{I} := \langle \bar{a}_i : i \in I \rangle$  be a sequence of elements of M of the same finite arity  $n < \omega$ . Let  $\chi \leq \mu$  be infinite cardinals such that<sup>11</sup>  $|\mathbf{I}| \geq \chi$ .

(1) For  $A \subseteq |M|$ , we let  $\operatorname{Av}_{\chi}(\mathbf{I}/A; M)$  (the  $\chi$ -average of  $\mathbf{I}$  over A in M) be the set of quantifier-free first-order formulas  $\phi(\bar{x})$  over A such that  $\ell(\bar{x}) = n$  and  $|\{i \in I \mid M \models \neg \phi[\bar{a}_i]\}| < \chi$ .

<sup>&</sup>lt;sup>11</sup>We sometimes think of **I** as just the set of its elements (i.e. as if it was only ran **I**), e.g. we write  $|\mathbf{I}|$  instead of  $|\operatorname{ran} \mathbf{I}|$  and  $\mathbf{I} \subseteq {}^{n}A$  instead of ran  $\mathbf{I} \subseteq {}^{n}A$ .

- (2) We say that  $\mathbf{I}$  is  $(\chi, \mu)$ -convergent in M if  $|\mathbf{I}| \ge \mu$  and for every  $A \subseteq |M|$ ,  $p := \operatorname{Av}_{\chi}(\mathbf{I}/A; M)$  is complete over A (i.e. for every quantifier-free formula  $\phi(\bar{x})$  over A with  $\ell(\bar{x}) = n$ , either  $\phi(\bar{x}) \in p$  or  $\neg \phi(\bar{x}) \in p$ ).
- (3) Let  $A, B \subseteq |M|$  and let p be a set of quantifier-free formulas over B (all of the same arity  $n < \omega$ ). We say that p is  $(\chi, \mu)$ -averageable over A in M if there exists a sequence  $\mathbf{I} \subseteq {}^{n}A$  that is  $(\chi, \mu)$ -convergent in M and with  $p = \operatorname{Av}_{\chi}(\mathbf{I}/B; M)$ .

REMARK 16.6.5. In the above notation, the usual notion of average from the first-order framework [She90, Definition III.1.5] can be written  $\operatorname{Av}_{\aleph_0}(\mathbf{I}/A; \mathfrak{C})$ , modulo the fact that here all the formulas are quantifier-free.

REMARK 16.6.6 (Monotonicity).

- (1) Since the formulas under consideration are quantifier-free, we have the following monotonicity properties: if  $M_0 \subseteq M$  and  $A, \mathbf{I} \subseteq |M_0|$ , then  $\operatorname{Av}_{\chi}(\mathbf{I}/A; M_0) = \operatorname{Av}_{\chi}(\mathbf{I}/A; M)$ . Similarly, if  $\mathbf{I}$  is  $(\chi, \mu)$ -convergent in M, then it is  $(\chi, \mu)$ -convergent in  $M_0$ , and if  $A \subseteq B \subseteq |M_0|$  and p is a quantifier-free type over B that is  $(\chi, \mu)$ -averageable over A in M, then it is  $(\chi, \mu)$ -averageable over A in  $M_0$ .
- (2) If p over B as in (1) is  $(\chi, \mu)$ -averageable over A in M, then whenever  $A \subseteq A' \subseteq B_0 \subseteq B$ , we have that  $p \upharpoonright B_0$  is  $(\chi, \mu)$ -averageable over A' in M.

DEFINITION 16.6.7 (V.A.4.1 in [She09b]). Let **K** be a universal class and let  $\chi \leq \mu$  be infinite cardinals. For  $M, N \in \mathbf{K}$ , we write  $M \leq^{\chi,\mu} N$  if  $M \subseteq N$  and for every  $\bar{c} \in {}^{<\omega}|N|$ , the quantifier-free type of  $\bar{c}$  over M in N,  $\operatorname{tp}_{qf}(\bar{c}/M; N)$ , is  $(\chi, \mu)$ -averageable over M.

Note that if  $M, N \in \mathbf{K}_{<\mu}$ , then we never have  $M \leq^{\chi,\mu} N$ . From now on we assume:

Hypothesis 16.6.8.

- (1)  $\mathbf{K} = (K, \subseteq)$  is a universal class with arbitrarily large models.
- (2) χ ≥ LS(**K**) is such that **K** does not have the order property of length χ<sup>+</sup>.
  (3) Set μ := 2<sup>2<sup>χ</sup></sup>.

DEFINITION 16.6.9. Let  $\mathbf{K}^0 := (K, \leq^{\chi^+, \mu^+}).$ 

The following is the key structure theorem for universal classes: from failure of the order property, Shelah [She09b, Chapter V.B] shows that one can make  $\mathbf{K}^0$  into a weak AEC satisfying AxFri<sub>1</sub>. Note that by Fact 16.6.3 one can take  $\chi, \mu < \beth_{(2^{\text{LS}(\mathbf{K})})^+}$ .

Fact 16.6.10.

- (1)  $\mathbf{K}^0$  is a weak AEC with  $LS(\mathbf{K}^0) \leq \mu^+$ .
- (2) For  $M \in K$  and  $A \subseteq |M|$ , let  $\operatorname{cl}^{M}(A)$  be the closure of A under the functions of M. We can define a 4-ary relation  $\downarrow$  on K by  $M_1 \stackrel{M_3}{\downarrow} M_2$  if

and only if all of the following conditions are satisfied:

- (a)  $M_0 \leq_{\mathbf{K}^0} M_1$  and  $M_0 \leq_{\mathbf{K}^0} M_2$ .
- (b)  $M_1 \subseteq M_3$  and  $M_2 \subseteq M_3$ .
- (c)  $\mathrm{cl}^{M_3}(M_1 \cup M_2) \leq_{\mathbf{K}^0} M_3.$

(d) For any  $\bar{c} \in {}^{<\omega}|M_1|$ ,  $\operatorname{tp}_{\operatorname{of}}(\bar{c}/M_2; M_3)$  is  $(\chi^+, \mu^+)$ -averageable over  $M_0$ . We then have that  $(\mathbf{K}^{0}, \downarrow, cl)$  satisfies AxFri<sub>1</sub>. Moreover cl is algebraic (see Definition 16.5.22) and  $\downarrow$  is  $\mu^+$ -based (see Definition 16.4.12).

**PROOF.** That  $(\mathbf{K}^0, \downarrow, cl)$  satisfies AxFri<sub>1</sub> and has Löwenheim-Skolem-Tarski number bounded by  $\mu^+$  is the content of [She09b, V.B.2.9]. Since cl is just closure under the functions, it is clearly algebraic. That  $\downarrow$  is  $\mu^+$ -based is observed (but not explicitly proven) in [She09b, V.C.5.7]. We give the proof here.

Claim:  $\downarrow$  is  $\mu^+$ -based.

Proof of Claim:

First, we show:

<u>Subclaim</u>: For any cardinal  $\lambda$ ,  $\mathbf{K}^0$  is  $(\leq \lambda, \mu^+)$ -smooth. That is, if  $\langle M_i : i < \mu^+ \rangle$ is increasing in  $\mathbf{K}^0$  and  $M \in \mathbf{K}^0$  is such that  $M_i \leq_{\mathbf{K}^0} M$  for all  $i < \mu^+$ , then  $\bigcup_{i<\mu^+} M_i \leq_{\mathbf{K}^0} M.$ 

Proof of Subclaim:

In [She09b, V.A.4.4], it is shown that for any  $N, N' \in \mathbf{K}^0$ ,  $N \leq_{\mathbf{K}^0} N'$  if and only if  $N \preceq_{\Delta} N'$ , where  $\Delta$  is a certain fragment of  $\mathbb{L}_{\mu^+,\mu^+}$ . The result now follows from the basic properties of  $\Delta$ -elementary substructure.  $\dagger_{Subclaim}$ 

Let  $M \leq_{\mathbf{K}^0} M^*$  and let  $A \subseteq |M^*|$  be given. By definition of  $\leq_{\mathbf{K}^0} = \leq^{\chi^+, \mu^+}$ , for each  $\bar{c} \in {}^{<\omega}|M^*|$  there exists  $\mathbf{I}^{\bar{c}} \subseteq {}^{\ell(\bar{c})}|M|$  that is  $(\chi^+, \mu^+)$ -convergent and so that  $\operatorname{Av}(\mathbf{I}^{\bar{c}}/M; M^*) = \operatorname{tp}_{\mathrm{af}}(\bar{c}/M; M^*).$  Without loss of generality,  $|\mathbf{I}^{\bar{c}}| \leq \mu^+$ .

We build increasing  $\langle M_i^0 : i < \mu^+ \rangle$ ,  $\langle M_i^1 : i < \mu^+ \rangle$  such that for all  $i < \mu^+$ :

- (1)  $M_i^0 \leq_{\mathbf{K}^0} M.$ (2)  $M_i^0 \leq_{\mathbf{K}^0} M_i^1 \leq_{\mathbf{K}^0} M^*.$ (3)  $\|M_i^1\| \leq |A| + \mu^+.$

- (4)  $|M| \cap |M_i^1| \subseteq |M_0^{i+1}|.$ (5) For all  $\bar{c} \in {}^{<\omega}M_i^1, \mathbf{I}^{\bar{c}} \subseteq |M_{i+1}^0|.$

This is enough: let  $N_0 := \bigcup_{i < \mu^+} M_i^0$ ,  $N_1 := \bigcup_{i < \mu^+} M_i^1$ . By the claim,  $N_0 \leq_{\mathbf{K}^0}$  $N_1 \leq_{\mathbf{K}^0} M^*$  and by requirements (1) and (4),  $M \cap N_1 = N_0$ . Finally,  $N_1 \downarrow^{M^*} M$  by definition of  $\mathbf{I}^{\bar{c}}$  and requirement (5).

This is possible: assume that  $\langle M_i^{\ell} : j < i \rangle$  have been defined for  $\ell = 0, 1$ . Let  $M_{i,0}^0 := \bigcup_{j < i} M_j^0$ ,  $M_{i,0}^1 := \bigcup_{j < i} M_j^1$ . Use that  $\mathrm{LS}(\mathbf{K}^0) \leq \mu^+$  to pick  $M_i^0$ such that  $M_i^0 \leq_{\mathbf{K}^0} M$ ,  $|M| \cap M_{i,0}^1 \subseteq |M_i^0|$ ,  $\mathbf{I}^{\bar{c}} \subseteq |M_i^0|$  for all  $\bar{c} \in {}^{<\omega}|M_{i,0}^1|$ , and  $||M_i^0|| \leq |A| + \mu^+$ . Note that by coherence,  $M_i^0 \leq_{\mathbf{K}^0} M_i^0$ . Now pick  $M_i^1$  such that  $M_i^1 \leq_{\mathbf{K}^0} M^*, \ A \cup |M_{i,0}^1| \cup |M_i^0| \subseteq M_i^1, \ \text{and} \ \|M_i^1\| \leq |A| + \mu^+.$  It is easy to check that this satisfies all the requirements.  $\dagger_{\text{Claim}}$ 

Note that  $\mathbf{K}^0$  has amalgamation (Remark 16.4.4). However, we do not know if it satisfies joint embedding, so we partition  $\mathbf{K}^{0}$  into disjoint AECs, each of which has joint embedding. We will then concentrate on just one of these AECs. This trick appears in [She87b, Section II.3].

DEFINITION 16.6.11. For  $M, N \in \mathbf{K}^0$ , write  $M \sim N$  if they can be  $\leq_{\mathbf{K}^0}$ embedded inside a common model. This is an equivalence relation and the equivalence classes partition  $\mathbf{K}^0$  into disjoint weak AECs  $\langle \mathbf{K}_i^0 : i \in I \rangle$  that have amalgamation and joint embedding. There is only a set of such classes, so there exists<sup>12</sup>  $i \in I$  such that  $\mathbf{K}_i^0$  has arbitrarily large models. Let  $\mathbf{K}^* := (\mathbf{K}_i^0)_{\geq u^+}$ .

From now on, we will work with  $\mathbf{K}^*$ . We note a few trivial properties of independence there:

Lemma 16.6.12.

- (1)  $\mathbf{K}^*$  is a weak AEC with amalgamation, joint embedding, and arbitrarily large models.
- (2)  $LS(\mathbf{K}^*) = \mu^+$ .
- (3) **K** and  $\mathbf{K}^*$  are compatible (recall Definition 16.3.1).
- (4)  $(\mathbf{K}^*, \bigcup \upharpoonright \mathbf{K}^*, \mathrm{cl})$  satisfies AxFri<sub>1</sub>, where for  $M \in \mathbf{K}^*$ ,  $\mathrm{cl}^M$  is closure under the functions of M and  $\bigcup \upharpoonright \mathbf{K}^*$  is the natural restriction of  $\bigcup$  (from Fact 16.6.10) to  $\mathbf{K}^*$ .

PROOF. Straightforward.

NOTATION 16.6.13. We abuse notation and write  $\downarrow$  for  $\downarrow \upharpoonright \mathbf{K}^*$  (where again  $\downarrow$  is from Fact 16.6.10).

- LEMMA 16.6.14. (1) If  $A \stackrel{M_3}{\longrightarrow} B$  and  $\bar{c} \in {}^{<\omega}A$ , then  $\operatorname{tp}_{qf}(\bar{c}/B; M_3)$  is  $(\chi^+, \mu^+)$ -averageable over  $M_0$ .
- (2) cl is algebraic.
- (3)  $\perp$  is LS(**K**)-based.

PROOF. (1) follows directly from the definition of  $\downarrow$ . For the rest, cl is algebraic because cl satisfies this property in  $\mathbf{K}^0$  (Fact 16.6.10). Similarly in  $\mathbf{K}^0$ ,  $\downarrow$  is  $\mu^+$ -based (Fact 16.6.10) and it is straightforward to check that this carries over to  $\mathbf{K}^*$ .

Next, we study what happens if smoothness fails in  $\mathbf{K}^*$ . Recall that our goal is to see that this is incompatible with categoricity (in a high-enough cardinal). Shelah has shown [**She09b**, V.C.2.6], that failure of smoothness implies that  $\mathbf{K}^*$ has  $2^{\lambda}$ -many nonisomorphic models at every high-enough *regular* cardinal  $\lambda$ . So in particular  $\mathbf{K}^*$  cannot be categorical in a regular cardinal. However we are also interested in the singular case. Shelah states as an exercise [**She09b**, V.C.4.13] that  $\mathbf{K}^*$  has (at least)  $2^{<\lambda}$ -many nonisomorphic models if  $\lambda$  is singular. However we have been unable to prove it.

Instead, we aim to see that failure of smoothness implies that  $\mathbf{K}^*$  has many *types*, i.e. it is Galois unstable in some suitable cardinals. This will contradict Lemma 16.3.4. The argument is similar to [She09b, V.E.3.15], which shows that failure of superstability (in the sense that there is an increasing chain  $\langle M_i : i < \delta \rangle$  and a type  $p \in gS(\bigcup_{i < \delta} M_i)$  that forks over every  $M_i$ ,  $i < \delta$ ) implies unstability at

<sup>&</sup>lt;sup>12</sup>There could be many and for our purpose the choice of *i* does not matter. Moreover *i* is unique if **K** is categorical in some  $\lambda \ge \mu^+$ .
suitable cardinals. The extra difficulty here is that smoothness fails, but the hard work in constructing the tree has already been done in Theorem 16.5.40.

First observe that any failure of smoothness must be witnessed by a small chain:

LEMMA 16.6.15. If  $\mathbf{K}^*$  is  $(\leq \mathrm{LS}(\mathbf{K}^*), \leq \mathrm{LS}(\mathbf{K}^*)^+)$ -smooth (recall Definition 16.2.10), then  $\mathbf{K}^*$  is smooth, i.e. it is an AEC.

**PROOF.** By Lemma 16.6.14.(3),  $\mathbf{K}^*$  is  $LS(\mathbf{K}^*)$ -based, so apply Fact 16.4.13.

We now show that failure of smoothness implies unstability at some not too high cardinal. A technical subtlety is that we can only show ( $< \omega$ )-unstability, i.e. there are many types of some fixed finite length. In this framework, we do not know whether this implies that there are also many types of length one (see also Remark 16.3.6).

THEOREM 16.6.16. Assume that  $\mathbf{K}^*$  is not  $(\leq \mathrm{LS}(\mathbf{K}^*), \leq \mathrm{LS}(\mathbf{K}^*)^+)$ -smooth. Let  $\kappa \leq \mathrm{LS}(\mathbf{K}^*)^+$  be least such that  $(\leq \mathrm{LS}(\mathbf{K}^*), \leq \kappa)$ -smoothness fails. If  $\lambda \geq \mathrm{LS}(\mathbf{K}^*)^+$  is such that  $\lambda = \lambda^{<\kappa}$  and  $\lambda < \lambda^{\kappa}$ , then  $\mathbf{K}^*$  is  $(< \omega)$ -unstable in  $\lambda$ .

PROOF. Fix an increasing sequence  $\langle M_i : i \leq \kappa \rangle$  such that  $||M_i|| \leq \mathrm{LS}(\mathbf{K}^*)^+$ for all  $i \leq \kappa$  and  $\bigcup_{i < \kappa} M_i \not\leq_{\mathbf{K}^*} M_{\kappa}$ . Without loss of generality (using minimality of  $\kappa$ ) the sequence is continuous below  $\kappa$ , i.e.  $M_i = \bigcup_{j < i} M_j$  for every  $i < \kappa$ . Let  $N \in \mathbf{K}^*$  and  $\langle M_{\eta}, f_{\eta} \mid \eta \in \leq \kappa \lambda \rangle$  be as given by Theorem 16.5.40 (where  $\delta, \mathbf{K}$ there stands for  $\kappa, \mathbf{K}^*$  here; note that cl is algebraic by Lemma 16.6.14.(2) so the hypotheses of the theorem hold).

By definition of  $\leq_{\mathbf{K}^*}$  (so really of  $\leq_{\mathbf{K}^0}$ , see Definitions 16.6.9 and 16.6.7), we have that  $\bigcup_{i < \kappa} M_i \not\leq^{\chi^+, \mu^+} M_{\kappa}$ . By definition of  $\leq^{\chi^+, \mu^+}$ , there exists  $\bar{c} \in {}^{<\omega}|M_{\kappa}|$ such that  $q := \operatorname{tp}_{qf}(\bar{c}/\bigcup_{i < \kappa} M_i; M_{\kappa})$  is not  $(\chi^+, \mu^+)$ -averageable over  $\bigcup_{i < \kappa} M_i$  in  $M_{\kappa}$ . For  $\eta \in {}^{\kappa}\lambda$ , let  $\bar{c}_{\eta} := f_{\eta}(\bar{c})$ .

Note that by (1) in Theorem 16.5.40, for all  $\eta \in {}^{\leq \kappa}\lambda$ ,  $||M_{\eta}|| = ||M_{\ell(\eta)}|| \leq LS(\mathbf{K}^*)^+ \leq \lambda$ , so fix  $M \leq_{\mathbf{K}^*} N$  such that  $||M|| = \lambda$  and  $\bigcup_{\eta \in {}^{<\kappa}\lambda} |M_{\eta}| \subseteq |M|$ . For  $\eta \in {}^{\kappa}\lambda$ , let  $p_{\eta} := \operatorname{gtp}_{\mathbf{K}^*}(\bar{c}_{\eta}/M; N)$ .

Because  $\lambda < \lambda^{\kappa}$ , it is enough to prove the following:

<u>Claim</u>: For  $\eta, \nu \in {}^{\kappa}\lambda$ , if  $\eta \neq \nu$ , then  $p_{\eta} \neq p_{\nu}$ .

<u>Proof of claim</u>: Let  $\alpha < \kappa$  be least such that  $\eta \upharpoonright (\alpha + 1) \neq \nu \upharpoonright (\alpha + 1)$ . By (3) in Theorem 16.5.40 and the monotonicity property of  $\overline{\bigcup}$  (see Lemma 16.4.9) we have that  $\bar{c}_{\eta} \underset{M_{\nu \downarrow \alpha}}{\overset{N}{\bigcup}} M_{\nu}$ . By monotonicity again,  $\bar{c}_{\eta} \underset{M_{\nu \downarrow \alpha}}{\overset{N}{\bigcup}} \underset{M_{\nu \uparrow \alpha}}{\overset{N}{\bigcup}} \bigcup_{\beta < \kappa} M_{\nu \restriction \beta}$ . Now assume for a con-

tradiction that  $p_{\eta} = p_{\nu}$ . Then by monotonicity and invariance,  $\bar{c}_{\nu} \bigcup_{M_{\nu \uparrow \alpha}}^{N} \bigcup_{\beta < \kappa} M_{\nu \restriction \beta}$ 

so  $\bar{c}_{\nu} \bigcup_{M_{\nu \restriction \alpha}}^{M_{\nu}} \bigcup_{\beta < \kappa} M_{\nu \restriction \beta}$ . Applying  $f_{\nu}^{-1}$  to this, we get that  $\bar{c} \bigcup_{M_{\alpha}}^{M_{\kappa}} \bigcup_{i < \kappa} M_i$ . In particular, by Lemma (16.6.14).(1), q is  $(\chi^+, \mu^+)$ -averageable over  $M_{\alpha}$  in  $M_{\kappa}$ . By Remark 16.6.6, q is  $(\chi^+, \mu^+)$ -averageable over  $\bigcup_{i < \kappa} M_i$  in  $M_{\kappa}$ . This contradicts the choice of  $\bar{c}$ .  $\dagger_{\text{Claim}}$ .

### 16.7. Categoricity in universal classes

In this section, we derive the main theorem of this chapter. First, we explain why, in a universal class, categoricity (in some  $\lambda > LS(\mathbf{K})$ ) implies failure of the

order property. Note that Shelah argues [She09b, Claim V.B.2.6] that if K has the order property, then it has  $2^{\mu}$ -many models of size  $\mu$  (for any  $\mu > \text{LS}(\mathbf{K})$ ). In particular, this violates categoricity but Shelah's construction of many models is very technical and when categoricity is assumed there is an easier proof. Note that we do not even need to work with Galois types and can use syntactic (first-order) quantifier-free types instead.

LEMMA 16.7.1. Assume that a universal class **K** is categorical in a  $\lambda > LS(\mathbf{K})$ . Then **K** does not have the order property (recall Definition 16.6.1).

PROOF. If **K** does not have arbitrarily large models, then **K** does not have the order property. Now assume that **K** has arbitrarily large models. We can use Ehrenfeucht-Mostowski models and the standard argument (due to Morley, see [**Mor65**, Theorem 3.7]) shows that if  $M \in K_{\lambda}$ ,  $\mu \in [\text{LS}(\mathbf{K}), \lambda)$ , and  $A \subseteq |M|$  is such that  $|A| \leq \mu$ , then M realizes at most  $\mu$ -many first-order syntactic quantifier-free types over A. However if **K** had the order property, we would be able to build a set  $A \subseteq |M|$  with  $|A| \leq \text{LS}(\mathbf{K})$  but with at least  $\text{LS}(\mathbf{K})^+$  (syntactic quantifier-free) types over A realized in M (using Dedekind cuts, see e.g. the proof of Fact 3.5.12). This is a contradiction.

Next, we deduce more structure from categoricity:

THEOREM 16.7.2. Let **K** be a universal class. If **K** is categorical in some  $\lambda \geq \beth_{h(\mathbf{K})}$ , then there exists  $\mathbf{K}^*$  such that:

- (1)  $\mathbf{K}^*$  is an AEC.
- (2)  $\operatorname{LS}(\mathbf{K}) \leq \operatorname{LS}(\mathbf{K}^*) < h(\mathbf{K}).$
- (3) **K** and  $\mathbf{K}^*$  are compatible (recall Definition 16.3.1).
- (4)  $\mathbf{K}^*$  has amalgamation, joint embedding, and arbitrarily large models.
- (5)  $\mathbf{K}^*$  is  $LS(\mathbf{K}^*)$ -tame.

PROOF. Let **K** be a universal class and let  $\lambda \geq \beth_{h(\mathbf{K})}$  be such that **K** is categorical in  $\lambda$ . By Fact 16.2.20, **K** has arbitrarily large models. By Lemma 16.7.1, **K** does not have the order property. By Fact 16.6.3, we can fix  $\chi \in [\mathrm{LS}(\mathbf{K}), h(\mathbf{K}))$  such that **K** does not have the order property of length  $\chi^+$ . Thus Hypothesis 16.6.8 is satisfied, and so Shelah's structure theorem for universal classes (Fact 16.6.10) applies. Let  $\mathbf{K}^*$  be as in Definition 16.6.11. We have to check that it has all the required properties. First,  $\mathbf{K}^*$  is a weak AEC with amalgamation, joint embedding, and arbitrarily large models (Lemma 16.6.12.(1)). Moreover (Lemma 16.6.12.(2)),  $\mathrm{LS}(\mathbf{K}) \leq \mathrm{LS}(\mathbf{K}^*) = \mu^+ = (2^{2^{\chi}})^+ < h(\mathbf{K})$ . Also, **K** and  $\mathbf{K}^*$  are compatible (Lemma 16.6.12.(3)). This takes care of (2), (3), and (4) in the statement of Theorem 16.7.2. Combining Lemma 16.4.14.(2) and Lemma 16.6.14, we obtain that  $\mathbf{K}^*$  is  $\mathrm{LS}(\mathbf{K}^*)$ -tame, so (5) also holds.

It remains to see (1):  $\mathbf{K}^*$  is an AEC, i.e. it satisfies the smoothness axiom. Suppose not. Then by Lemma 16.6.15, there is a small counter-example:  $\mathbf{K}^*$  is not  $(\leq \mathrm{LS}(\mathbf{K}^*), \leq \mathrm{LS}(\mathbf{K}^*)^+)$ -smooth. Let  $\kappa \leq \mathrm{LS}(\mathbf{K}^*)^+$  be least such that  $\mathbf{K}^*$  is not  $(\leq \mathrm{LS}(\mathbf{K}^*), \leq \kappa)$ -smooth. Note that  $\kappa$  is regular. Let  $\lambda_0 := \beth_{\kappa}(\mathrm{LS}(\mathbf{K}^*))$ . Note:

- $\lambda_0 \geq \mathrm{LS}(\mathbf{K}^*)^+$ .
- $\lambda_0 = \lambda_0^{<\kappa}$  and  $\lambda_0 < \lambda_0^{\kappa}$  (because cf  $\lambda_0 = \kappa$ ).
- Since  $\kappa \leq \mathrm{LS}(\mathbf{K}^*) < h(\mathbf{K})$ , we have that  $\lambda_0 \leq \beth_{\mathrm{LS}(\mathbf{K}^*)+\kappa} < \beth_{h(\mathbf{K})} \leq \lambda$ . Similarly,  $\lambda_0^+ < \lambda$ .

By Lemma 16.3.4 (where  $\mathbf{K}^1, \mathbf{K}^2, \mu, \lambda$  there stand for  $\mathbf{K}, \mathbf{K}^*, \lambda_0, \lambda$  here, note that we are using that  $\lambda_0^+ < \lambda$ ),  $\mathbf{K}^*$  is  $(< \omega)$ -stable in  $\lambda_0$ . However Theorem 16.6.16 (where  $\lambda$  there stands for  $\lambda_0$  here) says that  $\mathbf{K}^*$  is  $(< \omega)$ -unstable in  $\lambda_0$ , a contradiction.

Finally, we have all the results we need to prove the main theorem:

THEOREM 16.7.3. Let **K** be a universal class. If **K** is categorical in *some*  $\lambda \geq \beth_{h(\mathbf{K})}$ , then there exists  $\chi < \beth_{h(\mathbf{K})}$  such that **K** is categorical in all  $\lambda' \geq \chi$ . Moreover,  $\mathbf{K}_{>\chi}$  has amalgamation.

PROOF. Let  $\mathbf{K}^*$  be as given by Theorem 16.7.2. In particular,  $\mathbf{K}^*$  is tame and has amalgamation. By Fact 16.2.23,  $\mathbf{K}$  has primes, so we can use Theorem 16.3.8, compatibility, and the categoricity transfer theorem for tame AECs with primes (Fact 16.2.25.(2b)). That is, by Theorem 16.3.9 (where  $\mathbf{K}^1, \mathbf{K}^2$  there stand for  $\mathbf{K}, \mathbf{K}^*$  here),  $\mathbf{K}^*$  is categorical in all  $\lambda' \geq \chi := h(\mathrm{LS}(\mathbf{K}^*))$ . By compatibility (recalling that  $\mathrm{LS}(\mathbf{K}) \leq \mathrm{LS}(\mathbf{K}^*)$ ),  $\mathbf{K}$  is also categorical in all  $\lambda' \geq \chi$ . Finally, since  $\mathrm{LS}(\mathbf{K}^*) < h(\mathbf{K})$ , we have that  $\chi = h(\mathrm{LS}(\mathbf{K}^*)) = \beth_{(2^{\mathrm{LS}(\mathbf{K}^*)})^+} < \beth_{h(\mathbf{K})}$ .

For the moreover part, note that  $\chi^{\text{LS}(\mathbf{K},\mathbf{K}^*)} = \chi^{\mathbf{LS}(\mathbf{K}^*)} = \chi$  so by Lemma 16.3.7,  $\mathbf{K}_{\geq\chi} = \mathbf{K}^*_{\geq\chi}$ . Since the latter has amalgamation, so does the former.

REMARK 16.7.4. In fact,  $\mathbf{K}_{\geq\chi}$  satisfies much more than amalgamation. This is because  $\mathbf{K}_{\geq\chi}$  is a locally universal class (see Definition 8.2.19). Thus it is fully  $\chi$ -tame and short (see Corollary 8.3.8) and admits a global notion of independence (for types over arbitrary sets) that is similar to forking in a first-order superstable theory (see Section 8.8).

PROOF OF THEOREM 16.0.9 AND COROLLARY 16.0.10. Let  $\psi$  be a universal  $\mathbb{L}_{\omega_1,\omega}$  sentence. The class **K** of models of  $\psi$  is a universal class (Fact 16.2.4) with  $h(\mathbf{K}) = \beth_{\omega_1}$  (see Remark 16.2.17 and Fact 16.2.15). Now apply Theorem 16.7.3.  $\Box$ 

REMARK 16.7.5. By Fact 16.2.4 and Remark 16.2.17, Theorem 16.0.9 and Corollary 16.0.10 apply more generally to any universal class in a countable vocabulary.

# CHAPTER 17

# Saturation and solvability in abstract elementary classes with amalgamation

This chapter is based on [Vase].

### Abstract

THEOREM 17.0.6. Let **K** be an abstract elementary class (AEC) with amalgamation and no maximal models. Let  $\lambda > LS(\mathbf{K})$ . If **K** is categorical in  $\lambda$ , then the model of cardinality  $\lambda$  is Galois-saturated.

This answers a question asked independently by Baldwin and Shelah. We deduce several corollaries: **K** has a unique limit model in each cardinal below  $\lambda$ , (when  $\lambda$  is big-enough) **K** is weakly tame below  $\lambda$ , and the thresholds of several existing categoricity transfers can be improved.

We also prove a downward transfer of solvability (a version of superstability introduced by Shelah):

COROLLARY 17.0.7. Let **K** be an AEC with amalgamation and no maximal models. Let  $\lambda > \mu > \text{LS}(\mathbf{K})$ . If **K** is solvable in  $\lambda$ , then **K** is solvable in  $\mu$ .

### 17.1. Introduction

17.1.1. Motivation. Morley's categoricity theorem [Mor65] states that if a countable theory has a unique model of *some* uncountable cardinality, then it has a unique model in *all* uncountable cardinalities. The method of proof led to the development of stability theory, now a central area of model theory. In the mid seventies, Shelah conjectured a similar statement for classes of models of an  $\mathbb{L}_{\omega_1,\omega}$ -theory [She90, Open problem D.3(a)] and more generally for abstract elementary classes (AECs) [She09a, Conjecture N.4.2]<sup>1</sup>. A key step in Morley's proof was to show that the model in the categoricity cardinal is saturated. In this chapter, we lift this step to the framework of AECs which satisfy the amalgamation property and have no maximal models.

In this context, saturation is defined in terms of Galois (orbital types). Shelah [She87b, II.3.10] (see also [Gro02, Theorem 6.7]) has justified this definition by showing that (with the hypotheses of amalgamation and no maximal models, which we make for the remainder of this section) this notion of saturation is equivalent to being model-homogeneous<sup>2</sup> (in particular there can be at most one saturated model of a given cardinality). In a milestone paper, Shelah [She99] has shown that (again

<sup>&</sup>lt;sup>1</sup>see the introduction of Chapter 8 for more on the history of the conjecture.

<sup>&</sup>lt;sup>2</sup>where  $M \in \mathbf{K}$  is model-homogeneous if for every  $M_0, N \in \mathbf{K}$  with  $M_0 \leq_{\mathbf{K}} M, M_0 \leq_{\mathbf{K}} N$ , and ||N|| < ||M||, there is a **K**-embedding  $f : N \xrightarrow[M_0]{} M$ .

assuming amalgamation and no maximal models) a downward analog of Morley's categoricity theorem holds *if* the starting categoricity cardinal is high-enough *and* a successor. One reason<sup>3</sup> for making the successor assumption was exactly to show that the model in the categoricity cardinal was saturated. Indeed, Shelah observes [**She99**, Claim 1.7.(b)] that if **K** is categorical in  $\lambda$  and cf  $\lambda > LS(\mathbf{K})$ , then the model of cardinality  $\lambda$  is cf  $\lambda$ -saturated. In particular, if  $\lambda$  is regular then the model of cardinality  $\lambda$  is saturated. Shelah [**She09a**, Question IV.7.11] and independently Baldwin [**Bal09**, Problem D.1.(2)] have asked whether the model of cardinality  $\lambda$  is singular. The present chapter answers positively (this is Theorem 17.0.6 from the abstract, proven here as Corollary 17.4.11).

17.1.2. Earlier work. Shelah and Villaveces (see Fact 17.4.4) have shown that (regardless of the cofinality of  $\lambda$ ) categoricity in  $\lambda$  implies that a certain local superstability condition (see Definition 17.4.1) holds below  $\lambda$ . In Theorem 4.5.6, we showed that the local superstability condition implies stability (defined in terms of the number of Galois types) in all cardinals if the class **K** is LS(**K**)-tame (a locality property for Galois types introduced by Grossberg and VanDieren [**GV06b**]). Therefore if **K** is LS(**K**)-tame and categorical in  $\lambda > \text{LS}(\mathbf{K})$ , then **K** is stable in  $\lambda$  and hence the model of cardinality  $\lambda$  is saturated. This gives a new proof (even for uncountable theories) of the corresponding first-order fact. However without assuming tameness, we *cannot* in general conclude stability in the categoricity cardinal  $\lambda$  (there is a counterexample due to Hart and Shelah and analyzed in details by Baldwin and Kolesnikov [**HS90, BK09**]), thus different ideas are needed.

Shelah [**She09a**, Theorem IV.7.8] claims that the model of cardinality  $\lambda$  is  $\mu^+$ -saturated (for  $\mu \geq \text{LS}(\mathbf{K})$ ) if  $2^{2^{\mu}} + \aleph_{\mu^{+4}} \leq \lambda$ . We have not fully verified Shelah's proof, which uses PCF theory as well as the theory of Ehrenfeucht-Mostowski (EM) models<sup>4</sup>.

With VanDieren (Corollary 10.7.4), we showed that the model of cardinality  $\lambda$  is  $\mu^+$ -saturated if  $\lambda \geq \beth_{(2^{\mu^+})^+}$ . Assuming the generalized continuum hypothesis (GCH),  $\beth_{(2^{\mu^+})^+} = \aleph_{\mu^{+3}}$  so the bound is better than Shelah's (but if GCH fails badly then Shelah's bound is better). We conclude that if  $\lambda = \beth_{\lambda}$  then the model of cardinality  $\lambda$  is saturated (and assuming Shelah's bound, " $\lambda = \beth_{\lambda}$ " can be replaced by " $\lambda = \aleph_{\lambda}$  and  $\lambda$  is strong limit"). We show here that these hypotheses are not needed (the simplest new case is when  $\mathrm{LS}(\mathbf{K}) = \aleph_0$  and  $\lambda = \aleph_{\omega}$ ).

17.1.3. Description of the proof. The proof uses the symmetry property for splitting first isolated by VanDieren [Van16a]. It follows from an earlier result of VanDieren [Van16b] that if symmetry holds in a successor cardinal  $\mu$  then the model in the categoricity cardinal  $\lambda$  is  $\mu$ -saturated. Further if symmetry in  $\mu$  fails then **K** must satisfy a variant of the order property (defined in terms of Galois types) of length  $\lambda$  (Lemma 10.5.3). It turns out that if the length of this order property is bigger than  $\gamma := \beth_{(2^{\mu})^+}$  then **K** is unstable below  $\lambda$  and this contradicts categoricity. The aforementioned result with VanDieren used that  $\lambda \geq \gamma$ . The key argument of this chapter (Theorem 17.3.4) shows that **K** will have the order property of length  $\gamma$  even when  $\lambda < \gamma$ .

<sup>&</sup>lt;sup>3</sup>but not the only one, see the discussion on  $[\mathbf{BVd}, p. 20]$ .

<sup>&</sup>lt;sup>4</sup>Shelah even claims that it is enough to assume amalgamation and no maximal models in a small subset of cardinal below  $\lambda$ , but we are unable to verify Shelah's claim that "(i) + (iii) suffices" in clause  $(e)(\beta)$  of the theorem.

The main ingredient is a little known result of Shelah [**She99**, Claim 4.15] proving from categoricity in  $\lambda$  that any sequence of length  $\lambda$  contains a *strictly* indiscernible subsequence. Here, indiscernible is as usual defined in terms of Galois types and an indiscernible sequence is strict when (roughly) it can be extended to a longer indiscernible sequence of arbitrary size. For the convenience of the reader (and because Shelah omits several details), we give a full proof of Shelah's claim here (Fact 17.2.5).

17.1.4. Solvability. We can generalize Fact 17.2.5 using a weakening of categoricity called *solvability* (see Definition 17.3.1 here). Solvability was introduced by Shelah in [She09a, Chapter IV] as a possible definition of superstability in the AEC framework (it is equivalent to superstability in the first-order case, see Theorem 9.4.9). Shelah has asked [She09a, Question N.4.4] whether the solvability spectrum satisfies an analog of the categoricity conjecture. Inspired by this question, we showed with Grossberg (Theorem 9.5.4) that the solvability spectrum is either bounded or a tail *provided that* the AEC is tame (and has amalgamation and no maximal models). As an application of the main result of this chapter, we show here *without assuming tameness* (but still using amalgamation and no maximal models) that the solvability spectrum satisfies a *downward* analog of Shelah's categoricity conjecture (this is Corollary 17.0.7 from the abstract proven here as Corollary 17.5.1). Assuming tameness, we can also improve the threshold cardinal of our aforementioned work with Grossberg (Corollary 17.5.3).

17.1.5. Other applications. Other applications of our result can be obtained by taking known theorems that assumed that the model in the categoricity cardinal had some degree of saturation, and removing this saturation assumption from the hypotheses of the theorem! Several consequences are listed in Section 17.5. Especially notable is that uniqueness of limit models<sup>5</sup> holds everywhere below the categoricity cardinal (Corollary 17.5.7.(2). This gives a proof of the (in)famous [**SV99**, Theorem 3.3.7] (where a gap was identified in VanDieren's Ph.D. thesis [**Van02**]) provided that the class has full amalgamation. The original statement assuming only density of amalgamation bases remains open but we also make progress toward it, improving a recent result of VanDieren [**Van**] and fixing a gap of [**Van06**] isolated in [**Van13**]. This is presented in Section 17.5.4.

17.1.6. Notes. The background required to read the core (i.e. the first four sections) of this chapter is only a modest knowledge of AECs (for example Chapters 4 and 8 of [Bal09]) although we rely on (as black boxes) several facts and definitions from the recent literature (especially [Van16a, Van16b] and Chapter 10). To understand some of the applications in Section 17.5, a more solid background (described in the chapters and papers referenced there) may be needed.

### 17.2. Extracting strict indiscernibles

Everywhere in this chapter,  $\mathbf{K}$  denotes a fixed AEC (not necessarily satisfying amalgamation or no maximal models). We assume that the reader is familiar with the definitions of amalgamation, no maximal models, Galois types, and (Galois) saturation. We will use the notation from the preliminaries of Chapter 2.

<sup>&</sup>lt;sup>5</sup>The reader can consult [GVV16] for more background and motivation on limit models.

In particular,  $\operatorname{gtp}(\overline{b}/A; N)$  denotes the Galois types of  $\overline{b}$  over the set A as computed inside N (so we make use of Galois types over sets, defined as for Galois types over models; note also that the definition does not assume amalgamation). We let  $\operatorname{gS}^{\alpha}(A; N)$  denote the set of all Galois types of sequences of length  $\alpha$  over Acomputed in N, and let  $\operatorname{gS}^{\alpha}(M)$  denote the set of all Galois types of sequences of length  $\alpha$  over M (computed in any extension N of M). When  $\alpha = 1$ , we omit it.

When working with EM models, we will use the notation from [She09a, Chapter IV]:

DEFINITION 17.2.1. [She09a, Definition IV.0.8] For  $\mu \geq \text{LS}(\mathbf{K})$ , let  $\Upsilon_{\mu}[\mathbf{K}]$  be the set of  $\Phi$  proper for linear orders (that is,  $\Phi$  is a set  $\{p_n : n < \omega\}$ , where  $p_n$ is an *n*-variable quantifier-free type in a fixed vocabulary  $\tau(\Phi)$  and the types in  $\Phi$ can be used to generate a  $\tau(\Phi)$ -structure  $\text{EM}(I, \Phi)$  for each linear order I; that is,  $\text{EM}(I, \Phi)$  is the closure under the functions of  $\tau(\Phi)$  of the universe of I and for any  $i_0 < \ldots < i_{n-1}$  in  $I, i_0 \ldots i_{n-1}$  realizes  $p_n$ ) with:

- (1)  $|\tau(\Phi)| \le \mu$ .
- (2) If *I* is a linear order of cardinality  $\lambda$ ,  $\operatorname{EM}_{\tau(\mathbf{K})}(I, \Phi) \in \mathbf{K}_{\lambda+|\tau(\Phi)|+\mathrm{LS}(\mathbf{K})}$ , where  $\tau(\mathbf{K})$  is the vocabulary of **K** and  $\operatorname{EM}_{\tau(\mathbf{K})}(I, \Phi)$  denotes the reduct of  $\operatorname{EM}(I, \Phi)$  to  $\tau(\mathbf{K})$ . Here we are implicitly also assuming that  $\tau(\mathbf{K}) \subseteq \tau(\Phi)$ .
- (3) For  $I \subseteq J$  linear orders,  $\text{EM}_{\tau(\mathbf{K})}(I, \Phi) \leq_{\mathbf{K}} \text{EM}_{\tau(\mathbf{K})}(J, \Phi)$ .
- We call  $\Phi$  as above an *EM blueprint*.

The following follows from Shelah's presentation theorem. We will use it without explicit mention.

FACT 17.2.2. Let  $\mu \geq \text{LS}(\mathbf{K})$ . **K** has arbitrarily large models if and only if  $\Upsilon_{\mu}[\mathbf{K}] \neq \emptyset$ .

The next notions (due to Shelah) generalize the concept of an indiscernible sequence in a first-order theory. We prefer not to work inside a monster model (one reason is that some of our application will assume only weak versions of amalgamation, e.g. the Shelah-Villaveces context [**SV99**]), so give more localized definitions here (but assuming a monster model the definitions below coincide with Shelah's).

DEFINITION 17.2.3 (Indiscernibles, Definition 4.1 in [She99]). Let **K** be an AEC. Let  $N \in \mathbf{K}$ . Let  $\alpha$  be a non-zero cardinal,  $\theta$  be an infinite cardinal, and let  $\langle \bar{a}_i : i < \theta \rangle$  be a sequence of distinct elements with  $\bar{a}_i \in {}^{\alpha}|N|$  for all  $i < \theta$ . Let  $A \subseteq |N|$  be a set.

- (1) We say that  $\langle \bar{a}_i : i < \theta \rangle$  is indiscernible over A in N if for every  $n < \omega$ , every  $i_0 < \ldots < i_{n-1} < \theta$ ,  $j_0 < \ldots < j_{n-1} < \theta$ ,  $\operatorname{gtp}(\bar{a}_{i_0} \ldots \bar{a}_{i_n}/A; N) =$  $\operatorname{gtp}(\bar{a}_{j_0} \ldots \bar{a}_{j_n}/A; N)$ . When  $A = \emptyset$ , we omit it and just say that  $\langle \bar{a}_i : i < \theta \rangle$  is indiscernible in N.
- (2) We say that  $\langle \bar{a}_i : i < \theta \rangle$  is strictly indiscernible in N if there exists an EM blueprint  $\Phi$  (whose vocabulary is allowed to have arbitrary size) and a map f so that, letting  $N' := \text{EM}_{\tau(\mathbf{K})}(\theta, \Phi)$ :
  - (a)  $f: N \to N'$  is a **K**-embedding. For  $i < \theta$ , let  $\bar{b}_i := f(\bar{a}_i)$ .
  - (b) If for  $i < \theta$ ,  $\bar{b}_i = \langle b_{i,j} : j < \alpha \rangle$ , then for all  $j < \alpha$  there exists a unary  $\tau(\Phi)$ -function symbol  $\rho_j$  such that for all  $i < \theta$ ,  $b_{i,j} = \rho_i^{N'}(i)$ .
- (3) Let  $A \subseteq |N|$ . We say that  $\langle \bar{a}_i : i < \theta \rangle$  is strictly indiscernible over A in N if there exists an enumeration  $\bar{a}$  of A such that  $\langle \bar{a}_i \bar{a} : i < \theta \rangle$  is strictly indiscernible in N.

Because the compactness theorem is not available, indiscernible sequences may in general fail to extend to arbitrarily length. The point of strict indiscernibles is to correct that defect:

FACT 17.2.4. Assume that  $\mathbf{I} := \langle \bar{a}_i : i < \theta \rangle$  is strictly indiscernible over A in N. Then:

- (1) **I** is indiscernible over A in N.
- (2) For every  $\theta' \geq \theta$ , there exists  $N' \geq_{\mathbf{K}} N$  and  $\langle \bar{a}_i : i \in \theta' \setminus \theta \rangle$  such that  $\langle \bar{a}_i : i < \theta' \rangle$  is strictly indiscernible over A in N'.

PROOF SKETCH.

- (1) Because I is indiscernible (in the first-order sense, in the vocabulary  $\tau(\Phi)$ ) inside EM(I,  $\Phi$ ), and this transfers to Galois types in **K**.
- (2) Use the (first-order) compactness theorem on the theory of the  $\text{EM}(\theta, \Phi)$ , expanded with constant symbols for the sequence witnessing the strict indiscernibility.

The following fact is key to all the subsequent results. It shows that inside an EM model (generated by an ordinal), one can extract a *strictly* indiscernible subsequence from any long-enough sequence. It is due to Shelah and appears as [She99, Claim 4.15]. For the convenience of the reader, we give a full proof.

FACT 17.2.5. Let **K** be an AEC with arbitrarily large models and let  $LS(\mathbf{K}) < \theta \leq \lambda$  be cardinals with  $\theta$  regular. Let  $\kappa < \theta$  be a (possibly finite) cardinal. Let  $\Phi \in \Upsilon_{LS(\mathbf{K})}[\mathbf{K}]$  be an EM blueprint for **K**.

Let  $N := \mathrm{EM}_{\tau(\mathbf{K})}(\lambda, \Phi)$ . Let  $M \in \mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$  be such that  $M \leq_{\mathbf{K}} N$ . Let  $\langle \bar{a}_i : i < \theta \rangle$  be a sequence of distinct elements such that for all  $i < \theta$ ,  $\bar{a}_i \in {}^{\kappa}|N|$ .

If  $\theta_0^{\kappa} < \theta$  for all  $\theta_0 < \theta$ , then there exists  $w \subseteq \theta$  with  $|w| = \theta$  such that  $\langle \bar{a}_i : i \in w \rangle$  is strictly indiscernible over M in N.

REMARK 17.2.6. We do *not* assume amalgamation (we will work entirely inside  $\text{EM}_{\tau(\mathbf{K})}(\lambda, \Phi)$ ).

REMARK 17.2.7. The main case for us is  $\kappa < \aleph_0$ , where the cardinal arithmetic assumption holds trivially and the proof is simpler.

REMARK 17.2.8. We are assuming that  $||M|| \leq \text{LS}(\mathbf{K})$  only to simplify the notation: if  $\mu := ||M|| \in (\text{LS}(\mathbf{K}), \theta)$ , we can just replace  $\mathbf{K}$  by  $\mathbf{K}_{\geq \mu}$ .

PROOF OF FACT 17.2.5. First we claim that one can assume without loss of generality that  $\kappa < \text{LS}(\mathbf{K})$ . Assume that the statement of the lemma has been proven for that case. If  $\kappa > \text{LS}(\mathbf{K})$  one can replace  $\mathbf{K}$  with  $\mathbf{K}_{\geq\kappa}$  (and increase M) so assume that  $\kappa \leq \text{LS}(\mathbf{K})$ . Now if  $\kappa = \text{LS}(\mathbf{K})$ , then  $2^{\text{LS}(\mathbf{K})} = \kappa^{\kappa} < \theta$  so we can replace  $\mathbf{K}$  by  $\mathbf{K}_{\geq \text{LS}(\mathbf{K})^+}$  and work there. Thus assume without loss of generality that  $\kappa < \text{LS}(\mathbf{K})$ .

Pick  $u \subseteq \lambda$  such that  $|u| = \theta$ ,  $M \leq_{\mathbf{K}} N_0 := \mathrm{EM}_{\tau(\mathbf{K})}(u, \Phi)$ , and  $\bar{a}_i \in {}^{\kappa}|N_0|$  for all  $i < \theta$ . Increasing M if necessary, we can assume without loss of generality that  $M = \mathrm{EM}_{\tau(\mathbf{K})}(u', \Phi)$  for some  $u' \subseteq u$  with  $|u'| = \mathrm{LS}(\mathbf{K})$ .

For each  $i < \theta$ , we can also pick  $u_i \subseteq u$  with  $|u_i| < \kappa^+ + \aleph_0$  such that  $\bar{a}_i \in \kappa | \operatorname{EM}_{\tau(\mathbf{K})}(u_i, \Phi)|$ . Without loss of generality  $u = u' \cup \bigcup_{i < \theta} u_i$ . By the pigeonhole principle, we can without loss of generality fix an ordinal  $\alpha < \kappa^+ + \aleph_0$  such that

gtp $(u_i) = \alpha$  for all  $i < \theta$ . List  $u_i$  in increasing order as  $\bar{u}_i := \langle u_{i,j} : j < \alpha \rangle$ . By pruning further (using that  $\mathrm{LS}(\mathbf{K})^{\kappa} < \theta$ ), we can assume without loss of generality that for each  $i, i' < \theta$  and  $j < \alpha$ , the u'-cut of  $u_{i,j}$  and  $u_{i',j}$  are the same (i.e. for any  $\gamma \in u', \gamma < u_{i,j}$  if and only if  $\gamma < u_{i',j}$ ).

Pruning again with the  $\Delta$ -system lemma, we can assume without loss of generality that  $\langle u_i : i < \theta \rangle$  forms a  $\Delta$ -system (see Definition II.1.4 and Theorem II.1.6 in [**Kun80**]; at that point we are using that  $\theta_0^{\kappa} < \theta$  for all  $\theta_0 < \theta$ ). All these pruning steps ensure that  $\langle \bar{u}_i : i < \theta \rangle$  is indiscernible over u' in the vocabulary of linear orders.

Now list  $\bar{a}_i$  as  $\langle a_{i,j} : j < \kappa \rangle$ . Fix  $i < \theta$ . Since  $\bar{a}_i \in {}^{\kappa} | \operatorname{EM}_{\tau(\mathbf{K})}(u_i)|$ , for each  $j < \kappa$  there exists a  $\tau(\Phi)$ -term  $\rho_{i,j}$  of arity  $n := n_{i,j}$  and  $j_0^{i,j} < \ldots < j_{n-1}^{i,j} < \alpha$  such that  $a_{i,j} = \rho_{i,j} \left( u_{i,j_0^{i,j}} \ldots u_{i,j_{n-1}^{i,j}} \right)$ . By the pigeonhole principle applied to the map  $i \mapsto \langle (\rho_{i,j}, n_{i,j}, j_0^{i,j}, \ldots, j_{n_{i,j}-1}^{i,j}) : j < \kappa \rangle$  (using that  $\operatorname{LS}(\mathbf{K})^{\kappa} < \theta$ ), we can assume without loss of generality that these depend only on j, i.e.  $\rho_{i,j} = \rho_j$ ,  $n_{i,j} = n_j$ , and  $j_{\ell}^{i,j} = j_{\ell}^j$ .

Let  $\bar{u}'$  be an enumeration of u', and let  $\bar{a}'_i := \bar{a}_i \bar{u}'$ . We are assuming that  $\kappa < \mathrm{LS}(\mathbf{K})$  so  $\ell(\bar{a}'_i) < \mathrm{LS}(\mathbf{K})$ . Let  $\bar{b}_i$  be  $\bar{a}'_i$  followed by  $a_{i,0}$  repeated  $\mathrm{LS}(\mathbf{K})$ -many times (we only do this to make the order type of each element of our sequence  $\mathrm{LS}(\mathbf{K})$  and hence simplify the notation). Then  $\ell(\bar{b}_i) = \mathrm{LS}(\mathbf{K})$ . As before, say  $\bar{b}_i = \langle b_{i,j} : j < \mathrm{LS}(\mathbf{K}) \rangle$ . Let  $u'_i := u_i \cup u'$ . Let  $\bar{u}'_i$  be an enumeration of  $u'_i$  of type  $\mathrm{LS}(\mathbf{K})$  (so not necessarily increasing). Say  $\bar{u}'_i = \langle u'_{i,j} : j < \mathrm{LS}(\mathbf{K}) \rangle$ .

By the pruning carried out previously and the definition of u', we have that for each  $i < \theta$  and each  $j < \text{LS}(\mathbf{K})$ , there exists a  $\tau(\Phi)$ -term  $\rho_j$  of arity  $n_j$  and  $j_0^j < \ldots < j_{n-1}^j < \text{LS}(\mathbf{K})$  (the point is that they do *not* depend on i) such that  $b_{i,j} = \rho_j(u'_{i,j_0^j}, \ldots, u'_{i,j_{n_j-1}})$ . We will build an EM blueprint  $\Psi$  witnessing that  $\langle \bar{b}_i : i < \theta \rangle$  is strictly indiscernible.

For each *n*-ary  $\tau(\Phi)$ -term  $\rho$ , each  $\gamma_0 < \ldots < \gamma_{n-1} < \mathrm{LS}(\mathbf{K})$ , and each  $i_0 < \ldots < i_{n-1} < \theta$ , we define a function  $g = g_{\rho,\gamma_0\ldots\gamma_{n-1},i_0\ldots i_{n-1}}$  as follows: for  $i < \theta$ ,  $j < \mathrm{LS}(\mathbf{K})$ , let  $g(u'_{i,j}) := \rho(u'_{i+i_0,j+\gamma_0},\ldots,u'_{i+i_{n-1},j+\gamma_{n-1}})$ . We naturally extend g to have domain  $N_0 = \mathrm{EM}(u, \Phi)$  (recall from the beginning of the proof that  $u = u' \cup \bigcup_{i < \theta} u_i$ ). The vocabulary of  $\Psi$  will consist of the vocabulary of  $\tau(\Phi)$  together with a unary function symbol for each  $g_{\rho,\gamma_0\ldots\gamma_{n-1},i_0\ldots i_{n-1}}$ . For  $n < \omega$ , let  $p_n := \mathrm{tp}_{\tau(\Psi)}(u'_{0,0}u'_{1,0}\ldots u'_{n-1,0}/\emptyset; N_0)$  and let  $\Psi := \{p_n : n < \omega\}$ . Then  $\Psi$  is as desired.

We will use this fact to study lengths of the (Galois) order property (recall Definition 2.4.3). An easy consequence of Fact 17.2.5 is that if a long-enough order property holds, then we can assume that the sequence witnessing it is strictly indiscernible, and hence extend it:

THEOREM 17.2.9. Let **K** be an AEC with arbitrarily large models and let  $LS(\mathbf{K}) < \lambda$ . Let  $\kappa < \lambda$  be a (possibly finite) cardinal. Let  $\Phi \in \Upsilon_{LS(\mathbf{K})}[\mathbf{K}]$  be an EM blueprint for **K**.

Let  $N := \text{EM}_{\tau(\mathbf{K})}(\lambda, \Phi)$ . If N has the  $(\kappa, \text{LS}(\mathbf{K}))$ -order property of length  $(\text{LS}(\mathbf{K})^{\kappa})^+$  and  $\text{LS}(\mathbf{K})^{\kappa} < \lambda$ , then **K** has the  $(\kappa, \text{LS}(\mathbf{K}))$ -order property (of *any* length).

PROOF. Set  $\theta := (\mathrm{LS}(\mathbf{K})^{\kappa})^+$ . Fix  $\langle \bar{a}_i : i < \theta \rangle$  and A witnessing that N has the  $(\kappa, \mathrm{LS}(\mathbf{K}))$ -order property of length  $\theta$ . Using the Löwenheim-Skolem-Tarski axiom, pick  $M \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  such that  $A \subseteq |M|$  and  $M \leq_{\mathbf{K}} N$ . By Fact 17.2.5, there exists  $w \subseteq \theta$  such that  $|w| = \theta$  and  $\langle \bar{a}_i : i \in w \rangle$  is strictly indiscernible over M in N. Without loss of generality,  $w = \theta$ . Let  $\theta' \geq \theta$  be an arbitrary cardinal. By Fact 17.2.4.(2), we can find  $N' \geq_{\mathbf{K}} N$  and  $\langle \bar{a}_i : i \in \theta' \setminus \theta \rangle$  such that  $\mathbf{I} := \langle \bar{a}_i : i < \theta' \rangle$  is strictly indiscernible over M in N'. By Fact 17.2.4.(1),  $\mathbf{I}$  is indiscernible over M in N'. We claim that  $\mathbf{I}$  witnesses that N' has the  $(\kappa, \mathrm{LS}(\mathbf{K}))$ -order property of length  $\theta'$ . Indeed, if  $i_0 < i_1 < \theta', j_0 < j_1 < \theta'$ , then by indiscernibility,  $p := \mathrm{gtp}(\bar{a}_{i_0}\bar{a}_{i_1}/M; N') = \mathrm{gtp}(\bar{a}_0\bar{a}_1/M; N')$  and  $q := \mathrm{gtp}(\bar{a}_{j_1}\bar{a}_{j_0}/M; N') =$  $\mathrm{gtp}(\bar{a}_1\bar{a}_0/M; N')$ . Because the original sequence  $\langle \bar{a}_i : i < \theta \rangle$  was witnessing the  $(\kappa, \mathrm{LS}(\mathbf{K}))$ -order property, we have that  $p \neq q$ , as desired.  $\Box$ 

REMARK 17.2.10. By appending an enumeration of the base set to each element of the sequence, we get that the  $(\kappa, \mu)$ -order property implies the  $(\kappa + \mu)$ -order property. However Theorem 17.2.9 applies more easily to the  $(\kappa, \mu)$ -order property: think for example of the case  $\kappa < \aleph_0$ , when we always have that  $\mathrm{LS}(\mathbf{K})^{\kappa} = \mathrm{LS}(\mathbf{K}) < \lambda$ .

### 17.3. Solvability and failure of the order property

We recall Shelah's definition of solvability [She09a, Definition IV.1.4], and mention a more convenient notation for it with only one cardinal parameter. We also introduce semisolvability, which only asks for the EM model to be universal (instead of superlimit). Both variations are equivalent to superstability in the firstorder case (see Corollary 9.5.3). Shelah writes that solvability is perhaps the true analog of superstability in abstract elementary classes [She09a, N§4(B)].

DEFINITION 17.3.1. Let  $LS(\mathbf{K}) \leq \mu \leq \lambda$ .

- (1)  $M \in \mathbf{K}$  is universal in  $\lambda$  if  $M \in \mathbf{K}_{\lambda}$  and for any  $N \in \mathbf{K}_{\lambda}$  there exists  $f: N \to M$ .
- (2) [She09a, Definition IV.0.5]  $M \in \mathbf{K}$  is superlimit in  $\lambda$  if:
  - (a) M is universal in  $\lambda$ .
  - (b) M has a proper extension.
  - (c) For any limit ordinal  $\delta < \lambda^+$  and any increasing continuous chain  $\langle M_i : i \leq \delta \rangle$  in  $\mathbf{K}_{\lambda}$ , if  $M \cong M_i$  for all  $i < \delta$ , then  $M \cong M_{\delta}$ .
- (3) [She09a, Definition IV.1.4.(1)] We say that  $\Phi$  witnesses  $(\lambda, \mu)$ -solvability if:
  - (a)  $\Phi \in \Upsilon_{\mu}[\mathbf{K}]$ .
  - (b) If I is a linear order of size  $\lambda$ , then  $\text{EM}_{\tau(\mathbf{K})}(I, \Phi)$  is superlimit in  $\lambda$ .
- (4)  $\Phi$  witnesses  $(\lambda, \mu)$ -semisolvability if:
  - (a)  $\Phi \in \Upsilon_{\mu}[\mathbf{K}]$
  - (b) If I is a linear order of size  $\lambda$ , then  $\text{EM}_{\tau(\mathbf{K})}(I, \Phi)$  is universal in  $\lambda$ .
- (5) **K** is  $(\lambda, \mu)$ -[semi]solvable if there exists  $\Phi$  witnessing  $(\lambda, \mu)$ -[semi]solvability.
- (6) **K** is  $\lambda$ -[semi]solvable (or [semi]solvable in  $\lambda$ ) if **K** is ( $\lambda$ , LS(**K**))-[semi]solvable.

REMARK 17.3.2. By a straightforward argument (similar to the proof of Corollary 17.4.11.(2) here), superlimit models must be unique.

REMARK 17.3.3. Because superlimit models are universal, if **K** is  $(\lambda, \mu)$ -solvable, then **K** is  $(\lambda, \mu)$ -semisolvable. Also, the model in a categoricity cardinal must be

superlimit (if it has a proper extension), so if **K** has arbitrarily large models and is categorical in  $\lambda \geq LS(\mathbf{K})$ , then **K** is solvable in  $\lambda$ .

The reader not especially interested in solvability can simply remember the last remark and read "categorical" instead of "solvable" whenever appropriate.

We can combine Theorem 17.2.9 with semisolvability (we are still not assuming amalgamation):

THEOREM 17.3.4. Let  $\lambda > \text{LS}(\mathbf{K})$ . If **K** is semisolvable in  $\lambda$ , then for any cardinals  $\mu, \kappa$ , and any  $M \in \mathbf{K}_{\lambda}$ , M does not have the  $(\kappa, \mu)$ -order property of length  $((\mu + \text{LS}(\mathbf{K}))^{\kappa})^+$ .

REMARK 17.3.5. The statement is interesting only when  $(\mu + \text{LS}(\mathbf{K}))^{\kappa} < \lambda$ , but is still vacuously true otherwise (no M of size  $\lambda$  can witness an order property of length longer than  $\lambda$ ). The main case for us is  $\kappa < \aleph_0, \mu \in [\text{LS}(\mathbf{K}), \lambda)$ , where the result tells us that the  $(\kappa, \mu)$  order property of length  $\mu^+$  must fail.

REMARK 17.3.6. A similar result is [She09a, Claim IV.1.5.(2)]. There the conclusion is weaker (only the  $(\kappa, \mu)$ -order property fails, nothing is said about the length), and the hypothesis uses solvability instead of semisolvability. The proof relies on Shelah's construction of many models from the order property.

PROOF OF THEOREM 17.3.4. Replacing  $\mu$  by  $(\mu + \mathrm{LS}(\mathbf{K}))^{\kappa}$  if necessary, we can assume without loss of generality that  $\mu \geq \mathrm{LS}(\mathbf{K})$  and  $\mu = \mu^{\kappa}$ . If  $\mu \geq \lambda$  the result is vacuously true (see Remark 17.3.5) so assume without loss of generality that  $\mu < \lambda$ . Replacing  $\mathbf{K}$  by  $\mathbf{K}_{\geq \mu}$  if necessary, we can also assume without loss of generality that  $\mu = \mathrm{LS}(\mathbf{K})$ . Fix  $M \in \mathbf{K}_{\lambda}$  and assume for a contradiction that M has the  $(\kappa, \mu)$ -order property of length  $\mu^+$ .

Let  $\Phi$  be an EM blueprint witnessing semisolvability. By definition of semisolvability, we can embed M inside  $\mathrm{EM}_{\tau(\mathbf{K})}(\lambda, \Phi)$ , hence  $\mathrm{EM}_{\tau(\mathbf{K})}(\lambda, \Phi)$  has the  $(\kappa, \mu)$ -order property of length  $\mu^+$ . So assume without loss of generality that  $M = \mathrm{EM}_{\tau(\mathbf{K})}(\lambda, \Phi)$ . By Theorem 17.2.9, **K** has the  $(\kappa, \mu)$ -order property. As in the proof of Fact 3.5.12 (first observed by Shelah [**She99**, Claim 4.7.(2)]), we can build a linear order I of cardinality  $\lambda$  and a set  $A \subseteq N := \mathrm{EM}_{\tau(\mathbf{K})}(I, \Phi)$  such that  $|A| = \mu$  but  $|\mathbf{gS}^{\kappa}(A; N)| > \mu$ .

Now by an argument of Morley [Mor65, Theorem 3.7] (similar to the pruning done in the proof of Fact 17.2.5), for any  $A \subseteq |M|$ , if  $|A| = \mu$ , then  $|gS^{\kappa}(A; M)| = \mu$ . This is a contradiction because N embeds inside M (recall that M is universal by the semisolvability assumption).

#### 17.4. Solvability and saturation

In this section, we prove Theorem 17.0.6 from the abstract (the model in the categoricity cardinal is saturated). We will rely on the following local version of superstability, already implicit in [SV99] and since then studied in many papers, e.g. [Van06, GVV16, Van16a], Chapters 6, 7, 9. We quote Definition 6.10.1:

DEFINITION 17.4.1. **K** is  $\mu$ -superstable (or superstable in  $\mu$ ) if:

- (1)  $\mu \geq \text{LS}(\mathbf{K})$ .
- (2)  $\mathbf{K}_{\mu}$  is nonempty, has joint embedding, amalgamation, and no maximal models.
- (3) **K** is stable in  $\mu$ .

(4) There are no long splitting chains in  $\mu$ :

For any limit ordinal  $\delta < \mu^+$ , for every sequence  $\langle M_i \mid i < \delta \rangle$  of models of cardinality  $\mu$  with  $M_{i+1}$  universal over  $M_i$  and for every  $p \in$  $gS(\bigcup_{i < \delta} M_i)$ , there exists  $i < \delta$  such that p does not  $\mu$ -split over  $M_i$ .

We will also use the concept of symmetry for splitting isolated in [Van16a]:

DEFINITION 17.4.2. For  $\mu \geq \text{LS}(\mathbf{K})$ , we say that  $\mathbf{K}$  has  $\mu$ -symmetry (or symmetry in  $\mu$ ) if whenever models  $M, M_0, N \in \mathbf{K}_{\mu}$  and elements a and b satisfy the conditions (1)-(4) below, then there exists  $M^b$  a limit model over  $M_0$ , containing b, so that  $\text{gtp}(a/M^b)$  does not  $\mu$ -split over N.

- (1) M is universal over  $M_0$  and  $M_0$  is a limit model over N.
- (2)  $a \in |M| \setminus |M_0|$ .
- (3)  $gtp(a/M_0)$  is non-algebraic and does not  $\mu$ -split over N.
- (4) gtp(b/M) is non-algebraic and does not  $\mu$ -split over  $M_0$ .

REMARK 17.4.3. We will only use the consequences of Definitions 17.4.1 and 17.4.2, not their exact content.

By an argument of Shelah and Villaveces [**SV99**, Theorem 2.2.1] (see also Chapter 20), superstability holds below the categoricity (or just semisolvability) cardinal:

FACT 17.4.4 (The Shelah-Villaveces Theorem). Let  $\lambda > LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\lambda}$  has amalgamation and no maximal models. If  $\mathbf{K}$  is semisolvable in  $\lambda$ , then  $\mathbf{K}$  is superstable in any  $\mu \in [LS(\mathbf{K}), \lambda)$ .

REMARK 17.4.5. Here and below, we are assuming amalgamation and no maximal models but only (strictly) below  $\lambda$ . In at least one case ( $\lambda \geq \beth_{(2^{\chi})^+}$  where  $\chi > \text{LS}(\mathbf{K})$  is a measurable cardinal [**SK96, Bon14b**]), these assumptions are known to follow (inside  $\mathbf{K}_{\geq\chi}$  for the example just mentioned) from categoricity in  $\lambda$  but they are *not* known to hold above  $\lambda$ .

It is also known that failure of symmetry implies the order property. The proof of Lemma 10.5.3 gives:

FACT 17.4.6. Let  $\lambda > \mu \geq LS(\mathbf{K})$ . Assume that **K** is superstable in every  $\chi \in [\mu, \lambda)$ . If **K** does *not* have  $\mu$ -symmetry, then **K** has the  $(2, \mu)$ -order property of length  $\lambda$  (recall Definition 2.4.3).

REMARK 17.4.7. We have *not* explicitly assumed amalgamation and no maximal models, as this is implied (at the relevant cardinals) by the definition of superstability.

We conclude that  $\mu$ -symmetry follows from categoricity (or just semisolvability) in some  $\lambda > \mu$ . This improves on Corollary 10.7.2 which asked for the model of cardinality  $\lambda$  to be  $\mu^+$ -saturated (we will see next that this saturation also follows).

COROLLARY 17.4.8. Let  $\lambda > LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\lambda}$  has amalgamation and no maximal models. If **K** is semisolvable in  $\lambda$ , then for any  $\mu \in [LS(\mathbf{K}), \lambda)$ , **K** has  $\mu$ -symmetry.

PROOF. By Fact 17.4.4, **K** is superstable in any  $\mu \in [LS(\mathbf{K}), \lambda)$ . Fix such a  $\mu$ . Suppose for a contradiction that **K** does not have  $\mu$ -symmetry. By Fact 17.4.6, **K** has the  $(2, \mu)$ -order property of length  $\lambda$ . In particular, **K** has the  $(2, \mu)$ -order

property of length  $\mu^+$ . This contradicts Theorem 17.3.4 (where  $\kappa$  there stands for 2 here).

We will make strong use of the relationship between symmetry and chains of saturated models (due to VanDieren):

FACT 17.4.9 (Theorem 1 in [Van16b]). If K is  $\mu$ -superstable,  $\mu^+$ -superstable, and has  $\mu^+$ -symmetry, then the union of any increasing chain of  $\mu^+$ -saturated models is  $\mu^+$ -saturated.

We have arrived to Theorem 17.0.6 from the abstract. We first prove a lemma:

LEMMA 17.4.10. Let  $\lambda > LS(\mathbf{K})$ . If for every  $\mu \in [LS(\mathbf{K}), \lambda)$ ,  $\mathbf{K}$  is  $\mu$ -superstable and has  $\mu$ -symmetry, then  $\mathbf{K}$  has a saturated model of cardinality  $\lambda$ .

PROOF. If  $\lambda$  is a successor, then we can build the desired model using stability below  $\lambda$ , so assume that  $\lambda$  is limit.

Let  $\delta := \operatorname{cf} \lambda$ . Fix an increasing sequence  $\langle \mu_i : i < \delta \rangle$  cofinal in  $\lambda$  such that  $\operatorname{LS}(\mathbf{K}) \leq \mu_0$ . We build an increasing chain  $\langle M_i : i < \delta \rangle$  in  $\mathbf{K}_{\lambda}$  such that for all  $i < \delta$ ,  $M_i$  is  $\mu_i^+$ -saturated. This is enough since then it is easy to check that  $\bigcup_{i < \delta} M_i$  is saturated. This is possible: Using Fact 17.4.9, for any  $i < \delta$ , any union of an increasing chain of  $\mu_i^+$ -saturated models is  $\mu_i^+$ -saturated (note that  $\mu_i^+ < \lambda$  as  $\lambda$  is limit). Thus it is straightforward to carry out the construction.

COROLLARY 17.4.11. Let  $\lambda > LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\lambda}$  has amalgamation and no maximal models.

- (1) If **K** is semisolvable in  $\lambda$ , then **K** has a saturated model of cardinality  $\lambda$ .
- (2) If  $\Phi$  is an EM blueprint witnessing that **K** is solvable in  $\lambda$ , then for any linear order I of cardinality  $\lambda$ ,  $\text{EM}_{\tau(\mathbf{K})}(I, \Phi)$  is saturated.
- (3) If **K** has arbitrarily large models and is categorical in  $\lambda$ , then the model of cardinality  $\lambda$  is saturated.

Proof.

- (1) By Fact 17.4.4 and Corollary 17.4.8, **K** is superstable and has symmetry in any  $\mu \in [LS(\mathbf{K}), \lambda)$ . Now apply Lemma 17.4.10.
- (2) We show more generally that if **K** is semisolvable in  $\lambda$  and M is superlimit in  $\lambda$ , then M is saturated. We build increasing continuous chains  $\langle M_i : i \leq \lambda \rangle$ ,  $\langle N_i : i \leq \lambda \rangle$  in  $\mathbf{K}_{\lambda}$  such that for any  $i < \lambda$ :
  - (a)  $M_i \cong M$ .
  - (b)  $M_i \leq_{\mathbf{K}} N_i \leq_{\mathbf{K}} M_{i+1}$ .
  - (c)  $N_{i+1}$  is saturated.

This is possible by the first part (noting that the saturated model must be universal). This is enough: because M is superlimit,  $M \cong M_{\lambda} = N_{\lambda}$ . Further,  $N_{\lambda}$  must be saturated: if  $\lambda$  is a successor this is clear and if  $\lambda$ is limit this is because for any  $\mu < \lambda$  the union of any increasing chain of  $\mu$ -saturated models is  $\mu$ -saturated. Since  $N_{\lambda}$  is saturated, M is also saturated, as desired.

(3) By Remark 17.3.3, **K** is solvable in  $\lambda$ , so apply the previous part.

#### 17.5. Applications

17.5.1. Solvability transfers. We can now prove Corollary 17.0.7 from the abstract.

COROLLARY 17.5.1 (Downward solvability transfer). Let  $\lambda > LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\lambda}$  has amalgamation and no maximal models. If  $\mathbf{K}$  is solvable in  $\lambda$ , then there exists an EM blueprint  $\Psi$  which witnesses that  $\mathbf{K}$  is solvable in  $\mu$  for any  $\mu \in (LS(\mathbf{K}), \lambda]$ .

PROOF. Let  $\Phi$  be an EM blueprint witnessing that **K** is solvable in  $\lambda$ . By Corollary 17.4.11, EM<sub> $\tau(\mathbf{K})$ </sub>( $J, \Phi$ ) is saturated for any linear order J of cardinality  $\lambda$ . We now use [**She99**, Subfact 6.8] (a full proof is given in [**Shea**], the online version of [**She99**]). It says that there exists an EM blueprint  $\Psi \in \Upsilon_{\mathbf{LS}(\mathbf{K})}[\mathbf{K}]$  such that:

- (1) For any linear order I there exists a linear order J with,  $\text{EM}_{\tau(\mathbf{K})}(I, \Psi) = \text{EM}_{\tau(\mathbf{K})}(J, \Phi)$ . In particular,  $\Psi$  still witnesses that  $\mathbf{K}$  is solvable in  $\lambda$ .
- (2) For any  $\mu \in (LS(\mathbf{K}), \lambda]$  and any linear order I of cardinality  $\mu, EM_{\tau(\mathbf{K})}(I, \Psi)$  is saturated.

By Fact 17.4.4 and Corollary 17.4.8, **K** is superstable and has symmetry in every  $\mu \in [\text{LS}(\mathbf{K}), \lambda)$ . Now let  $\mu \in (\text{LS}(\mathbf{K}), \lambda]$ . We want to see that  $\Psi$  witnesses solvability in  $\mu$ . By the above,  $\Psi$  witnesses solvability in  $\lambda$ , so assume that  $\mu < \lambda$ . Using Fact 17.4.9 it is straightforward to see that the union of any increasing chain of  $\mu$ -saturated models will be  $\mu$ -saturated. In other words, the saturated model of cardinality  $\mu$  is superlimit and therefore  $\Psi$  witnesses that **K** is solvable in  $\mu$ .  $\Box$ 

REMARK 17.5.2. It is natural to ask what happens if  $\mu = \text{LS}(\mathbf{K})$ . In that case, if  $\Psi$  witnesses solvability in  $\text{LS}(\mathbf{K})^+$  we can find a linear order J of size  $\text{LS}(\mathbf{K})$ such that  $\text{EM}_{\tau(\mathbf{K})}(J,\Psi)$  is limit (see the proof of [She99, Lemma I.6.3]). This implies that  $\text{EM}_{\tau(\mathbf{K})}(I \times J, \Psi)$  is limit for any linear order I of size at most  $\text{LS}(\mathbf{K})$ (here  $I \times J$  is ordered with the lexicographical ordering). The class of linear orders of the form  $I \times J$  is an AEC with arbitrarily large models and hence has an EM blueprint. Composing this blueprint with  $\Psi$ , we can find a blueprint  $\Psi'$  such that  $\text{EM}_{\tau(\mathbf{K})}(I, \Psi') = \text{EM}_{\tau(\mathbf{K})}(I \times J, \Psi)$  for any linear order I. In particular,  $\Psi'$  also witnesses solvability in ( $\text{LS}(\mathbf{K}), \lambda$ ]. Moreover,  $\text{EM}_{\tau(\mathbf{K})}(I, \Psi')$  is limit for any linear order I of cardinality  $\text{LS}(\mathbf{K})$ . This implies that  $\Psi'$  witnesses *semisolvability* in  $\text{LS}(\mathbf{K})$ , but it is not clear that the limit model is superlimit (even though it is unique), see Question 10.6.12. Therefore we do not know if  $\Psi'$  witnesses solvability in  $\text{LS}(\mathbf{K})$ , but it will if there is any superlimit in  $\text{LS}(\mathbf{K})$ .

Assuming tameness, we can also get an upward transfer. Note that here only semisolvability is assumed so also the downward part of Corollary 17.5.3 is interesting.

COROLLARY 17.5.3. Assume that **K** is LS(**K**)-tame and has amalgamation and no maximal models. Write  $\mu_0 := (\beth_{\omega+2}(\text{LS}(\mathbf{K})))^+$ . If **K** is semisolvable in  $\lambda$  for some  $\lambda > 2^{\text{LS}(\mathbf{K})}$ , then **K** is  $(\mu, \mu_0)$ -solvable for all  $\mu \ge \mu_0$ .

REMARK 17.5.4. This improves on the threshold from Theorem 9.5.4: there  $\mu_0$  was around  $\beth_{(2^{\text{LS}(\mathbf{K})})^+}$ . We quote freely from there in the proof.

REMARK 17.5.5. In the conclusion, the same blueprint will witness  $(\mu, \mu_0)$ -solvability for all  $\mu$ .

PROOF OF COROLLARY 17.5.3. In the proof of Theorem 9.5.4, the only reason for the threshold to be around  $\beth_{(2^{LS}(\mathbf{K}))^+}$  was a bound on a cardinal  $\chi_0$  so that **K** does not have the LS(**K**)-order property of length  $\chi_0$ . Now using Theorem 17.3.4, we get that **K** does not have the LS(**K**)-order property of length  $(2^{LS(\mathbf{K})})^+$ . Following Section 9.4, we obtain that **K** is  $(\mu, \mu_0)$ -solvable for all  $\mu \ge \mu_0$ .

17.5.2. Structure of categorical AECs with amalgamation. Directly from existing results and Corollary 17.4.11, we obtain a good understanding of the structure below the categoricity cardinal of an AEC with amalgamation. For the convenience of the reader, we have added a few statements that we have already proven. We quote freely and refer the reader to the sources for more motivation on the results. We will use the following notation from [Bal09, Chapter 14]:

NOTATION 17.5.6.  $H_1 := \beth_{(2^{LS(\mathbf{K})})^+}$ .

COROLLARY 17.5.7. Let  $\lambda > LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\lambda}$  has amalgamation and no maximal models. If  $\mathbf{K}$  is semisolvable in  $\lambda$ , then:

- (1) For any  $\mu \in [LS(\mathbf{K}), \lambda)$ , **K** is  $\mu$ -superstable and has  $\mu$ -symmetry.
- (2) For any  $\mu \in [LS(\mathbf{K}), \lambda)$ , any  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$ , if  $M_1$  and  $M_2$  are limit over  $M_0$ , then  $M_1 \cong_{M_0} M_2$ .
- (3) For any  $\mu \in [LS(\mathbf{K}), \lambda)$ , the union of any increasing chain of  $\mu$ -saturated models is  $\mu$ -saturated.
- (4) If **K** is solvable in  $\lambda$ , then there exists an EM blueprint  $\Psi \in \Upsilon_{\mathrm{LS}(\mathbf{K})}[\mathbf{K}]$  such that  $\mathrm{EM}_{\tau(\mathbf{K})}(I, \Psi)$  is saturated for any linear order I of cardinality in  $(\mathrm{LS}(\mathbf{K}), \lambda]$ .
- (5) If **K** is solvable in  $\lambda$  and either cf  $\lambda > \text{LS}(\mathbf{K})$  or  $\lambda \ge H_1$ , then there exists  $\chi < \min(\lambda, H_1)$  such that:
  - (a) **K** is  $(\chi, < \lambda)$ -weakly tame.
  - (b) For any  $\mu \in (\chi, \lambda)$ , there is a type-full good  $\mu$ -frame with underlying AEC the saturated models in  $\mathbf{K}_{\mu}$ .

PROOF. Item (1) is Fact 17.4.4 and Corollary 17.4.8. As for (2), let  $\mu \in [LS(\mathbf{K}), \lambda)$ . By the previous part, **K** is  $\mu$ -superstable and has  $\mu$ -symmetry. By the main result of [Van16a], this implies uniqueness of limit models as stated here.

Items (3) and (4) are part of the proof of Corollary 17.5.1. As for (5a), we use the relevant facts (due to Shelah) which assumes that the model in the categoricity (or just solvability) cardinal is saturated. They appear in [She09a, Claim IV.7.2] and [She99, Main claim II.2.3] (depending on whether cf  $\lambda > LS(\mathbf{K})$  or  $\lambda \geq H_1$ ), see also Theorem 15.2.4. Now (5b) follows from (5a) by Theorem 10.6.4.

REMARK 17.5.8. Corollary 17.5.7.(2) proves [SV99, Theorem 3.3.7] with the additional assumption that the class has amalgamation and improves on Corollary 10.7.3 which assumed that the categoricity cardinal  $\lambda$  was "big-enough". See Section 17.5.4 for more on the uniqueness of limit models.

17.5.3. Some categoricity transfers. We mention improvements on several existing categoricity transfers. The partial downward transfer below improves on Corollary 10.7.7 and Corollary 15.3.7. The essence of the proof is a powerful omitting type theorem of Shelah [She99, Lemma II.1.6]. Indeed the result is already implicit in [She99] when the categoricity cardinal  $\lambda$  is regular (see also [Bal09, Theorem 14.9]).

COROLLARY 17.5.9. Let **K** be an AEC with arbitrarily large models and let  $\lambda > \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\lambda}$  has amalgamation and no maximal models. If **K** is categorical in  $\lambda$ , then there exists  $\chi < H_1$  such that **K** is categorical in any cardinal of the form  $\beth_{\delta}$ , where  $\delta$  is divisible by  $\chi$  and  $\beth_{\delta} < \lambda$ .

PROOF. By (the proof of) Corollary 15.3.7, using that the model of categoricity  $\lambda$  is saturated (Corollary 17.4.11).

We can also improve the thresholds of Shelah's proof of the eventual categoricity conjecture in AECs with amalgamation [She09a, Theorem IV.7.12] assuming the weak generalized continuum hypothesis. Shelah showed (assuming an unpublished claim) that in an AEC with amalgamation, categoricity in some  $\lambda \geq \beth_{(2^{\aleph_{LS}(\mathbf{K})^+})^+}$ 

implies categoricity in all  $\lambda' \ge \beth_{(2^{\aleph_{\mathrm{LS}(\mathbf{K})^+}})^+}$ .

Shelah's proof was revisited and expanded on in Section 15.5, from which we quote. Here, we improve the main lemma to:

LEMMA 17.5.10. Assume an unpublished claim of Shelah (Claim 15.5.2). Assume that **K** has arbitrarily large models. Let  $\lambda \ge \mu > \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\lambda}$  has amalgamation. If:

- (1) **K** is categorical in  $\lambda$ .
- (2)  $\mu$  is a limit cardinal with cf  $\mu > \text{LS}(\mathbf{K})$ .
- (3) For unboundedly many  $\chi < \mu$ ,  $2^{\chi^{+n}} < 2^{\chi^{+(n+1)}}$  for all  $n < \omega$ .

Then there exists  $\mu_* < \mu$  such that **K** is categorical in any  $\lambda' \ge \min(\lambda, \beth_{(2^{\mu_*})^+})$ .

PROOF. As in the proof of Fact 15.5.10, using that we know that the model of categoricity  $\lambda$  is saturated (it is shown there that we can assume without loss of generality that  $\mathbf{K}_{<\lambda}$  has no maximal models).

We deduce that one can start with  $\lambda \geq \aleph_{\mathrm{LS}(\mathbf{K})^+}$  instead of  $\lambda \geq \beth_{(2^{\aleph_{\mathrm{LS}(\mathbf{K})^+}})^+}$ .

COROLLARY 17.5.11. Assume an unpublished claim of Shelah (Claim 15.5.2) and  $2^{\mu} < 2^{\mu^+}$  for all cardinals  $\mu$ . Assume that **K** has arbitrarily large models. Let  $\lambda \geq \aleph_{\mathrm{LS}(\mathbf{K})^+}$  be such that  $\mathbf{K}_{<\lambda}$  has amalgamation. If **K** is categorical in  $\lambda$ , then **K** is categorical in any  $\lambda' \geq \min(\lambda, \beth_{(2^{<\aleph_{\mathrm{LS}(\mathbf{K})^+})^+})$ .

PROOF. Set 
$$\mu := \aleph_{\text{LS}(\mathbf{K})^+}$$
 in Lemma 17.5.10.

We showed in Corollary 15.5.9 that if **K** is  $LS(\mathbf{K})$ -tame and has amalgamation, then categoricity in some  $\lambda \geq H_1$  implies categoricity in all  $\lambda' \geq H_1$  (still assuming weak GCH and Shelah's unpublished claim). In Corollary 15.8.7, we showed that it was consistent (using additional cardinal arithmetic assumptions) that one could replace tameness by just weak tameness. Here we prove it unconditionally.

COROLLARY 17.5.12. Assume an unpublished claim of Shelah (Claim 15.5.2) and there exists  $\mu < \aleph_{\mathrm{LS}(\mathbf{K})^+}$  such that  $2^{\mu^{+n}} < 2^{\mu^{+(n+1)}}$  for all  $n < \omega$ . Assume that **K** is (LS(**K**),  $< H_1$ )-weakly tame and has arbitrarily large models. Let  $\lambda \ge \aleph_{\mathrm{LS}(\mathbf{K})^+}$  be such that  $\mathbf{K}_{<\lambda}$  has amalgamation. If **K** is categorical in  $\lambda$ , then there exists  $\chi < H_1$  such that **K** is categorical in any  $\lambda' \ge \min(\lambda, \chi)$ .

PROOF. Proceed as in the proof of Corollary 15.8.7 (as before, we can assume without loss of generality that  $\mathbf{K}_{<\lambda}$  has no maximal models, hence the model of cardinality  $\lambda$  is saturated). We use the better transfer we have just proven (Corollary 17.5.11).

We also obtain more information on the author's categoricity transfer in universal classes (Chapters 8, 16). There it was shown (Theorem 16.7.3) that if a universal class **K** is categorical in some  $\lambda \geq \beth_{H_1}$ , then it is categorical in all  $\lambda' \geq \beth_{H_1}$ . The reason that the threshold is  $\beth_{H_1}$  rather than  $H_1$  is that the proof works inside an auxiliary AEC **K**<sup>\*</sup> whose Löwenheim-Skolem-Tarski number is around  $H_1$ . A closer look at the proof reveals that  $LS(\mathbf{K}^*)$  is related to the length of a failure of the order property, so we can use Theorem 17.3.4 to improve the bound on  $LS(\mathbf{K}^*)$ . We are unable to do so unconditionally so will assume that the class has no maximal models:

LEMMA 17.5.13. Let **K** be a universal class. Set  $\mu := 2^{2^{\text{LS}(\mathbf{K})}}$ ,  $\chi_1 := \beth_{\mu^{++}}$ ,  $\chi_2 := \beth_{(2^{\mu^+})^+}$ . Let  $\lambda > \chi_1$ . If **K** is categorical in  $\lambda$  and  $\mathbf{K}_{<\lambda}$  has no maximal models<sup>6</sup>, then there exists  $\chi < \chi_2$  such that **K** is categorical (and has amalgamation) in all  $\lambda' \ge \min(\lambda, \chi)$ .

PROOF SKETCH. First observe that  $\chi_1 \geq H_1$ , so **K** has arbitrarily large models. Second, by Theorem 17.3.4 and the no maximal models hypothesis, for any  $\kappa < \aleph_0$ , **K** does not have the  $(\kappa, \text{LS}(\mathbf{K}))$ -order property of length  $\text{LS}(\mathbf{K})^+$ .

We now follow the proof of Theorems 16.7.2, 16.7.3. We define an auxiliary class  $\mathbf{K}^*$  which will have Löwenheim-Skolem-Tarski number  $(2^{2^{\chi_0}})^+$ , where  $\chi_0 \geq \mathrm{LS}(\mathbf{K})$  is least such that  $\mathbf{K}$  does not have a syntactic version of the order property of length  $\chi_0^+$ . It is straightforward to see that if  $\mathbf{K}$  has the order property (in the sense there) of length  $\chi_0^+$ , then for some  $\kappa < \aleph_0 \mathbf{K}$  has the  $(\kappa, 0)$ -order property (in the sense of Definition 2.4.3) of length  $\chi_0^+$ . This means that  $\chi_0 = \mathrm{LS}(\mathbf{K})$ , and hence  $\mathrm{LS}(\mathbf{K}^*) = \mu^+$ .

Now  $\mathbf{K}^*$  may not satisfy the smoothness axiom of AECs and to ensure this the proof of Theorem 16.7.2 uses categoricity in a  $\lambda$  with  $(\beth_{\mu^{++}})^+ < \lambda$ . However if  $\lambda = (\beth_{\mu^{++}})^+$ , then  $\lambda$  is regular so by [She87b, Theorem IV.1.11] (building many models in the categoricity cardinal from failure of smoothness) we also get that  $\mathbf{K}^*$  is an AEC. Therefore  $\mathbf{K}^*$  is an AEC whenever  $\lambda > \beth_{\mu^{++}} = \chi_1$ . and we can then continue exactly as in the proof of Theorem 16.7.3.

A more quotable version of Lemma 17.5.13 is below. Compared to Theorem 16.7.3,  $\exists_{\exists_{(2^{LS}(\mathbf{K}))}^+}$  is replaced by the much lower  $\exists_{\exists_5(LS(\mathbf{K}))}$ .

COROLLARY 17.5.14. Let **K** be a universal class with no maximal models. If **K** is categorical in *some*  $\lambda \geq \beth_{\beth_5(LS(\mathbf{K}))}$ , then **K** is categorical in *all*  $\lambda' \geq \beth_{\beth_5(LS(\mathbf{K}))}$ .

PROOF. In the statement of Lemma 17.5.13,  $\chi_1 < \chi_2 \leq \beth_{\beth_5(LS(\mathbf{K}))}$ .

REMARK 17.5.15. One can ask what happens if instead of no maximal models, one makes the stronger assumption of amalgamation below the categoricity cardinal. Then we obtain the best possible result as in Corollary 15.4.11 (this is proven using amalgamation also above the categoricity cardinal, but we can use Theorem 8.4.16 to get away with just amalgamation below).

<sup>&</sup>lt;sup>6</sup>It suffices to assume that for every  $M \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})^+}$  there exists  $N \in \mathbf{K}_{\lambda}$  with  $M \leq_{\mathbf{K}} N$ .

17.5.4. More on the uniqueness of limit models. The main result of [SV99] claims that assuming no maximal models and GCH (with instances of  $\Diamond$ ), limit models of cardinality  $\mu$  are unique for any  $\mu$  below the categoricity cardinal  $\lambda$ . In VanDieren's Ph.D. thesis [Van02], two additional hypotheses that it seemed the proof needed were identified (see [Van06] or the recent [Van] for background and terminology):

- (1) The union of any increasing chain of limit models in  $\mathbf{K}_{\mu}$  of length less than  $\mu^+$  is a limit model.
- (2) Reduced towers in  $\mathbf{K}_{\mu}$  are continuous.

In [Van06, Theorem III.10.3], VanDieren claimed to prove (2) assuming (1). VanDieren later found a gap [Van13] and fixed the gap assuming in addition that  $\lambda = \mu^+$ . In [Van, Theorem 5], VanDieren improved this condition to  $\lambda = \mu^{+n}$ , for some natural number  $n \ge 1$ , and characterized [Van, Theorem 1] when (1) holds (showing in particular that when  $\mu$  is a successor it is *equivalent* to the uniqueness of limit models). Here we prove the original statement of [Van06, Theorem III.10.3] (still using (1)). This also shows that we can remove the hypothesis " $\lambda = \mu^{+n}$ " from VanDieren's characterization.

The key is the next result which generalizes Corollary 17.4.8 and improves [Van, Theorem 5]. Note that we do not assume (1). Note also that, below, we use only the model-theoretic consequences (in the context described above [SV99]) of GCH and appropriate instances of  $\Diamond$ . Finally, note that we have replaced the assumption of categoricity in  $\lambda$  mentioned above by the weaker assumption of semisolvability in  $\lambda$  (see Remark 17.3.3).

COROLLARY 17.5.16. Let  $\lambda > LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\lambda}$  has no maximal models. Assume that  $\mathbf{K}$  semisolvable in  $\lambda$  and fix  $\mu \in [LS(\mathbf{K}), \lambda)$ . If in  $\mathbf{K}_{\mu}$  amalgamation bases are dense, universal extensions exist, and limit models are amalgamation bases, then  $\mathbf{K}$  has  $\mu$ -symmetry (Definition 17.4.2 has to be slightly adapted, see [Van, Definition 7]).

PROOF SKETCH. We follow the proof of Corollary 17.4.8. Fact 17.4.4 was originally proven in the context here, and the proof of Fact 17.4.6 shows that failure of  $\mu$ -symmetry implies the  $(2, \mu)$ -order property of length  $\mu^+$  (in that case it is not clear that the construction can be continued all the way to  $\lambda$ ). Theorem 17.3.4 (with  $\kappa$  there standing for 2 here) and the no maximal models hypothesis shows that **K** cannot have the  $(2, \mu)$ -order property of length  $\mu^+$ , so symmetry in  $\mu$  must hold.

We obtain the desired proof of [Van06, Theorems II.9.1, III.10.3]. This also generalizes Corollary 17.5.7.(2).

COROLLARY 17.5.17. Let  $\lambda > LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\lambda}$  has no maximal models. Assume that  $\mathbf{K}$  is semisolvable in  $\lambda$  and fix  $\mu \in [LS(\mathbf{K}), \lambda)$ . If in  $\mathbf{K}_{\mu}$ amalgamation bases are dense, universal extensions exist, limit models are amalgamation bases, and (1) above holds<sup>7</sup>, then reduced towers in  $\mathbf{K}_{\mu}$  are continuous. In particular, whenever  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$  are such that  $M_1$  and  $M_2$  are limit models over  $M_0$ , we have that  $M_1 \cong_{M_0} M_2$ .

<sup>&</sup>lt;sup>7</sup>or just: the union of any increasing chain of limit models in  $\mathbf{K}_{\mu}$  of length less than  $\mu^{+}$  is an *amalgamation base*.

PROOF. By Corollary 17.5.16, **K** has symmetry in  $\mu$ . By [**Van**, Theorem 3], reduced towers in  $\mathbf{K}_{\mu}$  are continuous. As pointed out in [**Van13**], the proof of [**Van06**, Theorem II.9.1] now goes through to prove the uniqueness of limit models.

REMARK 17.5.18. We can similarly use Corollaries 17.5.16 and 17.5.17 to replace the assumption " $\lambda = \mu^{+n}$ " in [Van, Theorem 1] by only " $\lambda > \mu$ ".

# CHAPTER 18

# Good frames in the Hart-Shelah example

This chapter is based on  $[\mathbf{BVc}]$  and is joint work with Will Boney. We would like to thank the referees for comments that helped improve the presentation of this chapter.

#### Abstract

For a fixed natural number  $n \geq 1$ , the Hart-Shelah example is an abstract elementary class (AEC) with amalgamation that is categorical exactly in the infinite cardinals less than or equal to  $\aleph_n$ .

We investigate recently-isolated properties of AECs in the setting of this example. We isolate the exact amount of type-shortness holding in the example and show that it has a type-full good  $\aleph_{n-1}$ -frame which fails the existence property for uniqueness triples. This gives the first example of such a frame. Along the way, we develop new tools to build and analyze good frames.

### 18.1. Introduction

In his milestone two-volume book on classification theory for abstract elementary classes (AECs) [She09a, She09b], Shelah introduces a central definition: good  $\lambda$ -frames. These are an axiomatic notion of forking for types of singletons over models of cardinality  $\lambda$  (see [She09a, II.2.1] or Definition 18.2.7 here). One can think of the statement "an AEC K has a good  $\lambda$ -frame" as meaning that K is well-behaved in  $\lambda$ , where "well-behaved" in this context means something similar to superstability in the context of first-order model theory. With this in mind, a key question is:

QUESTION 18.1.1 (The extension question). Assume an AEC K has a good  $\lambda$ -frame. Under what conditions does it (or a subclass of saturated models) have a good  $\lambda^+$ -frame?

Shelah's answer in [**She09a**, II] involves two dividing lines: the existence property for uniqueness triples, and smoothness of a certain ordering  $\leq_{\mathbf{K}_{\lambda^+}}^{\mathrm{NF}}$  (see Definitions 18.2.10, 18.2.13). Shelah calls a good frame satisfying the first property weakly successful and a good frame satisfying both properties is called successful. Assuming instances of the weak diamond, Shelah shows [**She09a**, II.5.9] that the failure of the first property implies many models in  $\lambda^{++}$ . In [**She09a**, II.8.7] (see also [**JS13**, 7.1.3]), Shelah shows that if the first property holds, then the failure of the second implies there exists  $2^{\lambda^{++}}$  many models in  $\lambda^{++}$ .

However, Shelah does not give any examples showing that these two properties can fail (this is mentioned as part of the "empty half of the glass" in Shelah's introduction [**She09a**, N.4(f)]). The present chapter investigates these dividing lines

in the specific setup of the Hart-Shelah example [**HS90**]. For a fixed<sup>1</sup>  $n \in [3, \omega)$ , the Hart-Shelah example is an AEC  $\mathbf{K}^n$  that is categorical exactly in the interval  $[\aleph_0, \aleph_{n-2}]$ . It was investigated in details by Baldwin and Kolesnikov [**BK09**] who proved that  $\mathbf{K}^n$  has (disjoint) amalgamation, is (Galois) stable exactly in the infinite cardinals less than or equal to  $\aleph_{n-3}$ , and is  $(<\aleph_0, \leq\aleph_{n-3})$ -tame (i.e. Galois types over models of size at most  $\aleph_{n-3}$  are determined by their restrictions to finite sets, see Definition 18.2.1).

The Hart-Shelah example is a natural place to investigate good frames, since it has good behavior only below certain cardinals (around  $\aleph_{n-3}$ ). Boney has shown [**Bon14a**, 10.2] that  $\mathbf{K}^n$  has a good  $\aleph_k$  frame for any  $k \leq n-3$ , but cannot have one above since stability is part of the definition of a good frame. Therefore at  $\aleph_{n-3}$ , the last cardinal when  $\mathbf{K}^n$  has a good frame, the answer to the extension question must be negative, so one of the two dividing lines above must fail, i.e. the good frame is *not* successful. The next question is: *which* of these properties fails? We show that the first property must fail: the frame is not weakly successful. In fact, we give several proofs (Theorem 18.6.6, Corollary 18.7.4). On the other hand, we show that the frames strictly below  $\aleph_{n-3}$  are successful<sup>2</sup>. This follows both from a concrete analysis of the Hart-Shelah example (Theorem 18.6.3) and from abstract results in the theory of good frames (Theorem 18.5.1).

Regarding the abstract theory, a focus of recent research has been the interaction of locality properties and frames. For example, Boney [**Bon14a**] (see also the slight improvements in Corollary 5.6.9) has shown that amalgamation and *tame*ness (a locality property for types isolated by Grossberg and VanDieren [**GV06b**]) implies a positive answer to the extension question (in particular, the Hart-Shelah example is not ( $\aleph_{n-3}, \aleph_{n-2}$ )-tame<sup>3</sup>). A relative of tameness is type-shortness, introduced by Boney in [**Bon14b**, 3.2]: roughly, it says that types of sequences are determined by their restriction to small subsequences. Sufficient amount of typeshortness implies (with a few additional technical conditions) that a good frame is weakly successful (Section 6.11).

As already mentioned, Baldwin and Kolesnikov have shown that the Hart-Shelah example is  $(<\aleph_0, \leq \aleph_{n-3})$ -tame (see Fact 18.3.2); here (Theorem 18.4.1) we refine their argument to show that it is also  $(<\aleph_0, <\aleph_{n-3})$ -type short over models of size less than or equal to  $\aleph_{n-3}$  (i.e. types of sequences of length less than  $\aleph_{n-3}$  are determined by their finite restrictions, see Definition 18.2.1). We prove that this is optimal: the result *cannot* be extended to types of length  $\aleph_{n-3}$  (see Corollary 18.8.12).

We can also improve Boney's aforementioned construction of a good  $\aleph_k$ -frame (when  $k \leq n-3$ ) in the Hart-Shelah example: the good frame built there is not type-full: forking is only defined for a certain (dense family) of basic types. We prove here that the good frame extends to a type-full one. This uses abstract constructions of good frames from Chapter 4 (as well as results of VanDieren on the symmetry property [Van16a]) when  $k \geq 1$ . When k = 0 we have to work more and develop new general tools to build good frames (see Section 18.8).

<sup>&</sup>lt;sup>1</sup>Note that our indexing follows [HS90] and [BK09] rather than [Bon14a].

 $<sup>^{2}</sup>$ While there are no known examples, it is conceivable that there is a good frame that is not successful but can still be extended.

 $<sup>^{3}</sup>$ This was already noticed by Baldwin and Kolesnikov using a different argument [**BK09**, 6.8].

The following summarizes our main results:

THEOREM 18.1.2. Let  $n \in [3, \omega)$  and let  $\mathbf{K}^n$  denote the AEC induced by the Hart-Shelah example. Then:

- (1)  $\mathbf{K}^n$  is  $(\langle \aleph_0, \langle \aleph_{n-3} \rangle)$ -type short over  $\leq \aleph_{n-3}$ -sized models and  $(\langle \aleph_0, \leq \aleph_{n-3} \rangle)$ -tame for  $(\langle \aleph_{n-3} \rangle)$ -length types.
- (2)  $\mathbf{K}^n$  is not  $(\langle \aleph_{n-3}, \aleph_{n-3})$ -type short over  $\aleph_{n-3}$ -sized models.
- (3) For any  $k \leq n-3$ , there exists a unique type-full good  $\aleph_k$ -frame  $\mathfrak{s}$  on  $\mathbf{K}^n$ . Moreover:
  - (a) If k < n 3,  $\mathfrak{s}$  is successful good<sup>+</sup>.
  - (b) If k = n 3,  $\mathfrak{s}$  is not weakly successful.

Proof.

- (1) By Theorem 18.4.1.
- (2) By Corollary 18.8.12.
- (3) By Theorems 18.5.1 and Corollary 18.8.11. Note also that by canonicity (Fact 18.2.20),  $\mathfrak{s}$  is unique, so extends  $\mathfrak{s}^{k,n}$  (see Definition 18.3.3).
  - (a) By Theorem 18.6.3,  $\mathfrak{s}^{k,n}$  is successful. By Lemma 18.5.2,  $\mathfrak{s}^{k,n}$  is good<sup>+</sup>. Now apply Facts 18.2.20 and 18.2.17.
  - (b) By Proposition 18.6.6,  $\mathfrak{s}^{k,n}$  is not weakly successful and since  $\mathfrak{s}$  extends  $\mathfrak{s}^{k,n}$ ,  $\mathfrak{s}$  is not weakly successful either.

We discuss several open questions. First, one can ask whether the aforementioned second dividing line can fail:

QUESTION 18.1.3 (See also 7.1 in [Jar16]). Is there an example of a good  $\lambda$ -frame that is weakly successful but not successful?

Second, one can ask whether there is any example at all of a good frame where the forking relation can be defined only for certain types<sup>4</sup>:

QUESTION 18.1.4. Is there an example of a good  $\lambda$ -frame that does *not* extend to a type-full frame?

We have not discussed  $good^+$  in our introduction: it is a technical property of frames that allows one to extend frames without changing the order (see the background in Section 18.2). No negative examples are known.

QUESTION 18.1.5. Is there an example of a good  $\lambda$ -frame that is not good<sup>+</sup>? Is there an example that is successful but not good<sup>+</sup>?

In a slightly different direction, we also do not know of an example of a good frame failing symmetry:

QUESTION 18.1.6 (See also Question 10.4.14). Is there an example of a triple  $(\mathbf{K}, \perp, \mathbf{gS}^{bs})$  satisfying all the requirements from the definition of a good  $\lambda$ -frame except symmetry?

<sup>&</sup>lt;sup>4</sup>After the initial circulation of this chapter in July 2016, it was found that an example of Shelah [**She09b**, VII.5.7] has a good frame that cannot be extended to be type-full, see [**Vasd**, Section 5].

In the various examples, the proofs of symmetry either uses disjoint amalgamation (as in [She09a, II.3.7]) or failure of the order property (see e.g. Theorem 3.5.13). Recently we (Corollary 17.4.8) have shown that symmetry follows from (amalgamation, no maximal models, and) *solvability* in any  $\mu > \lambda$  (see [She09a, IV.1.4(1)]; roughly it means that the union of a short chain of saturated model of cardinality  $\mu$  is saturated, and there is a "constructible" witness). We do not know of an example of a good  $\lambda$ -frame where solvability in every  $\mu > \lambda$  fails.

The background required to read this chapter is a solid knowledge of AECs (including most of the material in [Bal09]). Familiarity with good frames and the Hart-Shelah example would be helpful, although we have tried to give a self-contained presentation and quote all the black boxes we need.

### 18.2. Preliminaries: The abstract theory

Everywhere in this chapter,  $\mathbf{K}$  denotes a fixed AEC (that may or may not have structural properties such as amalgamation or arbitrarily large models). We assume the reader is familiar with concepts such as amalgamation, Galois types, tameness, type-shortness, stability, saturation, and splitting (see for example the first twelve chapters of [**Bal09**]). Our notation is standard and is described in the preliminaries of Chapter 2.

On tameness and type-shortness, we use the notation from [Bon14b, 3.1,3.2]:

DEFINITION 18.2.1. Let  $\lambda \geq LS(\mathbf{K})$  and let  $\kappa, \mu$  be infinite cardinals<sup>5</sup>

- (1) **K** is  $(< \kappa, \lambda)$ -tame for  $\mu$ -length types if for any  $M \in \mathbf{K}_{\lambda}$  and distinct  $p, q \in \mathrm{gS}^{\mu}(M)$ , there exists  $A \subseteq |M|$  with  $|A| < \kappa$  such that  $p \upharpoonright A \neq q \upharpoonright A$ . When  $\mu = 1$  (i.e. we are only interested in types of length one), we omit it and just say that **K** is  $(< \kappa, \lambda)$ -tame.
- (2) **K** is  $(< \kappa, \mu)$ -type short over  $\lambda$ -sized models if for any  $M \in \mathbf{K}_{\lambda}$  and distinct  $p, q \in gS^{\mu}(M)$ , there exists  $I \subseteq \mu$  with  $|I| < \kappa$  and  $p^{I} \neq q^{I}$ .

We similarly define variations such as "**K** is  $(< \kappa, \leq \mu)$ -type short over  $\leq \lambda$ -sized models.

18.2.1. Superstability and symmetry. We will rely on the following local version of superstability, already implicit in [SV99] and since then studied in many papers, e.g. [Van06, GVV16, Van16a], Chapters 6, 7, 9. We quote Definition 6.10.1:

DEFINITION 18.2.2. **K** is  $\mu$ -superstable (or superstable in  $\mu$ ) if:

- (1)  $\mu \geq \text{LS}(\mathbf{K})$ .
- (2)  $\mathbf{K}_{\mu}$  is nonempty, has joint embedding, amalgamation, and no maximal models.
- (3) **K** is stable in  $\mu$ .
- (4) There are no long splitting chains in  $\mu$ :

For any limit ordinal  $\delta < \mu^+$ , for every sequence  $\langle M_i \mid i < \delta \rangle$  of models of cardinality  $\mu$  with  $M_{i+1}$  universal over  $M_i$  and for every  $p \in$  $gS(\bigcup_{i < \delta} M_i)$ , there exists  $i < \delta$  such that p does not  $\mu$ -split over  $M_i$ .

We will also use the concept of symmetry for splitting isolated in [Van16a]:

<sup>&</sup>lt;sup>5</sup>As opposed to Boney's original definition, we allow  $\kappa \leq LS(\mathbf{K})$  by making use of Galois types over sets, see the preliminaries of Chapter 2.

DEFINITION 18.2.3. For  $\mu \geq \text{LS}(\mathbf{K})$ , we say that  $\mathbf{K}$  has  $\mu$ -symmetry (or symmetry in  $\mu$ ) if whenever models  $M, M_0, N \in \mathbf{K}_{\mu}$  and elements a and b satisfy the conditions (1)-(4) below, then there exists  $M^b$  a limit model over  $M_0$ , containing b, so that  $\text{gtp}(a/M^b)$  does not  $\mu$ -split over N.

- (1) M is universal over  $M_0$  and  $M_0$  is a limit model over N.
- (2)  $a \in |M| \setminus |M_0|$ .
- (3)  $gtp(a/M_0)$  is non-algebraic and does not  $\mu$ -split over N.
- (4) gtp(b/M) is non-algebraic and does not  $\mu$ -split over  $M_0$ .

By an argument of Shelah and Villaveces [**SV99**, 2.2.1] (see also Chapter 20), superstability holds below a categoricity cardinal.

FACT 18.2.4 (The Shelah-Villaveces Theorem). Let  $\lambda > LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{<\lambda}$  has amalgamation and no maximal models. If  $\mathbf{K}$  has arbitrarily large models and is categorical in  $\lambda$ , then  $\mathbf{K}$  is superstable in any  $\mu \in [LS(\mathbf{K}), \lambda)$ .

REMARK 18.2.5. We will only use the result when  $\lambda$  is a successor (in fact  $\lambda = \mu^+$ , where  $\mu$  is the cardinal where we want to derive superstability). In this case there is an easier proof due to Shelah. See [She99, I.6.3] or [Bal09, 15.3].

VanDieren [**Van16a**] has shown that (in an AEC with amalgamation and no maximal models) symmetry in  $\mu$  follows from categoricity in  $\mu^+$ . This was improved in Corollary 10.7.3 and recently in Corollary 17.4.8, but we will only use VanDieren's original result:

FACT 18.2.6. If **K** is  $\mu$ -superstable and categorical in  $\mu^+$ , then **K** has symmetry in  $\mu$ .

18.2.2. Good frames. Good  $\lambda$ -frames were introduced by Shelah in [She09a, II] as a bare-bone axiomatization of superstability. We give a simplified definition here:

DEFINITION 18.2.7 (II.2.1 in [She09a]). A good  $\lambda$ -frame is a triple  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \downarrow, gS^{bs})$  where:

- (1)  $\mathbf{K}$  is an AEC such that:
  - (a)  $\lambda \geq LS(\mathbf{K})$ .
  - (b)  $\mathbf{K}_{\lambda} \neq \emptyset$ .
  - (c)  $\mathbf{K}_{\lambda}$  has amalgamation, joint embedding, and no maximal models.
  - (d) **K** is stable<sup>6</sup> in  $\lambda$ .
- (2) For each  $M \in \mathbf{K}_{\lambda}$ ,  $\mathrm{gS}^{\mathrm{bs}}(M)$  (called the set of *basic types* over M) is a set of nonalgebraic Galois types over M satisfying the *density property*: if  $M <_{\mathbf{K}} N$  are both in  $\mathbf{K}_{\lambda}$ , there exists  $a \in |N| \setminus |M|$  such that  $\mathrm{gtp}(a/M; N) \in \mathrm{gS}^{\mathrm{bs}}(M)$ .
- (3)  $\downarrow$  is an (abstract) independence relation on the basic types satisfying invariance, monotonicity, extension existence, uniqueness, continuity, local character, and symmetry (see [She09a, II.2.1] for the full definition of these properties).

We say that  $\mathfrak{s}$  is *type-full* [She09a, III.9.2(1)] if for any  $M \in \mathbf{K}_{\lambda}$ , gS<sup>bs</sup>(M) is the set of *all* nonalgebraic types over M. Rather than explicitly using the relation  $\downarrow$ ,

<sup>&</sup>lt;sup>6</sup>In Shelah's original definition, only the set of basic types is required to be stable. However full stability follows, see [**She09a**, II.4.2].

we will say that gtp(a/M; N) does not fork over  $M_0$  if  $a \stackrel{N}{\downarrow} M$  (this is well-defined by the invariance and monotonicity properties). We say that a good  $\lambda$ -frame  $\mathfrak{s}$  is on **K** if the underlying AEC of  $\mathfrak{s}$  is  $\mathbf{K}_{\lambda}$ , and similarly for other variations.

REMARK 18.2.8. We will *not* use the axiom (B) [She09a, II.2.1] requiring the existence of a superlimit model of size  $\lambda$ . In fact many papers (e.g. [JS13]) define good frames without this assumption.

REMARK 18.2.9. We gave a shorter list of properties that in Shelah's original definition, but the other properties follow, see [She09a, II.2].

The next technical property is of great importance in Chapter II and III of [She09a]. The definition below follows [JS13, 4.1.5].

DEFINITION 18.2.10. Let  $\lambda \geq LS(\mathbf{K})$ .

- (1) For  $M_0 \leq_{\mathbf{K}} M_\ell$  all in  $\mathbf{K}_{\lambda}$ ,  $\ell = 1, 2$ , an amalgam of  $M_1$  and  $M_2$  over  $M_0$  is a triple  $(f_1, f_2, N)$  such that  $N \in \mathbf{K}_{\lambda}$  and  $f_\ell : M_\ell \xrightarrow[M_0]{} N$ .
- (2) Let  $(f_1^x, f_2^x, N^x)$ , x = a, b be amalgams of  $M_1$  and  $M_2$  over  $M_0$ . We say  $(f_1^a, f_2^a, N^a)$  and  $(f_1^b, f_2^b, N^b)$  are equivalent over  $M_0$  if there exists  $N_* \in \mathbf{K}_{\lambda}$  and  $f^x : N^x \to N_*$  such that  $f^b \circ f_1^b = f^a \circ f_1^a$  and  $f^b \circ f_2^b = f^a \circ f_2^a$ , namely, the following commutes:



Note that being "equivalent over  $M_0$ " is an equivalence relation ([**JS13**, 4.3]).

- (3) Let  $\mathfrak{s} = (\mathbf{K}_{\lambda}, \, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  be a good  $\lambda$ -frame on **K**.
  - (a) A triple (a, M, N) is a uniqueness triple (for  $\mathfrak{s}$ ) if  $M \leq_{\mathbf{K}} N$  are both in  $\mathbf{K}_{\lambda}, a \in |N| \setminus |M|, \operatorname{gtp}(a/M; N) \in \operatorname{gS}^{\operatorname{bs}}(M)$ , and for any  $M_1 \geq_{\mathbf{K}} M$  in  $\mathbf{K}_{\lambda}$ , there exists a unique (up to equivalence over M) amalgam  $(f_1, f_2, N_1)$  of N and  $M_1$  over M such that  $\operatorname{gtp}(f_1(a)/f_2[M_1]; N_1)$  does not fork over M.
  - (b)  $\mathfrak{s}$  has the existence property for uniqueness triples (or is weakly successful) if for any  $M \in \mathbf{K}_{\lambda}$  and any  $p \in \mathrm{gS}^{\mathrm{bs}}(M)$ , one can write  $p = \mathrm{gtp}(a/M; N)$  with (a, M, N) a uniqueness triple.

The importance of the existence property for uniqueness triples is that it allows us to extend the nonforking relation to types of models (rather than just types of length one). This is done by Shelah in [She09a, II.6] but was subsequently simplified in [JS13], so we quote from the latter.

DEFINITION 18.2.11. Let  $\mathfrak{s}$  be a weakly successful good  $\lambda$ -frame on  $\mathbf{K}$ , with  $\mathbf{K}$  categorical in  $\lambda$ .

- (1) [JS13, 5.3.1] Define a 4-ary relation NF<sup>\*</sup> = NF<sup>\*</sup><sub>\$\$</sub> on  $\mathbf{K}_{\lambda}$  by NF<sup>\*</sup>( $N_0, N_1, N_2, N_3$ ) if there is  $\alpha^* < \lambda^+$  and for  $\ell = 1, 2$  there are increasing continuous sequences  $\langle N_{\ell,i} : i \leq \alpha^* \rangle$  and a sequence  $\langle d_i : i < \alpha^* \rangle$  such that:
  - (a)  $\ell < 4$  implies  $N_0 \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} N_3$ .
  - (b)  $N_{1,0} = N_0, N_{1,\alpha^*} = N_1, N_{2,0} = N_2, N_{2,\alpha^*} = N_3.$
  - (c)  $i \leq \alpha^*$  implies  $N_{1,i} \leq_{\mathbf{K}} N_{2,i}$ .
  - (d)  $d_i \in |N_{1,i+1}| \setminus |N_{1,i}|.$
  - (e)  $(d_i, N_{1,i}, N_{1,i+1})$  is a uniqueness triple.
  - (f)  $gtp(d_i/N_{2,i}; N_{2,i+1})$  does not fork over  $N_{1,i}$ .
- (2) [JS13, 5.3.2] Define a 4-ary relation NF = NF<sub>\$\$</sub> on  $\mathbf{K}_{\lambda}$  by NF( $M_0, M_1, M_2, M_3$ ) if there are models  $N_0, N_1, N_2, N_3$  such that  $N_0 = M_0, \ \ell < 4$  implies  $M_{\ell} \leq_{\mathbf{K}} N_{\ell}$  and NF<sup>\*</sup>( $N_0, N_1, N_2, N_3$ ).

By [JS13, 5.5.4], NF satisfies several of the basic properties of forking:

FACT 18.2.12. If NF( $M_0, M_1, M_2, M_3$ ), then  $M_1 \cap M_2 = M_0$ . Moreover, NF respects  $\mathfrak{s}$  and satisfies monotonicity, existence, weak uniqueness, symmetry, and long transitivity (see [**JS13**, 5.2.1] for the definitions).

Shelah [**She09a**, III.1.1] says a weakly successful good frame is *successful* if an ordering  $\leq_{\mathbf{K}_{\lambda^+}}^{\mathrm{NF}}$  defined in terms of the relation NF induces an AEC. We quote the full definition from [**JS13**].

DEFINITION 18.2.13. Let  $\mathfrak{s}$  be a weakly successful good  $\lambda$ -frame on  $\mathbf{K}$ , with  $\mathbf{K}$  categorical in  $\lambda$ .

- (1) [**JS13**, 6.1.2] Define a 4-ary relation  $\widehat{NF} = \widehat{NF}_{\mathfrak{s}}$  on **K** by  $\widehat{NF}(N_0, N_1, M_0, M_1)$  if:
  - (a)  $\ell < 2$  implies that  $N_n \in \mathbf{K}_{\lambda}, M_n \in \mathbf{K}_{\lambda^+}$ .
  - (b) There is a pair of increasing continuous sequences  $\langle N_{0,\alpha} : \alpha < \lambda^+ \rangle$ ,  $\langle N_{1,\alpha} : \alpha \leq \lambda^+ \rangle$  such that for every  $\alpha < \lambda^+$ , NF $(N_{0,\alpha}, N_{1,\alpha}, N_{0,\alpha+1}, N_{1,\alpha+1})$ and for  $\ell < 2$ ,  $N_{0,\ell} = N_{\ell}$ ,  $M_{\ell} = N_{\ell,\lambda^+}$ .
- and for  $\ell < 2$ ,  $N_{0,\ell} = N_\ell$ ,  $M_\ell = N_{\ell,\lambda^+}$ . (2) [**JS13**, 6.1.4] For  $M_0 \leq_{\mathbf{K}} M_1$  both in  $\mathbf{K}_{\lambda^+}$ ,  $M_0 \leq_{\mathbf{K}_{\lambda^+}}^{\mathrm{NF}} M_1$  if there exists  $N_0, N_1 \in \mathbf{K}_{\lambda}$  such that  $\widehat{\mathrm{NF}}(N_0, N_1, M_0, M_1)$
- $N_{0}, N_{1} \in \mathbf{K}_{\lambda} \text{ such that } \widehat{\mathrm{NF}}(N_{0}, N_{1}, M_{0}, M_{1}).$ (3)  $[\mathbf{JS13}, 10.1.1] \mathfrak{s} \text{ is successful if } \leq_{\mathbf{K}_{\lambda^{+}}}^{\mathrm{NF}} \text{ satisfies smoothness on the saturated}$ models in  $\mathbf{K}_{\lambda^{+}}$ : whenever  $\delta < \lambda^{++}$  is limit,  $\langle M_{i} : i \leq \delta \rangle$  is a  $\leq_{\mathbf{K}_{\lambda^{+}}}^{\mathrm{NF}}$ increasing continuous sequence of saturated models of cardinality  $\lambda^{+}$ , and  $N \in \mathbf{K}_{\lambda^{+}}$  is saturated such that  $i < \delta$  implies  $M_{i} \leq_{\mathbf{K}_{\lambda^{+}}}^{\mathrm{NF}} N$ , then  $M_{\delta} \leq_{\mathbf{K}_{\lambda^{+}}}^{\mathrm{NF}}$ N.

The point of successful good frames is that they can be extended to a good  $\lambda^+$ -frame on the class of saturated model of cardinality  $\lambda^+$  (see [**JS13**, 10.1.9]). The ordering of the class will be  $\leq_{\mathbf{K}_{\lambda^+}}^{\mathrm{NF}}$ . Shelah also defines what it means for a frame to be good<sup>+</sup>. If the frame is successful, being good<sup>+</sup> implies that  $\leq_{\mathbf{K}_{\lambda^+}}^{\mathrm{NF}}$  is just  $\leq_{\mathbf{K}}$  and simplifies several arguments [**She09a**, III.1.3, III.1.8]:

DEFINITION 18.2.14. A good  $\lambda$ -frame  $\mathfrak{s}$  on **K** is *good*<sup>+</sup> when the following is *impossible*:

There exists an increasing continuous  $\langle M_i : i < \lambda^+ \rangle$ ,  $\langle N_i : i < \lambda^+ \rangle$ , a basic type  $p \in gS(M_0)$ , and  $\langle a_i : i < \lambda^+ \rangle$  such that for any  $i < \lambda^+$ :

(1)  $i < \lambda^+$  implies that  $M_i \leq_{\mathbf{K}} N_i$  and both are in  $\mathbf{K}_{\lambda}$ .

(2)  $a_{i+1} \in |M_{i+2}|$  and  $gtp(a_{i+1}/M_{i+1}; M_{i+2})$  is a nonforking extension of p, but  $gtp(a_{i+1}/N_0; N_{i+2})$  is not.

(3)  $\bigcup_{j < \lambda^+} M_j$  is saturated.

FACT 18.2.15. Let  $\mathfrak{s}$  be a successful good  $\lambda$ -frame on **K**. The following are equivalent:

- (1)  $\mathfrak{s}$  is good<sup>+</sup>.
- (2) For  $M, N \in \mathbf{K}_{\lambda^+}$  both saturated,  $M \leq_{\mathbf{K}_{\lambda^+}}^{\mathrm{NF}} N$  if and only if  $M \leq_{\mathbf{K}} N$ .

PROOF. (1) implies (2) is [She09a, III.1.8]. Let us see that (2) implies (1): Suppose for a contradiction that  $\langle M_i : i < \lambda^+ \rangle$ ,  $\langle N_i : i < \lambda^+ \rangle$ , p,  $\langle a_i : i < \lambda^+ \rangle$ witness that  $\mathfrak{s}$  is not good<sup>+</sup>. Write  $M_{\lambda^+} := \bigcup_{i < \lambda^+} M_i$ ,  $N_{\lambda^+} := \bigcup_{i < \lambda^+} N_i$ . Using [JS13, 6.1.6], we have that there exists a club  $C \subseteq \lambda^+$  such that for any i < j both in C, NF( $M_i, M_j, N_i, N_j$ ). In particular (by monotonicity), NF( $M_i, M_{i+2}, N_i, N_{i+2}$ ). Pick any  $i \in C$ . Because NF respects  $\mathfrak{s}$  (Fact 18.2.12), gtp( $a_{i+1}/N_i; N_{i+2}$ ) does not fork over  $M_i$ . By the properties of  $\langle a_i : i < \lambda^+ \rangle$ , gtp( $a_{i+1}/M_{i+1}; M_{i+2}$ ) is a nonforking extension of p. By transitivity, gtp( $a_{i+1}/N_i; N_{i+2}$ ) also is a nonforking extension of p, contradicting the definition of good<sup>+</sup>.

FACT 18.2.16 (III.1.8 in [She09a]). Let  $\mathfrak{s}$  be a successful good<sup>+</sup>  $\lambda$ -frame on **K**. Then there exists a good  $\lambda^+$ -frame  $\mathfrak{s}^+$  with underlying AEC the saturated models in **K** of size  $\lambda^+$  (ordered with the strong substructure relation inherited from **K**).

We will also use that successful good<sup>+</sup> frame can be extended to be type-full.

FACT 18.2.17 (III.9.6(2B) in [She09a]). If  $\mathfrak{s}$  is a successful good<sup>+</sup>  $\lambda$ -frame on **K** and **K** is categorical in  $\lambda$ , then there exists a *type-full* successful good<sup>+</sup>  $\lambda$ -frame  $\mathfrak{t}$  with underlying class  $\mathbf{K}_{\lambda}$ .

The next result derives good frames from some tameness and categoricity. The statement is not optimal (e.g. categoricity in  $\lambda^+$  can be replaced by categoricity in any  $\mu > \lambda$ ) but suffices for our purpose.

FACT 18.2.18. Assume that **K** has amalgamation and arbitrarily large models. Let  $LS(\mathbf{K}) < \lambda$  be such that **K** is categorical in both  $\lambda$  and  $\lambda^+$ . Let  $\kappa \leq LS(\mathbf{K})$  be an infinite regular cardinal such that  $LS(\mathbf{K}) = LS(\mathbf{K})^{<\kappa}$  and  $\lambda = \lambda^{<\kappa}$ .

If **K** is  $(LS(\mathbf{K}), \leq \lambda)$ -tame, then there is a type-full good  $\lambda$ -frame  $\mathfrak{s}$  on **K**. If in addition **K** is  $(LS(\mathbf{K}), \leq \lambda)$ -tame for  $(< \kappa)$ -length types and  $(< \kappa, \leq \lambda)$ -type-short over  $\lambda$ -sized models, then  $\mathfrak{s}$  is weakly successful.

PROOF. By Facts 18.2.4 and 18.2.6, **K** is superstable in any  $\mu \in [LS(\mathbf{K}), \lambda]$ , and has  $\lambda$ -symmetry. By Theorem 10.6.4, there is a type-full good  $\lambda$ -frame  $\mathfrak{s}$  on  $\mathbf{K}_{\lambda}$ . The last sentence is by Corollary 14.3.13.

Fact 18.2.18 gives a criteria for when a good frame is *weakly* successful, but when is it successful? This is answered by the next result, due to Adi Jarden [**Jar16**, 7.19] (note that the conjugation hypothesis there follows from [**She09a**, III.1.21]).

FACT 18.2.19. Let  $\mathfrak{s}$  be a weakly successful good  $\lambda$ -frame on **K**. If **K** is categorical in  $\lambda$ , has amalgamation in  $\lambda^+$ , and is  $(\lambda, \lambda^+)$ -tame, then  $\mathfrak{s}$  is successful good<sup>+</sup>.

We will also make use of the following result, which tells us that if the AEC is categorical, there can be at most one good frame (Theorem 6.9.7):

FACT 18.2.20 (Canonicity of categorical good frames). Let  $\mathfrak{s}$  and  $\mathfrak{t}$  be good  $\lambda$ -frame on **K** with the same basic types. If **K** is categorical in  $\lambda$ , then  $\mathfrak{s} = \mathfrak{t}$ .

### 18.3. Preliminaries: Hart-Shelah

DEFINITION 18.3.1. Fix  $n \in [2, \omega)$ . Let  $\mathbf{K}^n$  be the AEC from the Hart-Shelah example. This class is  $\mathbb{L}_{\omega_1,\omega}$ -definable and a model in  $\mathbf{K}^n$  consists of the following:

- *I*, some arbitrary index set
- $K = [I]^3$  with a membership relation for I
- *H* is a copy of  $\mathbb{Z}_2$  with addition
- $G = \bigoplus_{u \in K} \mathbb{Z}_2$  with the evaluation map from  $G \times K$  to  $\mathbb{Z}_2$
- $G^*$  is a set with a projection  $\pi_{G^*}$  onto K such that there is a 1-transitive action of G on each stalk  $G_u^* = \pi_{G^*}^{-1}(u)$ ; we denote this action by  $t_G(u, \gamma, x, y)$ for  $u \in K$ ,  $\gamma \in G$ , and  $x, y \in G_u^*$
- $H^*$  is a set with a projection  $\pi_{H^*}$  onto K such that there is a 1-transitive action of  $\mathbb{Z}_2$  on each stalk  $H_u^* = \pi_{H^*}^{-1}(u)$  denoted by  $t_H$
- Q is a (n+1)-ary relation on  $(G^*)^n \times H^*$  satisfying the following:
  - We can permute the first n elements (the one from  $G^*$ ) and preserve holding.
  - If  $Q(x_1, \ldots, x_n, y)$  holds, then the indices of their stalks are *compatible*, which means the following:  $x_{\ell} \in G^*_{u_{\ell}}$  and  $y \in H^*_v$  such that  $\{u_1, \ldots, u_n, v\}$  are all *n* element sets of some n+1 element subset of Ι.
  - -Q is preserved by "even" actions in the following sense: suppose
    - \*  $u_1, \ldots, u_n, v \in K$  are compatible

    - \*  $x_{\ell}, x'_{\ell} \in G^*_{u_{\ell}}$  and  $y, y' \in H^*_v$ \*  $\gamma_{\ell} \in G$  and  $\ell \in \mathbb{Z}_2$  are the unique elements that send  $x_{\ell}$  or yto  $x'_{\ell}$  or y'

then the following are equivalent

\* 
$$Q(x_1,\ldots,x_n,y)$$
 if and only if  $Q(x'_1,\ldots,x'_n,y')$ 

$$* \gamma_1(v) + \dots + \gamma_n(v) + \ell = 0 \mod 2$$

For  $M, N \in \mathbf{K}^n$ ,  $M \leq_{\mathbf{K}^n} N$  if and only if  $M \prec_{\mathbb{L}_{\omega_1,\omega}} N$ .

FACT 18.3.2 ([**BK09**]). Let  $n \in [2, \omega)$ .

- (1)  $\mathbf{K}^n$  has disjoint amalgamation, joint embedding, and arbitrarily large models.
- (2)  $\mathbf{K}^n$  is model-complete: For  $M, N \in \mathbf{K}^n, M \leq_{\mathbf{K}^n} N$  if and only if  $M \subseteq N$ .
- (3) For any infinite cardinal  $\lambda$ ,  $\mathbf{K}^n$  is categorical in  $\lambda$  if and only if  $\lambda \leq \aleph_{n-2}$ .
- (4)  $\mathbf{K}^n$  is not stable in any  $\lambda \geq \aleph_{n-2}$ .
- (5) If  $n \ge 3$ , then  $\mathbf{K}^n$  is  $(\langle \aleph_0, \leq \aleph_{n-3})$ -tame, but it is not  $(\aleph_{n-3}, \aleph_{n-2})$ -tame.

Note that the entire universe of a model of  $\mathbf{K}^n$  is determined by the index I, so if  $M \subseteq N$ , then  $I(M) \subseteq I(N)$ . Thus it is natural to define a frame whose basic types are just the types of elements in I and nonforking is just nonalgebraicity. The following definition appears in the proof of [Bon14a, 10.2]:

DEFINITION 18.3.3. Let  $n \in [3, \omega)$ . For  $k \leq n-3$ , let  $\mathfrak{s}^{k,n} = (\mathbf{K}_{\aleph_k}^n, \downarrow, \mathrm{gS}^{\mathrm{bs}})$  be defined as follows:

- $p \in gS^{bs}(M)$  if and only if p = gtp(a/M; N) for  $a \in I(N) \setminus I(M)$ .
- $gtp(a/M_1; M_2)$  does not fork over  $M_0$  if and only if  $a \in I(M_2) \setminus I(M_1)$ .

REMARK 18.3.4. By [Bon14a, 10.2],  $\mathfrak{s}^{k,n}$  is a good  $\aleph_k$ -frame.

The notion of a solution is key to analyzing models of  $\mathbf{K}^n$ .

DEFINITION 18.3.5 (2.1 and 2.3 in [**BK09**]). Let  $M \in \mathbf{K}^n$ .

(1) h = (f,g) is a solution for  $W \subseteq K(M)$  if and only if  $f \in \prod_{u \in W} G_u^*(M)$ and  $g \in \prod_{u \in W} H_u^*(M)$  such that, for all compatible  $u_1, \ldots, u_n, v \in W$ , we have

$$M \vDash Q(f(u_1), \dots, f(u_n), g(v))$$

- (2) h = (f,g) is a solution over  $A \subseteq I(M)$  if and only if it is a solution for  $[A]^n$ .
- (3) h = (f, g) is a solution for M if and only if it is a solution for K(M).

Given  $f: M \cong N$  and solutions  $h^M$  for M and  $h^N$ , we say that  $h^M$  and  $h^N$  are conjugate by f if

$$f^N = f \circ f^M \circ f^{-1}$$
 and  $g^N = f \circ g^M \circ f^{-1}$ 

We write this as  $h^N = f \circ h^M \circ f^{-1}$ .

A key notion is that of extending and amalgamating solutions.

DEFINITION 18.3.6 (2.9 in [**BK09**]).

- (1) A solution h = (f, g) extends another solution h' = (f', g') if  $f' \subseteq f$  and  $g' \subseteq g$ .
- (2) We say that  $\mathbf{K}^n$  has k-amalgamation for solutions over sets of size  $\lambda$  if given any  $M \in \mathbf{K}^n$ ,  $A \subseteq I(M)$  of size  $\lambda$ ,  $\{b_1, \ldots, b_n\} \subseteq I(M)$ , and solutions  $h_w$  over  $A \cup \{b_i \mid i \in w\}$  for every  $w \in [\{b_1, \ldots, b_b\}]^{n-1}$  such that  $\bigcup_w h_w$  is a function, there is a solution h for  $A \cup \{b_i \mid i \leq n\}$  that extends all  $h_w$ .

0-amalgamation is often referred to simply as the existence of solutions and 1-amalgamation is the extension of solutions.

Forgetting the Q predicate,  $M \in \mathbf{K}^n$  is a bunch of affine copies of  $G^M$ , so an isomorphism is determined by a bijection between the copies and picking a 0 from each affine copy. However, adding Q complicates this picture. Solutions are the generalization of picking 0's to  $\mathbf{K}^n$ . Thus, amongst the models of  $\mathbf{K}^n$  admitting solutions (which is at least  $\mathbf{K}^n_{\aleph_{n-2}}$ , see Fact 18.3.9), there is a strong, functorial correspondence between isomorphisms between M and N and pairs of solutions for M and N.

The following is implicit in [BK09], see especially Lemma 2.6 there.

FACT 18.3.7. We work in  $\mathbf{K}^n$ .

(1) Given  $f: M \cong N$  and a solution  $h^M$  of M, there is a unique solution  $h^N$  of N that is conjugate to  $h^M$  by f. Moreover, if  $f': M' \cong N'$  extends<sup>7</sup> f and  $h^{M'}$  is a solution of M' extending  $h^M$ , then the resulting  $h^{N'}$  extends  $h^N$ .

<sup>&</sup>lt;sup>7</sup>So  $M \leq_{\mathbf{K}^n} M'$  and  $N \leq_{\mathbf{K}^n} N'$ .

- (2) Given solutions  $h^M$  for M and  $h^N$  for N and a bijection  $h_0: I(M) \to$ I(N), there is a unique isomorphism  $f: M \cong N$  extending  $h_0$  such that  $h^{M}$  and  $h^{N}$  are conjugate by f. Moreover, if  $h^{M'}$  and  $h^{N'}$  are solutions for M' and N' that extend  $h^{M}$  and  $h^{N}$ , then the resulting f' extends f.
- (3) These processes are inverses of each other: if we have  $[f: M \cong N \text{ and a}]$ solution  $h^M$  of M/[solutions  $h^M$  and  $h^N$  for M and N and a bijection  $h_0: I(M) \to I(N)$  and then apply [(1) and then (2)]/[(2) and then (1)], then [the resulting isomorphism is f]/[the resulting solutions for N is  $h^N$ ],

LEMMA 18.3.8. Suppose  $M, N \in \mathbf{K}^n$  and  $f_0 : I(M) \to I(N)$  is an injection. Then there is a unique extension to  $f_1$  with domain  $M - (G^*(M) \cup H^*(M))$  that must be extended by any strong embedding extending  $f_0$ .

**PROOF.**  $M - (G^*(M) \cup H^*(M))$  is the definable closure of I(M), so the value of  $f_0$  on I(M) determines the value on  $M - (G^*(M) \cup H^*(M))$ .  $\square$ 

For the following, write  $\aleph_{-1}$  for finite.

FACT 18.3.9. Let  $n \in [2, \omega)$ ,  $k_0 < \omega$ , and  $k_1 \in \{-1\} \cup \omega$ . The following are equivalent:

- (1)  $\mathbf{K}^n$  has  $k_0$ -amalgamation of solutions over  $\aleph_{k_1}$ -sized sets.
- (2)  $k_0 + k_1 \le n 2$ .

PROOF. (1) implies (2) by the examples of [**BK09**, Section 6]. (2) implies (1) by combining [**BK09**, 2.11, 2.14]. 

We could do many more variations on the following, but I think this statement suffices for what we need to show.

DEFINITION 18.3.10. For  $n \in [2, \omega)$  and I an index set, the standard model for I is the unique  $M \in \mathbf{K}^n$  such that  $G^*(M) = K \times G_K$ , where  $K := [I]^n$ .

LEMMA 18.3.11. Let  $n \in [3, \omega)$ . Given any  $M \leq_{\mathbf{K}^n} N$  from  $\mathbf{K}^n_{\leq \aleph_{n-3}}$ , we may assume that they are standard. That is, if we write  $M^*$  for the standard model of I(M) and  $N^*$  for the standard model on I(N), then there is an isomorphism  $f: N \cong_{I(N)} N^*$  that restricts to an isomorphism  $M \cong_{I(M)} M^*$ .

**PROOF.** Find a solution  $h^M$  for M and extend it to a solution  $h^N$  for N; this is possible by Fact 18.3.9 since  $(n-3)+1 \le n-2$ . We have solutions  $h^{M^*}$  and  $h^{N^*}$ for  $M^*$  and  $N^*$  because they are the standard models and, thus, have solutions. Then Theorem 18.3.7 allows me to build an isomorphism between M and  $M^*$  and extend it to  $f: N \cong N^*$ , each of which extend the identity on I.  $\square$ 

### 18.4. Tameness and shortness

The following is a strengthening of [BK09, 5.1] to include type-shortness.

THEOREM 18.4.1. For  $n \in [3, \omega)$ ,  $\mathbf{K}^n$  is  $(\langle \aleph_0, \langle \aleph_{n-3} \rangle)$ -type short over  $\leq \aleph_{n-3}$ sized models and  $(\langle \aleph_0, \leq \aleph_{n-3})$ -tame for  $(\langle \aleph_{n-3})$ -length types. Moreover, these Galois types are equivalent to first-order existential (syntactic) types.

**PROOF.** For this proof, write  $tp_{\exists}$  for the first-order existential type. We prove the type-shortness claim. The tameness result follows from [**BK09**, 5.1]. Let  $M \in \mathbf{K}^n_{\leq \aleph_{n-3}}$  and  $M \leq_{\mathbf{K}^n} N^A, N^B$  with  $A \subseteq |N^A|, B \subseteq |N^B|$  of size

 $\leq \aleph_{n-4}$  (we use our convention from Fact 18.3.9 that  $\aleph_{-1}$  means finite) such that

 $\operatorname{tp}_{\exists}(A/M; N^A) = \operatorname{tp}_{\exists}(B/M; N^B)$ . By [**BK09**, 4.2], we can find minimal, full substructures  $M^A$  and  $M^B$ . Additionally, for each finite  $\bar{a} \in A$  and  $\bar{b} \in B$ , we can find minimal full substructures  $M^{\bar{a}}$  and  $M^{\bar{b}}$  in  $M^A$  and  $M^B$ . It's easy to see that  $M^A$ is the directed union of  $\{M^{\bar{a}} \mid \bar{a} \in A\}$  and similarly for  $M^B$ ; note that we don't necessarily have  $M^{\bar{a}}, M^{\bar{a}'} \subseteq |M^{\bar{a} \cup \bar{a}'}|$ , but they are in  $M^{M^{\bar{a}} \cup M^{\bar{a}'}}$ .

Set  $M_0 = M^A \cap M$ . We want to build  $f_0 : M^A \to_{M_0} N^B$  such that  $f_0(A) = B$ . Similarly, construct  $M^B$ . Note that

$$M_0 = M^A \cap M = \bigcup_{\bar{a} \in M} (M^{\bar{a}} \cap M_0) = \bigcup_{\bar{b} \in M} (M^b \cap M_0) = M^B \cap M_0$$

By assumption, we have  $\operatorname{tp}_{\exists}(A/M_0; M^A) = \operatorname{tp}_{\exists}(B/M_0; M^A)$ . Set  $X = \{\pi^{M^A}(x) \mid x \in A \cap G^*(M^A)\}$  and  $Y = \{\pi^{M^B}(x) \mid x \in B \cap G^*(M^B)\}$ , indexed appropriately.

Claim:  $\operatorname{tp}_{\exists}(AX/M_0; M^A) = \operatorname{tp}_{\exists}(BY/M_0; M^A)$ 

This is true because all of the added points are in the definable closure via an existential formula.

Thus, the induced partial map  $f: AX \to BY$  is  $\exists$ -elementary. By Fact 18.3.9, we have extensions of solutions. Let  $h^{M^A}$  be a solution for  $M^A$ . Then we can restrict this to  $h^X$  which is a solution for X. Then we can define a solution  $h^Y$  for Y by conjugating it with f. Finally, we can extend  $h^Y$  to a solution  $h^{M^B}$  for  $M^B$ . Since they satisfy the same existential type and the extensions are minimally constructed, we can define a bijection  $h_0: I(M^A) \to I(M^B)$  respecting the type. Given the two solutions and the bijection  $h_0$ , we can use Theorem 18.3.7 to find an isomorphism  $f_0: M^A \cong M^B$  extending  $h_0$  and making these solutions conjugate. By construction,  $f_0$  fixes  $M_0$  and sends A to B.

Resolve M as  $\langle M_i | i < \alpha \rangle$  starting with  $M_0$  so  $||M_i|| \leq \aleph_{n-4}$ . Then find increasing continuous  $\langle M_i^A, M_i^B | i < \alpha \rangle$  by setting  $M_0^A = M^A$  and  $M_{i+1}^A$  to be a disjoint amalgam<sup>8</sup> of  $M_{i+1}$  and  $M_i^A$  over  $M_i$ , and similarly for  $M_i^B$ .

Using extension of solutions, we can find an increasing chain of solutions  $\langle h^{M_i} | i < \alpha \rangle$  for  $M_i$ . Using 2-amalgamation of solutions over  $\leq \alpha_{n-4}$  sized sets<sup>9</sup>, we can find increasing chains of solutions  $\langle h^{M_i^A}, h^{M_i^B} | i < \alpha \rangle$  for  $M_i^A$  and  $M_i^B$ , respectively, such that  $h^{M_i^A}$  also extends  $h^{M_i}$ .

By another application of Theorem 18.3.7.(2), this gives us an increasing sequence of isomorphism  $\langle f_i : M_i^A \cong_{M_i} M_i^B | i < \alpha \rangle$ ; here we are using that  $I(M_{i+1}^A) - I(M_i^A) = I(M_{i+1}^B) - I(M_i^B)$ . At the top, we have that  $f_\alpha : M^A \cong_M M^B$ . This demonstrates that  $gtp(A/M; N^A) = gtp(B/M; N^B)$ .

Baldwin and Kolesnikov [**BK09**] have shown that tameness fails at the next cardinal and we will see later (Corollary 18.8.12) that  $\mathbf{K}^n$  is not ( $\langle \aleph_{n-3}, \aleph_{n-3} \rangle$ -type short over  $\aleph_{n-3}$ -sized models.

### 18.5. What the abstract theory tells us

We combine the abstract theory with the facts derived so far about the Hart-Shelah example.

<sup>&</sup>lt;sup>8</sup>Crucially, it is an amalgam such that  $I(M_{i+1}^A) = I(M_i^A) \cup I(M_{i+1})$  with the union disjoint over  $I(M_i)$ ; this is guaranteed by the second clause of the claim.

<sup>&</sup>lt;sup>9</sup>Crucially, this holds here, but fails at the next cardinal. Thus, we couldn't use this argument to get  $(<\aleph_0, \aleph_{n-3})$ -type shortness or over  $\aleph_{n-2}$  sized models.

We first give an abstract argument that in the Hart-Shelah example good frames below  $\aleph_{n-3}$  are weakly successful (in fact successful):

THEOREM 18.5.1. Let  $n \in [3, \omega)$ . For any  $k \in [1, n-3]$ , there is a type-full good  $\aleph_k$ -frame  $\mathfrak{s}$  on  $\mathbf{K}^n$ . Moreover,  $\mathfrak{s}$  (and therefore  $\mathfrak{s}^{k,n}$ ) is successful if k < n-3.

PROOF. Let  $\lambda := \aleph_k$ . First, assume that k < n - 3. By Fact 18.3.2,  $\mathbf{K}^n$  is categorical in  $\lambda$ ,  $\lambda^+$  and is  $(<\aleph_0, \leq \lambda^+)$ -tame. By Theorem 18.4.1,  $\mathbf{K}$  is  $(<\aleph_0, \lambda)$ -type-short over  $\lambda$ -sized models. Thus one can apply Fact 18.2.18 (where  $\kappa$  there stands for  $\aleph_0$  here) to get a weakly successful type-full good  $\lambda$ -frame  $\mathfrak{s}$  on  $\mathbf{K}^n$ . By Fact 18.2.19,  $\mathfrak{s}$  is actually successful. This implies that  $\mathfrak{s}^{k,n}$  is successful by canonicity (Fact 18.2.20).

Second, assume k = n - 3. We can still apply Fact 18.2.18 to get the existence of a type-full good  $\lambda$ -frame  $\mathfrak{s}$ , although we do not know it will be weakly successful (in fact this will fail, see Proposition 18.6.6). Then Fact 18.2.20 implies that  $\mathfrak{s}^{k,n}$  is  $\mathfrak{s}$  restricted to types in I.

Note that the case k = 0 is missing here, and will have to be treated differently (see Theorem 18.6.3 and Corollary 18.8.11). On the negative side, we show that  $\mathfrak{s}^{n-3,n}$  cannot be successful. First, we show that it is good<sup>+</sup> (Definition 18.2.14).

LEMMA 18.5.2. For  $n \in [3, \omega)$  and  $k \leq n - 3$ ,  $\mathfrak{s}^{k,n}$  is good<sup>+</sup>.

PROOF. Essentially this is because forking is trivial. In details, suppose that  $\mathfrak{s}^{k,n}$  is not good<sup>+</sup> and fix  $\langle M_i : i < \lambda^+ \rangle$ ,  $\langle N_i : i < \lambda^+ \rangle$ ,  $\langle a_i : i < \lambda^+ \rangle$  and p witnessing it. The set of  $i < \lambda^+$  such that  $M_{\lambda^+} \cap N_i = M_i$  is club, so pick such an i. Since  $\operatorname{gtp}(a_{i+1}/M_{i+1}; M_{i+2})$  is a nonforking extension of p, we know that  $a_{i+1} \in I(M_{i+2}) \setminus I(M_{i+1})$ . Because  $M_{\lambda^+} \cap N_i = M_i$ , we have that  $a_{i+1} \notin |N_i|$ . Since  $a_{i+1} \in I(M_{i+2})$ , also  $a_{i+1} \in I(N_{i+2})$ . Therefore  $\operatorname{gtp}(a_{i+1}/N_i; N_{i+2})$  does not fork over  $M_0$ , contradicting the defining assumption on  $\langle N_i : i < \lambda^+ \rangle$ .

COROLLARY 18.5.3. For  $n \in [3, \omega)$ ,  $\mathfrak{s}^{n-3,n}$  is not successful.

PROOF. Suppose for a contradiction that  $\mathfrak{s}^{n-3,n}$  is successful. Let  $\lambda := \aleph_{n-3}$ . By Fact 18.2.16, we can get a good  $\lambda^+$ -frame on the saturated models of  $\mathbf{K}_{\lambda^+}^n$ . Since  $\mathbf{K}^n$  is categorical in  $\lambda^+$ , this gives a good  $\lambda^+$ -frame on  $\mathbf{K}_{\lambda^+}^n$ . In particular,  $\mathbf{K}^n$  is stable in  $\lambda^+$ , contradicting Fact 18.3.2.

Notice that the proof gives no information as to which part of the definition of successful fails: i.e. whether  $\mathfrak{s}^{n-3,n}$  has the existence property for uniqueness triples (and then smoothness for  $\leq_{\mathbf{K}_{\lambda^+}}^{\mathrm{NF}}$  must fail) or not. To understand this, we take a closer look at uniqueness triples in the specific context of the Hart-Shelah example.

### 18.6. Uniqueness triples in Hart-Shelah

In this section, we show that the frame  $\mathfrak{s}^{n-3,n}$  is *not* weakly successful. This follows from the fact that the existence of uniqueness triples corresponds exactly to amalgamation of solutions.

The following says that it is sufficient to check one point extensions when trying to build uniqueness triples.

LEMMA 18.6.1. Let  $n \in [3, \omega)$  and let  $k \leq n - 3$ . The good  $\aleph_k$ -frame  $\mathfrak{s}^{k,n}$  (see Definition 18.3.3) is weakly successful if the following holds.

- (\*) Whenever  $M, M_a, M_b, M_{ab} \in \mathbf{K}_{\aleph_k}$  are such that:
  - (1)  $I(M_x) = I(M) \cup \{x\}$  for x = a, b, ab;
  - (2)  $M \leq_{\mathbf{K}^n} M_a, M_b$  and  $M_b \leq_{\mathbf{K}^n} M_{ab}$ ; and
  - (3) there is  $f_{\ell}: M_a \to_M M_{ab}$  such that  $f_{\ell}(a) = a$ .
  - Then there is  $f_*: M_{ab} \cong_{M_b} M_{ab}$  such that  $f_* \circ f_1 = f_2$

REMARK 18.6.2. By an easy renaming exercise, we could have the range of  $f_{\ell}$  be distinct one point extensions of  $M_b$  with  $f_{\ell}(a)$  being that point.

PROOF OF LEMMA 18.6.1. Suppose that (\*) holds. Let  $p = \operatorname{gtp}(a/M; N^+) \in \operatorname{gS}^{bs}(M)$  and find some<sup>10</sup>  $M_a \leq_{\mathbf{K}^n} N^+$  so  $I(M_a) = I(M) \cup \{a\}$ . We want to show that this is a uniqueness triple. To this end, suppose that we have  $N \succ M$ ,  $N \leq_{\mathbf{K}^n} M_\ell$ , and  $f_\ell : M_a \to_M M_\ell$  with  $f_\ell(a) \notin N$ . Enumerate  $I(N) - I(M) = \{a_i \mid i < \mu \leq \aleph_k\}$ ; Without loss of generality  $I(M_1) \cap I(M_2) = I(N)$ . Let  $M_\ell^- \leq_{\mathbf{K}^n} M_\ell$  be such that  $I(M_\ell^-) = \{f_\ell(a)\} \cup I(N)$ .

**Claim:** We can find  $f_{-}^*: M_1^- \cong_N M_2^-$  such that  $(f_{-}^*)^{-1} \circ f_1 = f_2$ .

This is enough: from the claim, we have  $M_1^- \leq_{\mathbf{K}^n} M_1$  and  $f_-^*: M_1^- \to M_2$ . The class has disjoint amalgamation by Fact 18.3.2, so find a disjoint amalgam  $N^*$  with maps  $g_\ell: M_\ell \to N^*$  such that  $g_1 \upharpoonright M_1^- = g_2 \circ f_-^*$ . This is the witness required to have that  $(a, M, M_a)$  is a uniqueness triple.

**Proof of the claim:** We can find resolutions  $\langle N_i : i < \mu \rangle$  and  $\langle M_i^{\ell} | i < \mu \rangle$  such that:

(1)  $M \leq_{\mathbf{K}^n} N_i \leq_{\mathbf{K}^n} M_i^{\ell} \leq_{\mathbf{K}^n} M_{\ell}^{-}$  and  $f_{\ell}(M_a) \leq_{\mathbf{K}^n} M_i^{\ell}$ ; and

(2) 
$$I(N) = I(M) \cup \{a_i \mid j < i\}$$
 and  $I(M_i^{\ell}) = I(N_i) \cup \{f_{\ell}(a)\}.$ 

The values of I for these models is specified, which determines K and G. Then  $G^*$  and  $H^*$  are just picked to be subsets of the larger models version that is closed under the relevant action. Since there are embeddings going everywhere, this can be done.

We build increasing, continuous  $f_i^* : M_i^1 \cong_{N_i} M_i^2$  such that  $f_i^* \circ f_1 = f_2$  by induction on  $i \ge 1$ .

- For i = 1, we use (\*) taking  $b = a_0$  (and using the renamed formulation). This gives  $f_1^* : M_1^1 \cong_{N_1} M_1^2$ .
- For i limit, we take unions of everything.
- For i = j + 1, we have an instance of (\*):



Then we can find  $f_{i+1}^*: M_{j+1}^1 \cong M_{j+1}^2$  that works.

We can now give a direct proof of Theorem 18.5.1 that also treats the case k = 0.

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 $<sup>{}^{10}</sup>M_a$  is not unique, but there is such an  $M_a$ 

THEOREM 18.6.3. Let  $n \in [3, \omega)$ . For any k < n - 3,  $\mathfrak{s}^{k,n}$  is successful.

PROOF. By Fact 18.2.19 (as in the proof of Theorem 18.5.1), it is enough to show that  $\mathfrak{s}^{k,n}$  is weakly successful. It suffices to show (\*) from Lemma 18.6.1. We start with a solution h on I(M). Working inside  $M_{ab}$ , we can find extensions  $h_a^1, h_a^2, h_b$  of h that are solutions for  $f_1(M_a), f_2(M_a), M_b$  by the extension property of solutions (which holds because 2-amalgamation does). Now, for  $\ell = 1, 2$ , amalgamate  $h_a^{\ell}$  and  $h_b$  over h into  $h_{ab}^{\ell}$ , which is a solution for  $M_{ab}$ . We use this to get a isomorphism  $f_*$ .

Set  $f_*$  to be the identity on  $I(M_{ab}) = I(M) \cup \{a, b\}$ . This determines its value on K, G, and  $\mathbb{Z}_2$ .

Let  $x \in G_u^*(M_{ab})$  for  $u \in K(M_{ab})$ . There is a unique  $\gamma \in G(M_{ab})$  such that  $t_{G^*}^{M_{ab}}(u, f_{ab}^1(u), x, \gamma)$ . Then, there is a unique  $y \in G^*_u(M_{ab})$  such that  $t_{G^*}^{M_{ab}}(u, f_{ab}^2(u), y, \gamma)$ . Set  $f_*(x) = y$ .

Let  $x \in H^*_u(M_{ab})$  for  $u \in K(M_{ab})$ . There is a unique  $n \in H(M_{ab})$  such that  $t^{M_{ab}}_{H^*}(u, f^1_{ab}(u), x, n)$ . Then, there is a unique  $y \in H^*_u(M_{ab})$  such that  $t^{M_{ab}}_{H^*}(u, f^2_{ab}(u), y, n)$ . Set  $f_*(x) = y$ .

This is a bijection on the universes, and clearly preserves all structure except maybe Q. So we show it preserves Q. It suffices to show one direction for positive instances of Q. So let  $u_1, \ldots, u_k, v$  be compatible from  $K(M_{ab})$  and  $x_j \in G^*_{u_j}(M_{ab}), y \in H^*_v(M_{ab})$  such that

$$M_{ab} \vDash Q(x_1, \ldots, x_k, y)$$

Note, by definition of solutions, we have

$$M_{ab} \models Q\left(f_{ab}^{1}(u_{1}), \dots, f_{ab}^{1}(u_{k}), g_{ab}^{1}(v)\right)$$
$$M_{ab} \models Q\left(f_{ab}^{2}(u_{1}), \dots, f_{ab}^{2}(u_{k}), g_{ab}^{2}(v)\right)$$

By the properties of Q, we get  $\gamma_i \in G(M_{ab})$  and  $n \in H(M_{ab})$  such that

- $\begin{array}{ll} (1) \ t_{G^*}^{M_{ab}}(u_j, f_{ab}^1(u_j), x_j, \gamma_j) \\ (2) \ t_{H^*}^{M_{ab}}(v, g_{ab}^1(v), y, n) \\ (3) \ \gamma_1(v) + \dots + \gamma_k(v) + n \equiv 0 \ \ \mathrm{mod} \ 2 \end{array}$

Then, by definition of  $f_*$ , we have

- (1)  $t_{G^*}^{M_{ab}}(u_j, f_{ab}^2(u_j), f_*(x_j), \gamma_j)$ (2)  $t_{H^*}^{M_{ab}}(v, g_{ab}^2(v), f_*(y), n)$

By the evenness of these shifts, we have that

$$M_{ab} \models Q(f_*(x_1), \dots, f_*(x_k), f_*(y))$$

Perfect.

The commutativity condition is easy to check.

The next two lemmas show that the uniqueness triples (if they exist) must be exactly the one point extensions. This can be seen from the abstract theory [She09a, III.3.5] but we give a direct proof here.

LEMMA 18.6.4. Let  $n \in [3, \omega)$  and let  $k \leq n-3$ . If  $(a, M, M^+)$  is a uniqueness triple of  $\mathfrak{s}^{k,n}$ , then  $I(M^+) = I(M) \cup \{a\}$ .

Recall (Definition 18.3.10) that the standard model is the one where  $G^*$  is literally equal to  $K \times G$ , so that we can easily recover 0's.

PROOF. Deny. By Lemma 18.3.11, without loss of generality, we have that M is the standard model on I(M) = X and  $M^+$  is the standard model on  $I(M^+) = X \cup X^+ \cup \{a\}$  (those unions are disjoint) with  $X^+$  nonempty. Set N to be the standard model on  $X \cup (2 \times X^+)$  and  $N_0, N_1$  to be standard models on  $X \cup 2 \times X^+ \cup \{a\}$ . For  $\ell = 0, 1$ , define  $f_\ell : M^+ \to_M N_\ell$  by

- (1)  $f_{\ell}$  is the identity on  $X \cup \{a\}$  and sends  $x \in X^+$  to  $(\ell, x)$ .
- (2) The above determines the map on K, H, and G.
- (3)  $(u, x) \in G^*(M^+)$  goes to  $(f_\ell(u), x) \in G^*(N_\ell)$ .
- (4)  $(u, n) \in H^*(M^+)$  goes to  $(f_{\ell}(u), n) \in H^*(N_{\ell})$ .

Then this is clearly a set-up for weak uniqueness. However, suppose there were a  $N^*$  with  $g_{\ell} : N_{\ell} \to_N N^*$  such that  $g_0 \circ f_0 = g_1 \circ f_1$ . Let  $x \in X^+$ . Then

$$f(0,x) = g_0(x) = f_0(g_0(x)) = f_1(g_1(x)) = f_1(1,x) = (1,x)$$

which is false.

LEMMA 18.6.5. Let  $n \in [3, \omega)$  and let  $k \leq n-3$ . Let  $M \leq_{\mathbf{K}^n} N$  both be in  $\mathbf{K}^n_{\aleph_k}$ . If  $\mathfrak{s}^{k,n}$  is weakly successful, then (a, M, N) is a uniqueness triple of  $\mathfrak{s}^{k,n}$  if and only if  $I(N) = I(M) \cup \{a\}$ .

PROOF. Lemma 18.6.4 gives one direction. Conversely, let (a, M, N) with  $I(N) = I(M) \cup \{a\}$ . Since  $\mathfrak{s}^{k,n}$  is weakly successful, there is some uniqueness triple (b, M', N') representing gtp(a/M; N). By Lemma 18.6.4, we must have  $I(N') = I(M') \cup \{b\}$ . By Lemma 18.3.11, we have  $(M, N) \cong (M', N')$  since they are both isomorphic to the standard model. This isomorphism must take a to b. Since  $(a, M, N) \cong (b, M', N')$ , the former is a uniqueness triple as well.  $\Box$ 

We deduce that  $\mathfrak{s}^{n-3,n}$  is not even weakly successful.

THEOREM 18.6.6. For  $n \in [3, \omega)$ ,  $\mathfrak{s}^{n-3,n}$  is not weakly successful.

PROOF. Let  $\lambda := \aleph_{k-3}$ . At this cardinal, 2-amalgamation of solutions over sets of size  $\lambda$  fails. To witness this, we have:

- *M* of size  $\lambda$  with solution h = (f, g)
- $M_a$  has a solution  $h_a = (f_a, g_a)$
- $M_b$  has a solution  $h_b = (f_b, g_b)$
- $M_{ab}$  has no solution that extends them both
- $I(M_x) = I(M) \cup \{x\}$  for x = a, b, ab

However,  $\lambda$  does have extension of solutions, so let  $h_{ab} = (f_{ab}, g_{ab})$  be a solution for  $M_{ab}$  that extends  $h_b$ .  $h_{ab}$  is a solution for  $I(M_a)$  in  $M_{ab}$ .<sup>11</sup> Set  $f_1 : M_a \to_M M_{ab}$  to be the identity. Define  $f_2 : M_a \to_M M_{ab}$  as follows:

- identity on  $I(M) \cup \{a\}$ , which determines it except on the affine stuff (in the sense of Lemma 18.3.8)
- Let  $x \in G_u^*(M_a)$  for  $u \in K(M_a)$ . Set  $f_2$  to send  $f_a(u)$  to  $f_{ab}(u)$  and the rest falls out by the G action
- Let  $x \in H_u^*(M_a)$  for  $u \in K(M_a)$ . Set  $f_2$  to send  $g_a(u)$  to  $g_{ab}(u)$  and the rest falls out by the G action.

<sup>&</sup>lt;sup>11</sup>Note that it isn't a solution in  $M_a$  as  $f_{ab}(u)$  might not be in  $M_a$  for  $u \in M_a$ .
This map commutes on M because if  $u \in K(M)$ , then  $f_{ab}(u) = f_a(u) = f(u)$ .

We claim that  $gtp(a/M; M_a)$  does not have a uniqueness triple. Suppose it does. By Lemma 18.6.5,  $(a, M, M_a)$  is one.

Suppose that we had  $N^*$  and  $g_{\ell} : M_{ab} \to_{M_b} N^*$  such that  $g_1 = g_2 f_2$  and  $g_1(a) = g_2(f_2(a))$  (recalling that  $f_1$  is the identity).

**Claim:** If  $u \in K(M_a)$ , then  $g_1(G_u^*(M_{ab})) = g_2(G_u^*(M_{ab}))$ .

There is  $\gamma_u \in G(M_{ab})$  such that  $f_{ab}(u) = f_a(u) + \gamma_u$ . Given  $x \in G^*_u(M_{ab})$ ,

$$g_1(x) = g_2(f_2(x)) = g_2(x + \gamma_u) = g_2(x) + \gamma_u$$

Thus  $g_1(G_u^*(M_{ab}))$  and  $g_2(G_u^*(M_{ab}))$  are both subsets of  $G_u^*(N^*)$  that have a 1-transitive action of  $G(M_{ab})$  and share points.

Now define  $h^+ = (f^+, g^+)$  on  $M_{ab}$  by

$$\begin{array}{lcl} f^+(u) & = & g_1^{-1} \circ g_2 \circ f_{ab}(u) \\ g^+(u) & = & g_1^{-1} \circ g_2 \circ g_{ab}(u) \end{array}$$

We claim  $h^+$  extends both  $h_a$  and  $h_b$ . If  $u \in K(M_b)$ , then  $f_{ab}(u) = f_b(u) \in M_b$ , so

$$f^{+}(u) = g_1^{-1} \circ g_2 \circ f_{ab}(u) = f_{ab}(u) = f_b(u)$$

since the  $g_{\ell}$ 's fix  $M_b$ . Suppose  $u \in K(M_a)$ . First note that  $g_1^{-1} \circ g_2 = f_2^{-1}$  by assumption. Also, since  $f_2(f_a(u)) = f_{ab}(u)$  and  $f_2$  respects the group action,  $f_2(f_{ab}(u)) = f_a(u)$ . Thus

$$f^+(u) = g_1^{-1} \circ g_2 \circ f_{ab}(u) = f_2^{-1} \circ f_{ab}(u) = f_a(u)$$

Similarly for  $g^+$ .

But this is our contradiction!  $h_a$  and  $h_b$  were not amalgamable, so there is no isomorphism.

### 18.7. Nonforking is disjoint amalgamation

Recall that if a good frame is weakly successful, one can define an independence relation NF for models (see Definition 18.2.11). We show here that NF in the Hart-Shelah example is just disjoint amalgamation, i.e. NF $(M_0, M_1, M_2, M_3)$  holds if and only if  $M_0 \leq_{\mathbf{K}^n} M_\ell \leq_{\mathbf{K}^n} M_3$  for  $\ell < 4$  and  $M_1 \cap M_2 = M_0$ . We deduce another proof of Theorem 18.6.6.

We will use the following weakening of [**BK09**, 4.2]

FACT 18.7.1. Let  $n \in [2, \omega)$ . If  $M_0, M_1 \leq_{\mathbf{K}^n} N$ , then there is  $M_2 \leq_{\mathbf{K}^n} N$  such that  $I(M_2) = I(M_0) \cup I(M_1)$  and  $M_0, M_1 \leq_{\mathbf{K}^n} M_2$ .

THEOREM 18.7.2. Let  $n \in [3, \omega)$  and let  $k \leq n-3$ . Let  $\lambda := \aleph_k$  and let  $M_0, M_1, M_2, M_3 \in \mathbf{K}^n_{\lambda}$  with  $M_0 \leq_{\mathbf{K}^n} M_\ell \leq_{\mathbf{K}^n} M_3$  for  $\ell < 4$ . If  $\mathfrak{s}^{k,n}$  is weakly successful, then  $\operatorname{NF}_{\mathfrak{s}^{k,n}}(M_0, M_1, M_2, M_3)$  if and only if  $M_1 \cap M_2 = M_0$ .

PROOF. Write NF for NF<sub>s<sup>k,n</sup></sub>. The left to right direction follows from the properties of NF (Fact 18.2.12). Now assume that  $M_1 \cap M_2 = M_0$ .

Write  $I(M_1) - I(M_0) = \{d_i \mid i < \alpha^*\}$ . By induction, build increasing, continuous  $M_{1,i} \leq_{\mathbf{K}^n} M_1$  for  $i < \alpha^*$  so  $I(M_{1,i}) = I(M_0) \cup \{d_j \mid j < i\}$ . Again by induction, build increasing continuous  $M_{2,i} \leq_{\mathbf{K}^n} M_3$  for  $i \leq \alpha^*$  such that

•  $I(M_{2,i}) = I(M_2) \cup \{d_i \mid j < i\}$ 

### • $M_{1,i} \leq_{\mathbf{K}^n} M_{2,i}$

The successor stage of this construction is possible by Fact 18.7.1 and the limit is easy. Now it's easy to see that  $gtp(d_i/M_{2,i}; M_{2,i+1})$  does not fork over  $M_{1,i}$ . Furthermore by Lemma 18.6.5,  $(d_i, M_{1,i}, M_{1,i+1})$  is a uniqueness triple. Thus letting  $M'_3 := M_{2,\alpha^*}$ , we have that NF<sup>\*</sup> $(M_0, M_1, M_2, M'_3)$ , so NF $(M_0, M_1, M_2, M'_3)$ . By the monotonicity property of NF, NF $(M_0, M_1, M_2, M_3)$  also holds.

We deduce another proof of Theorem 18.6.6. First we show that weakly successful implies successful in the context of Hart-Shelah:

LEMMA 18.7.3. Let  $n \in [3, \omega)$  and let  $k \leq n-3$ . If  $\mathfrak{s}^{k,n}$  is weakly successful, then  $\mathfrak{s}$  is successful (recall Definition 18.2.13). Moreover for  $M_0, M_1 \in \mathbf{K}_{\lambda^+}^n, M_0 \leq_{\mathbf{K}_{\lambda^+}}^{\mathrm{NF}} M_1$  if and only if  $M_0 \leq_{\mathbf{K}^n} M_1$ .

PROOF. This is straightforward from Definition 18.2.13 and Theorem 18.7.2.  $\hfill \Box$ 

COROLLARY 18.7.4. For  $n \in [3, \omega)$ ,  $\mathfrak{s}^{n-3,n}$  is not weakly successful.

PROOF. Assume for a contradiction that  $\mathfrak{s}^{n-3,n}$  is weakly successful. By Lemma 18.7.3,  $\mathfrak{s}^{n-3,n}$  is successful. This contradicts Corollary 18.5.3.

### **18.8.** A type-full good frame at $\aleph_0$

We have seen that when k < n-3,  $\mathfrak{s}^{k,n}$  is successful good<sup>+</sup> and therefore by Fact 18.2.17 extends to a type-full frame. When k = n-3,  $\mathfrak{s}^{k,n}$  is not successful, but by Theorem 18.5.1, it still extends to a type-full frame if  $k \ge 1$ . In this section, we complete the picture by building a type-full frame when k = 0 and n = 3.

Recall that (when  $n \geq 3$ )  $\mathbf{K}^n$  is a class of models of an  $\mathbb{L}_{\omega_1,\omega}$  sentence, categorical in  $\aleph_0$  and  $\aleph_1$ . Therefore by [She09a, II.3.4] (a generalization of earlier results in [She75a, She83a]), there will be a good  $\aleph_0$ -frame on  $\mathbf{K}^n$  provided that  $2^{\aleph_0} < 2^{\aleph_1}$ . Therefore the result we want is at least consistent with ZFC, but we want to use the additional structure of the Hart-Shelah example to remove the cardinal arithmetic hypothesis.

So we take here a different approach than Shelah's, giving new cases on when an AEC has a good  $\aleph_0$ -frame. As opposed to Shelah, we use Ehrenfeucht-Mostowski models (so assume that the AEC has arbitrarily large models).

We start by studying what limit models look like in the Hart-Shelah example: Recall that we're working in a zone where we have extensions of solutions.

THEOREM 18.8.1. Let  $n \in [3, \omega)$ . Let  $k \leq n-3$  and let  $M_0, M_1 \in \mathbf{K}_{\aleph_k}^n$ . Then  $M_1$  is universal over  $M_0$  if and only if  $|I(M_1) - I(M_0)| = ||M_1||$ . In particular,  $M_1$  is universal over  $M_0$  if and only if  $M_1$  is limit over  $M_0$ .

PROOF. First suppose that  $M_1$  is universal over  $M_0$ . We don't have maximal models, so let  $M_0 \leq_{\mathbf{K}^n} N_*$  be such that  $|I(N_*) - I(M_0)| = ||M_1||$ . We have that  $||N_*|| = ||M_1||$ , so there is an embedding  $f : N_* \to_{M_0} M_1$ . Then  $f(I(N_*)) \subseteq I(M_1)$ .

Now suppose that  $|I(M_1) - I(M_0)| = ||M_1||$  and let  $M_0 \leq_{\mathbf{K}^n} N_*$  with  $||N_*|| = ||M_1||$ . Let  $I^- \subseteq I(M_1) - I(M_0)$  be of size  $|I(N_*) - I(M_0)|$  and let  $M^- \leq_{\mathbf{K}^n} M_1$  have  $I(M^-) = I(M_0) \cup I(M^-)$ . Let (f,g) be a solution for  $M_0$ . Since we have extensions of solutions, we can extend this to solutions  $(f^-, g^-)$  on  $M^-$  and  $(f_*, g_*)$  on  $N_*$ . The whole point of solutions is that this allows us to build an isomorphism

between  $M^-$  and  $N_*$  over  $M_0$  by mapping the solutions to each other (see Theorem 18.3.7).

Shelah has defined a similar property  $[She09a, 1.3(2)]^{12}$ :

DEFINITION 18.8.2. **K** is  $\lambda$ -saturative (or saturative in  $\lambda$ ) if for any  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$  all in  $\mathbf{K}_{\lambda}$ , if  $M_1$  is limit over  $M_0$ , then  $M_2$  is limit over  $M_0$ .

So an immediate consequence of Theorem 18.8.1 is:

COROLLARY 18.8.3. Let  $n \in [3, \omega)$ . For any  $k \leq n-3$ ,  $\mathbf{K}^n$  is saturative in  $\aleph_k$ .

We will use the following consequence of being saturative:

LEMMA 18.8.4. Assume that  $LS(\mathbf{K}) = \aleph_0$ , and  $\mathbf{K}_{\aleph_0}$  has amalgamation, no maximal models, and is stable in  $\aleph_0$ . Let  $\langle M_i : i \leq \omega \rangle$  be an increasing continuous chain in  $\mathbf{K}_{\aleph_0}$ . If  $\mathbf{K}$  is categorical in  $\aleph_0$  and saturative in  $\aleph_0$ , then there exists an increasing continuous chain  $\langle N_i : i \leq \omega \rangle$  such that:

- (1) For  $i < \omega$ ,  $M_i$  is limit over  $N_i$ .
- (2) For  $i < \omega$ ,  $N_{i+1}$  is limit over  $N_i$ .
- (3)  $N_{\omega} = M_{\omega}$ .

PROOF. Let  $\{a_n : n < \omega\}$  be an enumeration of  $|M_{\omega}|$ . We will build  $\langle N_i : i \leq \omega \rangle$  satisfying (1) and (2) above and in addition that for each  $i < \omega$ ,  $\{a_n : n < i\} \cap |M_i| \subseteq |N_i|$ . Clearly, this is enough.

This is possible. By categoricity in  $\aleph_0$ , any model of size  $\aleph_0$  is limit, so pick any  $N_0 \in \mathbf{K}_{\aleph_0}$  such that  $M_0$  is limit over  $N_0$ . Now assume inductively that  $N_i$  has been defined for  $i < \omega$ . Since  $\mathbf{K}$  is saturative in  $\aleph_0$ ,  $M_{i+1}$  is limit over  $N_i$ . Since all limit models of the same cofinality are isomorphic,  $M_{i+1}$  is in particular  $(\aleph_0, \omega \cdot \omega)$ -limit over  $N_i$ . Fix an increasing continuous sequence  $\langle M_{i+1,j} : j \leq \omega \cdot \omega \rangle$  witnessing it:  $M_{i+1,0} = N_i$ ,  $M_{i+1,\omega \cdot \omega} = M_{i+1}$ , and  $M_{i+1,j+1}$  is universal over  $M_{i+1,j}$  for all  $j < \omega \cdot \omega$ . Now pick  $j < \omega \cdot \omega$  big enough so that  $\{a_n : n < i+1\} \cap |M_{i+1}| \subseteq |M_{i+1,j}|$ . Let  $N_{i+1} := M_{i+1,j+\omega}$ .

REMARK 18.8.5. We do not know how to replace  $\aleph_0$  by an uncountable cardinal in the argument above: it is not clear what to do at limit steps.

To build the good frame, we will also use the transitivity property of splitting:

DEFINITION 18.8.6. We say that **K** satisfies transitivity in  $\mu$  (or  $\mu$ -transitivity) if whenever  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$ ,  $M_1$  is limit over  $M_0$  and  $M_2$  is limit over  $M_1$ , if  $p \in gS(M_2)$  does not  $\mu$ -split over  $M_1$  and  $p \upharpoonright M_1$  does not  $\mu$ -split over  $M_0$ , we have that p does not  $\mu$ -split over  $M_0$ .

The following result of Shelah [She99, 7.5] is key:

FACT 18.8.7. Let  $\mu \geq \text{LS}(\mathbf{K})$ . Assume that  $\mathbf{K}_{\mu}$  has amalgamation and no maximal models. If **K** has arbitrarily large models and is categorical in  $\mu^+$ , then **K** has transitivity in  $\mu$ .

We will also use two lemmas on splitting isolated by VanDieren [Van06, I.4.10, I.4.12].

<sup>&</sup>lt;sup>12</sup>Shelah defines saturative as a property of frames, but it depends only on the class.

FACT 18.8.8. Let  $\mu \geq LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{\mu}$  has amalgamation, no maximal models, and is stable in  $\mu$ . Let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  all be in  $\mathbf{K}_{\mu}$  such that M is universal over  $M_0$ .

- (1) Weak extension: If  $p \in gS(M)$  does not  $\mu$ -split over  $M_0$ , then there exists  $q \in gS(N)$  extending p and not  $\mu$ -splitting over  $M_0$ . Moreover q is algebraic if and only if p is algebraic.
- (2) Weak uniqueness: If  $p, q \in gS(N)$  do not  $\mu$ -split over  $M_0$  and  $p \upharpoonright M = q \upharpoonright M$ , then p = q.

We are now ready to build the good frame:

THEOREM 18.8.9. If:

- (1) **K** is superstable in  $\aleph_0$ .
- (2) **K** has symmetry in  $\aleph_0$ .
- (3) **K** has transitivity in  $\aleph_0$ .
- (4) **K** is categorical in  $\aleph_0$ .
- (5) **K** is saturative in  $\aleph_0$ .

Then there exists a type-full good  $\aleph_0$ -frame with underlying class  $\mathbf{K}_{\aleph_0}$ .

PROOF. By the superstability assumption,  $\mathbf{K}_{\aleph_0}$  has amalgamation and no maximal models and is stable in  $\aleph_0$ . By the categoricity assumption,  $\mathbf{K}_{\aleph_0}$  also has joint embedding. It remains to define an appropriate forking notion. For  $M \leq_{\mathbf{K}} N$  both in  $\mathbf{K}_{\aleph_0}$ , let us say that  $p \in \mathrm{gS}(N)$  does not fork over M if there exists  $M_0 \in \mathbf{K}_{\aleph_0}$ such that M is universal over  $M_0$  and p does not  $\aleph_0$ -split over  $M_0$ . We check that it has the required properties (see Definition 18.2.7):

- (1) Invariance, monotonicity: Straightforward.
- (2) Extension existence: By the weak extension property of splitting (Fact 18.8.8).
- (3) Uniqueness: Let  $M \leq_{\mathbf{K}} N$  both be in  $\mathbf{K}_{\aleph_0}$  and let  $p, q \in \mathrm{gS}(N)$  be nonforking over M such that  $p \upharpoonright M = q \upharpoonright M$ . Using the extension property, we can make N bigger if necessary to assume without loss of generality that N is limit over M. By categoricity, M is limit. Pick  $\langle M_i : i \leq \omega \rangle$  increasing continuous witnessing it (so  $M_\omega = M$  and  $M_{i+1}$  is universal over  $M_i$  for all  $i < \omega$ ). By the superstability assumption, there exists  $i < \omega$  such that  $p \upharpoonright M$  does not  $\aleph_0$ -split over  $M_i$  and there exists  $j < \omega$  such that  $q \upharpoonright M$  does not  $\aleph_0$ -split over  $M_j$ . Let  $i^* := i + j$ . Then both  $p \upharpoonright M$  and  $q \upharpoonright M$  do not  $\aleph_0$ -split over  $M_{i^*}$ . By  $\aleph_0$ -transitivity, both p and q do not  $\aleph_0$ -split over  $M_{i^*}$ . Now use the weak uniqueness property of splitting (Fact 18.8.8).
- (4) Continuity: In the type-full context, this follows from local character (see [She09a, II.2.17(3)]).
- (5) Local character: Let  $\delta < \omega_1$  be limit and let  $\langle M_i : i \leq \delta \rangle$  be increasing continuous in  $\mathbf{K}_{\aleph_0}$ . Let  $p \in \mathrm{gS}(M_\delta)$ . We want to see that there exists  $i < \delta$  such that p does not fork over  $M_i$ . We have that cf  $\delta = \omega$ , so without loss of generality  $\delta = \omega$ . Let  $\langle N_i : i \leq \omega \rangle$  be as given by Lemma 18.8.4 (we are using saturativity here). By superstability, there exists  $i < \omega$  such that p does not  $\aleph_0$ -split over  $N_i$ . Because  $M_i$  is limit (hence universal) over  $M_i$ , this means that p does not fork over  $M_i$ , as desired.
- (6) Symmetry: by  $\aleph_0$ -symmetry (see Theorem 10.4.13).

COROLLARY 18.8.10. Assume that  $LS(\mathbf{K}) = \aleph_0$ . If:

(1) **K** has amalgamation in  $\aleph_0$ .

- (2) **K** is categorical in  $\aleph_0$ .
- (3) **K** is saturative in  $\aleph_0$ .
- (4) **K** has arbitrarily large models and is categorical in  $\aleph_1$ .

Then there exists a type-full good  $\aleph_0$ -frame with underlying class  $\mathbf{K}_{\aleph_0}$ .

PROOF. It is enough to check that the hypotheses of Theorem 18.8.9 are satisfied. First note that **K** has no maximal models in  $\aleph_0$  because it has a model in  $\aleph_1$  (by solvability) and is categorical in  $\aleph_0$ . Therefore by Fact 18.2.4, **K** is  $\aleph_0$ superstable. By Fact 18.2.6, **K** has  $\aleph_0$ -symmetry. Finally by Fact 18.8.7, **K** has  $\aleph_0$ -transitivity.

COROLLARY 18.8.11. For  $n \in [3, \omega)$ , there exists a type-full good  $\aleph_0$ -frame on  $\mathbf{K}^n$ .

PROOF. By Fact 18.3.2 and Corollary 18.8.3,  $\mathbf{K}^n$  satisfies the hypotheses of Corollary 18.8.10.

The argument also allows us to prove that Theorem 18.4.1 is optimal, even when n = 3:

COROLLARY 18.8.12. For  $n \in [3, \omega)$ ,  $\mathbf{K}^n$  is not  $(\langle \aleph_{n-3}, \aleph_{n-3} \rangle$ -type short over  $\aleph_{n-3}$ -sized models.

PROOF. Let  $\lambda := \aleph_{n-3}$ . By Theorem 18.5.1 (or Corollary 18.8.11 if  $\lambda = \aleph_0$ ), there is a type-full good  $\lambda$ -frame  $\mathfrak{s}$  on  $\mathbf{K}_{\lambda}$ . Assume for a contradiction that  $\mathbf{K}^n$  is  $(<\lambda,\lambda)$ -type short over  $\lambda$ -sized models. We will prove that  $\mathfrak{s}$  is weakly successful. This will imply (by Fact 18.2.20 and the definition of uniqueness triples) that  $\mathfrak{s}^{n-3,n}$ is weakly successful, contradicting Theorem 18.6.6. First observe that by Theorem 18.4.1,  $\mathbf{K}^n$  must be  $(<\aleph_0, \lambda)$ -type short over  $\lambda$ -sized models.

We now consider two cases.

- If  $\lambda > \aleph_0$ , then (recalling Facts 18.3.2 and 18.2.20) by Fact 18.2.18 (where  $\kappa$  there stands for  $\aleph_0$  here),  $\mathfrak{s}$  is weakly successful, which is the desired contradiction.
- If  $\lambda = \aleph_0$ , we proceed similarly: For  $M \leq_{\mathbf{K}} N$  both in  $\mathbf{K}_{\aleph_0}$  and  $p \in gS^{\alpha}(N)$  with  $\alpha < \aleph_1$ , let us say that p does not fork over M if for every finite  $I \subseteq \alpha$  there exists  $M_0 \leq_{\mathbf{K}} M$  with M universal over  $M_0$  such that  $p^I$  does not  $\mu$ -split over  $M_0$ . As in the proof of Theorem 18.8.9 (noting that in Fact 18.8.7 transitivity holds for any type of finite length), this nonforking relation has the uniqueness property for types of finite length. By the shortness assumption, it has it for types of length at most  $\aleph_0$  too. It is easy to see that nonforking satisfies local character for  $(< \aleph_0)$ -length types over  $(\aleph_0, \aleph_1)$ -limits and has the left  $(< \aleph_0)$ -witness property (see Definition 14.3.7). Therefore by Lemmas 14.3.8 and 14.3.9 it reflects down (see Definition 14.3.7). By Corollary 14.3.11,  $\mathfrak{s}$  is weakly successful, as desired.

### CHAPTER 19

# Toward a stability theory of tame abstract elementary classes

This chapter is based on [Vash].

### Abstract

We initiate a systematic investigation of the abstract elementary classes that have amalgamation, satisfy tameness (a locality property for orbital types), and are stable (in terms of the number of orbital types) in some cardinal. Assuming the singular cardinal hypothesis (SCH), we prove a full characterization of the (highenough) stability cardinals, and connect the stability spectrum with the behavior of saturated models.

We deduce (in ZFC) that if a class is stable on a tail of cardinals, then it has no long splitting chains (the converse is known). This indicates that there is a clear notion of superstability in this framework.

We also present an application to homogeneous model theory: for D a homogeneous diagram in a first-order theory T, if D is both stable in |T| and categorical in |T| then D is stable in all  $\lambda \geq |T|$ .

### 19.1. Introduction

**19.1.1.** Motivation and history. Abstract elementary classes (AECs) are partially ordered classes  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  which satisfy several of the basic category-theoretic properties of classes of the form  $(\operatorname{Mod}(T), \preceq)$  for T a first-order theory. They were introduced by Saharon Shelah in the late seventies [She87a] and encompass infinitary logics such as  $\mathbb{L}_{\lambda^+,\omega}(Q)$  as well as several algebraic examples. One of Shelah's test questions is the eventual categoricity conjecture: an AEC categorical in *some* high-enough cardinal should be categorical in *all* high-enough cardinals.

Toward an approximation, work of Makkai and Shelah [**MS90**] studied classes of models of an  $\mathbb{L}_{\kappa,\omega}$  theory categorical in a high-enough cardinal, when  $\kappa$  is a strongly compact cardinal. They proved [**MS90**, 1.13] that such a class has (eventual) amalgamation, joint embedding, and no maximal models. Thus one can work inside a monster model and look at the corresponding orbital types. Makkai and Shelah proved that the orbital types correspond to certain syntactic types, implying in particular that two orbital types are equal if all their restrictions of size less than  $\kappa$  are equal. They then went on to develop some theory of superstability and concluded that categoricity in some high-enough successor implies categoricity in all high-enough cardinals.

A common theme of recent work on AECs is to try to replace large cardinal hypotheses with their model-theoretic consequences. For example, regardless of whether there are large cardinals, many classes of interests have a monster model and satisfy a locality property for their orbital types (see the introduction to [**GV06b**] or the list of examples in the recent survey [**BVd**]). Toward that end, Grossberg and VanDieren made the locality property isolated by Makkai and Shelah (and later also used by Shelah in another work [**She99**]) into a definition: Call an AEC  $\mu$ -tame if its orbital types are determined by their  $\mu$ -sized restrictions. Will Boney [**Bon14b**] has generalized the first steps in the work of Makkai and Shelah to AECs, showing that tameness follows from a large cardinal axiom (amalgamation also follows if one assumes categoricity). Earlier, Shelah had shown that Makkai and Shelah's downward part of the transfer holds assuming amalgamation (but not tameness) [**She99**] and Grossberg and VanDieren used Shelah's proof (their actual initial motivation for isolating tameness) to show that the upward part of the transfer holds in tame AECs with amalgamation.

Recently, the superstability theory of tame AECs with a monster model has seen a lot of development (see [Bon14a] and Chapters 4,7, 10, and 9) and one can say that most of Makkai and Shelah's work has been generalized to the tame context (see also [Bal09, D.9(3)]). New concepts not featured in the Makkai and Shelah paper, such as good frames and limit models, have also seen extensive studies (e.g. in the previously-cited papers and in Shelah's book [She09a]). The theory of superstability for AECs has had several applications, including a full proof of Shelah's eventual categoricity conjecture in universal classes, see Chapter 16.

While we showed with Grossberg in Chapter 9 that several possible definitions of superstability are all equivalent in the tame case, it was still open whether stability on a tail of cardinals implied these possible definitions (e.g. locality of forking).

The present chapter answers positively (see Corollary 19.4.24) by developing the theory of *strictly stable* tame AECs with a monster model. We emphasize that this is *not* the first work on strictly stable AECs. In their paper introducing tameness [**GV06b**], Grossberg and VanDieren proved several fundamental results (see also [**BKV06**]). Shelah [**She99**, §4,§5] has made some important contributions without even assuming tameness; see also his work on universal classes [**She09b**, V.E]. Several recent works [**BG**, **BVa**] (as well as Chapters 2 and 7 in this thesis) establish results on independence, the first stability cardinal, chains of saturated models, and limit models. The present chapter aims to put these works together and improve some of their results using either the superstability machinery mentioned above or (in the case of Shelah's tameness-free results) assuming tameness.

19.1.2. Outline of the main results. Fix an LS(K)-tame AEC K with a monster model. Assume that K is stable (defined by counting Galois types) in some cardinal. Let  $\underline{\chi}(\mathbf{K})$  be the class of regular cardinals  $\chi$  such that for all high-enough stability cardinals  $\mu$ , any type over the union of a  $(\mu, \chi)$ -limit chain  $\langle M_i : i < \chi \rangle$  does not  $\mu$ -split over some  $M_i$ . Note that we do not know whether  $\underline{\chi}(\mathbf{K})$  must be an end segment of regular cardinals or whether it can have gaps (we can give a locality condition implying that it is an end segment, see Corollary 19.2.6 and Theorem 19.3.7).

Using results from the theory of averages in tame AECs (developed in Chapters 7, 9), we show assuming the singular cardinal hypothesis (SCH<sup>1</sup>) that for all highenough cardinals  $\mu$ , **K** is stable in  $\mu$  if and only if cf  $\mu \in \chi(\mathbf{K})$  (see Corollary

<sup>&</sup>lt;sup>1</sup>That is, for every infinite singular cardinal  $\lambda$ ,  $\lambda^{cf \lambda} = 2^{cf \lambda} + \lambda^+$ .

19.4.22). The right to left direction is implicit in Theorem 4.5.7 but the left to right direction is new. A consequence of the proof of Corollary 19.4.22 is that stability on a tail implies that  $\underline{\chi}(\mathbf{K})$  contains all regular cardinals (Corollary 19.4.24, note that this is in ZFC).

We then prove that  $\underline{\chi}(\mathbf{K})$  connects the stability spectrum with the behavior of saturated models: assuming SCH, a stable tame AEC with a monster model has a saturated in a high-enough  $\lambda$  if and only if  $[\lambda = \lambda^{<\lambda} \text{ or } \mathbf{K} \text{ is stable in } \lambda]$ . In ZFC, we deduce that having saturated models on a tail of cardinals implies superstability (Corollary 19.5.9). We conclude with Theorem 19.6.3, giving (in ZFC) several equivalent definitions of  $\underline{\chi}(\mathbf{K})$ , in terms of uniqueness of limit models, existence of saturated models, or the stability spectrum. Sections 19.7-19.11 adapt the study of strict stability from [She99] to the tame context and use a weak continuity property for splitting (as assumed in [BVa]) to improve on some of the results mentioned earlier. Section 17.5 gives a quick application to homogeneous model theory: categoricity in |T| and stability in |T| imply stability in all  $\lambda \geq |T|$ .

The reader may ask how SCH is used in the above results. Roughly, it makes cardinal arithmetic well-behaved enough that for any big-enough cardinal  $\lambda$ , **K** will either be stable in  $\lambda$  or in unboundedly many cardinals below  $\lambda$ . This is connected to defining the locality cardinals in  $\underline{\chi}(\mathbf{K})$  using chains rather than as the least cardinal  $\kappa$  for which every type does not fork over a set of size less than  $\kappa$  (indeed, in AECs it is not even clear what exact form such a definition should take). Still several results of this chapter hold (in ZFC) for "most" cardinals, and the role of SCH is only to deduce that "most" means "all".

By a result of Solovay [Sol74], SCH holds above a strongly compact. Thus our results which assume SCH hold also above a strongly compact. This shows that a stability theory (not just a superstability theory) can be developed in the context of the Makkai and Shelah paper, partially answering [She00, 6.15].

**19.1.3.** Future work. We believe that an important test question is whether the aforementioned SCH hypothesis can be removed:

QUESTION 19.1.1. Let  $\mathbf{K}$  be an LS( $\mathbf{K}$ )-tame AEC with a monster model. Can one characterize the stability spectrum in ZFC?

By the present work, the answer to Question 19.1.1 is positive assuming the existence of large cardinals.

Apart from  $\underline{\chi}(\mathbf{K})$ , several other cardinal parameters  $(\lambda(\mathbf{K}), \lambda'(\mathbf{K}), \text{ and } \bar{\kappa}_{(\mathbf{K})})$ are defined in this chapter. Under some assumptions, we can give loose bounds on these cardinals (see e.g. Theorem 19.11.3) but focus on eventual behavior. We believe it is a worthy endeavor (analog to the study of the behavior of the stability spectrum below  $2^{|T|}$  in first-order) to try to say something more on these cardinals.

19.1.4. Notes. The background required to read this chapter is a solid knowledge of tame AECs (as presented for example in [Bal09]). Familiarity with Chapter 4 would be very helpful. Results from the recent literature which we rely on can be used as black boxes.

Note that at the beginning of several sections, we make global hypotheses assumed throughout the section. In the statement of the main results, these global hypotheses will be repeated.

#### **19.2.** Preliminaries

#### 19.2.1. Basic notation.

19.2.2. Monster model, Galois types, and tameness. We say that an AEC K has a monster model if it has amalgamation, joint embedding, and arbitrarily large models. Equivalently, it has a (proper class sized) model-homogeneous universal model  $\mathfrak{C}$ . When **K** has a monster model, we fix such a  $\mathfrak{C}$  and work inside it. Note that for our purpose amalgamation is the only essential property. Once we have it, we can partition **K** into disjoint pieces, each of which has joint embedding (see for example [Bal09, 16.14]). Further, for studying the eventual behavior of K assuming the existence of arbitrarily large models is natural.

We use the notation of Chapter 2 for Galois types. In particular,  $gtp(\bar{b}/A; N)$ denotes the Galois type of the sequence b over the set A, as computed in  $N \in \mathbf{K}$ . In case **K** has a monster model  $\mathfrak{C}$ , we write  $gtp(\overline{b}/A)$  instead of  $gtp(\overline{b}/A;\mathfrak{C})$ . In this case,  $gtp(b/A) = gtp(\bar{c}/A)$  if and only if there exists an automorphism f of  $\mathfrak{C}$  fixing A such that  $f(\bar{b}) = \bar{c}$ .

Observe that the definition of Galois types is completely semantic. Tameness is a locality property for types isolated by Grossberg and VanDieren [GV06b] that, when it holds, allows us to recover some of the syntactic properties of first-order types. For a cardinal  $\mu \geq LS(\mathbf{K})$ , we say that an AEC **K** with a monster model is  $\mu$ -tame if whenever  $gtp(b/M) \neq gtp(c/M)$ , there exists  $M_0 \in \mathbf{K}_{<\mu}$  such that  $M_0 \leq_{\mathbf{K}} M$  and  $\operatorname{gtp}(b/M_0) \neq \operatorname{gtp}(c/M_0)$ . When assuming tameness in this chapter, we will usually assume that K is LS(K)-tame. Indeed if K is  $\mu$ -tame we can just replace **K** by  $\mathbf{K}_{\geq \mu}$ . Then  $\mathrm{LS}(\mathbf{K}_{\geq \mu}) = \mu$ , so  $\mathbf{K}_{\geq \mu}$  will be  $\mathrm{LS}(\mathbf{K}_{\geq \mu})$ -tame.

Concepts such as stability and saturation are defined as in the first-order case but using Galois type (see Chapter 2). For example, an AEC  $\mathbf{K}$  with a monster model is stable in  $\mu$  if  $|gS(M)| \leq \mu$  for every  $M \in \mathbf{K}_{\mu}$ . For  $\mu > LS(\mathbf{K})$ , a model  $M \in \mathbf{K}$  is *µ*-saturated if every Galois type over a  $\leq_{\mathbf{K}}$ -substructure of M of size less than  $\mu$  is realized in M. In the literature, these are often called "Galois stable" and "Galois saturated", but we omit the "Galois" prefix since there is no risk of confusion in this chapter. We will also make use of the order property from Definition 2.4.3.

**19.2.3.** Independence relations. Recall [Gro] that an abstract class (AC) is a partial order  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  where K is a class of structures in a fixed vocabulary  $\tau(\mathbf{K}), \mathbf{K}$  is closed under isomorphisms, and  $M \leq_{\mathbf{K}} N$  implies  $M \subseteq N$ . In this chapter, an *independence relation* will be a pair  $(\mathbf{K}, \downarrow)$ , where:

- (1)  $\mathbf{K}$  is a coherent<sup>2</sup> abstract class with amalgamation.
- (2)  $\downarrow$  is a 4-ary relation so that:
  - (a)  $\downarrow (M, A, B, N)$  implies  $M \leq_{\mathbf{K}} N, A, B \subseteq |N|, |A| \leq 1$ . We write  $A \bigcup_{M}^{N} B.$
  - (b)  $\downarrow$  satisfies invariance, normality, and monotonicity (see 6.3.6 for the definitions).

<sup>&</sup>lt;sup>2</sup>that is, whenever  $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$  and  $M_0 \leq_{\mathbf{K}} M_2$ , we have that  $M_0 \leq_{\mathbf{K}} M_1$ .

(c) We also ask that  $\downarrow$  satisfies base monotonicity: if  $A \underset{M_0}{\downarrow}^N B, M_0 \leq_{\mathbf{K}} N$ 

$$M \leq_{\mathbf{K}} N$$
, and  $|M| \subseteq B$ , then  $A \bigcup_{M}^{N} B$ .

Note that this definition differs slightly from that in Definition 6.3.6: there additional parameters are added controlling the size of the left and right hand side, and base monotonicity is assumed. Here, the size of the left hand side is at most 1 and the size of the right hand side is not bounded. So in the terminology of Chapter 6, we are defining a ( $\leq 1, [0, \infty)$ )-independence relation with base monotonicity.

When  $\mathbf{i} = (\mathbf{K}, \downarrow)$  is an independence relation and  $p \in gS(B; N)$  (we make use of Galois types over sets, see Definition 2.2.17), we say that p does not  $\mathbf{i}$ -fork over M if  $a \downarrow B$  for some (any) a realizing p in N. When  $\mathbf{i}$  is clear from context, we

omit it and just say that p does not fork over M.

The following independence notion is central. It was introduced by Shelah in [She99, 3.2].

DEFINITION 19.2.1. Let **K** be a coherent abstract class with amalgamation, let  $M \leq_{\mathbf{K}} N$ ,  $p \in \mathrm{gS}(N)$ , and let  $\mu \geq ||M||$ . We say that  $p \mu$ -splits over M if there exists  $N_1, N_2 \in \mathbf{K}_{\leq \mu}$  and f such that  $M \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} N$  for  $\ell = 1, 2, f : N_1 \cong_M N_2$ , and  $f(p \upharpoonright N_1) \neq p \upharpoonright N_2$ .

For  $\lambda$  an infinite cardinal, we write  $i_{\mu-\text{spl}}(\mathbf{K}_{\lambda})$  for the independence relation with underlying class  $\mathbf{K}_{\lambda}$  and underlying independence notion non  $\mu$ -splitting.

**19.2.4.** Universal orderings and limit models. Work inside an abstract class **K**. For  $M <_{\mathbf{K}} N$ , we say that N is universal over M (and write  $M <_{\mathbf{K}}^{\text{univ}} N$ ) if for any  $M' \in \mathbf{K}$  with  $M \leq_{\mathbf{K}} M'$  and ||M'|| = ||M||, there exists  $f : M' \xrightarrow[M]{} N$ . For a cardinal  $\mu$  and a limit ordinal  $\delta < \mu^+$ , we say that N is  $(\mu, \delta)$ -limit over M if there exists an increasing continuous chain  $\langle N_i : i \leq \delta \rangle$  such that  $N_0 = M$ ,  $N_{\delta} = N$ , and for any  $i < \delta$ ,  $N_i$  is in  $\mathbf{K}_{\mu}$  and  $N_{i+1}$  is universal over  $N_i$ . For  $A \subseteq \mu^+$  a set of limit ordinals, we say that N is  $(\mu, A)$ -limit over M if there exists  $\gamma \in A$  such that N is  $(\mu, \gamma)$ -limit over M.  $(\mu, \geq \delta)$ -limit means  $(\mu, [\delta, \mu^+) \cap \text{REG})$ -limit. We will use without mention the basic facts about limit models in AECs: existence (assuming stability and a monster model) and uniqueness when they have the same cofinality. See [**GVV16**] for an introduction to the theory of limit models.

**19.2.5.** Locality cardinals for independence. One of the main object of study of this chapter is  $\underline{\chi}(\mathbf{K})$  (see Definition 19.4.6), which roughly is the class of regular cardinals  $\chi$  such that for any increasing continuous chain  $\langle M_i : i \leq \chi \rangle$  where each model is universal over the previous one and for any  $p \in \mathrm{gS}(M_{\chi})$  there exists  $i < \chi$  such that p does not  $||M_i||$ -split over  $M_i$ . Interestingly, we cannot rule out the possibility that there are gaps in  $\underline{\chi}(\mathbf{K})$ , i.e. although we do not have any examples, it is conceivable that there are regular  $\chi_0 < \chi_1 < \chi_2$  such that chains of length  $\chi_0$  and  $\chi_2$  have the good property above but chains of length  $\chi_1$  do not). This is why we follow Shelah's approach from [She99] (see in particular the remark on top of p. 275 there) and define *classes* of locality cardinals, rather than directly taking a minimum (as in for example [GV06b, 4.3]). We give a sufficient locality condition implying that there are no gaps in  $\chi(\mathbf{K})$  (see Theorem 19.3.7).

The cardinals  $\underline{\kappa}^{wk}$  are already in [She99, 4.8], while  $\underline{\kappa}^{cont}$  is used in the proof of the Shelah-Villaveces theorem [SV99, 2.2.1], see also Chapter 20.

DEFINITION 19.2.2 (Locality cardinals). Let i be an independence relation. Let R be a partial order on  $\mathbf{K}$  extending  $\leq_{\mathbf{K}}$ .

- (1)  $\underline{\kappa}(\mathbf{i}, R)$  is the set of regular cardinals  $\chi$  such that whenever  $\langle M_i : i < \chi \rangle$  is an *R*-increasing chain<sup>3</sup>,  $N \in \mathbf{K}$  is such that  $M_i \leq_{\mathbf{K}} N$  for all  $i < \chi$ , and  $p \in \mathrm{gS}(\bigcup_{i < \chi} |M_i|; N)$ , there exists  $i < \chi$  such that p does not fork over  $M_i$ .
- (2)  $\underline{\kappa}^{\text{wk}}(\mathbf{i}, R)$  is the set of regular cardinals  $\chi$  such that whenever  $\langle M_i : i < \chi \rangle$ is an *R*-increasing chain,  $N \in \mathbf{K}$  is such that  $M_i \leq_{\mathbf{K}} N$  for all  $i < \chi$ , and  $p \in \text{gS}(\bigcup_{i < \chi} |M_i|; N)$ , there exists  $i < \chi$  such that  $p \upharpoonright M_{i+1}$  does not fork over  $M_i$ .
- (3)  $\underline{\kappa}^{\text{cont}}(\mathbf{i}, R)$  is the set of regular cardinals  $\chi$  such that whenever  $\langle M_i : i < \chi \rangle$ is an *R*-increasing chain,  $N \in \mathbf{K}$  is such that  $M_i \leq_{\mathbf{K}} N$  for all  $i < \chi$ , and  $p \in \mathrm{gS}(\bigcup_{i < \chi} |M_i|; N)$ , if  $p \upharpoonright M_i$  does not fork over  $M_0$  for all  $i < \chi$ , then p does not fork over  $M_0$ .

When R is  $\leq_{\mathbf{K}}$ , we omit it. In this chapter, R will mostly be  $<_{\mathbf{K}}^{\text{univ}}$  (see Section 19.2.4).

REMARK 19.2.3. The behavior at singular cardinals has some interests (see for example Lemma 20.2.6(4)), but we focus on regular cardinals in this chapter.

Note that  $\underline{\kappa}^{wk}(\mathbf{i}, R)$  is an end segment of regular cardinals (so it has no gaps): if  $\chi_0 < \chi_1$  are regular cardinals and  $\chi_0 \in \underline{\kappa}^{wk}(\mathbf{i}, R)$ , then  $\chi_1 \in \underline{\kappa}^{wk}(\mathbf{i}, R)$ . In section 19.3 we will give conditions under which  $\underline{\kappa}(\mathbf{i}, R)$  and  $\underline{\kappa}^{cont}(\mathbf{i}, R)$  are also end segments. In this case, the following cardinals are especially interesting (note the absence of line under  $\kappa$ ):

DEFINITION 19.2.4.  $\kappa(\mathfrak{i}, R)$  is the least regular cardinal  $\chi \in \underline{\kappa}(\mathfrak{i}, R)$  such that for any regular cardinals  $\chi' > \chi$ , we have that  $\chi' \in \underline{\kappa}(\mathfrak{i}, R)$ . Similarly define  $\kappa^{\mathrm{wk}}(\mathfrak{i}, R)$ and  $\kappa^{\mathrm{cont}}(\mathfrak{i}, R)$ .

The following is given by the proof of Lemma 20.2.6(1):

FACT 19.2.5. Let  $i = (\mathbf{K}, \bigcup)$  be an independence relation. Let R be a partial order on  $\mathbf{K}$  extending  $\leq_{\mathbf{K}}$ .

We have that  $\underline{\kappa}^{wk}(\mathfrak{i}, R) \cap \underline{\kappa}^{cont}(\mathfrak{i}, R) \subseteq \underline{\kappa}(\mathfrak{i}, R)$ .

COROLLARY 19.2.6. Let  $\mathbf{i} = (\mathbf{K}, \downarrow)$  be an independence relation. Let R be a partial order on  $\mathbf{K}$  extending  $\leq_{\mathbf{K}}$ . If  $\kappa^{\text{cont}}(\mathbf{i}, R) = \aleph_0$  (i.e.  $\underline{\kappa}^{\text{cont}}(\mathbf{i}, R)$  contains all the regular cardinals), then  $\underline{\kappa}(\mathbf{i}, R) = \underline{\kappa}^{\text{wk}}(\mathbf{i}, R)$  and both are end segments of regular cardinals.

PROOF. Directly from Fact 19.2.5 (using that by definition  $\underline{\kappa}^{wk}(\mathfrak{i}, R)$  is always and end segment of regular cardinals).

REMARK 19.2.7. The conclusion of Fact 19.2.5 can be made into an equality assuming that i satisfies a weak transitivity property (see the statement for splitting and  $R = <_{\mathbf{K}}^{\text{univ}}$  in Proposition 4.3.7). This is not needed in this chapter.

<sup>&</sup>lt;sup>3</sup>that is,  $M_i R M_j$  for all  $i < j < \chi$ .

#### 19.3. Continuity of forking

In this section, we aim to study the locality cardinals and give conditions under which  $\underline{\kappa}^{\text{cont}}$  contains all regular cardinals. We work in an AEC with amalgamation and stability in a single cardinal  $\mu$ :

Hypothesis 19.3.1.

- (1) **K** is an AEC,  $\mu \ge \text{LS}(\mathbf{K})$ .
- (2)  $\mathbf{K}_{\mu}$  has amalgamation, joint embedding, and no maximal models in  $\mu$ . Moreover **K** is stable in  $\mu$ .
- (3)  $\mathbf{i} = (\mathbf{K}_{\mu}, \downarrow)$  is an independence relation.

REMARK 19.3.2. The results of this section generalize to AECs that may not have full amalgamation in  $\mu$ , but only satisfy the properties from [**SV99**]: density of amalgamation bases, existence of universal extensions, and limit models being amalgamation bases.

We will usually assume that i has the weak uniqueness property:

DEFINITION 19.3.3. i has weak uniqueness if whenever  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  are all in  $\mathbf{K}_{\mu}$  with M universal over  $M_0$ ,  $p, q \in \mathrm{gS}(N)$  do not fork over  $M_0$ , and  $p \upharpoonright M = q \upharpoonright M$ , then p = q.

The reader can think of i as non- $\mu$ -splitting (Definition 19.2.1), where such a property holds [Van06, I.4.12]. We state a more general version:

FACT 19.3.4 (6.2 in **[GV06b]**).  $i_{\mu-\text{spl}}(\mathbf{K}_{\mu})$  has weak uniqueness. More generally, let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  all be in  $\mathbf{K}_{\geq\mu}$  with  $M_0 \in \mathbf{K}_{\mu}$ . Assume that M is universal over  $M_0$  and  $\mathbf{K}$  is  $(\mu, ||N||)$ -tame (i.e. types over models of size ||N|| are determined by their restrictions of size  $\mu$ ).

Let  $p, q \in gS(N)$ . If p, q both do not  $\mu$ -split over  $M_0$  and  $p \upharpoonright M = q \upharpoonright M$ , then p = q.

Interestingly, weak uniqueness implies a weak version of extension:

LEMMA 19.3.5 (Weak extension). Let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  all be in  $\mathbf{K}_{\mu}$ . Assume that M is universal over  $M_0$ . Let  $p \in \mathrm{gS}(M)$  and assume that p does not fork over  $M_0$ .

If i has weak uniqueness, then there exists  $q \in gS(N)$  extending p such that q does not fork over  $M_0$ .

PROOF. We first prove the result when M is  $(\mu, \omega)$ -limit over  $M_0$ . In this case we can write  $M = M_{\omega}$ , where  $\langle M_i : i \leq \omega \rangle$  is increasing continuous with  $M_{i+1}$ universal over  $M_i$  for each  $i < \omega$ .

universal over  $M_i$  for each  $i < \omega$ . Let  $f: N \xrightarrow{M_1} M$ . Let  $q := f^{-1}(p)$ . Then  $q \in gS(N)$  and by invariance q does not fork over  $M_0$ . It remains to show that q extends p. Let  $q_M := q \upharpoonright M$ . We want to see that  $q_M = p$ . By monotonicity,  $q_M$  does not fork over  $M_0$ . Moreover,  $q_M \upharpoonright M_1 = p \upharpoonright M_1$ . By weak uniqueness, this implies that  $q_M = p$ , as desired.

In the general case (when M is only universal over  $M_0$ ), let  $M' \in \mathbf{K}_{\mu}$  be  $(\mu, \omega)$ limit over  $M_0$ . By universality, we can assume that  $M_0 \leq_{\mathbf{K}} M' \leq_{\mathbf{K}} M$ . By the special case we have just proven, there exists  $q \in \mathrm{gS}(N)$  extending  $p \upharpoonright M'$  such that q does not fork over  $M_0$ . By weak uniqueness, we must have that also  $q \upharpoonright M = p$ , i.e. q extends p. We will derive continuity from weak uniqueness and the following locality property<sup>4</sup>, a weakening of locality from [**Bal09**, 11.4]:

DEFINITION 19.3.6. Let  $\chi$  be a regular cardinal. We say that an AEC **K** with a monster model is weakly  $\chi$ -local if for any increasing continuous chain  $\langle M_i : i \leq \chi \rangle$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \chi$ , if  $p, q \in gS(M_\chi)$  are such that  $p \upharpoonright M_i = q \upharpoonright M_i$  for all  $i < \chi$ , then p = q. We say that **K** is weakly  $(\geq \chi)$ -local if **K** is weakly  $\chi'$ -local for all regular  $\chi' \geq \chi$ .

Note that any  $(<\aleph_0)$ -tame AEC (such as an elementary class, an AEC derived from homogeneous model theory, or even a universal class [**Bonc**] (see also Theorem 8.3.6) is weakly  $(\geq \aleph_0)$ -local.

THEOREM 19.3.7. Let  $\chi < \mu^+$  be a regular cardinal. If **K** is weakly  $\chi$ -local and  $\mathfrak{i}$  has weak uniqueness, then  $\chi \in \underline{\kappa}^{\text{cont}}(\mathfrak{i}, <_{\mathbf{K}}^{\text{univ}})$ .

PROOF. Let  $\langle M_i : i \leq \chi \rangle$  be increasing continuous in  $\mathbf{K}_{\mu}$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \chi$ . Let  $p \in \mathrm{gS}(M_{\chi})$  and assume that  $p \upharpoonright M_i$  does not for over  $M_0$  for all  $i < \chi$ . Let  $q \in \mathrm{gS}(M_{\chi})$  be an extension of  $p \upharpoonright M_1$  such that q does not fork over  $M_0$ . This exists by weak extension (Lemma 19.3.5). By weak uniqueness,  $p \upharpoonright M_i = q \upharpoonright M_i$  for all  $i < \chi$ . By weak  $\chi$ -locality, p = q, hence p does not fork over  $M_0$ , as desired.

In the rest of this chapter, we will often look at  $\mu$ -splitting. The following notation will be convenient:

DEFINITION 19.3.8. Define  $\underline{\kappa}(\mathbf{K}_{\mu}, <_{\mathbf{K}}^{\mathrm{univ}}) := \underline{\kappa}(\mathfrak{i}_{\mu-\mathrm{spl}}(\mathbf{K}_{\mu}), <_{\mathbf{K}}^{\mathrm{univ}})$ . Similarly define the other variations in terms of  $\underline{\kappa}^{\mathrm{wk}}$  and  $\underline{\kappa}^{\mathrm{cont}}$ . Also define  $\kappa(\mathbf{K}_{\mu}, <_{\mathbf{K}}^{\mathrm{univ}})$  and its variations.

Note that any independence relation with weak uniqueness is extended by nonsplitting. This is essentially observed in Lemma 3.4.2 but we give a full proof here for the convenience of the reader.

LEMMA 19.3.9. Assume that i has weak uniqueness.

- (1) Let  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$  all be in  $\mathbf{K}_{\mu}$  such that  $M_1$  is universal over  $M_0$ and M is universal over  $M_1$ . Let  $p \in \mathrm{gS}(M)$ . If p does not fork over  $M_0$ , then p does not  $\mu$ -split over  $M_1$ .
- (2)  $\underline{\kappa}(\mathfrak{i}, <_{\mathbf{K}}^{\mathrm{univ}}) \subseteq \underline{\kappa}(\mathbf{K}_{\mu}, <_{\mathbf{K}}^{\mathrm{univ}})$ , and similarly for  $\underline{\kappa}^{\mathrm{wk}}$  and  $\underline{\kappa}^{\mathrm{cont}}$ .

Proof.

- (1) Let  $N_1, N_2 \in \mathbf{K}_{\mu}$  and  $f: N_1 \cong_{M_1} N_2$  be such that  $M_1 \leq_{\mathbf{K}} N_{\ell} \leq_{\mathbf{K}} M$  for  $\ell = 1, 2$ . We want to see that  $f(p \upharpoonright N_1) = p \upharpoonright N_2$ . By monotonicity,  $p \upharpoonright N_{\ell}$  does not fork over  $M_0$  for  $\ell = 1, 2$ . Consequently,  $f(p \upharpoonright N_1)$  does not fork over  $M_0$ . Furthermore,  $f(p \upharpoonright N_1) \upharpoonright M_1 = p \upharpoonright M_1 = (p \upharpoonright N_2) \upharpoonright M_1$ . Applying weak uniqueness, we get that  $f(p \upharpoonright N_1) = p \upharpoonright N_2$ .
- (2) Follows from the first part.

 $<sup>^{4}</sup>$ In an earlier version, we derived continuity without any locality property but our argument contained a mistake.

#### 19.4. The stability spectrum of tame AECs

For an AEC **K** with a monster model, we define the *stability spectrum of* **K**,  $\operatorname{Stab}(\mathbf{K})$  to be the class of cardinals  $\mu \geq \operatorname{LS}(\mathbf{K})$  such that **K** is stable in  $\mu$ . We would like to study it assuming tameness. From earlier work, the following is known about  $\operatorname{Stab}(\mathbf{K})$  in tame AECs:

FACT 19.4.1. Let  $\mathbf{K}$  be an LS( $\mathbf{K}$ )-tame AEC with a monster model.

- (1) (Theorem 2.4.15) If  $\text{Stab}(\mathbf{K}) \neq \emptyset$ , then  $\min(\text{Stab}(\mathbf{K})) < H_1$  (recall Definition 2.2.2).
- (2)  $[\mathbf{GV06b}, 6.4]^5$  If  $\mu \in \mathrm{Stab}(\mathbf{K})$  and  $\lambda = \lambda^{\mu}$ , then  $\lambda \in \mathrm{Stab}(\mathbf{K})$ .
- (3) [**BKV06**, 1] If  $\mu \in \text{Stab}(\mathbf{K})$ , then  $\mu^+ \in \text{Stab}(\mathbf{K})$ .
- (4) (Lemma 4.5.5) If  $\langle \mu_i : i < \delta \rangle$  is strictly increasing in Stab(**K**) and cf  $\delta \in \underline{\kappa}(\mathbf{K}_{\mu_0}, <_{\mathbf{K}}^{\mathrm{univ}})$ , then  $\sup_{i < \delta} \mu_i \in \mathrm{Stab}(\mathbf{K})$ .

Let us say that **K** is *stable* if  $\text{Stab}(\mathbf{K}) \neq \emptyset$ . In this case, it is natural to give a name to the first stability cardinal:

DEFINITION 19.4.2. For **K** an AEC with a monster model, let  $\lambda(\mathbf{K}) := \min(\operatorname{Stab}(\mathbf{K}))$ (if  $\operatorname{Stab}(\mathbf{K}) = \emptyset$ , let  $\lambda(\mathbf{K}) := \infty$ ).

From Fact 19.4.1, if **K** is an LS(**K**)-tame AEC with a monster model, then  $\lambda(\mathbf{K}) < \infty$  implies that  $\lambda(\mathbf{K}) < H_1$ .

We will also rely on the following basic facts:

FACT 19.4.3 (Proposition 3.3.12). Let **K** be an LS(**K**)-tame AEC with a monster model. For  $M \leq_{\mathbf{K}} N$ ,  $p \in gS(N)$ ,  $\mu \in [||M||, ||N||]$ ,  $p \mu$ -splits over M if and only if p ||M||-splits over M.

FACT 19.4.4 (3.3 in [She99]). Let **K** be an AEC with a monster model. Assume that **K** is stable in  $\mu \geq \text{LS}(\mathbf{K})$ . For any  $M \in \mathbf{K}_{\geq \mu}$  and any  $p \in \text{gS}(M)$ , there exists  $M_0 \leq_{\mathbf{K}} M$  with  $M_0 \in \mathbf{K}_{\mu}$  such that p does not  $\mu$ -split over  $M_0$ .

It is natural to look at the sequence  $\langle \underline{\kappa}(\mathbf{K}_{\mu}, <^{\mathrm{univ}}_{\mathbf{K}}) : \mu \in \mathrm{Stab}(\mathbf{K}) \rangle$ . From Section 4.4, we have that:

FACT 19.4.5. Let **K** be an LS(**K**)-tame AEC with a monster model. If  $\mu < \lambda$  are both in Stab(**K**), then  $\underline{\kappa}(\mathbf{K}_{\mu}, <_{\mathbf{K}}^{\mathrm{univ}}) \subseteq \underline{\kappa}(\mathbf{K}_{\lambda}, <_{\mathbf{K}}^{\mathrm{univ}})$ .

Thus we define:

DEFINITION 19.4.6. For **K** an LS(**K**)-tame AEC with a monster model, let  $\underline{\chi}(\mathbf{K}) := \bigcup_{\mu \in \text{Stab}(\mathbf{K})} \underline{\kappa}(\mathbf{K}_{\mu(\mathbf{K})}, <_{\mathbf{K}}^{\text{univ}})$ . Let  $\chi(\mathbf{K})$  be the least regular cardinal  $\chi$  such that  $\chi' \in \underline{\chi}(\mathbf{K})$  for any regular  $\chi' \geq \chi$ . Set  $\underline{\chi}(\mathbf{K}) := \emptyset$  and  $\chi(\mathbf{K}) := \infty$  if  $\lambda(\mathbf{K}) = \infty$ .

REMARK 19.4.7. By Fact 19.4.4,  $\chi(\mathbf{K}) \leq \lambda(\mathbf{K})^+$ . Assuming continuity of splitting, we can prove that  $\chi(\mathbf{K}) \leq \lambda(\mathbf{K})$  (see Theorem 19.11.3).

REMARK 19.4.8. If **K** comes from a first-order theory, then  $\underline{\chi}(\mathbf{K})$  is the set of regular cardinals greater than or equal to  $\kappa_r(T)$ , see Corollary 19.4.18.

<sup>&</sup>lt;sup>5</sup>Grossberg and VanDieren's proof shows that the assumption there that  $\mu > H_1$  can be removed, see [Bal09, Theorem 12.10].

Fact 19.4.4 implies more generally that  $[\lambda(\mathbf{K}), \infty) \cap \text{REG} \subseteq \underline{\kappa}(\mathbf{K}_{\mu}, <_{\mathbf{K}}^{\text{univ}})$  for any stability cardinal  $\mu$ . Thus we can let  $\lambda'(\mathbf{K})$  be the first place where the sequence of  $\underline{\kappa}(\mathbf{K}_{\mu}, <_{\mathbf{K}}^{\text{univ}})$  stabilizes. One can think of it as the first "well-behaved" stability cardinal.

DEFINITION 19.4.9. For **K** an LS(**K**)-tame AEC with a monster model, let  $\lambda'(\mathbf{K})$  be the least stability cardinal  $\lambda$  such that  $\underline{\kappa}(\mathbf{K}_{\mu}, <_{\mathbf{K}}^{\text{univ}}) \subseteq \underline{\kappa}(\mathbf{K}_{\lambda}, <_{\mathbf{K}}^{\text{univ}})$  for all  $\mu \in \text{Stab}(\mathbf{K})$ . When  $\lambda(\mathbf{K}) = \infty$ , we set  $\lambda'(\mathbf{K}) = \infty$ .

We do not know whether  $\lambda'(\mathbf{K}) = \lambda(\mathbf{K})$ . In fact, while we know that  $\lambda'(\mathbf{K}) < \infty$ if  $\lambda(\mathbf{K}) < \infty$ , we are unable to give any general bound at all on  $\lambda'(\mathbf{K})$ . Assuming continuity of splitting, we can show that  $\lambda'(\mathbf{K}) < h(\lambda(\mathbf{K}))$  (see Theorem 19.11.3).

In this section, we prove what we can on  $\underline{\chi}(\mathbf{K})$  without assuming continuity of splitting. Section 19.11 will prove more assuming continuity of splitting.

We will use the following fact, whose proof relies on the machinery of averages for tame AECs:

FACT 19.4.10 (Theorem 7.5.16). Let  $\mathbf{K}$  be an LS( $\mathbf{K}$ )-tame AEC with a monster model.

There exists a stability cardinal  $\chi_0 < H_1$  such that for any  $\mu > \mu_0 \ge \chi_0$ , if:

(1) **K** is stable in unboundedly many cardinals below  $\mu$ .

(2) **K** is stable in  $\mu_0$  and cf  $\delta \in \underline{\kappa}(\mathbf{K}_{\mu_0}, <^{\text{univ}}_{\mathbf{K}})$ 

then whenever  $\langle M_i : i < \delta \rangle$  is an increasing chain of  $\mu$ -saturated models, we have that  $\bigcup_{i < \delta} M_i$  is  $\mu$ -saturated.

The following is the key result.

THEOREM 19.4.11. Let **K** be an LS(**K**)-tame AEC with a monster model. Let  $\chi_0 < H_1$  be as given by Fact 19.4.10. For any  $\mu > \chi_0$ , if **K** is stable in  $\mu$  and in unboundedly many cardinals below  $\mu$ , then cf  $\mu \in \underline{\kappa}(\mathbf{K}_{\mu}, <_{\mathbf{K}}^{\mathrm{univ}})$ .

The proof of Theorem 19.4.11 will use the lemma below, which improves on Lemma 9.3.17.

LEMMA 19.4.12. Let **K** be an LS(**K**)-tame AEC with a monster model. Let  $\delta$  be a limit ordinal and let  $\langle M_i : i \leq \delta \rangle$  be an increasing continuous sequence. If  $M_{\delta}$  is  $(\text{LS}(\mathbf{K}) + \delta)^+$ -saturated, then for any  $p \in \text{gS}(M_{\delta})$ , there exists  $i < \delta$  such that p does not  $||M_i||$ -split over  $M_i$ .

PROOF. Assume for a contradiction that  $p \in gS(M_{\delta})$  is such that  $p ||M_i||$ -splits over  $M_i$  for every  $i < \delta$ . Then for every  $i < \delta$  there exists  $N_1^i, N_2^i, f_i$  such that  $M_i \leq_{\mathbf{K}} N_{\ell}^i \leq_{\mathbf{K}} M, \ell = 1, 2, f_i : N_1^i \cong_{M_i} N_2^i$ , and  $f_i(p \upharpoonright N_1^i) \neq p \upharpoonright N_2^i$ . By tameness, there exists  $M_1^i \leq_{\mathbf{K}} N_1^i, M_2^i \leq_{\mathbf{K}} N_2^i$  both in  $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$  such that  $f_i[M_1^i] = M_2^i$  and  $f_i(p \upharpoonright M_1^i) \neq p \upharpoonright M_2^i$ .

Let  $N \leq_{\mathbf{K}} M$  have size  $\mu := \mathrm{LS}(\mathbf{K}) + \delta$  and be such that  $M_{\ell}^i \leq_{\mathbf{K}} N$  for  $\ell = 1, 2$  and  $i < \delta$ .

Since  $M_{\delta}$  is  $\mu^+$ -saturated, there exists  $b \in |M_{\delta}|$  realizing  $p \upharpoonright N$ . Let  $i < \delta$  be such that  $b \in |M_i|$ . By construction, we have that  $f_i(p \upharpoonright M_1^i) \neq p \upharpoonright M_2^i$  but on the other hand  $p \upharpoonright M_{\ell}^i = \operatorname{gtp}(b/M_{\ell}^i; M)$  and  $f_i(p \upharpoonright M_1^i) = \operatorname{gtp}(b/M_2^i; M)$ , since  $f_i(b) = b$ (it fixes  $M_i$ ). This is a contradiction.

Before proving Theorem 19.4.11, we show that Fact 19.4.10 implies saturation of long-enough limit models:

THEOREM 19.4.13. Let **K** be an LS(**K**)-tame AEC with a monster model. Let  $\chi_0 < H_1$  be as given by Fact 19.4.10. Let  $\mu > \mu_0 \ge \chi_0$  be such that **K** is stable in  $\mu_0, \mu$ , and in unboundedly many cardinals below  $\mu$ .

Then any  $(\mu, \underline{\kappa}(\mathbf{K}_{\mu_0}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mu^+)$ -limit model (see Section 19.2.4) is saturated. In particular, there is a saturated model of cardinality  $\mu$ .

PROOF. Assume for simplicity that  $\mu$  is limit (if  $\mu$  is a successor cardinal, the proof is completely similar). Let  $\gamma := cf \mu$  and let  $\langle \mu_i : i < \gamma \rangle$  be increasing cofinal in  $\mu$  such that **K** is stable in  $\mu_i$  for all  $i < \gamma$ . By Fact 19.4.1(3), **K** is stable in  $\mu_i^+$  for all  $i < \gamma$ . Let  $\delta \in \underline{\kappa}(\mathbf{K}_{\mu_0}, <_{\mathbf{K}}^{\mathrm{univ}}) \cap \mu^+$ . By Fact 19.4.10, for all  $i < \gamma$ , the union of a chain of  $\mu_i$ -saturated models of length  $\delta$  is  $\mu_i$ -saturated. It follows that the  $(\mu, \delta)$ -limit model is saturated. Indeed, for each fixed  $i < \gamma$ , we can build an increasing continuous chain  $\langle M_j : j \leq \delta \rangle$  such that for all  $j < \delta$ ,  $M_j \in \mathbf{K}_{\mu}$ ,  $M_{j+1}$  is universal over  $M_j$ , and  $M_{j+1}$  is  $\mu_i$ -saturated. By what has just been observed,  $M_{\delta}$  is  $\mu_i$ -saturated, and is a  $(\mu, \delta)$ -limit model. Now apply uniqueness of limit models of the same length.

To see the "in particular" part, assume again that  $\mu$  is limit (if  $\mu$  is a successor, the  $(\mu, \mu)$ -limit model is saturated). Then without loss of generality,  $\mu_0 > \lambda(\mathbf{K})$ , so by Fact 19.4.4,  $\lambda(\mathbf{K})^+ \in \underline{\kappa}(\mathbf{K}_{\mu_0}, <_{\mathbf{K}}^{\mathrm{univ}})$ . Thus the  $(\mu, \lambda(\mathbf{K})^+)$ -limit model is saturated.

PROOF OF THEOREM 19.4.11. Let  $\mu > \chi_0$  be such that **K** is stable in  $\mu$  and in unboundedly many cardinals below  $\mu$ . Let  $\delta := \operatorname{cf} \mu$ .

By Theorem 19.4.13, there is a saturated model M of cardinality  $\mu$ . Using that **K** is stable in unboundedly many cardinals below  $\mu$ , one can build an increasing continuous resolution  $\langle M_i : i \leq \delta \rangle$  such that  $M_{\delta} = M$  and for  $i < \delta$ ,  $M_i \in \mathbf{K}_{<\mu}$ ,  $M_{i+1}$  is universal over  $M_i$ . By a back and forth argument, this shows that M is  $(\mu, \delta)$ -limit. By Lemma 19.4.12,  $\delta \in \underline{\kappa}(\mathbf{K}_{\mu}, <_{\mathbf{K}}^{\mathrm{univ}})$ , as desired.

COROLLARY 19.4.14. Let **K** be an LS(**K**)-tame AEC with a monster model. Let  $\chi_0$  be as given by Fact 19.4.10. For any  $\mu \ge \lambda'(\mathbf{K}) + \chi_0^+$  such that **K** is stable in unboundedly many cardinals below  $\mu$ , the following are equivalent:

- (1) **K** is stable in  $\mu$ .
- (2) cf  $\mu \in \chi(\mathbf{K})$ .

PROOF. (1) implies (2) is by Theorem 19.4.11 and (2) implies (1) is by Facts 19.4.1(3),(4).  $\hfill \Box$ 

It is natural to ask whether Corollary 19.4.14 holds for *arbitrary* high-enough  $\mu$ 's (i.e. without assuming stability in unboundedly many cardinals below  $\mu$ ). At present, the answer we can give is sensitive to cardinal arithmetic: Fact 19.4.1 does not give us enough tools to answer in ZFC. There is however a large class of cardinals on which there is no cardinal arithmetic problems. This is already implicit in Section 4.5.

DEFINITION 19.4.15. A cardinal  $\mu$  is  $\theta$ -closed if  $\lambda^{\theta} < \mu$  for all  $\lambda < \mu$ . We say that  $\mu$  is almost  $\theta$ -closed if  $\lambda^{\theta} \leq \mu$  for all  $\lambda < \mu$ .

LEMMA 19.4.16. Let **K** be an LS(**K**)-tame AEC with a monster model. If  $\mu$  is almost  $\lambda(\mathbf{K})$ -closed, then either  $\mu = \mu^{\lambda(\mathbf{K})}$  and **K** is stable in  $\mu$ , or **K** is stable in unboundedly many cardinals below  $\mu$ .

PROOF. If  $\mu^{\lambda(\mathbf{K})} = \mu$ , then **K** is stable in  $\mu$  by Fact 19.4.1(2). Otherwise,  $\mu$  is  $\lambda(\mathbf{K})$ -closed. Thus for any  $\mu_0 < \mu$ ,  $\mu_1 := \mu_0^{\lambda(\mathbf{K})}$  is such that  $\mu_1 < \mu$  and  $\mu_1^{\lambda(\mathbf{K})} = \mu_1$ , hence **K** is stable in  $\mu_1$ . Therefore **K** is stable in unboundedly many cardinals below  $\mu$ .

We have arrived to the following application of Corollary 19.4.14:

COROLLARY 19.4.17 (Eventual stability spectrum for closed cardinals). Let **K** be an LS(**K**)-tame AEC with a monster model. Let  $\chi_0 < H_1$  be as given by Fact 19.4.10. Let  $\mu$  be almost  $\lambda(\mathbf{K})$ -closed with  $\mu \geq \lambda'(\mathbf{K}) + \chi_0^+$ . Then **K** is stable in  $\mu$  if and only if cf  $\mu \in \chi(\mathbf{K})$ .

PROOF. If **K** is stable in unboundedly many cardinals below  $\mu$ , this is Corollary 19.4.14. Otherwise by Lemma 19.4.16, **K** is stable in  $\mu$  and  $\mu^{\lambda(\mathbf{K})} = \mu$ . In particular, cf  $\mu > \lambda(\mathbf{K})$ , so by Fact 19.4.4, cf  $\mu \in \chi(\mathbf{K})$ .

COROLLARY 19.4.18. Let **K** be the class of models of a first-order stable theory T ordered by  $\leq$ . Then  $\underline{\chi}(\mathbf{K})$  is an end-segment and  $\chi(\mathbf{K}) = \kappa_r(T)$  (the least regular cardinal greater than or equal to  $\kappa(T)$ ).

PROOF. Let  $\chi$  be a regular cardinal and let  $\mu := \beth_{\chi}(\lambda'(\mathbf{K}))$ . If  $\chi \ge \kappa(T)$ ,  $\mu = \mu^{<\kappa(T)}$  so by the first-order theory  $\mathbf{K}$  is stable in  $\mu$ . By Corollary 19.4.17,  $\chi \in \underline{\chi}(\mathbf{K})$ . Conversely, if  $\chi \in \underline{\chi}(\mathbf{K})$  then by Corollary 19.4.17,  $\mathbf{K}$  is stable in  $\mu$ , hence  $\mu = \mu^{<\kappa(T)}$ , so  $\chi \ge \kappa(T)$ .

Note that the class of almost  $\lambda(\mathbf{K})$ -closed cardinals forms a club, and on this class Corollary 19.4.17 gives a complete (eventual) characterization of stability. We do not know how to analyze the cardinals that are *not* almost  $\lambda(\mathbf{K})$ -closed in ZFC. Using hypotheses beyond ZFC, we can see that all big-enough cardinals are almost  $\lambda(\mathbf{K})$ -closed. For ease of notation, we define the following function:

DEFINITION 19.4.19. For  $\mu$  an infinite cardinal,  $\theta(\mu)$  is the least cardinal  $\theta$  such that any  $\lambda \geq \theta$  is almost  $\mu$ -closed. When such a  $\theta$  does not exist, we write  $\theta(\mu) = \infty$ .

If  $\lambda$  is a strong limit cardinal, then  $2^{\lambda} = \lambda^{cf \lambda}$  and so if  $2^{\lambda} > \lambda^{+}$  we have that  $\lambda^{+}$  is not almost cf  $\lambda$ -closed. Foreman and Woodin [**FW91**] have shown that it is consistent with ZFC and a large cardinal axiom that  $2^{\lambda} > \lambda^{+}$  for all infinite cardinals  $\lambda$ . Therefore it is possible that  $\theta(\aleph_{0}) = \infty$  (and hence  $\theta(\mu) = \infty$  for any infinite cardinal  $\mu$ ). However, we have:

FACT 19.4.20. Let  $\mu$  be an infinite cardinal.

- (1) If SCH holds, then  $\theta(\mu) = 2^{\mu}$ .
- (2) If  $\kappa > \mu$  is strongly compact, then  $\theta(\mu) \leq \kappa$ .

PROOF. The first fact follows from basic cardinal arithmetic (see [Jec03, 5.22]), and the third follows from a result of Solovay (see [Sol74] or [Jec03, 20.8]).  $\Box$ 

The following easy lemma will be used in the proof of Theorem 19.5.8:

LEMMA 19.4.21. Let **K** be an LS(**K**)-tame AEC with a monster model. If  $\mu > \theta(\lambda(\mathbf{K}))$  and  $\mu$  is limit, then **K** is stable in unboundedly many cardinals below  $\mu$ .

PROOF. Let  $\mu_0 \in [\theta(\lambda(\mathbf{K})), \mu)$ . As  $\mu$  is limit,  $\mu_0^+ < \mu$  and  $\mu_0^+$  is almost  $\lambda(\mathbf{K})$ closed. In particular,  $\mu_0^{\lambda(\mathbf{K})} \leq \mu_0^+$ . By Fact 19.4.1(2), **K** is stable in either  $\mu_0$  or  $\mu_0^+$ , as needed.

COROLLARY 19.4.22. Let **K** be an LS(**K**)-tame AEC with a monster model and let  $\chi_0 < H_1$  be as given by Fact 19.4.10. For any  $\mu \ge \lambda'(\mathbf{K}) + \chi_0^+ + \theta(\lambda(\mathbf{K}))$ , **K** is stable in  $\mu$  if and only if cf  $\mu \in \chi(\mathbf{K})$ .

PROOF. By Corollary 19.4.17 and the definition of  $\theta(\lambda(\mathbf{K}))$ .

A particular case of Theorem 19.4.11 derives superstability from stability in a tail of cardinals. The following concept is studied already in [She99, 6.3].

DEFINITION 19.4.23. An AEC K is  $\mu$ -superstable if:

- (1)  $\mu \geq \text{LS}(\mathbf{K})$ .
- (2)  $\mathbf{K}_{\mu}$  is non-empty, has amalgamation, joint embedding, and no maximal models.
- (3) **K** is stable in  $\mu$ .
- (4)  $\kappa(\mathbf{K}_{\mu}, <_{\mathbf{K}}^{\mathrm{univ}}) = \aleph_0$  (i.e.  $\underline{\kappa}(\mathbf{K}_{\mu}, <_{\mathbf{K}}^{\mathrm{univ}})$  consists of all the regular cardinals).

This definition has been well-studied and has numerous consequences in tame AECs, such as the existence of a well-behaved independence notion (a good frame), the union of a chain of  $\lambda$ -saturated being  $\lambda$ -saturated, or the uniqueness of limit models (see for example Chapter 9 for a survey and history). Even though in tame AECs Definition 19.4.23 is (eventually) equivalent to all these consequences (see Chapter 9), it was not known whether it followed from stability on a tail of cardinals. We show here that it does (note that this is a ZFC result).

COROLLARY 19.4.24. Let  $\mathbf{K}$  be an LS( $\mathbf{K}$ )-tame AEC with a monster model. The following are equivalent.

- (1)  $\chi(\mathbf{K}) = \aleph_0$  (i.e.  $\chi(\mathbf{K})$  consists of all regular cardinals).
- (2) **K** is  $\lambda'(\mathbf{K})$ -superstable.
- (3) **K** is stable on a tail of cardinals.

The proof uses that  $\mu$ -superstability implies stability in every  $\mu' \ge \mu$  (this is a straightforward induction using Fact 19.4.1, see Theorem 4.5.6). We state a slightly stronger version:

FACT 19.4.25 (Proposition 6.10.10). Let **K** be a  $\mu$ -tame AEC with amalgamation. If **K** is  $\mu$ -superstable, then **K** is  $\mu$ '-superstable for every  $\mu' \ge \mu$ .

PROOF OF COROLLARY 19.4.24. If (1) holds, then (2) holds by definition of  $\chi(\mathbf{K})$ . By Fact 19.4.25, this implies stability in every  $\mu \geq \lambda'(\mathbf{K})$ , so (3). Now if (3) holds then by Corollary 19.4.14 we must have that  $\chi(\mathbf{K}) = \aleph_0$ , so (1) holds.

Corollary 19.4.24 and the author's earlier work with Grossberg (Chapter 9) justify the following definition for tame AECs:

DEFINITION 19.4.26. Let **K** be an LS(**K**)-tame AEC with a monster model. We say that **K** is *superstable* if  $\chi(\mathbf{K}) = \aleph_0$ .

#### 19.5. The saturation spectrum

Theorem 19.4.13 shows that there is a saturated model in many stability cardinals. It is natural to ask whether this generalizes to all stability cardinals, and whether the converse is true, as in the first-order case. We show here that this holds assuming SCH, but prove several ZFC results along the way. Some of the proofs are inspired from the ones in homogeneous model theory (due to Shelah [She75c], see also the exposition in [GL02]).

The following is standard and will be used without comments.

FACT 19.5.1. Let **K** be an AEC with a monster model. If  $LS(\mathbf{K}) < \mu \leq \lambda = \lambda^{<\mu}$ , then **K** has a  $\mu$ -saturated model of cardinality  $\lambda$ .

In particular, **K** has a saturated model in  $\lambda$  if  $\lambda = \lambda^{<\lambda}$ .

We turn to studying what we can say about  $\lambda$  when  ${\bf K}$  has a saturated model in  $\lambda.$ 

THEOREM 19.5.2. Let **K** be an LS(**K**)-tame AEC with a monster model. Let  $LS(\mathbf{K}) < \lambda$ . If **K** has a saturated model of cardinality  $\lambda$  and **K** is stable in unboundedly many cardinals below  $\lambda$ , then **K** is stable in  $\lambda$ .

PROOF. By Fact 19.4.1(3), we can assume without loss of generality that  $\lambda$  is a limit cardinal. Let  $\delta := \operatorname{cf} \lambda$ . Pick  $\langle \lambda_i : i \leq \delta \rangle$  strictly increasing continuous such that  $\lambda_{\delta} = \lambda$ ,  $\lambda_0 \geq \operatorname{LS}(\mathbf{K})$ , and  $i < \delta$  implies that **K** is stable in  $\lambda_{i+1}$ . Let  $M \in \mathbf{K}_{\lambda}$ be saturated and let  $\langle M_i : i \leq \delta \rangle$  be an increasing continuous resolution of M such that for each  $i < \delta$ ,  $M_i \in \mathbf{K}_{\lambda_i}$  and  $M_{i+2}$  is universal over  $M_{i+1}$ .

<u>Claim</u>: For any  $p \in gS(M)$ , there exists  $i < \delta$  such that p does not  $\lambda$ -split over  $M_i$ .

<u>Proof of Claim</u>: If  $\delta > \lambda_1$ , then the result follows from Facts 19.4.3 and 19.4.4. If  $\delta \leq \lambda_1$ , then this is given by Lemma 19.4.12.  $\dagger_{\text{Claim}}$ 

Now assume for a contradiction that **K** is not stable in  $\lambda$  and let  $\langle p_i : i < \lambda^+ \rangle$  be distinct members of gS(M) (the saturated model must witness instability because it is universal). By the claim, for each  $i < \lambda^+$  there exists  $j_i < \delta$  such that p does not  $\lambda$ -split over  $M_{j_i}$ . By the pigeonhole principle, without loss of generality  $j_i = j_0$  for each  $i < \lambda^+$ . Now  $|gS(M_{j_0})| \leq |gS(M_{j_0+2})| = ||M_{j_0+2}|| < \lambda$ , so by the pigeonhole principle again, without loss of generality  $p_i \upharpoonright M_{j_0+2} = p_j \upharpoonright M_{j_0+2}$  for all  $i < j < \lambda^+$ . By weak uniqueness of non- $\lambda$ -splitting and tameness, this implies that  $p_i = p_j$ , a contradiction.

We have not used the full strength of the assumption that **K** is stable in unboundedly many cardinals below  $\lambda$ . For example, the same argument as in Theorem 19.5.2 proves:

THEOREM 19.5.3. Let **K** be an LS(**K**)-tame AEC with a monster model. Let  $LS(\mathbf{K}) < \lambda$ , with  $\lambda$  a singular cardinal, and let  $M \in \mathbf{K}_{\lambda}$  be a saturated model. If for all  $M_0 \leq_{\mathbf{K}} M$  with  $||M_0|| < ||M||, |gS(M_0)| < \lambda$  (this happens if e.g.  $\lambda$  is strong limit), then **K** is stable in  $\lambda$ .

We can also prove in ZFC that existence of a saturated model at a cardinal  $\lambda < \lambda^{<\lambda}$  implies that the class is stable. We first recall the definition of another locality cardinal:

DEFINITION 19.5.4 (4.4 in [**GV06b**]). For **K** a LS(**K**)-tame AEC with a monster model, define  $\bar{\kappa}_{\ell}$ **K**) to be the least cardinal  $\mu > \text{LS}(\mathbf{K})$  such that for any  $M \in \mathbf{K}$  and any  $p \in \mathrm{gS}(M)$ , there exists  $M_0 \in \mathbf{K}_{<\mu}$  with  $M_0 \leq_{\mathbf{K}} M$  such that p does not  $||M_0||$ -split over  $M_0$ . Set  $\bar{\kappa}(\mathbf{K}) = \infty$  if there is no such cardinal.

We have that stability is equivalent to boundedness of  $\bar{\kappa}(\mathbf{K})$ :

THEOREM 19.5.5. Let  $\mathbf{K}$  be an LS( $\mathbf{K}$ )-tame AEC with a monster model. The following are equivalent:

- (1)  $\mathbf{K}$  is stable.
- (2)  $\bar{\kappa}(\mathbf{K}) < H_1$ .
- (3)  $\bar{\kappa}_{(\mathbf{K})} < \infty$ .

PROOF. (1) implies (2) is because by Fact 19.4.4,  $\bar{\kappa}_{(\mathbf{K})} \leq \lambda(\mathbf{K})^+$  and by Fact 19.4.1(1),  $\lambda(\mathbf{K})^+ < H_1$ . (2) implies (3) is trivial. To see that (3) implies (1), let  $\mu := \bar{\kappa}_{(\mathbf{K})}$ . Pick any  $\lambda_0 \geq \mathrm{LS}(\mathbf{K})$  such that  $\lambda_0 = \lambda_0^{<\mu}$  (e.g.  $\lambda_0 = 2^{\mu}$ ), and pick any  $\lambda > \lambda_0$  such that  $\lambda^{\lambda_0} = \lambda$  (e.g.  $\lambda = 2^{\lambda_0}$ ). We claim that  $\mathbf{K}$  is stable in  $\lambda$ . Let  $M \in \mathbf{K}_{\lambda}$ , and extend it to  $M' \in \mathbf{K}_{\lambda}$  that is  $\mu$ -saturated. It is enough to see that  $|\mathrm{gS}(M')| = \lambda$ , so without loss of generality M = M'. Suppose that  $|\mathrm{gS}(M)| > \lambda$  and let  $\langle p_i : i < \lambda^+ \rangle$  be distinct members. By definition of  $\mu$ , for each  $i < \lambda^+$  there exists  $M_i \in \mathbf{K}_{<\mu}$  such that  $M_i \leq_{\mathbf{K}} M$  and p does not  $||M_i||$ -split over  $M_i$ . Since  $\lambda = \lambda^{<\mu}$ , we can assume without loss of generality that  $M_i = M_0$  for all  $i < \lambda^+$ . Further,  $|\mathrm{gS}(M_0)| \leq 2^{<\mu} \leq \lambda_0^{<\mu} = \lambda_0$ , so we can pick  $M'_0 \leq_{\mathbf{K}} M$  with  $M'_0 \in \mathbf{K}_{\lambda_0}$  such that  $M'_0 = p_j \upharpoonright M'_0$ . By tameness and weak uniqueness of non-splitting, we conclude that  $p_i = p_j$ , a contradiction.

We will use that failure of local character of splitting allows us to build a tree of types, see the proof of [**GV06b**, 4.6].

FACT 19.5.6. Let **K** be an LS(**K**)-tame AEC with a monster model. Let LS(**K**) <  $\mu$ , with  $\mu$  a regular cardinal. If  $\bar{\kappa}_{l}$ (**K**) >  $\mu$ , then there exists an increasing continuous tree  $\langle M_{\eta} : \eta \in {}^{\leq \mu}2 \rangle$ , and tree of types  $\langle p_{\eta} : \eta \in {}^{\leq \mu}2 \rangle$ , and sets  $\langle A_{\eta} : \eta \in {}^{\leq \mu}2 \rangle$  such that for all  $\eta \in {}^{<\mu}2$ :

- (1)  $M_{\eta} \in \mathbf{K}_{<\mu}$ .
- (2)  $p_n \in gS(M_n)$ .
- (3)  $A_{\eta} \subseteq |M_{\eta \frown 0}| \cap |M_{\eta \frown 1}|.$
- (4)  $|A_{\eta}| \leq \mathrm{LS}(\mathbf{K}).$
- (5)  $p_{\eta \frown 0} \upharpoonright A_{\eta} \neq p_{\eta \frown 1} \upharpoonright A_{\eta}$ .

THEOREM 19.5.7. Let **K** be an LS(**K**)-tame AEC with a monster model. Let  $LS(\mathbf{K}) < \lambda < \lambda^{<\lambda}$ . If **K** has a saturated model of cardinality  $\lambda$ , then  $\bar{\kappa}_{(\mathbf{K})} \leq \lambda^{+}$ . In particular, **K** is stable.

PROOF. The last sentence is Theorem 19.5.5. Now suppose for a contradiction that  $\bar{\kappa}_i(\mathbf{K}) > \lambda^+$ .

<u>Claim</u>:  $2^{<\lambda} = \lambda$ .

<u>Proof of Claim</u>: Suppose not and let  $\mu < \lambda$  be minimal such that  $2^{\mu} > \lambda$ . Then  $\mu$  is regular so let  $\langle M_{\eta} : \eta \in {}^{\leq \mu}2 \rangle$ ,  $\langle p_{\eta} : \eta \in {}^{\leq \mu}2 \rangle$ , and  $\langle A_{\eta} : \eta \in {}^{\leq \mu}2 \rangle$  be as given by Fact 19.5.6. Since  $2^{<\mu} \leq \lambda$ , we can use universality of M to assume without loss of generality that  $M_{\eta} \leq_{\mathbf{K}} M$  for each  $\eta \in {}^{<\mu}2$ . By continuity of the tree,  $M_{\eta} \leq_{\mathbf{K}} M$  for each  $\eta \in {}^{<\mu}2$ . By construction of the tree, each of these types has a different realization so in particular,  $2^{\mu} \leq \lambda$ , a contradiction.  $\dagger_{\text{Claim}}$ 

Now if there exists  $\mu < \lambda$  such that  $2^{\mu} = \lambda$ , then  $2^{\mu'} = \lambda$  for all  $\mu' \in [\mu, \lambda)$ , hence  $\lambda = \lambda^{<\lambda}$ , which we assumed was not true. Therefore  $\lambda$  is strong limit. Since  $\lambda < \lambda^{<\lambda}$ , this implies that  $\lambda$  is singular. By Theorem 19.5.3, **K** is stable in  $\lambda$ . By Fact 19.4.4,  $\bar{\kappa}(\mathbf{K}) \leq \lambda^+$ , as desired.

We have arrived to the following. Note that we need some set-theoretic hypotheses (e.g. assuming SCH,  $\theta(H_1) = 2^{H_1}$ , see Fact 19.4.20) to get that  $\theta(H_1) < \infty$  otherwise the result holds vacuously.

COROLLARY 19.5.8. Let **K** be an LS(**K**)-tame AEC with a monster model. Let  $\chi_0 < H_1$  be as given by Fact 19.4.10. Let  $\lambda > \chi_0 + \theta(\lambda(\mathbf{K}))$  (recall Definition 19.4.19). The following are equivalent:

- (1) **K** has a saturated model of cardinality  $\lambda$ .
- (2)  $\lambda = \lambda^{<\lambda}$  or **K** is stable in  $\lambda$ .

PROOF. First assume (2). If  $\lambda = \lambda^{<\lambda}$ , we get a saturated model of cardinality  $\lambda$  using Fact 19.5.1, so assume that **K** is stable in  $\lambda$ . If  $\lambda$  is a successor, the  $(\lambda, \lambda)$ -limit model is saturated, so assume that  $\lambda$  is limit. By Lemma 19.4.21, **K** is stable in unboundedly many cardinals below  $\lambda$ . By Theorem 19.4.13, **K** has a saturated model of cardinality  $\lambda$ .

Now assume (1) and  $\lambda < \lambda^{<\lambda}$ . By Theorem 19.5.7, **K** is stable. By Lemma 19.4.16, either **K** is stable in  $\lambda$ , or there are unboundedly many stability cardinals below  $\lambda$ . In the former case we are done and in the latter case, we can use Theorem 19.5.2.

When **K** is superstable (i.e.  $\chi(\mathbf{K}) = \aleph_0$ , see Definition 19.4.26), we obtain a characterization in ZFC.

COROLLARY 19.5.9. Let **K** be an LS(**K**)-tame AEC with a monster model. The following are equivalent:

- (1)  $\mathbf{K}$  is superstable.
- (2) **K** has a saturated model of size  $\lambda$  for every  $\lambda \ge \lambda'(\mathbf{K}) + \mathrm{LS}(\mathbf{K})^+$ .
- (3) There exists  $\mu$  such that **K** has a saturated model of size  $\lambda$  for every  $\lambda \geq \mu$ .

PROOF. (1) implies (2) is known (use Corollary 19.4.24 to see that **K** is  $\lambda'(\mathbf{K})$ superstable, then apply Corollary 10.6.9 together with [**Van16a**]), and (2) implies
(3) is trivial. Now assume (3). By Theorem 19.5.7, **K** is stable. We prove by
induction on  $\lambda \geq \mu^{\lambda(\mathbf{K})}$  that **K** is stable in  $\lambda$ . This implies superstability by
Corollary 19.4.24.

If  $\lambda = \mu^{\lambda(\mathbf{K})}$ , then  $\lambda^{\lambda(\mathbf{K})} = \lambda$  so **K** is stable in  $\lambda$  (see Fact 19.4.1(2)). Now if  $\lambda > \mu^{\lambda(\mathbf{K})}$ , then by the induction hypothesis **K** is stable in unboundedly many cardinals below  $\lambda$ , hence the result follows from Theorem 19.5.2.

### 19.6. Characterizations of stability

In Chapter 9, Grossberg and the author characterize superstability in terms of the behavior of saturated, limit, and superlimit models. We show that stability can be characterized analogously. In fact, we are able to give a list of statements equivalent to " $\chi \in \chi(\mathbf{K})$ ".

REMARK 19.6.1. Another important characterization of superstability in Chapter 9 was solvability: roughly, the existence of an EM blueprint generating superlimit models. We do not know if there is a generalization of solvability to stability.

Indeed it follows from the proof of [**SV99**, 2.2.1] that even an EM blueprint generating just universal (not superlimit) models would imply superstability (see also Chapter 20).

We see the next definition as the "stable" version of a superlimit model. Very similar notions appear already in [She87a].

DEFINITION 19.6.2. Let **K** be an AEC. For  $\chi$  a regular cardinal,  $M \in \mathbf{K}_{\geq \chi}$  is  $\chi$ -superlimit if:

- (1) M has a proper extension.
- (2) M is universal in  $\mathbf{K}_{||M||}$ .
- (3) For any increasing chain  $\langle M_i : i < \chi \rangle$ , if  $i < \chi$  implies  $M \cong M_i$ , then  $M \cong \bigcup_{i < \chi} M_i$ .

In Chapter 9, it was shown that one of the statements below holds for all  $\chi$  if and only if all of them hold for all  $\chi$ . The following characterization is a generalization to strictly stable AECs, where  $\chi$  is fixed at the beginning.

THEOREM 19.6.3. Let **K** be a (not necessarily stable)  $LS(\mathbf{K})$ -tame AEC with a monster model. Let  $\chi$  be a regular cardinal. The following are equivalent:

- (0)  $\chi \in \chi(\mathbf{K})$ .
- (1) For unboundedly many  $H_1$ -closed stability cardinals  $\mu$ , cf  $\mu = \chi$ .
- (2) For unboundedly many cardinals  $\mu$ , there exists a saturated  $(\mu, \chi)$ -limit model.
- (3) For unboundedly many μ, the union of any increasing chain of μ-saturated models of length χ is μ-saturated.
- (4) For unboundedly many stability cardinals μ, there is a χ-superlimit model of cardinality μ.
- (5) For unboundedly many  $H_1$ -closed cardinals  $\mu$  with cf  $\mu = \chi$ , there is a saturated model of cardinality  $\mu$ .

PROOF. We first show that each of the condition implies that **K** is stable. If (0) holds, then by definition of  $\underline{\chi}(\mathbf{K})$  we must have that **K** is stable. If (2) holds, then there exists in particular limit models and this implies stability. Also (1) and (4) imply stability by definition. If (5) holds, then we have stability by Theorem 19.5.7. Finally, assume that (3) holds. Build an increasing continuous chain of cardinals  $\langle \mu_i : i \leq \chi \rangle$  such that  $\chi + \mathrm{LS}(\mathbf{K}) < \mu_0$ , for each  $i \leq \chi$  any increasing chain of  $\mu_i$ -saturated models of length  $\chi$  is  $\mu_i$ -saturated, and  $2^{\mu_i} < \mu_{i+1}$  for all  $i < \chi$ . Let  $\mu := \mu_{\chi}$ . Build an increasing chain  $\langle M_i : i < \chi \rangle$  such that  $M_{i+1} \in \mathbf{K}_{2^{\mu_i}}$ and  $M_{i+1}$  is  $\mu_i$ -saturated. Now by construction  $M := \bigcup_{i < \chi} M_i$  is in  $\mathbf{K}_{\mu}$  and is saturated. Since cf  $\mu = \chi$ , we have that  $\mu < \mu^{\chi} \leq \mu^{<\mu}$ . By Theorem 19.5.7, **K** is stable. We have shown that we can assume without loss of generality that **K** is stable.

We now show that (3) is equivalent to (4). Indeed, if we have a  $\chi$ -superlimit at a stability cardinal  $\mu$ , then it must be saturated and witnesses that the union of an increasing chain of  $\mu$ -saturated models of length  $\chi$  is  $\mu$ -saturated. Conversely, We have shown in the first paragraph of this proof how to build a saturated model in a cardinal  $\mu$  such that the union of an increasing chain of  $\mu$ -saturated models of length  $\chi$  is  $\mu$ -saturated. Such a saturated model must be a  $\chi$ -superlimit.

We also have that (4) implies (2), as it is easy to see that a  $\chi$ -superlimit model in a stability cardinal  $\mu$  must be unique and also a  $(\mu, \chi)$ -limit model. Also, (0) implies (3) (Fact 19.4.10) and (2) implies (0) (Lemma 19.4.12). Therefore (0), (2), (3), (4) are all equivalent.

Now, (0) implies (5) (Theorem 19.4.13). Also, (5) implies (1): Let  $\mu$  be  $H_1$ closed such that cf  $\mu = \chi$  and there is a saturated model of cardinality  $\mu$ . By Lemma 19.4.16, either **K** is stable in  $\mu$  or stable in unboundedly many cardinals below  $\mu$ . In the latter case, Theorem 19.5.2 implies that **K** is stable in  $\mu$ . Thus **K** is stable in  $\mu$ , hence (1) holds.

It remains to show that (1) implies (0). Let  $\mu$  be an  $H_1$ -closed stability cardinal of cofinality  $\chi$ . By the proof of Lemma 19.4.16, **K** is stable in unboundedly many cardinals below  $\mu$ . By Theorem 19.4.11,  $\chi \in \chi(\mathbf{K})$ , so (0) holds.

### 19.7. Indiscernibles and bounded equivalence relations

We review here the main tools for the study of strong splitting in the next section: indiscernibles and bounded equivalence relations. All throughout, we assume:

HYPOTHESIS 19.7.1. K is an AEC with a monster model.

REMARK 19.7.2. By working more locally, the results and definitions of this section could be adapted to the amalgamation-less setup (see for example Definition 17.2.3).

DEFINITION 19.7.3 (Indiscernibles, 4.1 in [She99]). Let  $\alpha$  be a non-zero cardinal,  $\theta$  be an infinite cardinal, and let  $\langle \bar{a}_i : i < \theta \rangle$  be a sequence of distinct elements each of length  $\alpha$ . Let A be a set.

- (1) We say that  $\langle \bar{a}_i : i < \theta \rangle$  is indiscernible over A in N if for every  $n < \omega$ , every  $i_0 < \ldots < i_{n-1} < \theta$ ,  $j_0 < \ldots < j_{n-1} < \theta$ ,  $\operatorname{gtp}(\bar{a}_{i_0} \ldots \bar{a}_{i_n}/A) = \operatorname{gtp}(\bar{a}_{j_0} \ldots \bar{a}_{j_n}/A)$ . When  $A = \emptyset$ , we omit it and just say that  $\langle \bar{a}_i : i < \theta \rangle$  is indiscernible.
- (2) We say that  $\langle \bar{a}_i : i < \theta \rangle$  is strictly indiscernible if there exists an EM blueprint  $\Phi$  (whose vocabulary is allowed to have arbitrary size) an automorphism f of  $\mathfrak{C}$  so that, letting  $N' := \mathrm{EM}_{\tau(\mathbf{K})}(\theta, \Phi)$ :
  - (a) For all  $i < \theta$ ,  $\bar{b}_i := f(\bar{a}_i) \in {}^{\alpha}|N'|$ .
  - (b) If for  $i < \theta$ ,  $\bar{b}_i = \langle b_{i,j} : j < \alpha \rangle$ , then for all  $j < \alpha$  there exists a unary  $\tau(\Phi)$ -function symbol  $\rho_j$  such that for all  $i < \theta$ ,  $b_{i,j} = \rho_j^{N'}(i)$ .
- (3) Let A be a set. We say that  $\langle \bar{a}_i : i < \theta \rangle$  is strictly indiscernible over A if there exists an enumeration  $\bar{a}$  of A such that  $\langle \bar{a}_i \bar{a} : i < \theta \rangle$  is strictly indiscernible.

Any strict indiscernible sequence extends to arbitrary lengths: this follows from a use of first-order compactness in the EM language. The converse is also true. This follows from the more general extraction theorem, essentially due to Morley:

FACT 19.7.4. Let  $\mathbf{I} := \langle \bar{a}_i : i < \theta \rangle$  be distinct such that  $\ell(\bar{a}_i) = \alpha$  for all  $i < \theta$ . Let A be a set. If  $\theta \ge h(\mathrm{LS}(\mathbf{K}) + |\alpha| + |A|)$ , then there exists  $\mathbf{J} := \langle \bar{b}_i : i < \omega \rangle$  such that  $\mathbf{J}$  is strictly indiscernible over A and for any  $n < \omega$  there exists  $i_0 < \ldots < i_{n-1} < \theta$  such that  $\mathrm{gtp}(\bar{b}_0 \ldots \bar{b}_{n-1}/A) = \mathrm{gtp}(\bar{a}_{i_0} \ldots \bar{a}_{i_{n-1}}/A)$ .

FACT 19.7.5. Let  $\langle \bar{a}_i : i < \theta \rangle$  be indiscernible over A, with  $\ell(\bar{a}_i) = \alpha$  for all  $i < \theta$ . The following are equivalent:

(1) For any infinite cardinal  $\lambda$ , there exists  $\langle \bar{b}_i : i < \lambda \rangle$  that is indiscernible over A and such that  $\bar{b}_i = \bar{a}_i$  for all  $i < \theta$ .

- (2) For all infinite  $\lambda < h(\theta + |A| + |\alpha| + \text{LS}(\mathbf{K}))$  (recall Definition 2.2.2), there exists  $\langle \bar{b}_i : i < \lambda \rangle$  as in (1).
- (3)  $\langle \bar{a}_i : i < \theta \rangle$  is strictly indiscernible over A.

We want to study bounded equivalence relations: they are the analog of Shelah's finite equivalence relations from the first-order setup but here the failure of compactness compels us to only ask for the number of classes to be bounded (i.e. a cardinal). The definition for homogeneous model theory appears in [**HS00**, 1.4].

DEFINITION 19.7.6. Let  $\alpha$  be a non-zero cardinal and let A be a set. An  $\alpha$ -ary Galois equivalence relation on A is an equivalence relation E on  ${}^{\alpha}\mathfrak{C}$  such that for any automorphism f of  $\mathfrak{C}$  fixing A,  $\bar{b}E\bar{c}$  if and only if  $f(\bar{b})Ef(\bar{c})$ .

DEFINITION 19.7.7. Let  $\alpha$  be a non-zero cardinal, A be a set, and E be an  $\alpha$ -ary Galois equivalence relation on A.

- (1) Let c(E) be the number of equivalence classes of E.
- (2) We say that E is bounded if  $c(E) < \infty$  (i.e. it is a cardinal).
- (3) Let  $SE^{\alpha}(A)$  be the set of  $\alpha$ -ary bounded Galois equivalence relations over A (S stands for strong).

Remark 19.7.8.

$$|\mathrm{SE}^{\alpha}(A)| \le |2^{\mathrm{gS}^{\alpha+\alpha}(A)}| \le 2^{2^{|A|+\mathrm{LS}(\mathbf{K})+\alpha}}$$

The next two results appear for homogeneous model theory in [HS00, §1]. The main difference here is that strictly indiscernible and indiscernibles need not coincide.

LEMMA 19.7.9. Let  $E \in SE^{\alpha}(A)$ . Let **I** be strictly indiscernible over A. For any  $\bar{a}, \bar{b} \in \mathbf{I}$ , we have that  $\bar{a}E\bar{b}$ .

PROOF. Suppose not, say  $\neg(\bar{a}E\bar{b})$ . Fix any infinite cardinal  $\lambda \geq |\mathbf{J}|$ . By Theorem 19.7.5,  $\mathbf{I}$  extends to a strictly indiscernible sequence  $\mathbf{J}$  over A of cardinality  $\lambda$ . Thus  $c(E) \geq \lambda$ . Since  $\lambda$  was arbitrary, this contradicts the fact that E was bounded.

LEMMA 19.7.10. Let A be a set and  $\alpha$  be a non-zero cardinal. Let E be an  $\alpha$ -ary Galois equivalence relation over A. The following are equivalent:

(1) E is bounded.

(2)  $c(E) < h(|A| + \alpha + \mathrm{LS}(\mathbf{K})).$ 

PROOF. Let  $\theta := h(|A| + \alpha + \mathrm{LS}(\mathbf{K}))$ . If  $c(E) < \theta$ , E is bounded. Conversely if  $c(E) \ge \theta$  then we can list  $\theta$  non-equivalent elements as  $\mathbf{I} := \langle \bar{a}_i : i < \theta \rangle$ . By Fact 19.7.4, there exists a strictly indiscernible sequence over  $A \langle \bar{b}_i : i < \omega \rangle$  reflecting some of the structure of  $\mathbf{I}$ . In particular, for  $i < j < \omega$ ,  $\neg(\bar{b}_i E \bar{b}_j)$ . By Lemma 19.7.9, E cannot be bounded.

The following equivalence relation will play an important role (see [HS00, 4.7])

DEFINITION 19.7.11. For all A and  $\alpha$ , let  $E_{\min,A,\alpha} := \bigcap SE^{\alpha}(A)$ .

By Remark 19.7.8 and a straightforward counting argument, we have that  $E_{\min,A,\alpha} \in SE^{\alpha}(A)$ .

### 19.8. Strong splitting

We study the AEC analog of first-order strong splitting. It was introduced by Shelah in [She99, 4.11]. In the next section, the analog of first-order dividing will be studied. Shelah also introduced it [She99, 4.8] and showed how to connect it with strong splitting. After developing enough machinery, we will be able to connect Shelah's results on the locality cardinals for dividing [She99, 5.5] to the locality cardinals for splitting.

All throughout this section, we assume:

HYPOTHESIS 19.8.1. K is an AEC with a monster model.

DEFINITION 19.8.2. Let  $\mu$  be an infinite cardinal,  $A \subseteq B$ ,  $p \in gS(B)$ . We say that  $p \ (<\mu)$ -strongly splits over A if there exists a strictly indiscernible sequence  $\langle \bar{a}_i : i < \omega \rangle$  over A with  $\ell(\bar{a}_i) < \mu$  for all  $i < \omega$  such that for any b realizing p,  $gtp(b\bar{a}_0/A) \neq gtp(b\bar{a}_1/A)$ . We say that p explicitly  $(<\mu)$ -strongly splits over A if the above holds with  $\bar{a}_0\bar{a}_1 \in {}^{<\mu}B$ .

 $\mu$ -strongly splits means ( $\leq \mu$ )-strongly splits, which has the expected meaning.

REMARK 19.8.3. For  $\mu < \mu'$ , if p [explicitly] ( $< \mu$ )-strongly splits over A, then p [explicitly] ( $< \mu'$ )-strongly splits over A.

LEMMA 19.8.4 (Base monotonicity of strong splitting). Let  $A \subseteq B \subseteq C$  and let  $p \in gS(C)$ . Let  $\mu > |B \setminus A|$  be infinite. If  $p \ (< \mu)$ -strongly splits over B, then  $p \ (< \mu)$ -strongly splits over A.

PROOF. Let  $\langle \bar{a}_i : i < \omega \rangle$  witness the strong splitting over B. Let  $\bar{c}$  be an enumeration of  $B \setminus A$ . The sequence  $\langle \bar{a}_i \bar{c} : i < \omega \rangle$  is strictly indiscernible over A. Moreover, for any b realizing p,  $gtp(b\bar{c}\bar{a}_0/A) \neq gtp(b\bar{c}\bar{a}_1/A)$  if and only if  $gtp(b\bar{a}_0/A\bar{c}) \neq gtp(b\bar{a}_1/A\bar{c})$  if and only if  $gtp(b\bar{a}_0/B) \neq gtp(b\bar{a}_1/B)$ , which holds by the strong splitting assumption.

Lemma 19.8.4 motivates the following definition:

DEFINITION 19.8.5. For  $\lambda \geq \text{LS}(\mathbf{K})$ , we let  $i_{\lambda-\text{strong-spl}}(\mathbf{K}_{\lambda})$  be the independence relation whose underlying class is  $\mathbf{K}'$  and whose independence relation is non  $\lambda$ -strong-splitting.

Next, we state a key characterization lemma for strong splitting in terms of bounded equivalence relations. This is used in the proof of the next result, a kind of uniqueness of the non-strong-splitting extension. It appears already for homogeneous model theory in  $[\mathbf{HS00}, 1.12]$ 

DEFINITION 19.8.6. Let  $N \in \mathbf{K}$ ,  $A \subseteq |N|$ , and  $\mu$  be an infinite cardinal. We say that N is  $\mu$ -saturated over A if any type in  $gS^{<\mu}(A)$  is realized in N.

LEMMA 19.8.7. Let  $N \in \mathbf{K}$  and let  $A \subseteq |N|$ . Assume that N is  $(\aleph_1 + \mu)$ -saturated over A. Let  $p := \operatorname{gtp}(b/N)$ . The following are equivalent.

(1) p does not explicitly  $(< \mu)$ -strongly split over A.

(2) p does not  $(<\mu)$ -strongly split over A.

(3) For all  $\alpha < \mu$ , all  $\bar{c}$ ,  $\bar{d}$  in  $^{\alpha}|N|$ ,  $\bar{c}E_{\min,A,\alpha}\bar{d}$  implies  $\operatorname{gtp}(b\bar{c}/A) = \operatorname{gtp}(b\bar{d}/A)$ .

PROOF. If p explicitly  $(< \mu)$ -strongly splits over A, then  $p(< \mu)$ -strongly splits over A. Thus (2) implies (1).

#### 19.9. DIVIDING

If  $p \ (<\mu)$ -splits strongly over A, let  $\mathbf{I} = \langle \bar{a}_i : i < \omega \rangle$  witness it, with  $\bar{a}_i \in {}^{\alpha} |\mathfrak{C}|$ for all  $i < \omega$ . By Lemma 19.7.9,  $\bar{a}_0 E_{\min,A,\alpha} \bar{a}_1$ . However by the strong splitting assumption  $\operatorname{gtp}(b\bar{a}_0/A) \neq \operatorname{gtp}(b\bar{a}_1/A)$ . This proves (3) implies (2).

It remains to show that (1) implies (3). Assume (1). Assume  $\bar{c}$ ,  $\bar{d}$  are in  ${}^{\alpha}|N|$ such that  $\bar{c}E_{\min,A,\alpha}\bar{d}$ . Define an equivalence relation E on  ${}^{\alpha}|\mathfrak{C}|$  as follows.  $\bar{b}_0E\bar{b}_1$ if and only if  $\bar{b}_0 = \bar{b}_1$  or there exists  $n < \omega$  and  $\langle \mathbf{I}_i : i < n \rangle$  strictly indiscernible over A such that  $\bar{b}_0 \in I_0$ ,  $\bar{b}_1 \in I_{n-1}$  and for all i < n-1,  $\mathbf{I}_i \cap \mathbf{I}_{i+1} \neq \emptyset$ . Eis a Galois equivalence relation over A. Moreover if  $\langle \bar{a}_i : i < \theta \rangle$  are in different equivalence classes and  $\theta$  is sufficiently big, we can extract a strictly indiscernible sequence from it which will witness that all elements are actually in the same class. Therefore  $E \in SE^{\alpha}(A)$ .

Since  $\bar{c}E_{\min,A,\alpha}\bar{d}$ , we have that  $\bar{c}E\bar{d}$  and without loss of generality  $\bar{c} \neq \bar{d}$ . Let  $\langle \mathbf{I}_i : i < n \rangle$  witness equivalence. By saturation, we can assume without loss of generality that  $\mathbf{I}_i$  is in |M| for all i < n. Now use the failure of explicit strong splitting to argue that  $gtp(b\bar{c}/A) = gtp(b\bar{d}/A)$ .

LEMMA 19.8.8 (Toward uniqueness of non strong splitting). Let  $M \leq_{\mathbf{K}} N$  and let  $A \subseteq |M|$ . Assume that N is  $(\aleph_1 + \mu)$ -saturated over A and for every  $\alpha < \mu$ ,  $\bar{c} \in {}^{\alpha}|N|$ , there is  $\bar{d} \in {}^{\alpha}|M|$  such that  $\bar{d}E_{\min,A,\alpha}\bar{c}$ .

Let  $p, q \in gS(N)$  not  $(< \mu)$ -strongly split over A. If  $p \upharpoonright M = q \upharpoonright M$ , then  $p \upharpoonright B = q \upharpoonright B$  for every  $B \subseteq |N|$  with  $|B| < \mu$ .

PROOF. Say  $p = \operatorname{gtp}(a/N)$ ,  $q = \operatorname{gtp}(b/N)$ . Let  $\bar{c} \in {}^{<\mu}|N|$ . We want to see that  $\operatorname{gtp}(a/\bar{c}) = \operatorname{gtp}(b/\bar{c})$ . We will show that  $\operatorname{gtp}(a\bar{c}/A) = \operatorname{gtp}(b\bar{c}/A)$ . Pick  $\bar{d}$  in M such that  $\bar{c}E_{\min,A,\alpha}\bar{d}$ . Then by Lemma 19.8.7,  $\operatorname{gtp}(a\bar{c}/A) = \operatorname{gtp}(a\bar{d}/A)$ . Since  $p \upharpoonright M = q \upharpoonright M$ ,  $\operatorname{gtp}(a\bar{d}/A) = \operatorname{gtp}(b\bar{d}/A)$ . By Lemma 19.8.7 again,  $\operatorname{gtp}(b\bar{d}/A) = \operatorname{gtp}(b\bar{c}/A)$ . Combining these equalities, we get that  $\operatorname{gtp}(a\bar{c}/A) = \operatorname{gtp}(b\bar{c}/A)$ , as desired.  $\Box$ 

### 19.9. Dividing

HYPOTHESIS 19.9.1. K is an AEC with a monster model.

The following notion generalizes first-order dividing and was introduced by Shelah [She99, 4.8].

DEFINITION 19.9.2. Let  $A \subseteq B$ ,  $p \in gS(B)$ . We say that p divides over A if there exists an infinite cardinal  $\theta$  and a strictly indiscernible sequence  $\langle \bar{b}_i : i < \theta \rangle$ over A as well as  $\langle f_i : i < \theta \rangle$  automorphisms of  $\mathfrak{C}$  fixing A such that  $\bar{b}_0$  is an enumeration of B,  $f_i(\bar{b}_0) = \bar{a}_i$  for all  $i < \theta$ , and  $\langle f_i(p) : i < \theta \rangle$  is inconsistent.

It is clear from the definition that dividing induces an independence relation:

DEFINITION 19.9.3. For  $\lambda \geq LS(\mathbf{K})$ , we let  $i_{div}(\mathbf{K}_{\lambda})$  be the independence relation whose underlying class is  $\mathbf{K}'$  and whose independence relation is non-dividing.

The following fact about dividing was proven by Shelah in [She99, 5.5(2)]:

FACT 19.9.4. Let  $\mu_1 \ge \mu_0 \ge LS(\mathbf{K})$ . Let  $\alpha < \mu_1^+$  be a regular cardinal. If  $\mathbf{K}$  is stable in  $\mu_1$  and  $\mu_1^{\alpha} > \mu_1$ , then  $\alpha \in \underline{\kappa}^{wk}(\mathfrak{i}_{div}(\mathbf{K}_{\mu_0}))$  (recall Definition 19.2.2).

To see when strong splitting implies dividing, Shelah considered the following property:

DEFINITION 19.9.5. **K** satisfies  $(*)_{\mu,\theta,\sigma}$  if whenever  $\langle \bar{a}_i : i < \delta \rangle$  is a strictly indiscernible sequence with  $\ell(\bar{a}_i) < \mu$  for all  $i < \delta$ , then for any  $\bar{b}$  with  $\ell(\bar{b}) < \sigma$ , there exists  $u \subseteq \delta$  with  $|u| < \theta$  such that for any  $i, j \in \delta \setminus u$ ,  $\operatorname{gtp}(\bar{a}_i \bar{b}/\emptyset) = \operatorname{gtp}(\bar{a}_i \bar{b}/\emptyset)$ .

FACT 19.9.6 (4.12 in [She99]). Let  $\mu^* := \text{LS}(\mathbf{K}) + \mu + \sigma$ . If **K** does not have the  $\mu^*$ -order property (recall Definition 2.4.3), then  $(*)_{\mu^+,h(\mu^*),\sigma^+}$  holds.

LEMMA 19.9.7. Let  $A \subseteq B$ . Let  $p \in gS(B)$ . Assume that  $(*)_{|B|^+,\theta,\sigma}$  holds for some infinite cardinals  $\theta$  and  $\sigma$ .

If p explicitly |B|-strongly splits over A, then p divides over A.

PROOF. Let  $\mu := |B|$ . Let  $\langle \bar{a}_i : i < \omega \rangle$  witness the explicit strong splitting (so  $\ell(\bar{a}_i) = \mu$  for all  $i < \omega$  and  $\bar{a}_0 \in {}^{\mu}B$ ). Increase the indiscernible to assume without loss of generality that  $\bar{a}_0$  enumerates B and increase further to get  $\langle \bar{a}_i : i < \theta^+ \rangle$ . Pick  $\langle f_i : i < \theta^+ \rangle$  automorphisms of  $\mathfrak{C}$  fixing A such that  $f_0$  is the identity and  $f_i(\bar{a}_0\bar{a}_1) = \bar{a}_i\bar{a}_{i+1}$  for each  $i < \theta^+$ . We claim that  $\langle \bar{a}_i : i < \theta^+ \rangle$ ,  $\langle f_i : i < \theta^+ \rangle$  witness the dividing over A.

Indeed, suppose for a contradiction that b realizes  $f_i(p)$  for each  $i < \theta^+$ . In particular, b realizes  $f_0(p) = p$ . By  $(*)_{\mu^+,\theta,\sigma}$ , there exists  $i < \theta^+$  such that  $gtp(b\bar{a}_i/A) =$  $gtp(b\bar{a}_{i+1}/A)$ . Applying  $f_i^{-1}$  to this equation, we get that  $gtp(c\bar{a}_0/A) = gtp(c\bar{a}_1/A)$ , where  $c := f_i^{-1}(b)$ . But since b realizes  $f_i(p)$ , c realizes p. This contradicts the strong splitting assumption.

We have arrived to the following result:

LEMMA 19.9.8. Let  $\mu_1 \ge \mu_0 \ge LS(\mathbf{K})$  be such that  $\mathbf{K}$  is stable in both  $\mu_0$  and  $\mu_1$ . Assume further that  $\mathbf{K}$  does not have the  $\mu_0$ -order property.

Let  $\alpha < \mu_0^+$  be a regular cardinal. If  $\mu_1^{\alpha} > \mu_1$ , then:

$$\alpha \in \underline{\kappa}^{\mathrm{wk}}(\mathfrak{i}_{\mu_0 \operatorname{-strong-spl}}(\mathbf{K}_{\mu_0}), <^{\mathrm{univ}}_{\mathbf{K}})$$

PROOF. By Fact 19.9.6,  $(*)_{\mu_0^+, h(\mu_0), \mu_0^+}$  holds.

Now let  $\langle M_i : i < \alpha \rangle$  be  $\langle \mathbf{K}^{\text{univ}}$ -increasing in  $\mathbf{K}_{\lambda}$ . Let  $p \in \mathrm{gS}(\bigcup_{i < \alpha} M_i)$ . By Fact 19.9.4, there exists  $i < \alpha$  such that  $p \upharpoonright M_{i+1}$  does not divide over  $M_i$ . By Lemma 19.9.7,  $p \upharpoonright M_{i+1}$  does not explicitly  $\lambda$ -strongly split over  $M_i$ . By Lemma 19.8.7 (recall that  $M_{i+1}$  is universal over  $M_i$ ),  $p \upharpoonright M_{i+1}$  does not  $\lambda$ -strongly split over  $M_i$ .

Our aim in the next section will be to show that non-strong splitting has weak uniqueness. This will allow us to apply the results of Section 19.3 and (assuming enough locality) replace  $\underline{\kappa}^{wk}$  by  $\underline{\kappa}$ .

### 19.10. Strong splitting in stable tame AECs

HYPOTHESIS 19.10.1.  $\mathbf{K}$  is an LS( $\mathbf{K}$ )-tame AEC with a monster model.

Why do we assume tameness? Because we would like to exploit the uniqueness of strong splitting (Lemma 19.8.8), but we want to be able to conclude p = q, and not just  $p \upharpoonright B = q \upharpoonright B$  for every small B. This will allow us to use the tools of Section 19.3.

DEFINITION 19.10.2. For  $\mu \geq \text{LS}(\mathbf{K})$ , let  $\chi^*(\mu) \in [\mu^+, h(\mu)$  be the least cardinal  $\chi^*$  such that whenever A has size at most  $\mu$  and  $\alpha < \mu^+$  then  $c(E_{\min,A,\alpha}) < \chi^*$  (it exists by Lemma 19.7.10).

The following is technically different from the  $\mu$ -forking defined in Definition 4.4.2 (which uses  $\mu$ -splitting), but it is patterned similarly.

DEFINITION 19.10.3. For  $p \in gS(B)$ , we say that p does not  $\mu$ -fork over  $(M_0, M)$  if:

(1)  $M_0 \leq_{\mathbf{K}} M, |M| \subseteq B.$ 

(2)  $M_0 \in \mathbf{K}_{\mu}$ .

(3) M is  $\chi^*(\mu)$ -saturated over  $M_0$ .

(4) p does not  $\mu$ -strongly split over  $M_0$ .

We say that p does not  $\mu$ -fork over M if there exists  $M_0$  such that p does not  $\mu$ -fork over  $(M_0, M)$ .

The basic properties are satisfied:

Lemma 19.10.4.

- (1) (Invariance) For any automorphism f of  $\mathfrak{C}$ ,  $p \in \mathrm{gS}(B)$  does not  $\mu$ -fork over  $(M_0, M)$  if and only if f(p) does not  $\mu$ -fork over  $(f[M_0], f[M])$ .
- (2) (Monotonicity) Let  $M_0 \leq_{\mathbf{K}} M'_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} M'$ ,  $|M'| \subseteq B$ . Assume that  $M_0, M'_0 \in \mathbf{K}_{\mu}$  and M is  $\chi^*(\mu)$ -saturated over  $M'_0$ 
  - Let  $p \in gS(B)$  be such that p does not  $\mu$ -fork over  $(M_0, M)$ . Then:
  - (a) p does not  $\mu$ -fork over  $(M'_0, M)$ .
  - (b) p does not  $\mu$ -fork over  $(M_0, M')$ .

PROOF. Invariance is straightforward. We prove monotonicity. Assume that  $M_0, M'_0, M, M', B, p$  are as in the statement. First we have to show that p does not  $\mu$ -fork over  $(M'_0, M)$ . We know that p does not  $\mu$ -strongly split over  $M_0$ . Since  $M'_0 \in \mathbf{K}_{\mu}$ , Lemma 19.8.4 implies that p does not  $\mu$ -strongly split over  $M'_0$ , as desired.

Similarly, it follows directly from the definitions that p does not  $\mu$ -fork over  $(M_0, M')$ .

This justifies the following definition:

DEFINITION 19.10.5. For  $\lambda \geq \text{LS}(\mathbf{K})$ , we write  $\mathfrak{i}_{\mu\text{-forking}}(\mathbf{K}_{\lambda})$  for the independence relation with class  $\mathbf{K}_{\lambda}$  and independence relation induced by non- $\mu$ -forking.

We now want to show that under certain conditions  $i_{\mu\text{-forking}}(\mathbf{K}_{\lambda})$  has weak uniqueness (see Definition 19.3.3). First, we show that when two types do not fork over the same sufficiently saturated model, then the "witness"  $M_0$  to the  $\mu$ -forking can be taken to be the same.

LEMMA 19.10.6. Let M be  $\chi^*(\mu)$ -saturated. Let  $|M| \subseteq B$ . Let  $p, q \in gS(B)$ and assume that both p and q do not  $\mu$ -fork over M. Then there exists  $M_0$  such that both p and q do not  $\mu$ -fork over  $(M_0, M)$ .

PROOF. Say p does not fork over  $(M_p, M)$  and q does not fork over  $(M_q, M)$ . Pick  $M_0 \leq_{\mathbf{K}} M$  of size  $\mu$  containing both  $M_p$  and  $M_q$ . This works since M is  $\chi^*(\mu)$ -saturated and  $\chi^*(\mu) > \mu$ .

LEMMA 19.10.7. Let  $\mu \geq \mathrm{LS}(\mathbf{K})$ . Let  $M \in \mathbf{K}_{\geq \mu}$  and let B be a set with  $|M| \subseteq B$ . Let  $p, q \in \mathrm{gS}(B)$  and assume that  $p \upharpoonright M = q \upharpoonright M$ .

(1) (Uniqueness over  $\chi^*$ -saturated models) If M is  $\chi^*(\mu)$ -saturated and p, q do not  $\mu$ -fork over M, then p = q.

(2) (Weak uniqueness) Let  $\lambda > \chi^*(\mu)$  be a stability cardinal. Let  $M_0 \leq_{\mathbf{K}} M$  be such that  $M_0, M \in \mathbf{K}_{\lambda}$  and M is universal over  $M_0$ . If p, q do not  $\mu$ -fork over  $M_0$ , then p = q. In other words,  $i_{\mu\text{-forking}}(\mathbf{K}_{\lambda})$  has weak uniqueness.

# Proof.

- (1) By Lemma 19.10.6, we can pick  $M_0$  such that both p and q do not  $\mu$ -fork over  $(M_0, M)$ . By Lemma 19.8.8, p = q.
- (2) Using stability, we can build  $M' \in \mathbf{K}_{\lambda}$  that is  $\chi^{*}(\mu)$ -saturated with  $M_{0} \leq_{\mathbf{K}} M'$ . Without loss of generality (using universality of M over  $M_{0}$ ),  $M' \leq_{\mathbf{K}} M$ . By base monotonicity, both p and q do not  $\mu$ -fork over M'. Since  $p \upharpoonright M = q \upharpoonright M$ , we also have that  $p \upharpoonright M' = q \upharpoonright M'$ . Now use the first part.

The following theorem is the main result of this section, so we repeat its global hypotheses here for convenience.

THEOREM 19.10.8. Let  $\mathbf{K}$  be an LS( $\mathbf{K}$ )-tame AEC with a monster model.

Let  $\mu_0 \geq \mathrm{LS}(\mathbf{K})$  be a stability cardinal. Let  $\lambda > \chi^*(\mu_0)$  be another stability cardinal. For any  $\mu_1 \geq \mu_0$ , if **K** is stable in  $\mu_1$  then  $\mu_1^{<\kappa^{\mathrm{wk}}(\mathbf{K}_{\lambda},<\mathbf{K}^{\mathrm{univ}})} = \mu_1$  (recall Definition 19.3.8).

The proof will use the following fact (recall from Hypothesis 19.10.1 that we are working inside the monster model of a tame AEC):

FACT 19.10.9 (Theorem 2.4.15). The following are equivalent:

- (1)  $\mathbf{K}$  is stable.
- (2)  $\mathbf{K}$  does not have the LS( $\mathbf{K}$ )-order property.

PROOF OF THEOREM 19.10.8. We prove that for any regular cardinal  $\alpha < \lambda^+$ , if  $\mu_1^{\alpha} > \mu_1$  then  $\alpha \in \underline{\kappa}^{wk}(\mathbf{K}_{\lambda}, <^{univ}_{\mathbf{K}})$ . This suffices because the least cardinal  $\alpha$  such that  $\mu_1^{\alpha} > \mu_1$  is regular.

Note that by definition  $\underline{\kappa}^{wk}(\mathbf{K}_{\lambda}, <_{\mathbf{K}}^{univ})$  is an end segment of regular cardinals. Note also that by Lemma 19.10.7,  $\mathfrak{i} := \mathfrak{i}_{\mu_0\text{-forking}}(\mathbf{K}_{\lambda})$  has weak uniqueness and thus we can use the results from Section 19.3 also on  $\mathfrak{i}$ .

By Fact 19.4.3 and 19.4.4,  $\mu_0^+ \in \underline{\kappa}(\mathbf{K}_{\lambda}, <_{\mathbf{K}}^{\mathrm{univ}})$ . Therefore we may assume that  $\alpha < \mu_0^+$ .

By Fact 19.10.9, **K** does not have the LS(**K**)-order property. By Lemma 19.9.8,  $\alpha \in \underline{\kappa}^{wk}(\mathbf{i}_{\mu_0-strong-spl}(\mathbf{K}_{\mu_0}), <^{univ}_{\mathbf{K}})$ . As in Section 4.4, this implies that  $\alpha \in \underline{\kappa}^{wk}(\mathbf{i}_{\mu_0-forking}(\mathbf{K}_{\lambda}), <^{univ}_{\mathbf{K}})$ . But by Lemma 19.3.9, this means that  $\alpha \in \underline{\kappa}^{wk}(\mathbf{K}_{\lambda}, <^{univ}_{\mathbf{K}})$ .

## 19.11. Stability theory assuming continuity of splitting

In this section, we will assume that splitting has the weak continuity property studied in Section 19.3:

DEFINITION 19.11.1. For **K** an AEC with a monster model, we say that *splitting* has weak continuity if for any  $\mu \in \text{Stab}(\mathbf{K})$ ,  $\kappa^{\text{cont}}(\mathbf{K}_{\mu}, <^{\text{univ}}_{\mathbf{K}}) = \aleph_0$ .

Recall that Theorem 19.3.7 shows that splitting has weak continuity under certain locality hypotheses. In particular, this holds in any class from homogeneous model theory and any universal class.

Assuming continuity and tameness, we have that  $\chi(\mathbf{K})$  is an end-segment of regular cardinals (see Corollary 19.2.6). Therefore  $\chi(\mathbf{K})$  is simply the minimal cardinal in  $\chi(\mathbf{K})$ . We have the following characterization of  $\chi(\mathbf{K})$ :

THEOREM 19.11.2. Let **K** be a stable  $LS(\mathbf{K})$ -tame AEC with a monster model. If splitting has weak continuity, then  $\chi(\mathbf{K})$  is the maximal cardinal  $\chi$  such that for any  $\mu \geq LS(\mathbf{K})$ , if **K** is stable in  $\mu$  then  $\mu = \mu^{<\chi}$ .

PROOF. First, let  $\mu \geq LS(\mathbf{K})$  be a stability cardinal. By Theorem 19.10.8 (recalling Corollary 19.2.6),  $\mu^{<\chi(\mathbf{K})} = \mu$ .

Conversely, consider the cardinal  $\mu := \beth_{\chi(\mathbf{K})}(\lambda'(\mathbf{K}))$ . By Fact 19.4.1, **K** is stable in  $\mu$ . However of  $\mu = \chi(\mathbf{K})$  so  $\mu^{\chi(\mathbf{K})} > \mu$ . In other words, there does not exist a cardinal  $\chi > \chi(\mathbf{K})$  such that  $\mu^{<\chi} = \mu$ .

Still assuming continuity, we deduce an improved bound on  $\chi(\mathbf{K})$  (compared to Remark 19.4.7) and an explicit bound on  $\lambda'(\mathbf{K})$ :

THEOREM 19.11.3. Let  $\mathbf{K}$  be a stable  $LS(\mathbf{K})$ -tame AEC with a monster model and assume that splitting has weak continuity.

(1)  $\chi(\mathbf{K}) \leq \lambda(\mathbf{K}) < H_1$ .

(2) 
$$\lambda(\mathbf{K}) \leq \lambda'(\mathbf{K}) < h(\lambda(\mathbf{K})) < \beth_{H_1}$$
.

Proof.

- (1) That  $\lambda(\mathbf{K}) < H_1$  is Fact 19.4.1. Now by Theorem 19.11.2,  $\lambda(\mathbf{K})^{<\chi(\mathbf{K})} = \lambda(\mathbf{K})$  and hence  $\chi(\mathbf{K}) \leq \lambda(\mathbf{K})$ .
- (2) Let  $\lambda'$  be the least stability cardinal above  $\chi^*(\lambda(\mathbf{K}))$  (see Definition 19.10.2) We have that  $\lambda' < h(\lambda(\mathbf{K}))$ . We claim that  $\lambda'(\mathbf{K}) \leq \lambda'$ . Indeed by Theorem 19.10.8, for any stability cardinal  $\mu$ , we have that  $\mu^{<\kappa(\mathbf{K}_{\lambda'},<\mathbf{K}^{\mathrm{iniv}})} = \mu$ . We know that  $\chi(\mathbf{K})$  is the maximal cardinal with that property, but on the other hand we have that  $\chi(\mathbf{K}) \leq \kappa(\mathbf{K}_{\lambda'},<\mathbf{K}^{\mathrm{univ}})$  by definition. We conclude that  $\chi(\mathbf{K}) = \kappa(\mathbf{K}_{\lambda'},<\mathbf{K}^{\mathrm{univ}})$ , as desired.

Theorem 19.11.3 together with Corollary 19.4.24 partially answers Question 9.1.8, which asked whether the least  $\mu$  such that **K** is  $\mu$ -superstable must satisfy  $\mu < H_1$ . We know now that (assuming continuity of splitting)  $\mu \leq \lambda'(\mathbf{K}) < \beth_{H_1}$ , so there is a Hanf number for superstability but whether it is  $H_1$  (rather than  $\beth_{H_1}$ ) remains open.

We also obtain an analog of Corollary 19.4.22:

COROLLARY 19.11.4. Let **K** be an LS(**K**)-tame AEC with a monster model and assume that splitting has weak continuity. For any  $\mu \geq \lambda'(\mathbf{K}) + \theta(\lambda(\mathbf{K}))$ , **K** is stable in  $\mu$  if and only if  $\mu = \mu^{<\chi(\mathbf{K})}$ .

PROOF. The left to right direction follows from Theorem 19.11.2 and the right to left direction is by Fact 19.4.1 and the definition of  $\theta(\lambda(\mathbf{K}))$  (recalling that  $\mu = \mu^{<\chi(\mathbf{K})}$  implies that of  $\mu \ge \chi(\mathbf{K})$ ).

We emphasize that for the right to left directions of Corollary 19.4.22 to be nontrivial, we need  $\theta(\lambda(\mathbf{K})) < \infty$ , which holds under various set-theoretic hypotheses by Fact 19.4.20. This is implicit in Section 4.5. The left to right direction is new and does not need the boundedness of  $\theta(\lambda(\mathbf{K}))$  (Theorem 19.11.2).

**19.11.1.** On the uniqueness of limit models. It was shown in [**BVa**] that continuity of splitting implies a nice local behavior of limit models in stable AECs:

FACT 19.11.5 (Theorem 1 in [**BVa**]). Let **K** be an AEC and let  $\mu \geq LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{\mu}$  has amalgamation, joint embedding, no maximal models, and is stable in  $\mu$ . If:

 $\begin{array}{ll} (1) & \delta \in \underline{\kappa}(\mathbf{K}_{\mu}, <^{\mathrm{univ}}_{\mathbf{K}}) \cap \mu^{+}. \\ (2) & \kappa^{\mathrm{cont}}(\mathbf{K}_{\mu}, <^{\mathrm{univ}}_{\mathbf{K}}) = \aleph_{0}. \end{array}$ 

(3) **K** has  $(\mu, \delta)$ -symmetry.

Then whenever  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$  are such that both  $M_1$  and  $M_2$  are  $(\mu, \geq \delta)$ limit over  $M_0$  (recall Section 19.2.4), we have that  $M_1 \cong_{M_0} M_2$ .

We will not need to use the definition of  $(\mu, \delta)$ -symmetry, only the following fact, which combines [**BVa**, 18] and the proof of [**Van16a**, 2].

FACT 19.11.6. Let **K** be an AEC and let  $\mu \geq LS(\mathbf{K})$ . Assume that  $\mathbf{K}_{\mu}$  has amalgamation, joint embedding, no maximal models, and is stable in  $\mu$ . Let  $\delta < \mu^+$ be a regular cardinal. If whenever  $\langle M_i : i < \delta \rangle$  is an increasing chain of saturated models in  $\mathbf{K}_{\mu^+}$  we have that  $\bigcup_{i < \delta} M_i$  is saturated, then **K** has  $(\mu, \delta)$ -symmetry.

We can conclude that in tame stable AECs with weak continuity of splitting, any two big-enough  $(\geq \chi(\mathbf{K}))$ -limits are isomorphic.

THEOREM 19.11.7. Let **K** be an  $LS(\mathbf{K})$ -tame AEC with a monster model. Assume that splitting has weak continuity.

Let  $\chi_0 < H_1$  be as given by Fact 19.4.10. Then for any stability cardinal  $\mu \geq \lambda'(\mathbf{K}) + \chi_0$  and any  $M_0, M_1, M_2 \in \mathbf{K}_{\mu}$ , if both  $M_1$  and  $M_2$  are  $(\mu, \geq \chi(\mathbf{K}))$ limit over  $M_0$ , then  $M_1 \cong_{M_0} M_2$ .

PROOF. By Fact 19.4.10, we have that the union of an increasing chain of saturated models in  $\mathbf{K}_{\mu^+}$  of length  $\chi(\mathbf{K})$  is saturated. Therefore by Fact 19.11.6, **K** has  $(\mu, \chi(\mathbf{K}))$ -symmetry. Now apply Fact 19.11.5. 

We deduce the following improvement on Theorem 19.4.13 in case splitting has weak continuity:

COROLLARY 19.11.8. Let  $\mathbf{K}$  be an LS( $\mathbf{K}$ )-tame AEC with a monster model. Assume that splitting has weak continuity.

Let  $\chi_0 < H_1$  be as given by Fact 19.4.10. For any stability cardinal  $\mu \geq$  $\lambda'(\mathbf{K}) + \chi_0$ , there is a saturated model of cardinality  $\mu$ .

**PROOF.** There is a  $(\mu, \chi(\mathbf{K}))$ -limit model of cardinality  $\mu$ , and it is saturated by Theorem 19.11.7. 

In Fact 19.4.10, it is open whether Hypothesis (1) can be removed. We aim to show that it can, assuming continuity of splitting and SCH. We first revisit an argument of VanDieren [Van16b] to show that one can assume stability in  $\lambda$ instead of stability in unboundedly many cardinals below  $\lambda$ .

LEMMA 19.11.9. Let  $\mathbf{K}$  be an LS( $\mathbf{K}$ )-tame AEC with a monster model. Let  $\mu > \mathrm{LS}(\mathbf{K})$ . Assume that **K** is stable in both  $\mathrm{LS}(\mathbf{K})$  and  $\mu$ . Let  $\langle M_i : i < \delta \rangle$ be an increasing chain of  $\mu$ -saturated models. If  $\mathrm{cf}\,\delta \in \underline{\kappa}(\mathbf{K}_{\mathrm{LS}(\mathbf{K})}, <^{\mathrm{univ}}_{\mathbf{K}})$  and the  $(\mu, \delta)$ -limit model is saturated, then  $\bigcup_{i < \delta} M_i$  is  $\mu$ -saturated.

Let us say a little bit about the argument. VanDieren [Van16b] shows that superstability in  $\lambda$  and  $\mu := \lambda^+$  combined with the uniqueness of limit models in  $\lambda^+$ implies that unions of chains of  $\lambda^+$ -saturated models are  $\lambda^+$ -saturated. One can use VanDieren's argument to prove that superstability in unboundedly many cardinals below  $\mu$  implies that unions of chains of  $\mu$ -saturated models are  $\mu$ -saturated, and this generalizes to the stable case too. However the case that interests us here is when **K** is stable in  $\mu$  and not necessarily in unboundedly many cardinals below (the reader can think of  $\mu$  as being the successor of a singular cardinal of low cofinality). This is where tameness enters the picture: by assuming stability e.g. in  $LS(\mathbf{K})$  as well as  $LS(\mathbf{K})$ -tameness, we can transfer the locality of splitting upward and the main idea of VanDieren's argument carries through (note that continuity of splitting is not needed). Still several details have to be provided, so a full proof is given here.

PROOF OF LEMMA 19.11.9. For  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ , let us say that  $p \in gS(N)$ does not fork over  $(M_0, M)$  if M is  $||M_0||^+$ -saturated over  $M_0$  (recall Definition 19.8.6) and  $M_0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$ . Say that p does not fork over M if there exists  $M_0$  so that it does not fork over  $(M_0, M)$ .

Without loss of generality,  $\delta = \operatorname{cf} \delta < \mu$ . Let  $M_{\delta} := \bigcup_{i < \delta} M_i$ . Let  $N \leq_{\mathbf{K}} M_{\delta}$ with  $N \in \mathbf{K}_{<\mu}$ . Let  $p \in \mathrm{gS}(N)$ . We want to see that p is realized in  $M_{\delta}$ . We may assume without loss of generality that  $M_i \in \mathbf{K}_{\mu}$  for all  $i \leq \delta$ . Let  $q \in gS(M_{\delta})$  be an extension of p.

Since  $\delta \in \underline{\kappa}(\mathbf{K}_{\mathrm{LS}(\mathbf{K})}, <^{\mathrm{univ}}_{\mathbf{K}})$ , using Section 4.4 there exists  $i < \delta$  such that q does not fork over  $M_i$ . This means there exists  $M_i^0 \leq_{\mathbf{K}} M_i$  such that  $M_i^0 \in \mathbf{K}_{\mathrm{LS}(\mathbf{K})}$  and qdoes not fork over  $(M_i^0, M_i)$ . Without loss of generality, i = 0. Let  $\mu_0 := LS(\mathbf{K}) + \delta$ . Build  $\langle N_i : i \leq \delta \rangle$  increasing continuous in  $\mathbf{K}_{\mu_0}$  such that  $M_0^0 \leq_{\mathbf{K}} N_0, N \leq_{\mathbf{K}} N_{\delta}$ , and for all  $i \leq \delta$ ,  $N_i \leq_{\mathbf{K}} M_i$ . Without loss of generality,  $N = N_{\delta}$ .

We build an increasing continuous directed system  $\langle M_i^*, f_{i,j} : i \leq j < \delta \rangle$  such that for all  $i < \delta$ :

- (1)  $M_i^* \in \mathbf{K}_{\mu}$ .
- (2)  $N_i \leq_{\mathbf{K}} \tilde{M}_i^* \leq_{\mathbf{K}} M_i.$ (3)  $f_{i,i+1}$  fixes  $N_i.$
- (4)  $M_{i+1}^*$  is universal over  $M_i^*$ .

This is possible. Take  $M_0^* := M_0$ . At *i* limit, take  $M_i^{**}$  to be the a direct limit of the system fixing  $N_i$  and let  $g: M_i^{**} \xrightarrow{N_i} M_i$  (remember that  $M_i$  is saturated). Let  $M_i^* := g[M_i^{**}]$ , and define the  $f_{j,i}$ 's accordingly. At successors, proceed similarly and define the  $f_{i,j}$ 's in the natural way.

This is enough. Let  $(M^*_{\delta}, f_{i,\delta})_{i<\delta}$  be a direct limit of the system extending  $N_{\delta}$ (note: we do not know that  $M^*_{\delta} \leq_{\mathbf{K}} M_{\delta}$ ). We have that  $M^*_{\delta}$  is a  $(\mu, \delta)$ -limit model, hence is saturated. Now find a saturated  $\mathfrak{C} \in \mathbf{K}_{\mu}$  containing  $M_{\delta} \cup M_{\delta}^*$  and such that for each  $i < \delta$ , there exists  $f_{i,\delta}^*$  an automorphism of  $\mathfrak{C}$  extending  $f_{i,\delta}$  such that  $f_{i,\delta}^*[N_{\delta}] \leq_{\mathbf{K}} M_{\delta}^*$ . This is possible since  $M_{\delta}^*$  is universal over  $M_i^*$  for each  $i < \delta$ . Let  $N^* \leq_{\mathbf{K}} M^*_{\delta}$  be such that  $N^* \in \mathbf{K}_{\mu_0}$  and  $|N_{\delta}| \cup \bigcup_{i < \delta} |f^*_{i,\delta}[N_{\delta}]| \subseteq |N^*|$ .

<u>Claim</u>: For any saturated  $\widehat{M} \in \mathbf{K}_{\mu}$  with  $M_{\delta} \leq_{\mathbf{K}} \widehat{M}$ , there exists  $\hat{q} \in \mathrm{gS}(\widehat{M})$ extending q and not forking over  $(M_0^0, N_0)$ .

<u>Proof of Claim</u>: We know that  $M_0$  is saturated. Thus there exists  $f: M_0 \cong_{N_0}$  $\widehat{M}$ . Let  $\hat{q} := f(q \upharpoonright M_0)$ . We have that  $\hat{q} \in \mathrm{gS}(\widehat{M})$  and  $\hat{q}$  does not fork over  $(M_0^0, N_0)$ . Further,  $\hat{q} \upharpoonright N_0 = q \upharpoonright N_0$ . By uniqueness of nonforking (see Theorem 4.5.3),  $\hat{q} \upharpoonright M = q$ .  $\dagger_{\text{Claim}}$ 

By the claim, there exists  $\hat{q} \in \mathrm{gS}(\mathfrak{C})$  extending q and not forking over  $(M_0^0, N_0)$ . Because  $M_{\delta}^*$  is  $(\mu_0^+, \mu)$ -limit, there exists  $M^{**} \in \mathbf{K}_{\mu}$  saturated such that  $N^* \leq_{\mathbf{K}} M^{**} \leq_{\mathbf{K}} M_{\delta}^*$  and  $M_{\delta}^*$  is universal over  $M^{**}$ .

Since  $M_{\delta}^*$  is universal over  $M^{**}$ , there is  $b^* \in M_{\delta}^*$  realizing  $\hat{q} \upharpoonright M^{**}$ . Fix  $i < \delta$ and  $b \in M_i^*$  such that  $f_{i,\delta}(b) = b^*$ . We claim that b realizes p (this is enough since by construction  $M_i^* \leq_{\mathbf{K}} M_i \leq_{\mathbf{K}} M_{\delta}$ ). We show a stronger statement: b realizes  $\hat{q} \upharpoonright M'$ , where  $M' := (f_{i,\delta}^*)^{-1}[M^{**}]$ . This is stronger because  $N^* \leq_{\mathbf{K}} M^{**}$  so by definition of  $N^*$ ,  $N \leq_{\mathbf{K}} (f_{i,\delta}^*)^{-1}[N^*] \leq_{\mathbf{K}} M'$ . Work inside  $\mathfrak{C}$ . Since  $\hat{q}$  does not fork over  $(M_0^0, N_0)$ , also  $\hat{q} \upharpoonright M^{**} = \operatorname{gtp}(b^*/M^{**})$  does not fork over  $(M_0^0, N_0)$ . Therefore  $\operatorname{gtp}(b/M')$  does not fork over  $(M_0^0, N_0)$ . Moreover,  $\operatorname{gtp}(b/N_0) = \operatorname{gtp}(b^*/N_0) = \hat{q} \upharpoonright$  $N_0$ , since  $f_{i,\delta}$  fixes  $N_0$ . By uniqueness,  $\operatorname{gtp}(b/M') = \hat{q} \upharpoonright M'$ . In other words, brealizes  $\hat{q} \upharpoonright M'$ , as desired.  $\Box$ 

REMARK 19.11.10. It is enough to assume that amalgamation and the other structural properties hold only in  $\mathbf{K}_{[\mathrm{LS}(\mathbf{K}),\mu]}$ .

We have arrived to the second main result of this section. Note that the second case below is already known (Fact 19.4.10), but the others are new.

THEOREM 19.11.11. Let  $\mathbf{K}$  be an LS( $\mathbf{K}$ )-tame AEC with a monster model. Assume that splitting has weak continuity.

Let  $\chi_0 < H_1$  be as given by Fact 19.4.10. Let  $\lambda > \lambda'(\mathbf{K}) + \chi_0$  and let  $\langle M_i : i < \delta \rangle$  be an increasing chain of  $\lambda$ -saturated models. If  $\mathrm{cf} \ \delta \ge \chi(\mathbf{K})$ , then  $\bigcup_{i < \delta} M_i$  is  $\lambda$ -saturated provided that at least one of the following conditions hold:

- (1) **K** is stable in  $\lambda$ .
- (2) **K** is stable in unboundedly many cardinals below  $\lambda$ .
- (3)  $\lambda \ge \theta(\lambda(\mathbf{K}))$  (recall Definition 19.4.19).
- (4) SCH holds and  $\lambda \ge 2^{\lambda(\mathbf{K})}$ .

Proof.

- (1) We check that the hypotheses of Lemma 19.11.9 hold, with  $\mathbf{K}, \mu$  there standing for  $\mathbf{K}_{\geq \lambda'(\mathbf{K})}$ ,  $\lambda$  here. By definition and assumption,  $\mathbf{K}$  is stable in both  $\lambda'(\mathbf{K})$  and  $\lambda$ . Furthermore, cf  $\delta \in \underline{\kappa}(\mathbf{K}_{\lambda'(\mathbf{K})}, <_{\mathbf{K}}^{\mathrm{univ}})$  by definition of  $\lambda'(\mathbf{K})$  and  $\underline{\chi}(\mathbf{K})$ . Finally, any two  $(\lambda, \geq \mathrm{cf}\,\delta)$ -limit models are isomorphic by Theorem 19.11.7.
- (2) If  $\lambda$  is a successor, then **K** is also stable in  $\lambda$  by Fact 19.4.1(3) so we can use the first part. If  $\lambda$  is limit, then we can use the first part with each stability cardinal  $\mu \in (\lambda'(\mathbf{K}) + \chi_0, \lambda)$  to see that the union of the chain is  $\mu$ -saturated. As  $\lambda$  is limit, this implies that the union of the chain is  $\lambda$ -saturated.
- (3) By definition of  $\theta(\lambda(\mathbf{K}))$ ,  $\lambda$  is almost  $\lambda(\mathbf{K})$ -closed. By Lemma 19.4.16, either **K** is stable in  $\lambda$  or **K** is stable in unboundedly many cardinals below  $\lambda$ , so the result follows from the previous parts.
- (4) This is a special case of the previous part, see Fact 19.4.20.

# 19.12. Applications to existence and homogeneous model theory

We present here the following application of Lemma 19.4.12:

THEOREM 19.12.1. Let **K** be an AEC and let  $\lambda \geq \text{LS}(\mathbf{K})$ . Assume that **K** has amalgamation in  $\lambda$ , no maximal models in  $\lambda$ , is stable in  $\lambda$ , and is categorical in  $\lambda$ . Let  $\mu \leq \lambda$  be a regular cardinal.

If **K** is  $(< \mu)$ -tame, then **K** is  $\lambda$ -superstable (recall Definition 19.4.23). In particular, it has a model of cardinality  $\lambda^{++}$ .

REMARK 19.12.2. Here,  $(< \mu)$ -tameness is defined using Galois types over sets, see Definition 2.2.23.

Theorem 19.12.1 can be seen as a partial answer to the question "what stabilitytheoretic properties in  $\lambda$  imply the existence of a model in  $\lambda^{++}$ ?" (this question is in turn motivated by the problem [**She09a**, I.3.21] of whether categoricity in  $\lambda$  and  $\lambda^+$  should imply existence of a model in  $\lambda^{++}$ ). It is known that  $\lambda$ -superstability is enough (Theorem 6.8.9). Theorem 19.12.1 shows that in fact  $\lambda$ -superstability is implied by categoricity, amalgamation, no maximal models, stability, and tameness. In case  $\lambda = \aleph_0$ , Chapter 23 shows more: amalgamation in  $\aleph_0$ , no maximal models in  $\aleph_0$ , and stability in  $\aleph_0$  together imply ( $< \aleph_0$ )-tameness and  $\aleph_0$ -superstability.

Before proving Theorem 19.12.1, we state a corollary to homogeneous model theory (see [She70] or the exposition in [GL02]). The result is known in the first-order case [She90, VIII.0.3] but to the best of our knowledge, it is new in homogeneous model theory.

COROLLARY 19.12.3. Let D be a homogeneous diagram in a first-order theory T. If D is both stable and categorical in |T|, then D is stable in all  $\lambda \ge |T|$ .

PROOF. Let  $\mathbf{K}_D$  be the class of *D*-models of *T*. It is easy to check that it is an  $(<\aleph_0)$ -tame AEC with a monster model. By Theorem 19.12.1,  $\mathbf{K}_D$  is |T|-superstable. Now apply Fact 19.4.25.

PROOF OF THEOREM 19.12.1. The "in particular" part is by the proof of Theorem 6, which shows that  $\lambda$ -superstability implies no maximal models in  $\lambda^+$ . We now prove that **K** is  $\lambda$ -superstable. For this it is enough to show that  $\kappa(\mathbf{K}_{\lambda}, <_{\mathbf{K}}^{\text{univ}})$  $) = \aleph_0$ . So let  $\delta < \lambda^+$  be a regular cardinal. We want to see that  $\delta \in \underline{\kappa}(\mathbf{K}_{\lambda}, <_{\mathbf{K}}^{\text{univ}})$ . We consider two cases:

- <u>Case 1:</u>  $\delta < \lambda$ . Let  $\langle M_i : i \leq \delta \rangle$  be  $\langle \mathbf{K}^{\text{univ}}$ -increasing continuous in  $\mathbf{K}_{\lambda}$  and let  $p \in \mathrm{gS}(M_{\delta})$ . Then by categoricity  $M_{\delta}$  is  $(\lambda, \delta^+ + \mu)$ -limit, so the proof of Lemma 19.4.12 directly gives that there exists  $i < \delta$  such that p does not  $\lambda$ -split over  $M_i$ .
- <u>Case 2:  $\delta = \lambda$ </u>. Note first that this means  $\lambda$  is regular. By [She99, I.3.3(2)],  $\lambda \in \underline{\kappa}^{wk}(\mathbf{K}_{\lambda}, <_{\mathbf{K}}^{univ})$ . By assumption, **K** is  $(< \lambda)$ -tame, thus it is weakly  $\lambda$ -local (recall Definition 19.3.6). By Theorem 19.3.7,  $\lambda \in \underline{\kappa}^{cont}(\mathbf{K}_{\lambda}, <_{\mathbf{K}}^{univ})$ . By Fact 19.2.5,  $\delta = \lambda \in \underline{\kappa}(\mathbf{K}_{\lambda}, <_{\mathbf{K}}^{univ})$ , as desired.
## CHAPTER 20

# Superstability from categoricity in abstract elementary classes

This chapter is based [**BGVV17**] and is joint work with Will Boney, Rami Grossberg, and Monica VanDieren. We thank the referee for comments that helped improve the presentation of this work.

### Abstract

Starting from an abstract elementary class with no maximal models, Shelah and Villaveces have shown (assuming instances of diamond) that categoricity implies a superstability-like property for nonsplitting, a particular notion of independence. We generalize their result as follows: given any abstract notion of independence for Galois (orbital) types over models, we derive that the notion satisfies a superstability property provided that the class is categorical and satisfies a weakening of amalgamation. This extends the Shelah-Villaveces result (the independence notion there was splitting) as well as a result of Boney and Grossberg where the independence notion was coheir. The argument is in ZFC and fills a gap in the Shelah-Villaveces proof.

## 20.1. Introduction

**20.1.1. General motivation and history.** Forking is one of the central notions of model theory, discovered and developed by Shelah in the seventies for stable and NIP theories [She78]. One way to extend Shelah's first-order stability theory is to move beyond first-order. In the mid seventies, Shelah did this by starting the program of *classification theory for non-elementary classes* focusing first on classes axiomatizable in  $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$  [She75a] and later on the more general abstract elementary classes (AECs) [She87a]. Roughly, an AEC is a pair  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  satisfying some of the basic category-theoretic properties of  $(\text{Mod}(T), \prec)$  (but not the compactness theorem). Among the central problems, there are the decades-old categoricity and eventual categoricity conjectures of Shelah. In this chapter, we assume that the reader has a basic knowledge of AECs, see for example [Gro02] or [Bal09].

One key shift in this program is the move away from syntactic types (studied in the  $\mathbb{L}_{\lambda^+,\omega}$  context by [She72, GS86b, GS86a] and others) and towards a semantic notion of type, introduced in [She87b] and named *Galois type* by Grossberg [Gro02].<sup>1</sup> This has an easy definition when the class **K** has amalgamation, joint embedding and no maximal models, as these properties allow us to assume that all the elements of **K** we would like to discuss are substructures of a "monster" model

<sup>&</sup>lt;sup>1</sup>Shelah uses the name *orbital types* in some later papers.

 $\mathfrak{C} \in \mathbf{K}$ . In that case,  $\operatorname{gtp}(\overline{b}/A)$  is defined as the orbit of  $\overline{b}$  under the action of the group  $\operatorname{Aut}_A(\mathfrak{C})$  on  $\mathfrak{C}$ . One can also develop the notion of Galois type without the above assumption, however then the definition is more technical.

**20.1.2.** Independence, superstability, and no long splitting chains in **AECs.** In [She99] a first candidate for an independence relation was introduced: the notion of  $\mu$ -splitting (for  $M_0 \leq_{\mathbf{K}} M$  both in  $\mathbf{K}_{\mu}$ ,  $p \in \mathrm{gS}(M)$   $\mu$ -splits over  $M_0$  provided there are  $M_0 \leq_{\mathbf{K}} M_{\ell} \leq_{\mathbf{K}} M$ ,  $\ell = 1, 2$  and  $f : M_1 \cong_{M_0} M_2$  such that  $f(p \upharpoonright M_1) \neq p \upharpoonright M_2$ ).

This notion was used by Shelah to establish a downward version of his categoricity conjecture from a successor for classes having the amalgamation property. Later similar arguments [**GV06c**, **GV06a**] were used to derive a strong upward version of Shelah's conjecture for classes satisfying the additional locality property of (Galois) types called tameness.

In Chapter II of [She09a], Shelah introduced good  $\lambda$ -frames: an axiomatic definition of forking on Galois types over models of size  $\lambda$ . The notion is, by definition, required to satisfy basic properties of forking in superstable first-order theories (e.g. symmetry, extension, uniqueness, and local character). The theory of good  $\lambda$ -frames is well-developed and has had several applications to the categoricity conjecture (see Chapters III and IV of [She09a] and this thesis).

Constructions of good frames rely on weaker independence notions like nonsplitting, see e.g. Chapters 4 and 10. A key property of splitting in these constructions is that there is "no long splitting chains in  $\mathbf{K}_{\mu}$ ": if  $\langle M_i : i \leq \alpha \rangle$  is an increasing continuous chain in  $\mathbf{K}_{\mu}$  (so  $\alpha < \mu^+$  is a limit ordinal) and  $M_{i+1}$  is universal over  $M_i$ for each  $i < \alpha$ , then for any  $p \in gS(M_{\alpha})$  there exists  $i < \alpha$  so that p does not  $\mu$ -split over  $M_i$  (this is called *strong universal local character at*  $\alpha$  in the present chapter, see Definition 20.2.1). This can be seen as a replacement for the statement "every type does not fork over a finite set". The property is already studied in [She99], and has several nontrivial consequences: for example (assuming amalgamation, joint embedding, no maximal models, stability in  $\mu$ , and tameness), no long splitting chains in  $\mathbf{K}_{\mu}$  implies that  $\mathbf{K}$  is stable everywhere above  $\mu$  (Theorem 4.5.6) and has a good  $\mu^+$ -frame on the subclass of saturated models of cardinality  $\mu^+$  (Corollary 10.6.14). No long splitting chains has consequences for the uniqueness of limit models, another superstability-like property saying in essence that saturated models can be built in few steps (see for example [SV99, Van06, Van13, Van16a]).

Boney and Grossberg have explored another approach to independence by adapting the notion of coheir to AECs. They have shown that for classes satisfying amalgamation which are also tame and short (a strengthening of tameness, using the variables of a type instead of its parameters), failure of a certain order property implies that coheir has some basic properties of forking from a stable first-order theory. There the "no long coheir chain" property also has strong consequences (for example on the uniqueness of limit models [**BG**, Corollary 6.18]).

**20.1.3.** No long splitting chains from categoricity. It is natural to ask whether no long splitting chains (or no long coheir chains) in  $\mathbf{K}_{\mu}$  follows from categoricity above  $\mu$ . Shelah has shown that this holds for splitting (assuming amalgamation and no maximal models) if the categoricity cardinal has cofinality greater than  $\mu$  [She99, Lemma 6.3]. Without any cofinality restriction, a break-through was made in a paper of Shelah and Villaveces when they proved no long

#### 20.1. INTRODUCTION

splitting chains assuming no maximal models and instances of diamond [**SV99**, Theorem 2.2.1]. Later, Boney and Grossberg used the Shelah-Villaveces argument to derive the result in their context also for coheir [**BG**, Theorem 6.8]. It was also observed in earlier versions of Chapter 9 that the Shelah-Villaveces argument does not need diamond if one assumes full amalgamation. In conclusion we have:

FACT 20.1.1. Let **K** be an AEC with no maximal models. Let  $LS(\mathbf{K}) \leq \mu < \lambda$  and assume that **K** is categorical in  $\lambda$ .

- (1) [SV99, Theorem 2.2.1] If  $\Diamond_{S_{cf\mu}^{\mu^+}}$  holds then **K** has no long splitting chains in  $\mathbf{K}_{\mu}$ .
- (2) [**BG**, Theorem 6.8] If **K** has amalgamation,  $\kappa \in (LS(\mathbf{K}), \mu)$ , **K** does not have the weak  $\kappa$ -order property and is fully ( $< \kappa$ )-tame and short, then **K** has no long coheir chains in  $\mathbf{K}_{\mu}$ .
- (3) If **K** has amalgamation, then **K** has no long splitting chains in  $\mathbf{K}_{\mu}$ .

REMARK 20.1.2. Fact 20.1.1 has applications to more "concrete" frameworks than AECs. One can deduce from it (and the aforementioned fact that no long splitting chains implies stability on a tail in the presence of tameness) an alternate proof that a first-order theory T categorical above |T| is superstable. More generally, one obtains the same statement for the class K of models of a homogeneous diagram in T [She70]. The later was open for |T| uncountable and K categorical in  $\aleph_{\omega}(|T|)$  (see Section 11.4).

**20.1.4.** Gaps in the Shelah-Villaveces proof. In a preliminary version of [BG], the proof of Theorem 6.8 referred to the argument used in [SV99, Theorem 2.2.1]. The referee of [BG] insisted that the full argument necessary for Theorem 6.8 be included. After looking closely at the argument in [SV99], we concluded that there was a small gap in the division of cases and a need to specify the exact use of the club guessing principle that they imply.

More specifically, Shelah and Villaveces [**SV99**, Theorem 2.2.1] assume for a contradiction that no long splitting chains fails and can divide the situation into three cases, (a), (b), and (c). In the division into cases [**SV99**, Claim 2.2.3], just after the statement of property  $\otimes_i$ , Shelah and Villaveces claim that they can "repeat the procedure above" on a certain chain of models of length  $\mu$ . However the "procedure above" was used on a chain of length  $\sigma$ , where  $\sigma$  is a *regular* cardinal and regularity was used in the proof. As  $\mu$  is a potentially singular cardinal, there is a problem.

Once the division of cases is done, Shelah and Villaveces prove that cases (a), (b), (c) contradict categoricity. When proving this for (b), they use a club-guessing principle for  $\mu^+$  on the stationary set of points of cofinality  $\sigma$  (see Fact 20.2.9). The principle only holds when  $\sigma < \mu$ , so the case  $\sigma = \mu$  is missing.

**20.1.5.** Statement and discussion of the main theorem. In this chapter, we give a generalized, detailed, and corrected proof of Fact 20.1.1 that does not rely on any of the material in [SV99]. The key definitions are given at the start of the next section and the first seven hypotheses are collected in Hypothesis 20.2.3.

THEOREM 20.1.3 (Main Theorem). If:

(1)  $\mathbf{K}$  is an AEC.

(2)  $\mu \geq \text{LS}(\mathbf{K}).$ 

- (3) For every  $M \in \mathbf{K}_{\mu}$ , there exists an amalgamation base  $M' \in \mathbf{K}_{\mu}$  such that  $M \leq_{\mathbf{K}} M'$ .
- (4) For every amalgamation base  $M \in \mathbf{K}_{\mu}$ , there exists an amalgamation base  $M' \in \mathbf{K}_{\mu}$  such that M' is universal over M.
- (5) Every limit model in  $\mathbf{K}_{\mu}$  is an amalgamation base.
- (6)  $\downarrow$  is as in Definition 20.2.1 with  $\mathbf{K}^*$  the class of amalgamation bases in  $\mathbf{K}_{\mu}$  (ordered with the strong substructure relation inherited from  $\mathbf{K}$ ).
- (7)  $\downarrow$  satisfies invariance (I) and monotonicity (M).
- (8)  $\downarrow$  has weak universal local character at *some* cardinal  $\sigma < \mu^+$ .
- (9) **K** has an Ehrenfeucht-Mostowski (EM) blueprint  $\Phi$  with  $|\tau(\Phi)| \leq \mu$  such that every  $M \in \mathbf{K}_{[\mu,\mu^+]}$  embeds inside  $\mathrm{EM}_{\tau}(\mu^+, \Phi)$  (where we write  $\tau := \tau(\mathbf{K})$ ).

Then  $\hat{\downarrow}$  has strong universal local character at all limit ordinals  $\alpha < \mu^+$ .

REMARK 20.1.4. As in [SV99], when we say that M is an *amalgamation base* we mean that it is an amalgamation base in the class  $\mathbf{K}_{\|M\|}$ , i.e. we do not require that larger models can be amalgamated over M.

Some of the hypotheses of Theorem 20.1.3 may appear technical. Let us give a little more motivation.

- Hypotheses (3-5) are the statements that Shelah and Villaveces derive (assuming instances of diamond) from categoricity and no maximal models. It is well known that they hold in AECs with amalgamation.
- Hypothesis (4) implies stability in  $\mu$ .
- Hypothesis (8) can be seen as a consequence of stability (akin to "every type does not fork over a set of size at most  $\mu$ ").
- Hypothesis (9) follows from categoricity (see the proof of Corollary 20.1.5). In fact, it is strictly weaker: for a first-order theory T, (9) holds if and only if T is superstable by Section 9.5.

How are the gaps mentioned in Section 20.1.4 addressed in our proof of Theorem 20.1.3? The first gap (in the division into cases) is fixed in Lemma 20.2.6.(4). The second gap (in the use of the club guessing principle) is addressed here by a division into cases in the proof of Theorem 20.1.3 at the end of this chapter: there we use Lemma 20.2.8 only when  $\alpha < \sigma$ .

Before starting to prove Theorem 20.1.3, we give several contexts in which its hypotheses hold. This shows in particular that Fact 20.1.1 follows from Theorem 20.1.3.

COROLLARY 20.1.5. Let **K** be an AEC with arbitrarily large models. Let  $LS(\mathbf{K}) \leq \mu < \lambda$  and assume that **K** is categorical in  $\lambda$  and  $\mathbf{K}_{<\lambda}$  has no maximal models. Then:

- (1) If  $\diamond_{S^{\mu^+}_{ct\mu}}$  holds, then the hypotheses of Theorem 20.1.3 hold with  $\downarrow^*$  being non- $\mu$ -splitting.
- (2) If  $\mathbf{K}_{\mu}$  has amalgamation, then:
  - (a) The hypotheses of Theorem 20.1.3 hold with  $\downarrow$  being non- $\mu$ -splitting.

(b) If  $\kappa \in (LS(\mathbf{K}), \mu)$  is such that **K** does not have the weak  $\kappa$ -order property, then the hypotheses of Theorem 20.1.3 hold with  $\downarrow^*$  being  $(<\kappa)$ -coheir (see [**BG**]).

PROOF. Fix an EM blueprint  $\Psi$  for **K** (with  $|\tau(\Psi)| \leq \mu$ ). We first show that there exists an EM blueprint  $\Phi$  with  $|\tau(\Phi)| \leq \mu$  such that any  $M \in \mathbf{K}_{[\mu,\mu^+]}$ embeds inside  $\mathrm{EM}_{\tau}(\mu^+, \Phi)$ . Let  $M \in \mathbf{K}_{[\mu,\mu^+]}$ . Using no maximal models and categoricity, M embeds inside  $\mathrm{EM}_{\tau}(\lambda, \Psi)$ , and hence inside  $\mathrm{EM}_{\tau}(S, \Psi)$  for some  $S \subseteq \lambda$  with  $|S| \leq \mu^+$ . Therefore M also embeds inside  $\mathrm{EM}_{\tau}(\alpha, \Psi)$ , where  $\alpha :=$  $\mathrm{gtp}(S) < \mu^{++}$ . Now it is well known (see e.g. [Bal09, Claim 15.5]) that  $\alpha$  embeds inside  $\mathrm{EM}_{\tau}({}^{<\omega}\mu^+, \Phi)$ . The class  $\{{}^{<\omega}I \mid I$  is a linear order $\}$  is an AEC, therefore by composing EM blueprints there exists an EM blueprint  $\Phi$  for  $\mathbf{K}$  such that  $|\tau(\Phi)| \leq \mu$ and  $\mathrm{EM}_{\tau}(I, \Phi) = \mathrm{EM}_{\tau}({}^{<\omega}I, \Psi)$  for any linear order I. In particular, M embeds inside  $\mathrm{EM}_{\tau}(\mu^+, \Phi)$ , as desired.

As for the hypotheses on density of amalgamation bases, existence of universal extension, and limit models being amalgamation bases, in the first context this is proven in [**SV99**] (note that  $\diamondsuit_{S_{cf\mu}^{\mu+}}$  implies  $2^{\mu} = \mu^+$ ). When  $\mathbf{K}_{\mu}$  has full amalgamation, existence of universal extension is due to Shelah. It is stated (but not proven) in [**She99**, Lemma 2.2]; see [**Bal09**, Lemma 10.5] for a proof.

In all the contexts given, it is trivial that  $\downarrow$  satisfies (I) and (M). In the first context, it can be shown that non  $\mu$ -splitting has weak universal local character at any  $\sigma < \mu^+$  such that  $2^{\sigma} > \mu$  (see the proof of case (c) in [**SV99**, Theorem 2.2.1] or [**Bal09**, Lemma 12.2]). Of course, this also holds when  $\mathbf{K}_{\mu}$  has full amalgamation. As for ( $< \kappa$ )-coheir, it has weak universal local character at any  $\sigma < \mu^+$  such that  $2^{\sigma} > \kappa$ . This is given by the proof of [**BG**, Theorem 6.8] (note that using a back and forth argument, one can assume without loss of generality that any  $M_{i+1}$  in the chain is  $\kappa$ -saturated).

**20.1.6.** Other advantages of the main theorem. As should be clear from Corollary 20.1.5, another advantage of the main theorem is that it separates the combinatorial set theory from the model theory (it holds in ZFC) and also shows that there is nothing special about splitting in [SV99].

Some results here are of independent interest. For example, any independence relation satisfying invariance and monotonicity has (assuming categoricity) a certain continuity property (see Lemma 20.2.8).

## 20.2. Proof of the main theorem

We now define the weak framework for independence that we use.

DEFINITION 20.2.1. Let  $\mathbf{K}^*$  be an abstract class<sup>2</sup> and  $\stackrel{*}{\downarrow}$  be a 4-ary relation such that if  $a \stackrel{*}{\underset{M_0}{\downarrow}} {}^N M$  holds, then  $M_0 \leq_{\mathbf{K}^*} M \leq_{\mathbf{K}^*} N$  are all in  $\mathbf{K}^*$  and  $a \in |N|$ .

(1) The following are several properties we will assume about  $\downarrow$  (but we will always mention when we assume them).

<sup>&</sup>lt;sup>2</sup>That is, a partial order  $(K^*, \leq_{\mathbf{K}^*})$  such that  $K^*$  is a class of structures in a fixed vocabulary closed under isomorphisms,  $\leq_{\mathbf{K}^*}$  is invariant under isomorphisms, and  $M \leq_{\mathbf{K}^*} N$  implies that M is a substructure of N.

(a) ⊥ has invariance (I) if it is preserved under isomorphisms: if a ⊥ M<sub>0</sub><sup>N</sup>M and f: N ≅ N', then f(a) ⊥ f[M<sub>0</sub>]<sup>\*</sup>N' f[M].
(b) ⊥ has monotonicity (M) if:

- (i) If  $a \stackrel{*}{\underset{M_{0}}{\downarrow}} {}^{N}M$ ,  $M_{0} \leq_{\mathbf{K}^{*}} M'_{0} \leq_{\mathbf{K}^{*}} M' \leq_{\mathbf{K}^{*}} M$ , and  $N \leq_{\mathbf{K}^{*}} N'$ , then  $a \stackrel{*}{\underset{M'_{0}}{\downarrow}} {}^{N'}M'_{0}$ ; and: (ii) If  $a \stackrel{*}{\underset{M_{0}}{\downarrow}} {}^{N}M$ ,  $N' \leq_{\mathbf{K}^{*}} N$  is such that  $M \leq_{\mathbf{K}^{*}} N'$  and  $a \in |N'|$ , then  $a \stackrel{*}{\underset{M_{0}}{\downarrow}} {}^{N'}M$ . (2) (I) and (M) mean that this relation is really about Galois types, so we
- (2) (I) and (M) mean that this relation is really about Galois types, so we write gtp(a/M; N) does not \*-fork over  $M_0$  for  $a \stackrel{*}{\underset{M_0}{\downarrow}} {}^N M$ .
- (3) For a limit ordinal  $\alpha$ ,  $\downarrow$  has weak universal local character at  $\alpha$  if for any increasing continuous sequence  $\langle M_i \in \mathbf{K}^* \mid i \leq \alpha \rangle$  and any type  $p \in \mathrm{gS}(M_\alpha)$ , if  $M_{i+1}$  is universal over  $M_i$  for each  $i < \alpha$ , then there is some  $i_0 < \alpha$  such that  $p \upharpoonright M_{i_0+1}$  does not \*-fork over  $M_{i_0}$ .
- (4) For a limit ordinal  $\alpha$ ,  $\downarrow$  has strong universal local character at  $\alpha$  if for any increasing continuous sequence  $\langle M_i \in \mathbf{K}^* | i \leq \alpha \rangle$  and any type  $p \in \mathrm{gS}(M_\alpha)$ , if  $M_{i+1}$  is universal over  $M_i$  for each  $i < \alpha$ , then there is some  $i_0 < \alpha$  such that p does not \*-fork over  $M_{i_0}$ .

Remark 20.2.2.

- (1) In the setup of Fact 20.1.1.(1), non- $\mu$ -splitting on the class  $\mathbf{K}^*$  of amalgamation bases of cardinality  $\mu$  will have (I) and (M), see Fact 20.1.5.
- (2) If  $\alpha < \beta$  are limit ordinals and  $\downarrow$  has weak universal local character at  $\alpha$ , then  $\downarrow^*$  has weak universal local character at  $\beta$ , but this need not hold for strong universal local character (if say cf  $\beta < \text{cf } \alpha$ ).
- (3) If  $\stackrel{*}{\downarrow}$  has (M) and  $\stackrel{*}{\downarrow}$  has strong universal local character at cf  $\alpha$ , then  $\stackrel{*}{\downarrow}$  has strong universal local character at  $\alpha$ .
- (4) If  $\downarrow$  has (M), strong universal local character at  $\alpha$  implies weak universal local character at  $\alpha$ .
- (5) If (as will be the case in this note)  $\mathbf{K}^*$  is a class of structures of a fixed size  $\mu$ , then we only care about the properties when  $\alpha < \mu^+$ .

We collect the first seven hypotheses of Theorem 20.1.3 into a hypothesis that will be assumed for the rest of the chapter.

Hypothesis 20.2.3.

- (1)  $\mathbf{K}$  is an AEC.
- (2)  $\mu \geq \text{LS}(\mathbf{K}).$
- (3) For every  $M \in \mathbf{K}_{\mu}$ , there exists an amalgamation base  $M' \in \mathbf{K}_{\mu}$  such that  $M \leq_{\mathbf{K}} M'$ .

- (4) For every amalgamation base  $M \in \mathbf{K}_{\mu}$ , there exists an amalgamation base  $M' \in \mathbf{K}_{\mu}$  such that M' is universal over M.
- (5) Every limit model in  $\mathbf{K}_{\mu}$  is an amalgamation base.
- (6)  $\downarrow$  is as in Definition 20.2.1 with  $\mathbf{K}^*$  the class of amalgamation bases in  $\mathbf{K}_{\mu}$  (ordered with the strong substructure relation inherited from  $\mathbf{K}$ ).
- (7)  $\downarrow$  satisfies invariance (I) and monotonicity (M).

The proof of Theorem 20.1.3 can be decomposed into two steps. First, we study two more variations on local character: continuity and absence of alternations. We show that if strong local character fails but enough weak local character holds, then there must be some failure of continuity, or some alternations. Second, we show that categoricity (or more precisely the existence of a universal EM model in  $\mu^+$ ) implies continuity and absence of alternations. The first step uses the weak local character (but not categoricity, it is essentially forking calculus) but the second does not (but does use categoricity).

The precise definitions of continuity and alternations are as follows.

DEFINITION 20.2.4. Let  $\mathbf{K}^*$  and  $\stackrel{*}{\downarrow}$  be as in Definition 20.2.1 and let  $\alpha$  be a limit ordinal.

- (1)  $\downarrow$  has universal continuity at  $\alpha$  if for any increasing continuous sequence  $\langle M_i \in \mathbf{K}^* \mid i \leq \alpha \rangle$  and any type  $p \in \mathrm{gS}(M_\alpha)$ , if for each  $i < \alpha M_{i+1}$  is universal over  $M_i$  and  $p \upharpoonright M_i$  does not \*-fork over  $M_0$ , then p does not \*-fork over  $M_0$ .
- (2) For  $\delta < \mu^+$  a limit,  $\downarrow$  has no  $\delta$ -limit alternations at  $\alpha$  if for any increasing continuous sequence  $\langle M_i \in \mathbf{K}^* \mid i \leq \alpha \rangle$  with  $M_{i+1}$   $(\mu, \delta)$ -limit over  $M_i$  for all  $i < \alpha$  and any type  $p \in \mathrm{gS}(M_\alpha)$ , there exists  $i < \alpha$  such that the following fails:  $p \upharpoonright M_{2i+1}$  \*-forks over  $M_{2i}$  and  $p \upharpoonright M_{2i+2}$  does not \*-fork over  $M_{2i+1}$ . If this fails, we say that  $\stackrel{*}{\downarrow}$  has  $\delta$ -limit alternations at  $\alpha$ .

over  $M_{2i+1}$ . If this fails, we say that  $\bigcirc$  has 0 timet alternations at a.

Note that the failure of universal continuity and no  $\delta$ -limit alternation correspond respectively to cases (a) and (b) in the proof of [**SV99**, Theorem 2.2.1]. Case (c) there corresponds to failure of weak universal local character at  $\mu$  (which is assumed to hold here, see (8) of Theorem 20.1.3).

The following technical lemmas and proposition implement the first step described after the statement of Hypothesis 20.2.3. In particular, Proposition 20.2.7 below says that if we can prove weak local character at some  $\sigma$ , continuity and no alternations at *all*  $\alpha$ , then strong local character at all  $\alpha$  follows. Lemma 20.2.6 is a collection of preliminary steps toward proving Proposition 20.2.7. Lemma 20.2.5 is used separately in the proof of the main theorem (it says that weak universal local character implies the absence of alternations). Throughout, recall that we are assuming Hypothesis 20.2.3.

LEMMA 20.2.5. Let  $\sigma < \mu^+$  be a (not necessarily regular) cardinal and  $\delta < \mu^+$  be a limit ordinal. If  $\stackrel{*}{\downarrow}$  has weak universal local character at  $\sigma$ , then  $\stackrel{*}{\downarrow}$  has no  $\delta$ -limit alternations at  $\sigma$ .

PROOF. Fix  $\langle M_i : i \leq \alpha \rangle$ ,  $\delta$ , p as in the definition of having no  $\delta$ -limit alternations. Apply weak universal local character to the chain  $\langle M_{2i} : i \leq \alpha \rangle$ .

We now outline the proof of Proposition 20.2.7. Again, it may be helpful to remember that we will later prove that (in the context of Theorem 20.1.3) continuity holds at all lengths and that there are no alternations.

Two important basic results are

- continuity together with weak local character imply strong local character at *regular length* (Lemma 20.2.6.(1)); and
- it does not matter whether in the definition of weak and strong universal local character we require " $M_{i+1}$  limit over  $M_i$ " or " $M_{i+1}$  universal over  $M_i$ ," and the length of the limit models does not matter (Lemma 20.2.6.(2)).

The first of these is proven by contradiction, and the second is a straightforward argument using universality.

Assume for a moment we have strong universal local character at some limit length  $\gamma$ . Let us try to prove weak universal local character at (say)  $\omega$  (then we can use the first basic result to get the strong version, assuming continuity). By the second basic result, we can assume we are given an increasing continuous sequence  $\langle M_n : n \leq \omega \rangle$  with  $M_{n+1}$  ( $\mu, \gamma$ )-limit over  $M_n$  for all  $n < \omega$  and  $p \in gS(M_\omega)$ . By the strong universal local character assumption we know that  $p \upharpoonright M_{n+1}$  does not \*-fork over some intermediate model between  $M_n$  and  $M_{n+1}$ , so if we assume that  $p \upharpoonright M_{n+1}$  \*-forks over  $M_n$  for all  $n < \omega$ , we will end up getting alternations. This is the essence of Lemma 20.2.6.(5).

Thus to prove strong universal local character at *all* cardinals, it is enough to obtain it at *some* cardinal. Fortunately in the hypothesis of Proposition 20.2.7, we are already assuming weak universal local character at some  $\sigma$ . If  $\sigma$  is regular we are done by the first basic result, but unfortunately  $\sigma$  could be singular. In this case Lemma 20.2.6.(4) (using Lemma 20.2.6.(3) as an auxiliary claim) shows that failure of strong universal local character at  $\sigma$  implies alternations, even when  $\sigma$  is singular.

LEMMA 20.2.6. Let  $\alpha < \mu^+$  be a regular cardinal,  $\sigma < \mu^+$  be a (not necessarily regular) cardinal, and  $\delta < \mu^+$  be a limit ordinal.

- (1) If  $\downarrow$  has universal continuity at  $\alpha$  and weak universal local character at  $\alpha$ , then  $\downarrow$  has strong universal local character at  $\alpha$ .
- (2) We obtain an equivalent definition of weak [strong] universal local character at  $\sigma$ , if in Definition 20.2.1.(3) [20.2.1.(4)] we ask in addition that " $M_{i+1}$  is  $(\mu, \delta)$ -limit over  $M_i$ " for all  $i < \sigma$ .
- (3) Assume that  $\downarrow$  has weak universal local character at  $\sigma$ . Let  $\langle M_i : i \leq \sigma \rangle$  be increasing continuous in  $\mathbf{K}^*$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \sigma$ . For any  $p \in \mathrm{gS}(M_{\sigma})$  there exists a successor  $i < \sigma$  such that  $p \upharpoonright M_{i+1}$  does not \*-fork over  $M_i$ .
- (4) If  $\downarrow$  has universal continuity at  $\sigma$ , weak universal local character at  $\sigma$ , and no  $\delta$ -limit alternations at  $\omega$ , then  $\stackrel{*}{\downarrow}$  has strong universal local character at  $\sigma$ .

(5) Assume that  $\downarrow$  has strong universal local character at  $\sigma$ . If  $\downarrow$  does not have weak universal local character at  $\alpha$ , then  $\stackrel{*}{\downarrow}$  has  $\sigma$ -limit alternations at  $\alpha$ .

## Proof.

(1) Suppose that  $\langle M_i : i \leq \alpha \rangle$ , p is a counterexample.

**Claim:** For each  $i < \alpha$ , there exists  $j_i \in (i, \alpha)$  such that  $p \upharpoonright M_{j_i}$ \*-forks over  $M_i$ .

**Proof of Claim:** If  $i < \alpha$  is such that for all  $j \in (i, \alpha)$ ,  $p \upharpoonright M_j$  does not \*-fork over  $M_i$ , then applying universal continuity at  $\alpha$  on the chain  $\langle M_k : k \in [i, \alpha] \rangle$  we would get that p does not \*-fork over  $M_i$ , contradicting the choice of  $\langle M_i : i \leq \alpha \rangle$ , p.  $\dagger_{\text{Claim}}$ 

Now define inductively for  $i \leq \alpha$ ,  $k_0 := 0$ ,  $k_{i+1} := j_{k_i}$ , and when i is limit  $k_i := \sup_{j \leq i} k_j$ . Note that  $\langle k_i : i \leq \alpha \rangle$  is strictly increasing continuous and  $i < \alpha$  implies  $k_i < \alpha$  (this uses regularity of  $\alpha$ ; when  $\alpha$  is singular, see (4)).

Apply weak universal local character to the chain  $\langle M_{k_i} : i \leq \alpha \rangle$  and the type p. We get that there exists  $i < \alpha$  such that  $p \upharpoonright M_{k_{i+1}}$  does not \*-fork over  $M_{k_i}$ . This is a contradiction since  $k_{i+1} = j_{k_i}$  and we chose  $j_{k_i}$ so that  $p \upharpoonright M_{j_{k_i}}$  \*-forks over  $M_{k_i}$ .

(2) We prove the result for weak universal local character, and the proof for the strong version is similar. Fix  $\langle M_i^0 : i \leq \sigma \rangle$ , p witnessing failure of weak universal local character at  $\sigma$ . We build a witness of failure  $\langle M_i : i \leq \sigma \rangle$ , p such that  $M_{\sigma} = M_{\sigma}^0$ , and  $M_{i+1}$  is  $(\mu, \delta)$ -limit over  $M_i$  for each  $i < \alpha$ . Using existence of universal extensions, we can extend each  $M_i^0$  to  $M_i^*$ that is  $(\mu, \delta)$ -limit over  $M_i^0$ . Since  $M_{i+1}^0$  is universal over  $M_i^0$ , we can find  $f_i : M_{i+1}^* \to_{M_i^0} M_{i+1}^0$ . Since limit models are amalgamation bases,  $f_i(M_{i+1}^*)$  is an amalgamation base. Now set  $M_i^1 := M_i^0$  for  $i \leq \sigma$  limit or 0 and  $M_{i+1}^1 := f_i(M_{i+1}^*)$ . This is an increasing continuous chain of amalgamation bases with  $M_{i+1}^1$   $(\mu, \delta)$ -limit over  $M_i^1$ . Let  $M_i := M_{2i}^0$ .

This works: if there was an  $i < \sigma$  such that  $p \upharpoonright M_{i+1}$  does not \*fork over  $M_i$ , this would mean that  $p \upharpoonright M_{2i+2}^1$  does not \*-fork over  $M_{2i}^1$ , but since  $M_{2i}^1 \leq_{\mathbf{K}^*} M_{2i+1}^0 \leq_{\mathbf{K}^*} M_{2i+2}^0 \leq_{\mathbf{K}^*} M_{2i+2}^1$ , we have by (M) that  $p \upharpoonright M_{2i+2}^0$  does not \*-fork over  $M_{2i+1}^0$ , a contradiction.

- (3) Apply weak universal local character to the chain  $\langle M_{2i} : i < \sigma \rangle$  to get  $j < \sigma$  such that  $p \upharpoonright M_{2j+2}$  does not \*-fork over  $M_{2j}$ . By monotonicity, this implies that  $p \upharpoonright M_{2j+2}$  does not \*-fork over  $M_{2j+1}$ . Let i := 2j + 1.
- (4) Suppose not, and let  $\langle M_i : i \leq \sigma \rangle$ , p be a counterexample. By (2), without loss of generality  $M_{i+1}$  is  $(\mu, \delta)$ -limit over  $M_i$  for all  $i < \delta$ . As in the proof of (1), for each  $i < \sigma$ , there exists  $j_i \in [i, \sigma)$  such that  $p \upharpoonright M_{j_i}$ \*-forks over  $M_i$ . On the other hand, applying (3) to the chain  $\langle M_j :$  $j \in [j_i, \sigma] \rangle$ , for each  $i < \sigma$ , there exists a successor ordinal  $k_i \geq j_i$  such that  $p \upharpoonright M_{k_i+1}$  does not \*-fork over  $M_{k_i}$ . Define by induction on  $n \leq \omega$ ,  $m_0 := 0, m_{2n+1} := k_{m_{2n}}, m_{2n+2} := k_{m_{2n}} + 1$ , and  $m_{\omega} := \sup_{n < \omega} m_n$ . By

construction, the sequence  $\langle M_{m_n} : n \leq \omega \rangle$  witnesses that  $\overset{\circ}{\downarrow}$  has  $\delta$ -limit alternations at  $\omega$ , a contradiction.

(5) Let  $\gamma := \sigma \cdot \sigma$ . By (2), there exists  $\langle M_i : i \leq \alpha \rangle$ , p witnessing failure of weak universal local character at  $\alpha$  such that for all  $i < \alpha$ ,  $M_{i+1}$  is  $(\mu, \gamma)$ -limit over  $M_i$ . Let  $\langle M_{i,j} : j \leq \gamma \rangle$  witness that  $M_{i+1}$  is  $(\mu, \gamma)$ -limit over  $M_i$  (i.e. it is increasing continuous with  $M_{i,j+1}$  universal over  $M_{i,j}$ for all  $j < \gamma$ ,  $M_{i,0} = M_i$ , and  $M_{i,\delta} = M_{i+1}$ ). By strong universal local character at  $\sigma$ , for all  $i < \alpha$ , there exists  $j_i < \gamma$  such that  $p \upharpoonright M_{i+1}$  does not \*-fork over  $M_{i,j_i}$ . By replacing  $j_i$  by  $j_i + \sigma$  if necessary we can assume without loss of generality that cf  $j_i = \operatorname{cf} \sigma$ .

Observe also that for any  $i < \alpha$ ,  $p \upharpoonright M_{i+1,j_i}$  \*-forks over  $M_i$  (using (M) and the assumption that  $p \upharpoonright M_{i+1}$  \*-forks over  $M_i$ ). Therefore  $\langle M_0, M_{1,j_1}, M_2, M_{3,j_3}, \ldots \rangle$ , p witness that  $\stackrel{*}{\bigcup}$  has  $\sigma$ -limit alternations at  $\alpha$ .

PROPOSITION 20.2.7. Let  $\alpha < \mu^+$  be a regular cardinal and  $\sigma < \mu^+$  be a (not necessarily regular) cardinal. Assume that  $\stackrel{*}{\downarrow}$  has *weak* universal local character at  $\sigma$ . If  $\stackrel{*}{\downarrow}$  has universal continuity at  $\alpha$  and  $\sigma$ ,  $\stackrel{*}{\downarrow}$  has no  $\sigma$ -limit alternations at  $\omega$ , and  $\stackrel{*}{\downarrow}$  has no  $\sigma$ -limit alternations at  $\alpha$ , then  $\stackrel{*}{\downarrow}$  has strong universal local character at  $\alpha$ .

PROOF. By Lemma 20.2.6.(4),  $\stackrel{*}{\downarrow}$  has strong universal local character at  $\sigma$ . By the contrapositive of Lemma 20.2.6.(5),  $\stackrel{*}{\downarrow}$  has weak universal local character at  $\alpha$ . By Lemma 20.2.6.(1),  $\stackrel{*}{\downarrow}$  has strong universal local character at  $\alpha$ .

The next lemma corresponds to the second step outlined at the beginning of this section. Note that the added assumption is (9) from the hypotheses of Theorem 20.1.3 and recall we are assuming Hypothesis 20.2.3 throughout.

LEMMA 20.2.8. Assume **K** has an EM blueprint  $\Phi$  with  $|\tau(\Phi)| \leq \mu$  such that every  $M \in \mathbf{K}_{[\mu,\mu^+]}$  embeds inside  $\mathrm{EM}_{\tau}(\mu^+, \Phi)$ . Let  $\alpha < \mu^+$  be a regular cardinal. Then:

- (1)  $\downarrow$  has universal continuity at  $\alpha$ .
- (2) If in addition  $\alpha < \mu$ , then for any limit  $\gamma < \mu^+$ ,  $\downarrow^*$  has no  $\gamma$ -limit alternations at  $\alpha$ .

PROOF. Let  $\langle M_i \mid i \leq \alpha \rangle$  and p be as in the definition of universal continuity or  $\gamma$ -limit alternations. Let  $S^{\mu^+}_{\alpha} := \{\delta < \mu^+ \mid \text{cf } \delta = \alpha\}$ . We say that  $\bar{C} = \langle C_\delta \mid \delta \in S^{\mu^+}_{\alpha} \rangle$  is an  $S^{\mu^+}_{\alpha}$ -club sequence if each  $C_\delta \subseteq \delta$  is club. Clearly, club sequences exist: just take  $C_\delta := \delta$  (this will be enough for proving universal continuity). Shelah [**She94**] proves the existence of club-guessing club sequences in ZFC under various hypotheses (the specific result that we use will be stated later, see Fact 20.2.9). We will describe a construction of a sequence of models  $\bar{N}(\bar{C})$  based on a club sequence and then plug in the necessary club sequence in each case.

Given an  $S^{\mu^+}_{\alpha}$ -club sequence  $\bar{C}$ , enumerate  $C_{\delta} \cup \{\delta\}$  in increasing order as  $\langle \beta_{\delta,j} \mid j \leq \alpha \rangle.$ 

**Claim:** Let  $\gamma < \mu^+$  be a limit ordinal. We can build increasing, continuous  $\overline{N}(\overline{C}) = \langle N_i \in \mathbf{K}^* \mid i < \mu^+ \rangle$  such that for all  $i < \mu^+$ :

- (1)  $N_{i+1}$  is  $(\mu, \gamma)$ -limit over  $N_i$ ;
- (2) when  $i \in S_{\alpha}^{\mu^+}$ , there is  $g_i : M_{\alpha} \cong N_i$  such that  $g_i(M_j) = N_{\beta_{i,j}}$  for all  $j \leq \alpha$ ; and:
- (3) when  $i \in S^{\mu^+}_{\alpha}$ , there is  $a_i \in N_{i+1}$  that realizes  $g_i(p)$ .

Proof of Claim: Build the increasing continuous chain of models as follows: start with an amalgamation base  $N_0$ , which exists by Hypothesis 20.2.3.(3). Given an amalgamation base  $N_i$ , build  $N_{i+1}$  to be  $(\mu, \gamma)$ -limit over it. This exists by Hypothesis 20.2.3.(4) of Theorem 20.1.3), and  $N_{i+1}$  is an amalgamation base by Hypothesis 20.2.3.(5). At limits, it also guarantees we have an amalgamation base.

At limits i of cofinality  $\alpha$ , use the uniqueness of  $(\mu, \gamma)$ -limits models to find the desired isomorphisms: the weak version gives  $M_0 \cong M_{\beta_{i,0}}$ , and the strong (over the base) version allows this isomorphism to be extended to get an isomorphism  $g_i$ between  $\langle M_j \mid j \leq \alpha \rangle$  and  $\langle N_{\beta_{i,j}} \mid j \leq \alpha \rangle$  as described. Since  $N_{i+1}$  is universal over  $N_i$ , we there is some  $a_i \in N_{i+1}$  that realizes  $g_i(p)$ . <sup>†</sup>Claim

By assumption, we may assume that  $N := \bigcup_{i < \mu^+} N_i \leq_{\mathbf{K}^*} \mathrm{EM}_{\tau}(\mu^+, \Phi)$ . Thus, we can write  $a_i = \rho_i(\gamma_1^i, \ldots, \gamma_{n(i)}^i)$  with:

$$\gamma_1^i < \dots < \gamma_{m(i)}^i < i \le \gamma_{m(i)+1}^i < \dots < \gamma_{n(i)}^i < \mu^+$$

Now we begin to prove each part of the lemma. In each, we will find  $i_1 < i_2 \in$  $S^{\mu^+}_{\alpha}$  such that  $gtp(a_{i_1}/N_{i_1}; N)$  and  $gtp(a_{i_2}/N_{i_1}; N)$  are both the same (because of the EM structure) and different (because they exhibit different \*-forking behavior), which is our contradiction.

(1) Assume that  $p \upharpoonright M_j$  does not fork over  $M_0$ , for all  $j < \alpha$ . Let  $\bar{C}$  be an  $S^{\mu^+}_{\alpha}$ -club sequence, and set  $\langle N_i \in \mathbf{K}^* \mid i < \mu^+ \rangle = \bar{N}(\bar{C})$ as in the Claim (the value of  $\gamma$  doesn't matter here, e.g. take  $\gamma := \omega$ ). By Fodor's Lemma, there is a stationary subset  $S^* \subseteq S^{\mu^+}_{\alpha}$ , a term  $\rho_*$ ,  $m_*, n_* < \omega$  and ordinals  $\gamma_0^*, \ldots \gamma_{n_*}, \beta_{*,0}$  such that:

> For every  $i \in S^*$ , we have  $\rho_i = \rho_*$ ;  $n(i) = n_*$ ;  $m(i) = m_*$ ;  $\gamma_i^i = \gamma_i^*$ for  $j \leq m_*$ ; and  $\beta_{i,0} = \beta_{*,0}$ .

Set  $E := \{\delta < \mu^+ \mid \delta \text{ is limit and } EM_\tau(\delta, \Phi) \cap N = N_\delta\}$ . This is a club. Let  $i_1 < i_2$  both be in  $S^* \cap E$ . Then we have:

$$gtp(a_{i_1}/N_{i_1}) = gtp(\rho_*(\gamma_1^*, \dots, \gamma_{m_*}^*, \gamma_{m_*+1}^{i_1}, \dots, \gamma_{n_*}^{i_1})/N \cap EM_{\tau}(i_1, \Phi))$$
  
= gtp( $\rho_*(\gamma_1^*, \dots, \gamma_{m_*}^*, \gamma_{m_*+1}^{i_2}, \dots, \gamma_{n_*}^{i_2})/N \cap EM_{\tau}(i_1, \Phi)$ )  
= gtp( $a_{i_2}/N_{i_1}$ )

where all the types are computed inside N. This is because the only differences between  $a_{i_1}$  and  $a_{i_2}$  lie entirely above  $i_1$ .

We have that  $g_{i_1}: (N_{i_1}, N_{\beta_{*,0}}) \cong (M_{\alpha}, M_0)$  and that p \*-forks over  $M_0$ . Thus,  $gtp(a_{i_1}/N_{i_1}) = g_{i_1}(p)$  \*-forks over  $N_{\beta_{*,0}}$ . On the other hand,

 $C_{i_2}$  is cofinal in  $i_2$ , so there is  $j < \alpha$  such that  $\beta_{i_2,j} > i_1$  and, thus,  $N_{i_1} \leq_{\mathbf{K}^*} N_{\beta_{i_2,j}}$ . Again,  $g_{i_2} : (N_{\beta_{i_2,j}}, N_{\beta_{*,0}}) \cong (M_j, M_0)$  and  $p \upharpoonright M_j$  does not \*-fork over  $M_0$  by assumption. Thus,  $\operatorname{gtp}(a_{i_2}/N_{\beta_{i_2,j}}) = g_{i_2}(p \upharpoonright M_j)$ does not \*-fork over  $N_{\beta_{*,0}}$ . By monotonicity (M),  $\operatorname{gtp}(a_{i_2}/N_{i_1})$  does not \*-fork over  $N_{\beta_{*,0}}$ . Thus,  $\operatorname{gtp}(a_{i_1}/N_{i_1}) \neq \operatorname{gtp}(a_{i_2}/N_{i_2})$ , a contradiction.

- (2) Let χ be a big-enough cardinal and create an increasing, continuous elementary chain of models of set theory (𝔅<sub>i</sub> | i < μ<sup>+</sup>) such that for all i < μ<sup>+</sup>:
  - (a)  $\mathfrak{B}_i \prec (H(\chi), \in);$
  - (b)  $\|\mathfrak{B}_i\| = \mu;$
  - (c)  $\mathfrak{B}_0$  contains, as elements<sup>3</sup>,  $\Phi$ , EM( $\mu^+, \Phi$ ),  $h, \mu^+, \langle N_i | i < \mu^+ \rangle, S^{\mu^+}_{\alpha}, \langle a_i | i \in S^{\mu^+}_{\alpha} \rangle$ , and each  $f \in \tau(\Phi)$ ; and
  - (d)  $\mathfrak{B}_i \cap \mu^+$  is an ordinal.

We will use the following fact which was originally proven in [She94, III.2] (or see [AM10, Theorem 2.17] for a short proof).

FACT 20.2.9. Let  $\lambda$  be a cardinal such that  $\operatorname{cf} \lambda \geq \theta^{++}$  for some regular  $\theta$  and let  $S \subseteq S_{\theta}^{\lambda}$  be stationary. Then there is a S-club sequence  $\langle C_{\delta} | \delta \in S \rangle$  such that, if  $E \subseteq \lambda$  is club, then there are stationarily many  $\delta \in S$  such that  $C_{\delta} \subseteq E$ .

We have that  $\alpha < \mu$ , so we can apply Fact 20.2.9 with  $\lambda, \theta, S$  there standing for  $\mu^+, \alpha, S^{\mu^+}_{\alpha}$  here. Let  $\bar{C}$  be the  $S^{\mu^+}_{\alpha}$ -club sequence that the fact gives. Let  $\langle N_i \in K_{\mu} \mid i < \mu^+ \rangle = \bar{N}(\bar{C})$  be as in the Claim. Note that  $E := \{i < \mu^+ \mid \mathfrak{B}_i \cap \mu^+ = i\}$  is a club. By the conclusion of Fact 20.2.9, there is some  $i_2 \in S^{\mu^+}_{\alpha}$  such that  $C_{i_2} \subseteq E$ . We have  $a_{i_2} = \rho_{i_2}(\gamma_1^{i_2}, \ldots, \gamma_{n(i_2)}^{i_2})$ , with:

$$\gamma_1^{i_2} < \dots < \gamma_{m(i_2)}^{i_2} < i_2 \le \gamma_{m(i_2)+1}^{i_2} < \dots < \gamma_{n(i_2)}^{i_2}$$

Since the  $\beta_{i_2,j}$ 's enumerate a cofinal sequence in  $i_2$ , we can find  $j < \alpha$ such that  $\gamma_{m(i_2)}^{i_2} < \beta_{i_2,2j+1} < i$ . Recall that we have  $p \upharpoonright M_{2j+2}$  does not \*-fork over  $M_{2j+1}$  by assumption. Then  $(H(\chi), \in)$  satisfies the following formulas with parameters exactly the objects listed in item (2c) above and ordinals below  $\beta_{i_2,2j+2}$ :

$$\exists x, y_{m(i_2)+1}, \dots, y_{n(i)}. ("x \in S^{\mu^+}_{\alpha}")$$
  
 
$$\wedge "x > \beta_{i_2, 2j+1}" \wedge "y_k \in (x, \mu^+) \text{ are increasing ordinals"}$$
  
 
$$\wedge "a_x = \rho_{i_2}(\gamma^{i_2}_1, \dots, \gamma^{i_2}_{m(i_2)}, y_{m(i_2)+1}, \dots, y_{n(i_2)})")$$
  
 
$$\wedge "N_x \subset \text{EM}(x, \Phi)")$$

This is witnessed by  $x = i_2$  and  $y_k = \gamma_k^{i_2}$ . By elementarity,  $\mathfrak{B}_{\beta_2,2j+2}$ satisfies this formula as it contains all the parameters. Let  $i_1 \in (\beta_{i_2,2j+1}, \mu^+) \cap$ 

<sup>&</sup>lt;sup>3</sup>When we say that  $\mathfrak{B}_0$  contains a sequence as an element, we mean that it contains the function that maps an index to its sequence element.

 $\mathfrak{B}_{\beta_{i_2,2j+2}} = (\beta_{i_2,2j+1}, \beta_{i_2,2j+2})^4$  witness this, along with  $\gamma'_{m(i_2)+1} < \cdots < \gamma'_{n(i_2)} < \mu^+$ . Then we have:

$$a_{i_1} = \rho_{i_2}(\gamma_1^{i_2}, \dots, \gamma_{m(i_2)}^{i_2}, \gamma_{m(i_2)+1}', \dots, \gamma_{n(i_2)}')$$

with  $\beta_{i_2,2j+1} < \gamma_{m(i_2)+1}$ . We want to compare  $gtp(a_{i_2}/N_{i_1})$  and  $gtp(a_{i'}/N_{i_1})$ .

- From the elementarity, we get that  $N_{i_1} \subseteq \text{EM}_{\tau}(i_1, \Phi)$ . We also know that  $i_1 < \beta_{i_2, 2j+2} < \gamma_{m(i_2)+1}^{i_2}, \gamma_{m(i_2)+1}'$ . Thus, as before, the types are equal.
- We know that  $p \upharpoonright M_{2j+2}$  does not \*-fork over  $M_{2j+1}$ . Thus,  $\operatorname{gtp}(a_{i_2}/N_{\beta_{i_2,2j+2}})$  does not \*-fork over  $N_{\beta_{i_2,2j+1}}$ . Since we have  $N_{\beta_{i_2,2j+1}} \leq_{\mathbf{K}^*} N_{i_1} <_{\mathbf{K}^*}$   $N_{\beta_{i_2,2j+2}}$ , this gives  $\operatorname{gtp}(a_{i_2}/N_{i_1})$  does not \*-fork over  $N_{\beta_{i_2,2j+1}}$ .
- We have  $\beta_{i_2,2j+1} < i_1$ , so there is some  $k < \alpha$  such that  $\beta_{i_2,2j+1} < \beta_{i_1,k} < i'$ . By assumption, p \*-forks over  $M_k$ . Thus  $g_{i_1}(p)$  \*-forks over  $N_{\beta_{i_1,k}}$ . Therefore  $\operatorname{gtp}(a_{i_1}/N_{i_1})$  \*-forks over  $N_{\beta_{i_2,2j+1}} \leq_{\mathbf{K}^*} N_{\beta_{i_1,k}}$ . As before, these three statements contradict each other.

We now prove the main theorem, Theorem 20.1.3. Recall that the assumptions

of this theorem include the main context of this section (Hypothesis 20.2.3);  $\downarrow$  has weak universal local character somewhere; and **K** has an *EM* blueprint that every model embeds into.

PROOF OF THEOREM 20.1.3. Pick a cardinal  $\sigma < \mu^+$  such that  $\downarrow$  has weak universal local character at  $\sigma$  (exists by assumption (8)).

As announced at the beginning of this section, our proof of Theorem 20.1.3 really has two steps: a forking calculus step (implemented in Lemmas 20.2.5 and 20.2.6 and Proposition 20.2.7) and a set-theoretic step (implemented in Lemma 20.2.8). The claim below is key. The work done in the first step will show that the claim suffices, and the second step will prove the claim.

<u>Claim</u>: For any limit ordinal  $\gamma < \mu^+$  and any regular cardinal  $\alpha < \mu^+$ ,  $\stackrel{\circ}{\downarrow}$  has universal continuity at  $\alpha$  and no  $\gamma$ -limit alternations at  $\alpha$ .

By Proposition 20.2.7, the claim implies that  $\downarrow$  has strong universal local character at any regular  $\alpha < \mu^+$ . This suffices by Remark 20.2.2. It remains to prove the claim.

<u>Proof of Claim</u>: Universal continuity holds by Lemma 20.2.8. When  $\alpha < \sigma$ , Lemma 20.2.8 also gives that  $\stackrel{*}{\downarrow}$  has no  $\gamma$ -limit alternations at  $\alpha$ . Assume now that  $\alpha \geq \sigma$ . By Remark 20.2.2,  $\stackrel{*}{\downarrow}$  has weak universal local character at any limit  $\sigma' \in [\sigma, \mu^+)$ , so in particular in  $\alpha$ . By Lemma 20.2.5,  $\stackrel{*}{\downarrow}$  has no  $\gamma$ -limit alternations at  $\alpha$ , as desired.  $\dagger_{\text{Claim}}$ 

<sup>&</sup>lt;sup>4</sup>The equality here is the key use of club guessing.

## CHAPTER 21

## Quasiminimal abstract elementary classes

This chapter is based on [**Vasd**]. I thank John Baldwin, Will Boney, Levon Haykazyan, Jonathan Kirby, and Boris Zilber for helpful feedback on an early draft of this chapter. Finally, I thank an anonymous referee for comments that helped improve the chapter.

## Abstract

We propose the notion of a quasiminimal abstract elementary class (AEC). This is an AEC satisfying four semantic conditions: countable Löwenheim-Skolem-Tarski number, existence of a prime model, closure under intersections, and uniqueness of the generic orbital type over every countable model. We exhibit a correspondence between Zilber's quasimimimal pregeometry classes and quasiminimal AECs: any quasimimimal pregeometry class induces a quasiminimal AEC (this was known), and for any quasiminimal AEC there is a natural functorial expansion that induces a quasiminimal pregeometry class.

We show in particular that the exchange axiom is redundant in Zilber's definition of a quasiminimal pregeometry class. We also study a (non-quasiminimal) example of Shelah where exchange fails, and show that it has a good frame that cannot be extended to be type-full.

## 21.1. Introduction

Quasiminimal pregeometry classes were introduced by Zilber [**Zil05a**] in order to prove a categoricity theorem for the so-called pseudo-exponential fields. Quasiminimal pregeometry classes are a class of structures carrying a pregeometry satisfying several axioms. Roughly (see Definition 21.4.5) the axioms specify that the countable structures are quite homogeneous and that the generic type over them is unique (where types here are syntactic quantifier-free types). The original axioms included an "excellence" condition, but it has since been shown [**BHH**<sup>+</sup>14] that this follows from the rest. Zilber showed that a quasiminimal pregeometry class has at most one model in every uncountable cardinal, and in fact the structures are determined by their dimension. Note that quasiminimal pregeometry classes are typically non-elementary (see [**Kir10**, §5]): they are axiomatizable in  $\mathbb{L}_{\omega_1,\omega}(Q)$ (where Q is the quantifier "there exists uncountably many") but not even in  $\mathbb{L}_{\omega_1,\omega}$ .

The framework of abstract elementary classes (AECs) was introduced by Saharon Shelah [**She87a**] and encompasses for example classes of models of an  $\mathbb{L}_{\omega_1,\omega}(Q)$ theory. Therefore quasiminimal pregeometry classes can be naturally seen as AECs (see Fact 21.4.8). In this chapter, we show that a converse holds: there is a natural class of AECs, which we call the *quasiminimal AECs*, that corresponds to quasiminimal pregeometry classes. Quasiminimal AECs are required to satisfy four purely

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semantic properties (see Definition 21.4.1), the most important of which are that the AEC must, in a technical sense, be closed under intersections (this is called "admitting intersections", see Definition 21.3.1) and over each countable model M there must be a *unique* orbital (Galois) type that is not realized inside M.

It is straightforward (and implicit e.g. in [Kir10, §4], see also [HK16, Lemma 2.87]) to see that any quasiminimal pregeometry class is a quasiminimal AEC, but here we prove a converse (Theorem 21.4.21). We have to solve two difficulties:

- (1) The axioms of quasiminimal pregeometry classes are very syntactic because they are phrased in terms of quantifier-free types. For example, one of the axioms (II.(2) in Definition 21.4.5) specifies that the models must have some syntactic homogeneity.
- (2) Nothing in the definition of quasiminimal AECs says that the models must carry a pregeometry. It is not clear that the natural closure  $\operatorname{cl}^{M}(A)$  given by the intersections of all the **K**-substructures of M containing A satisfies exchange.

To get around the first difficulty, we use a recent joint work with Shelah (Chapter 23) together with the technique of adding relation symbols for small Galois types to the vocabulary (called the Galois Morleyization in Chapter 2). To get around the second difficulty, we develop new tools to prove the exchange axiom of pregeometries in any setup where we know that the other axioms of pregeometries hold. We show (Corollary 21.2.12) that any *homogeneous* closure space satisfying the finite character axiom of pregeometries also satisfies the exchange axiom (to the best of our knowledge, this is new<sup>1</sup>). As a consequence, the exchange axiom is redundant in the definition of a quasiminimal pregeometry class (Corollary 21.4.12)<sup>2</sup>.

An immediate corollary of the correspondence between quasiminimal AECs and quasiminimal pregeometry classes is that a quasiminimal AEC has at most one model in every uncountable cardinal (Corollary 21.4.22). This can be seen as a generalization of the fact that algebraically closed fields of a fixed characteristic are uncountably categorical (indeed, algebraically closed fields are closed under intersections and if F is a field, a, b are transcendental over F, then a and b satisfy the same type over F).

In the last section of this chapter, we study a (non-quasiminimal) example of Shelah that is quite well-behaved but where exchange fails. We point out that in this setup there is a good frame that cannot be extended to be type-full (Theorem 21.5.8). This answers a question of Boney and the author, see Question 18.1.4.

Throughout this chapter, we assume basic familiarity with AECs (see [Bal09]). We use the notation from Chapter 2. In particular,  $gtp(\bar{b}/A; N)$  denotes the Galois type of  $\bar{b}$  over A as computed in N.

<sup>&</sup>lt;sup>1</sup>Although related to the study of quasiminimal structures in Itai-Tsuboi-Wakai [**ITW04**] and later Pillay-Tanović [**PT11**], Corollary 21.2.12 is different. It gives a stronger conclusion using stronger hypotheses, see Remarks 21.2.3, 21.2.14.

<sup>&</sup>lt;sup>2</sup>Interestingly, exchange was initially not part of Zilber's definition of quasiminimal pregeometry classes (see [**Zil05b**,  $\S5$ ]) but was added later. Some sources claim that the axiom is necessary, see [**Bal09**, Remark 2.24] or [**Kir10**, p. 554], but this seems to be due to a related counterexample that does not fit in the framework of quasiminimal pregeometry classes (see the discussion in Remark 21.2.14).

#### 21.2. Exchange in homogeneous closure spaces

In this section, we study closure spaces, which are objects satisfying the monotonicity and transitivity axioms of pregeometries. We want to know whether they satisfy the exchange axiom when they are homogeneous. We give criterias for when this is the case (Corollary 21.2.12). To the best of our knowledge, this is new (but see Remarks 21.2.3, 21.2.14).

The following definition is standard, see e.g. [CR70].

DEFINITION 21.2.1. A closure space is a pair W = (X, cl), where:

- (1) X is a set.
- (2) cl:  $\mathcal{P}(X) \to \mathcal{P}(X)$  satisfies:
  - (a) Monotonicity: For any  $A \subseteq X$ ,  $A \subseteq cl(A)$ .
  - (b) Transitivity: For any  $A, B \subseteq X, A \subseteq cl(B)$  implies  $cl(A) \subseteq cl(B)$ .

We write |W| for X and  $\operatorname{cl}^W$  for cl (but when W is clear from context we might forget it). For  $a \in A$ , we will often write  $\operatorname{cl}(a)$  instead of  $\operatorname{cl}(\{a\})$ . Similarly, for sets  $A, B \subseteq |W|$  and  $a \in |W|$ , we will write  $\operatorname{cl}(Aa)$  instead of  $\operatorname{cl}(A \cup \{a\})$  and  $\operatorname{cl}(AB)$ instead of  $\operatorname{cl}(A \cup B)$ .

DEFINITION 21.2.2. Let W be a closure space.

- (1) For closure spaces  $W_1, W_2$ , we say that a function  $f : |W_1| \to |W_2|$  is an *isomorphism* if it is a bijection and for any  $A \subseteq |W_1|$ ,  $f[cl^{W_1}(A)] = cl^{W_2}(f[A])$ . When  $W_1 = W_2 = W$ , we say that f is an *automorphism* of W.
- (2) We say that  $A \subseteq |W|$  is closed if  $cl^W(A) = A$ .
- (3) For  $\mu$  an infinite cardinal, we say that W is  $\mu$ -homogeneous if for any closed set A with  $|A| < \mu$  and any  $a, b \in |W| \setminus A$ , there exists an automorphism of W that fixes A pointwise and sends a to b.
- (4) Let LS(W) be the least infinite cardinal  $\mu$  such that for any  $A \subseteq |W|$ ,  $|\operatorname{cl}^W(A)| \leq |A| + \mu$ .
- (5) Let  $\kappa(W)$  be the least infinite cardinal  $\kappa$  such that for any  $A \subseteq |W|$ ,  $a \in \operatorname{cl}^W(A)$  implies that there exists  $A_0 \subseteq A$  with  $|A_0| < \kappa$  and  $a \in \operatorname{cl}^W(A_0)$ . We say that W has *finite character* if  $\kappa(W) = \aleph_0$ .
- (6) We say that W has exchange over A if  $A \subseteq |W|$  and for any a, b, if  $a \in \operatorname{cl}^W(Ab) \setminus \operatorname{cl}^W(A)$ , then  $b \in \operatorname{cl}^W(Aa)$ . We say that W has exchange if it has exchange over every  $A \subseteq |W|$ .
- (7) We say that W is a *pregeometry* if it has finite character and exchange.

REMARK 21.2.3. The notion of homogeneity considered here is *not* the same as that considered in [**PT11**, §4]. There the notion is defined syntactically using first-order types and here we use automorphisms. The notion used here is stronger: two elements could satisfy the same first-order type but not the same type e.g. in an infinitary logic. This is used in the proof of Theorem 21.2.11.(3): if (I, <) is a dense linear order and b < c, then b and c will satisfy the same first-order type over  $(-\infty, b)$ , but there cannot be an automorphism sending b to c fixing  $(-\infty, b)$ . Thus  $I = \mathbb{Q} \times \omega_1$  cannot be a counterexample to Theorem 21.2.11.(3). In the proof of Theorem 21.4.11, we will build a (Galois) saturated model M and work with the pregeometry generated by a certain closure operator inside it. The (orbital) homogeneity of M will give homogeneity of the pregeometry in the strong sense given here. Remark 21.2.4.

- (1)  $\operatorname{LS}(W) \leq ||W|| + \aleph_0$  and  $\kappa(W) \leq ||W||^+ + \aleph_0$ . (2)  $\operatorname{LS}(W) \leq \kappa(W) \cdot \sup_{A \subseteq |W|, |A| < \kappa(W)} |\operatorname{cl}^W(A)|$ .

DEFINITION 21.2.5. For  $A \subseteq |W|$ , let  $W_A$  be the following closure space:  $|W_A| := |W| \setminus A$ , and  $\operatorname{cl}^{W_A}(B) := \operatorname{cl}^W(AB) \cap |W_A|$ .

REMARK 21.2.6. Let W be a closure space.

- (1) For  $\mu$  an infinite cardinal, if W is  $\mu$ -homogeneous,  $A \subseteq |W|$  and  $|A| < \mu$ , then  $W_A$  is  $\mu$ -homogeneous.
- (2) W has exchange over A if and only if  $W_A$  has exchange over  $\emptyset$ .
- (3) W has exchange if and only if W has exchange over every A with |A| < 1 $\kappa(W).$

Closure spaces where exchange always fails are studied in the literature under the names "antimatroid" or "convex geometry" [EJ85]. One of the first observation one can make is that there is a natural ordering in this context:

DEFINITION 21.2.7. Let W be a closure space. For  $a, b \in |W|$ , say  $a \leq b$  if  $a \in cl(b)$ . We say a < b if  $a \leq b$  but  $b \not\leq a$ .

REMARK 21.2.8. By the transitivity axiom,  $(|W|, \leq)$  is a pre-order. We will often think of it as a partial order, i.e. identify a, b such that  $a \leq b$  and  $b \leq a$ .

REMARK 21.2.9. Let W be a closure space where  $\emptyset$  is closed. Then W fails exchange over  $\emptyset$  if and only if there exists  $a, b \in |W|$  such that a < b.

To give conditions under which exchange follows from homogeneity, we will study the ordering (|W|, <). If exchange fails, it must be linear:

LEMMA 21.2.10. If W is  $LS(W)^+$ -homogeneous,  $\emptyset$  is closed, and W fails exchange over  $\emptyset$ , then  $(|W|, \leq)$  is (if we identify a, b with  $a \leq b$  and  $b \leq a$ ) a dense linear order without endpoints.

**PROOF.** Using failure of exchange, fix a, b such that  $a \in cl(b)$  but  $b \notin cl(a)$ . Let c, d be given such that  $d \not\leq c$ . Then  $d \notin cl(c)$ . We show that  $c \leq d$ , i.e.  $c \in cl(d)$ . By homogeneity, there exists an automorphism f of W taking c to a. Let d' := f(d). Then  $d' \notin cl(a)$ . Note that  $|cl(a)| \leq LS(W)$  so by  $LS(W)^+$ -homogeneity, there exists an automorphism g of W fixing cl(a) and sending d' to b. Applying  $f^{-1} \circ g^{-1}$ to the formula  $a \in cl(b)$ , we obtain  $c \in cl(d)$ , as desired.

We have shown that  $(|W|, \leq)$  is a linear order. That it is dense and without endpoints similarly follow from homogeneity.  $\square$ 

THEOREM 21.2.11. Let W be a  $LS(W)^+$ -homogeneous closure space where  $\emptyset$ is closed. Then W has exchange over  $\emptyset$  if at least one of the following conditions hold:

(1) 
$$||W|| < \aleph_0.$$

(2) 
$$||W|| \ge LS(W)^{++}$$
.  
(3)  $\kappa(W) = \aleph_0$ .

**PROOF.** Suppose for a contradiction that exchange over  $\emptyset$  fails.

For  $b \in |W|$ , write  $(-\infty, b) := \{a \in |W| : a < b\}$ , and similarly for  $(-\infty, b]$ . Note that if  $A \subseteq |W|$  is closed and  $a \in A$ , then by definition of  $\leq$  and the transitivity axiom,  $(-\infty, a] \subseteq A$ . Similarly, if  $b \notin A$  then by Lemma 21.2.10  $A \subseteq (-\infty, b)$ .

- (1) If  $||W|| < \aleph_0$ , then Lemma 21.2.10 directly gives a contradiction.
- (2) Let  $A \subseteq |W|$  be closed such that  $|A| \leq \mathrm{LS}(W)$  and let  $B \subseteq |W|$  be closed with  $A \subseteq B$  and  $|B| = \mathrm{LS}(W)^+$ . Let  $a \in A$  and let  $b \notin B$ . Then  $(-\infty, a) \subseteq A$  and  $B \subseteq (-\infty, b)$ . Therefore  $|(-\infty, a)| \leq \mathrm{LS}(W)$  and  $|(-\infty, b)| \geq \mathrm{LS}(W)^+$ . However by homogeneity there exists an automorphism of W sending a to b, a contradiction.
- (3) We first prove two claims.

<u>Claim 1</u>: If  $b \in |W|$ , then  $cl((-\infty, b)) = (-\infty, b]$ .

<u>Proof of Claim 1</u>: Let  $B := cl(-\infty, b)$ . First note that  $B \subseteq cl(b)$ , hence  $|B| \leq LS(W)$  (so we can apply homogeneity to it) and  $B \subseteq (-\infty, b]$ . By monotonicity,  $(-\infty, b) \subseteq B$ . Also, if  $B \neq (-\infty, b)$ , then  $b \in B$  (say  $c \in B \setminus (-\infty, b)$ . Then  $c \not\leq b$ , so by Lemma 21.2.10,  $b \leq c$ , so since B is closed  $b \in B$ ). Thus if  $b \notin B$ , then  $B = (-\infty, b)$ . This is impossible: take  $c \in |W|$  such that b < c (exists by Lemma 21.2.10). Then there is an automorphism of W taking b to c fixing B, which is impossible as b is a least upper bound of B but c is not. Therefore  $(-\infty, b] \subseteq B$ .  $\dagger_{Claim 1}$ 

<u>Claim 2</u>: If  $\langle A_i : i \in I \rangle$  is a non-empty collection of subsets of |W|, then  $\operatorname{cl}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} \operatorname{cl}(A_i)$ .

<u>Proof of Claim 2</u>: Clearly, the right hand side is contained in the left hand side. We show the other inclusion. Let  $A := \bigcup_{i \in I} A_i$ . Let  $a \in cl(A)$ . By finite character, there exists a finite  $A' \subseteq A$  such that  $a \in cl(A')$ . Since  $\emptyset$  is closed, A' cannot be empty. Say  $A' = \{a_0, \ldots, a_{n-1}\}$ , with  $a_0 \leq a_1 \leq \ldots \leq a_{n-1}$  (we are implicitly using Lemma 21.2.10). Then  $a \in cl(a_{n-1})$ . Pick  $i \in I$  such that  $a_{n-1} \in A_i$ . Then  $a \in cl(A_i)$ , as desired.  $\dagger_{\text{Claim 2}}$ 

Now pick any  $b \in |W|$ . Note that (using Lemma 21.2.10)  $(-\infty, b) = \bigcup_{a < b} (-\infty, a) = \bigcup_{a < b} (-\infty, a]$ . However on the one hand, by Claim 1,  $\operatorname{cl}(\bigcup_{a < b} (-\infty, a)) = \operatorname{cl}((-\infty, b)) = (-\infty, b]$  but on the other hand, by Claim 2,  $\operatorname{cl}(\bigcup_{a < b} (-\infty, a)) = \bigcup_{a < b} \operatorname{cl}((-\infty, a)) = \bigcup_{a < b} (-\infty, a] = (-\infty, b)$ , a contradiction.

COROLLARY 21.2.12. Let W be a  $(\kappa(W) + \mathrm{LS}(W)^+)$ -homogeneous closure space. If either  $\kappa(W) = \aleph_0$  or  $||W|| \notin [\aleph_0, \mathrm{LS}(W)^+]$ , then W has exchange.

PROOF. Let  $\mu := \kappa(W) + \mathrm{LS}(W)^+$ . By Remark 21.2.6, it is enough to see that W has exchange over every set A with  $|A| < \kappa(W)$ . Fix such an A. By Remark 21.2.6, it is enough to see that  $W' := W_A$  has exchange over  $\emptyset$ . Note that W' is  $\mu$ -homogeneous and  $\mathrm{LS}(W') \leq \mathrm{LS}(W)$ . Let  $B := \mathrm{cl}^{W'}(\emptyset)$ . We have that  $|B| \leq \mathrm{LS}(W)$ . Let  $W'' := W'_B$ . We show that W'' has exchange over  $\emptyset$ . Note that W'' is still  $\mu$ -homogeneous. Moreover  $\emptyset$  is closed in W''. Observe that  $\kappa(W) = \aleph_0$  implies that  $\kappa(W'') = \aleph_0$ ,  $||W''|| \leq ||W||$ , but  $||W|| \geq \mathrm{LS}(W)^{++}$  implies that  $||W''|| \geq \mathrm{LS}(W)^{++}$ . Therefore by Theorem 21.2.11, W'' has exchange over  $\emptyset$ , as desired.  $\Box$ 

We give a few examples showing that the hypotheses of Corollary 21.2.12 are near optimal:

Example 21.2.13.

#### 21. QUASIMINIMAL AECS

- (1) On any partial order  $\mathbb{P}$ , one can define a closure operator  $cl_1$  by  $cl_1(A) := \{b \in \mathbb{P} \mid \exists a \in A : b \leq a\}$ . The resulting closure space  $W_1$  has exchange over  $\emptyset$  if and only if there are no  $a, b \in \mathbb{P}$  with a < b. Note that if  $\mathbb{P}$  is e.g. a dense linear order, then  $W_1$  is not  $\aleph_1$ -homogeneous.
- (2) On the other hand, one can define  $\operatorname{cl}_2(A) := \operatorname{cl}_1(A) \cup \{b \in \mathbb{P} \mid \forall c(c < b \rightarrow c \in \operatorname{cl}_1(A))\}$ . This gives a closure space  $W_2$ . If  $\mathbb{P} = \mathbb{Q}$ , then  $W_2$  is  $\aleph_1$ -homogeneous and does not have exchange over  $\emptyset$  but note that  $\kappa(W) = \aleph_1$ , as witnessed by the fact that the statement " $0 \in \operatorname{cl}((-\infty, 0))$ " is not witnessed by a finite subset of  $(-\infty, 0)$ .
- (3) The closure space  $W_3$  induced by  $\mathbb{P} = \mathbb{Q} \times \omega_1$  and the closure operator cl<sub>2</sub> is also  $\aleph_1$ -homogeneous, satisfies  $\mathrm{LS}(W_3) = \aleph_0$ ,  $\kappa(W_3) = \aleph_1$ , and does not have exchange over  $\emptyset$ .

REMARK 21.2.14. In [**PT11**, §5], Pillay and Tanović (generalizing an earlier result of Itai, Tsuboi, and Wakai [**ITW04**, Proposition 2.8]) prove (roughly) that any quasiminimal structure (i.e. every definable set is either countable or co-countable) of size at least  $\aleph_2$  induces a pregeometry. This is a (more general) version of Corollary 21.2.12 for the case  $\kappa(W) = \aleph_0$ ,  $||W|| \ge \aleph_2$ , and  $\mathrm{LS}(W) = \aleph_0$  (note that one can see any such W as a structure by adding an *n*-ary function for the closure of each set of size *n*).

Note that in the Pillay-Tanović context the hypothesis that the size should be at least  $\aleph_2$  is needed: consider [**ITW04**, Example 2.2.(3a)] the structure M := $(\mathbb{Q} \times \omega_1, <)$ . The closure space induced by M is as in  $W_1$  from Example 21.2.13, so it does not have exchange. Note that M is homogeneous in the model-theoretic sense that every countable partial elementary mapping from M into M can be extended (and also in the syntactic sense of [**PT11**, §4], see Remark 21.2.3), but this does *not* make the corresponding closure space homogeneous in the sense of Definition 21.2.2.(3).

## 21.3. On AECs admitting intersections

We recall the definition of an AEC admitting intersections, first appearing in Baldwin and Shelah [**BS08**, Definition 1.2]. We give a few known facts and show (Theorem 21.3.5) that admitting intersections transfers up in AECs: if  $\mathbf{K}_{\lambda}$  admits intersections, then  $\mathbf{K}_{>\lambda}$  admits intersections.

As in [**Gro**], we call an *abstract class* a pair  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  where K is a class of structures in a fixed vocabulary  $\tau = \tau(\mathbf{K}), \leq_{\mathbf{K}}$  is a partial order on K, both K and  $\leq_{\mathbf{K}}$  are closed under isomorphisms, and for  $M, N \in K, M \leq_{\mathbf{K}} N$  implies  $M \subseteq N$ . We say that an abstract class is *coherent* if for  $M_0, M_1, M_2 \in \mathbf{K}, M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$  and  $M_0 \leq_{\mathbf{K}} M_2$  imply  $M_0 \leq_{\mathbf{K}} M_1$ .

Note that any AEC is a coherent abstract class, and if **K** is an AEC and  $\lambda$  is a cardinal, then  $\mathbf{K}_{\lambda}$  is also a coherent abstract class.

DEFINITION 21.3.1. Let **K** be a coherent abstract class. Let  $N \in \mathbf{K}$  and let  $A \subseteq |N|$ .

- (1) Let  $\operatorname{cl}^{N}(A)$  be the set  $\bigcap\{|M|: M \leq_{\mathbf{K}} N \land A \subseteq |M|\}$ . Note that  $\operatorname{cl}^{N}(A)$  induces a  $\tau(\mathbf{K})$ -substructure of N, so we will abuse notation and also write  $\operatorname{cl}^{N}(A)$  for this substructure.
- (2) We say that N admits intersections over A if  $\operatorname{cl}^{N}(A) \leq_{\mathbf{K}} N$  (more formally, there exists  $M \leq_{\mathbf{K}} N$  such that  $|M| = \operatorname{cl}^{N}(A)$ ).

- (3) We say that N admits intersections if it admits intersections over all  $A \subseteq |N|$ .
- (4) We say that **K** admits intersections if every  $N \in \mathbf{K}$  admits intersections.

REMARK 21.3.2. Let **K** be a coherent abstract class and let  $N \in \mathbf{K}$ . Then  $(|N|, \mathrm{cl}^N)$  is a closure space and any  $M \leq_{\mathbf{K}} N$  is closed.

The following characterization of admitting intersections in terms of the existence of a certain closure operator will be used often in this chapter. Part of it appears already (for AECs) in Theorem 8.2.10.

FACT 21.3.3. Let **K** be a coherent abstract class and let  $N \in \mathbf{K}$ . The following are equivalent:

- (1) N admits intersections.
- (2) For every non-empty collection S of K-substructures of N, we have that  $\bigcap S \leq_{\mathbf{K}} N$ .
- (3) There is a closure space W such that:
  - (a) |W| = |N|.
  - (b) The closed sets in W are exactly the sets of the form |M| for  $M \leq_{\mathbf{K}} N$ .

PROOF. That (1) is equivalent to (2) is an exercise in the definition, left to the reader (and not used in this chapter). Also, (1) implies (3) is clear: take  $W := (|N|, cl^N)$ . We prove that (3) implies (1). Let  $(W, cl^W)$  be a closure space on |N| such that the closed sets are exactly the **K**-substructures of N. Let  $A \subseteq |N|$  and let  $M := cl^W(A)$ . By assumption,  $M \leq_{\mathbf{K}} N$  so it suffices to see that  $M = cl^N(A)$ . Let  $M' \leq_{\mathbf{K}} N$  be such that  $A \subseteq |M'|$ . By assumption,  $M' \leq_{\mathbf{K}} M'$  is closed in W, so  $M = cl^W(A) \subseteq |M'|$ . By coherence,  $M \leq_{\mathbf{K}} M'$ . Since M' was arbitrary, it follows from the definition of  $cl^N$  that  $cl^N(A) = M$ , as desired.

The next result is observed (for AECs) in Proposition 8.2.13. The proof generalizes to coherent abstract classes.

FACT 21.3.4. Let **K** be a coherent abstract class and let  $M \leq_{\mathbf{K}} N$  both be in **K**. Let  $A \subseteq |M|$  If N admits intersections over A, then M admits intersections over A and  $cl^{M}(A) = cl^{N}(A)$ .

We now show that admitting intersections transfer up. This is quite routine using the characterization of Fact 21.3.3 but we give a full proof.

THEOREM 21.3.5. Let **K** be an AEC and let  $\lambda \geq \text{LS}(\mathbf{K})$ . Let  $N \in \mathbf{K}_{\geq \lambda}$ . If M admits intersections for all  $M \in \mathbf{K}_{\lambda}$  with  $M \leq_{\mathbf{K}} N$ , then N admits intersections. In particular if  $\mathbf{K}_{\lambda}$  admits intersections, then  $\mathbf{K}_{>\lambda}$  admits intersections.

PROOF. First note that if  $M \in \mathbf{K}_{<\lambda}$  and  $M \leq_{\mathbf{K}} N$ , then M also admits intersections by Fact 21.3.4. Now on to the proof. By Fact 21.3.3, it is enough to find a closure space W with universe |N| such that the closed sets of W are exactly the **K**-substructures of N. Define  $cl^{W} : \mathcal{P}(|N|) \to \mathcal{P}(|N|)$  as follows:

$$\mathrm{cl}^{W}(A) := \bigcup \{ \mathrm{cl}^{M}(A \cap |M|) : M \leq_{\mathbf{K}} N \land M \in \mathbf{K}_{\lambda} \}$$

We claim that  $W := (|N|, cl^W)$  is the required closure space. We prove this via a chain of claims:

<u>Claim 1</u>: For any  $M \in \mathbf{K}_{<\lambda}$  with  $M \leq_{\mathbf{K}} N$ ,  $\mathrm{cl}^M = \mathrm{cl}^W \upharpoonright \mathcal{P}(|M|)$ .

<u>Proof of Claim 1</u>: Let  $A \subseteq |M|$ . By definition,  $\operatorname{cl}^{M}(A) \subseteq \operatorname{cl}^{W}(A)$ . Conversely, let  $a \in \operatorname{cl}^{W}(A)$ . Pick  $M' \in \mathbf{K}_{\lambda}$  such that  $M' \leq_{\mathbf{K}} N$  and  $a \in \operatorname{cl}^{M'}(A \cap |M'|)$ . Pick  $M'' \in \mathbf{K}_{\lambda}$  with  $M' \leq_{\mathbf{K}} M''$  and  $M \leq_{\mathbf{K}} M''$ . By monotonicity and Fact 21.3.4,  $a \in \operatorname{cl}^{M''}(A) = \operatorname{cl}^{M}(A)$ , as desired.  $\dagger_{\operatorname{Claim 1}}$ 

<u>Claim 2</u>: If  $A' \subseteq cl^W(A)$  is such that  $|A'| \leq \lambda$ , then there exists  $A_0 \subseteq A$  such that  $|A_0| \leq \lambda$  and  $A' \subseteq cl^W(A_0)$ .

<u>Proof of Claim 2</u>: Pick  $M \in \mathbf{K}_{\lambda}$  witnessing that  $a \in \mathrm{cl}^{W}(A)$  and let  $A_{0} := A \cap |M|$ .  $\dagger_{\mathrm{Claim 2}}$ 

<u>Claim 3</u>: W is a closure space.

Proof of Claim 3:

- Monotonicity: Let  $A \subseteq |N|$ . We want to see that  $A \subseteq cl^{W}(A)$ . Let  $a \in A$ . Using the Löwenheim-Skolem-Tarski axiom, fix  $M \in \mathbf{K}_{\lambda}$  with  $M \leq_{\mathbf{K}} N$ and  $a \in |M|$ . Now since  $(|M|, cl^{M})$  is a closure space (Remark 21.3.2),  $a \in cl^{M}(a) \subseteq cl^{W}(A)$ , as needed.
- Transitivity: Let  $A, B \subseteq |N|$ . First note that  $A \subseteq B$  implies  $\operatorname{cl}^W(A) \subseteq \operatorname{cl}^W(B)$ . Now assume that  $A \subseteq \operatorname{cl}^W(B)$ . We show that  $\operatorname{cl}^W(A) \subseteq \operatorname{cl}^W(B)$ . Let  $a \in \operatorname{cl}^W(A)$ . By Claim 2, there exists  $A_0 \subseteq A$  with  $|A_0| \leq \lambda$  such that  $a \in \operatorname{cl}^W(A_0)$ . Since by assumption  $A_0 \subseteq \operatorname{cl}^W(B)$ , there exists by Claim 2 again  $B_0 \subseteq B$  such that  $|B_0| \leq \lambda$  and  $A_0 \subseteq \operatorname{cl}^W(B_0)$ . Pick  $M \in \mathbf{K}_{\lambda}$  such that  $A_0 \cup B_0 \subseteq |M|$  and  $M \leq_{\mathbf{K}} N$ . By Claim 1,  $\operatorname{cl}^W(A_0) = \operatorname{cl}^M(A_0)$ . By transitivity in the closure space  $(|M|, \operatorname{cl}^M), \operatorname{cl}^M(A_0) \subseteq \operatorname{cl}^M(B_0)$ . By Claim 1 again,  $\operatorname{cl}^M(B_0) = \operatorname{cl}^W(B_0)$ . By what has been said earlier,  $\operatorname{cl}^W(B_0) \subseteq \operatorname{cl}^W(B)$ . It follows that  $\operatorname{cl}^W(A_0) \subseteq \operatorname{cl}^W(B)$ , hence  $a \in \operatorname{cl}^W(B)$  as desired.

<sup>†</sup>Claim 3

<u>Claim 4</u>: If  $M \leq_{\mathbf{K}} N$ , then  $cl^{W}(M) = M$ .

<u>Proof of Claim 4</u>: Let  $a \in cl^W(M)$ . By Claim 2 and monotonicity we can pick  $M_0 \in \mathbf{K}_{\leq \lambda}$  with  $M_0 \leq_{\mathbf{K}} M$  such that  $a \in cl^W(M_0)$ . By Claim 1,  $a \in cl^{M_0}(M_0) = M_0 \leq_{\mathbf{K}} M$ . Therefore  $a \in |M|$ , as desired.  $\dagger_{\text{Claim 4}}$ 

<u>Claim 5</u>: For any  $A \subseteq |N|$ ,  $cl^W(A) \leq_{\mathbf{K}} N$ .

<u>Proof of Claim 5</u>: From Claim 2, it is easy to see that  $\operatorname{cl}^W(A) = \bigcup \{\operatorname{cl}^W(A_0) : A_0 \subseteq A \land |A_0| \leq \lambda\}$ . Therefore by the chain and coherence axioms of AECs, it is enough to show the claim when  $|A| \leq \lambda$ . Pick  $M \in \mathbf{K}_{\lambda}$  with  $M \leq_{\mathbf{K}} N$  and  $A \subseteq |M|$ . By Claim 1,  $\operatorname{cl}^W(A) = \operatorname{cl}^M(A)$ . By Fact 21.3.4,  $\operatorname{cl}^M(A) \leq_{\mathbf{K}} M$ . Since  $M \leq_{\mathbf{K}} N$ , the result follows.  $\dagger_{\text{Claim 5}}$ 

Putting together Claim 3, 4, and 5, we have the desired result.

We will use two facts about AECs admitting intersections in the next section. First, the closure operator has finite character (Proposition 8.2.13).

FACT 21.3.6. Let **K** be an AEC and let  $N \in \mathbf{K}$ . If N admits intersections, then  $\kappa((|N|, \mathrm{cl}^N)) = \aleph_0$ .

Second, Galois types can be characterized nicely (see [**BS08**, Lemma 1.3.(1)] or Proposition 8.2.16).

FACT 21.3.7. Let **K** be an AEC admitting intersections. Then  $\operatorname{gtp}(\bar{b}_1/A; N_1) = \operatorname{gtp}(\bar{b}_2/A; N_2)$  if and only if there exists  $f : \operatorname{cl}^{N_1}(A\bar{b}_1) \cong_A \operatorname{cl}^{N_2}(A\bar{b}_2)$  such that  $f(\bar{b}_1) = \bar{b}_2$ .

### 21.4. Quasiminimal AECs

In this section, we define quasiminimal AECs and show that they are essentially the same as quasiminimal pregeometry classes.

Following Shelah [**She09a**, II.1.9.(1A)], we will write  $gS^{na}(M)$  for the set of *nonalgebraic* types over M: that is, the set of  $p \in gS(M)$  such that p = gtp(a/M; N) with  $a \notin |M|$  (in the context of this chapter, there will be a unique nonalgebraic type which we will call the generic type). We say that  $M \in \mathbf{K}$  is prime if for any  $N \in \mathbf{K}$ , there exists  $f : M \to N$ .

DEFINITION 21.4.1. An AEC K is quasiminimal if:

- (1)  $LS(\mathbf{K}) = \aleph_0$ .
- (2) There is a prime model in  $\mathbf{K}$ .
- (3)  $\mathbf{K}_{<\aleph_0}$  admits intersections.

(4) (Uniqueness of the generic type) For any  $M \in \mathbf{K}_{\leq \aleph_0}$ ,  $|\mathbf{gS}^{\mathrm{na}}(M)| \leq 1$ .

We say that  $\mathbf{K}$  is *unbounded* if it satisfies in addition:

(5) There exists  $\langle M_i : i < \omega \rangle$  strictly increasing in **K**.

REMARK 21.4.2. It is possible for a quasiminimal AEC to have maximal countable models. However if it is unbounded it turns out it will not have any maximal models (this is a consequence of the equivalence with quasiminimal pregeometry classes, see Corollary 21.4.22).

REMARK 21.4.3. We obtain an equivalent definition if we replace axiom (2) by:

(2)'  $\mathbf{K} \neq \emptyset$  and  $\mathbf{K}_{\langle \aleph_0}$  has joint embedding.

Why? That (2) implies (2)' (modulo the other axioms) is given by Lemma 21.4.9. For the other direction, one can use joint embedding to see that  $\operatorname{cl}^{M}(\emptyset)$  is a prime model for any  $M \in \mathbf{K}_{\leq \aleph_{0}}$ .

REMARK 21.4.4. What happens if in Definition 21.4.1 one replaces  $\aleph_0$  with an uncountable cardinal  $\lambda$ ? Then the natural generalizations of Lemmas 21.4.9, 21.4.10, and Theorem 21.4.11 hold but we do not know whether Lemma 21.4.14 generalizes.

For the convenience of the reader, we repeat here the definition of a quasiminimal pregeometry class. We use the numbering and presentation from Kirby [**Kir10**], see there for more details on the terminology. We omit axiom III (excellence), since it has been shown [**BHH**<sup>+</sup>**14**] that it follows from the rest. We have added axiom 0.(3) that also appears in Haykazyan [**Hay16**, Definition 2.2] and corresponds to (2) in the definition of a quasiminimal AEC, as well as axiom 0.(1) which requires that the class be non-empty and that the vocabulary be countable (this can be assumed without loss of generality, see [**Kir10**, Proposition 5.2]).

As in Definition 21.4.1, we call the class *unbounded* if it has an infinite dimensional model (this is the nontrivial case that interests us here).

DEFINITION 21.4.5. A quasiminimal pregeometry class is a class C of pairs  $(H, cl_H)$ , where H is a  $\tau$ -structure (for a fixed vocabulary  $\tau = \tau(C)$ ) and  $cl_H : \mathcal{P}(|H|) \to \mathcal{P}(|H|)$  is a function satisfying the following axioms.

- 0: (1)  $|\tau(\mathcal{C})| \leq \aleph_0$  and  $\mathcal{C} \neq \emptyset$ .
  - (2) If  $(H, cl_H), (H', cl_{H'})$  are both  $\tau$ -structures with functions on their powersets and  $f : H \cong H'$  is also an isomorphism from  $(|H|, cl_H)$  onto  $(|H'|, cl_{H'})$ , then  $(H', cl_{H'}) \in \mathcal{C}$ .

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- (3) If  $(H, cl_H), (H', cl_{H'}) \in \mathcal{C}$ , then H and H' satisfy the same quantifier-free sentences.
- I: (1) For each  $(H, cl_H) \in C$ ,  $(|H|, cl_H)$  is a pregeometry such that the closure of any finite set is countable.
  - (2) If  $(H, cl_H) \in \mathcal{C}$  and  $X \subseteq |H|$ , then the  $\tau(\mathcal{C})$ -structure induced by  $cl_H(X)$  together with the appropriate restriction of  $cl_H$  is in  $\mathcal{C}$ .
  - (3) If  $(H, \operatorname{cl}_H), (H', \operatorname{cl}_{H'}) \in \mathcal{C}, X \subseteq |H|, y \in \operatorname{cl}_H(X), \text{ and } f : H \to H' \text{ is a partial embedding with } X \cup \{y\} \subseteq \operatorname{preim}(f), \text{ then } f(y) \in \operatorname{cl}_{H'}(f[X]).$
- II: Let  $(H, cl_H), (H', cl_{H'}) \in C$ . Let  $G \subseteq H$  and  $G' \subseteq H'$  be countable closed subsets or empty and let  $g: G \to G'$  be an isomorphism.
  - (1) If  $x \in |H|$  and  $x' \in |H'|$  are independent from G and G' respectively, then  $g \cup \{(x, x')\}$  is a partial embedding.
  - (2) If  $g \cup f : H \to H'$  is a partial embedding, f has finite preimage X, and  $y \in cl_H(X \cup G)$ , then there is  $y' \in H'$  such that  $g \cup f \cup \{(y, y')\}$  is a partial embedding.

We say that C is *unbounded* if it satisfies in addition:

- (1) C is closed under unions of increasing chains: If  $\delta$  is a limit ordinal and  $\langle (H_i, \operatorname{cl}_{H_i}) : i < \delta \rangle$  is increasing with respect to being a closed substructure (i.e. for each  $i < \delta$ ,  $H_i \subseteq H_{i+1}$  and  $\operatorname{cl}_{H_{i+1}} \upharpoonright \mathcal{P}(|H_i|) =$  $\operatorname{cl}_{H_i}$ ), then  $(H_{\delta}, \operatorname{cl}_{H_{\delta}}) \in C$ , where  $H_{\delta} = \bigcup_{i < \delta} H_i$  and  $\operatorname{cl}_{H_{\delta}}(X) =$  $\bigcup_{i < \delta} \operatorname{cl}_{H_i}(X \cap |H_i|)$ .
- (2) C contains an infinite dimensional model (i.e. there exists  $(H, cl_H) \in C$ with  $\langle a_i : i < \omega \rangle$  in H such that  $a_i \notin cl_H(\{a_j : j < i\})$  for all  $i < \omega$ ).

REMARK 21.4.6. If C is a quasimiminal pregeometry class,  $(H, cl_1), (H, cl_2) \in C$ , then by axiom I.(3) used with the identity embedding,  $cl_1 = cl_2$ . In other words, once C is fixed the pregeometry is determined by the structure (see also the discussion after [**Kir10**, Example 1.2]).

It is straightforward to show that quasiminimal pregeometry classes are (after forgetting the pregeometry and ordering them with "being a closed substructure") quasiminimal AECs. That they are AECs is noted in [**Kir10**, §4]. In fact, the exchange axiom is not necessary for this. We sketch a proof here for completeness.

DEFINITION 21.4.7. Let  $\mathcal{C}$  be a quasiminimal pregeometry class.

- (1) For  $(H, cl_H), (H', cl_{H'}) \in \mathcal{C}$ , we write  $(H, cl_H) \leq_{\mathcal{C}} (H', cl_{H'})$  if  $H \subseteq H'$ and  $cl_{H'} \upharpoonright \mathcal{P}(|H|) = cl_H$ .
- (2) Let  $\mathbf{K} = \mathbf{K}(\mathcal{C}) := (K(\mathcal{C}), \leq_{\mathbf{K}})$  be defined as follows:
  - (a)  $K(\mathcal{C}) := \{ M \mid \exists \operatorname{cl} : (M, \operatorname{cl}) \in \mathcal{C} \}.$
  - (b)  $M \leq_{\mathbf{K}} N$  if for some  $\operatorname{cl}_M, \operatorname{cl}_N, (M, \operatorname{cl}_M), (N, \operatorname{cl}_N) \in \mathcal{C}$  and  $(M, \operatorname{cl}_M) \leq_{\mathcal{C}} (N, \operatorname{cl}_N)$ .

FACT 21.4.8. If C satisfies all the axioms of a quasiminimal pregeometry class except that in I.(1)  $cl_H$  may not have exchange, then  $\mathbf{K}(C)$  is a quasiminimal AEC. Moreover C is unbounded if and only if  $\mathbf{K}(C)$  is unbounded.

PROOF. We will use Remark 21.4.6 without explicit mention. Let  $\mathbf{K} := \mathbf{K}(\mathcal{C})$ . By axioms 0, I, the finite character axiom of pregeometries, and (if  $\mathcal{C}$  is unbounded; note that if  $\mathcal{C}$  is bounded there are no infinite increasing chains) IV,  $\mathbf{K}$  is an AEC. Since the closure of any finite set is countable (axiom I.(1)) and  $|\tau(\mathcal{C})| \leq \aleph_0$  (axiom 0.(1)),  $\mathrm{LS}(\mathbf{K}) = \aleph_0$ . This proves that (1) in Definition 21.4.1 holds.

IV:

As for axiom (2), by axiom 0.(1),  $\mathcal{C} \neq \emptyset$ . Let  $(M, \operatorname{cl}_M) \in \mathcal{C}$  and let  $M_0 := \operatorname{cl}_M(\emptyset)$ . By axiom I.(2),  $(M_0, \operatorname{cl}_M \upharpoonright \mathcal{P}(|M_0|)) \in \mathcal{C}$ . We show that  $M_0$  is the desired prime model. Let  $N \in \mathbf{K}$ . This means that  $(N, \operatorname{cl}_N) \in \mathcal{C}$ . By axiom 0.(3), the empty map is a partial embedding from M into N. Using axiom II.(2)  $\omega$ -many times (see the proof of [**Kir10**, Theorem 2.1]), we can extend it to a map  $f_0 : M_0 = \operatorname{cl}_M(\emptyset) \cong$  $\operatorname{cl}_N(\emptyset)$ . Since  $\operatorname{cl}_N(\emptyset) \leq_{\mathbf{K}} N$  (for the same reason that  $M_0 \leq_{\mathbf{K}} M$ ),  $f_0$  witnesses that  $M_0$  embeds into N, as desired.

Why does  $\mathbf{K}_{\leq\aleph_0}$  admit intersections (axiom (3) in Definition 21.4.1)? This is by the characterization in Fact 21.3.3 (use the definition of  $\leq_{\mathbf{K}}$  and axiom I.(2).

Let us check axiom (4) in Definition 21.4.1. Let  $M \in \mathbf{K}_{\leq\aleph_0}$ . We want to show that  $|\mathbf{gS}^{\mathrm{na}}(M)| \leq 1$ . Let  $p_1, p_2 \in \mathbf{gS}^{\mathrm{na}}(M)$ . Say  $p_\ell = \mathrm{gtp}(a_\ell/M; N_\ell)$ ,  $\ell = 1, 2$ . We want to see that  $p_1 = p_2$ . Without loss of generality (since we have just seen that  $\mathbf{K}_{\leq\aleph_0}$  admits intersections),  $N_\ell = \mathrm{cl}^{N_\ell}(Ma_\ell)$ . We show that there exists  $f: N_1 \cong_M$  $N_2$  with  $f(a_1) = a_2$ . We use axiom II.(1), where G, G', H, H', g, x, x' there stand for  $M, M, N_1, N_2, \mathrm{id}_M, a_1, a_2$  here. We get that  $\mathrm{id}_M \cup \{a_1, a_2\}$  is a partial embedding from  $N_1$  to  $N_2$ . Now use axiom II.(2)  $\omega$ -many times (as in the second paragraph of this proof) to extend this partial embedding to an isomorphism  $f: N_1 \cong_M N_2$ . By construction, we will have that  $f(a_1) = a_2$ , as desired.

Finally, it is straightforward to see that (5) holds if and only if C is unbounded, as desired.

We now start going toward the other direction. For this we first prove a couple of lemmas about quasiminimal AECs. In particular, we want to show that they have amalgamation, joint embedding, are stable, tame, and that the closure operator induces a pregeometry on them.

LEMMA 21.4.9. If **K** is a quasiminimal AEC, then  $\mathbf{K}_{\leq\aleph_0}$  has amalgamation and joint embedding.

PROOF. We prove amalgamation, and joint embedding can then be obtained from the existence of the prime model and some renaming. By the "in particular" part of Theorem 8.4.14, it is enough to prove the so-called type extension property in  $\mathbf{K}_{<\aleph_0}$ . This is given by the following claim:

<u>Claim</u>: If  $M \leq_{\mathbf{K}} N$  are both in  $\mathbf{K}_{\leq\aleph_0}$  and  $p \in gS(M)$ , then there exists  $q \in gS(N)$  extending p.

<u>Proof of Claim</u>: Say p = gtp(a/M; N'). If  $a \in |M|$  (i.e. p is algebraic), let q := gtp(a/N; N). Assume now that  $a \notin |M|$ . If M = N, take q = p, so assume also that  $M <_{\mathbf{K}} N$ . Let  $b \in |N| \setminus |M|$  and let p' := gtp(b/M; N). By uniqueness of the generic type, p' = p. Therefore q := gtp(b/N; N) is as desired.  $\dagger_{\text{Claim}}$ 

LEMMA 21.4.10. If **K** is a quasiminimal AEC, then **K** is (Galois) stable in  $\aleph_0$ .

PROOF. By uniqueness of the generic type.

THEOREM 21.4.11. If **K** is a quasiminimal AEC, then **K** admits intersections and for any  $N \in \mathbf{K}$ ,  $(|N|, cl^N)$  is a pregeometry whose closed sets are exactly the **K**-substructures of N.

PROOF. That **K** admits intersection is Theorem 21.3.5. Now let  $N \in \mathbf{K}$  and let  $W := (|N|, \mathrm{cl}^N)$ . By Remark 21.3.2, W is a closure space and by Fact 21.3.3, its closed sets are exactly the **K**-substructures of N. By Fact 21.3.6,  $\kappa(W) = \aleph_0$ , i.e. W has finite character. It remains to see that W has exchange. Let  $a, b \in |N|$  and

let  $A \subseteq |N|$ . Assume that  $a \in cl^N(Ab) \setminus cl^N(A)$ . We want to see that  $b \in cl^N(Aa)$ . By finite character we can assume without loss of generality that  $|A| \leq \aleph_0$ . Using the Löwenheim-Skolem-Tarski axiom, we may also assume that  $N \in \mathbf{K}_{<\aleph_0}$ .

Using stability, let  $N' \in \mathbf{K}_{\leq \aleph_1}$  be such that  $N \leq_{\mathbf{K}} N'$  and N' is  $\aleph_1$ -saturated (this can be done even if there is a countable maximal model above N. In this case this will be the desired N'). Then  $W' := (|N'|, \operatorname{cl}^{N'})$  is a closure space with  $\kappa(W') = \aleph_0$  which (using uniqueness of the generic type) is  $\aleph_1$ -homogeneous. Therefore by Corollary 21.2.12, W' satisfies exchange. It follows immediately (see Fact 21.3.4) that W also satisfies exchange.

COROLLARY 21.4.12. If C satisfies all the axioms of a quasiminimal pregeometry class except that in I.(1)  $cl_H$  may not have exchange, then C is a quasiminimal pregeometry class.

PROOF. By Fact 21.4.8,  $\mathbf{K}(\mathcal{C})$  is a quasiminimal AEC. By Theorem 21.4.11,  $(|M|, \mathrm{cl}^M)$  is a pregeometry for every  $M \in \mathbf{K}$ . The result now follows from Remark 21.4.6.

To prove tameness, we will use:

FACT 21.4.13 (Corollary 23.4.13). If an AEC **K** is stable in  $\aleph_0$ , has amalgamation in  $\aleph_0$ , and has joint embedding in  $\aleph_0$ , then **K** is  $(<\aleph_0,\aleph_0)$ -tame for types of finite length. That is, if  $M \in \mathbf{K}_{\aleph_0}$  and  $p \neq q$  are both in  $gS^{<\omega}(M)$ , then there exists a finite  $A \subseteq |M|$  such that  $p \upharpoonright A \neq q \upharpoonright A$ .

LEMMA 21.4.14. If **K** is a quasiminimal AEC, then **K** is  $(\langle \aleph_0, \aleph_0 \rangle)$ -tame for types of finite length.

PROOF. This is a consequence of Fact 21.4.13 together with Lemmas 21.4.9 and 21.4.10.  $\hfill \Box$ 

We are now ready to state a correspondence between quasiminimal AECs and quasiminimal pregeometry classes. We will use the concept of a functorial expansion, which basically is an expansion of the vocabulary of the class that does not change anything about how the class behaves. Functorial expansions are defined in Definition 2.3.1.

DEFINITION 21.4.15. Let  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  be an abstract class. A functorial expansion of  $\mathbf{K}$  is a class  $\hat{K}$  satisfying the following properties:

- (1)  $\widehat{K}$  is a class of  $\widehat{\tau}$ -structures, where  $\widehat{\tau}$  is a fixed vocabulary extending  $\tau(\mathbf{K})$ .
- (2) The map  $\widehat{M} \mapsto \widehat{M} \upharpoonright \tau(\mathbf{K})$  is a bijection from  $\widehat{K}$  onto K. For  $M \in K$ , we will write  $\widehat{M}$  for the unique element of  $\widehat{K}$  whose reduct is M. When we write " $\widehat{M} \in \widehat{K}$ ", it is understood that  $M = \widehat{M} \upharpoonright \tau(K)$ .
- (3) Invariance: For  $M, N \in K$ , if  $f: M \cong N$ , then  $f: \widehat{M} \cong \widehat{N}$ .
- (4) Monotonicity: If  $M \leq_{\mathbf{K}} N$  are in K, then  $\widehat{M} \subseteq \widehat{N}$ .

REMARK 21.4.16 (Proposition 2.3.7). If  $\widehat{K}$  is a functorial expansion of  $\mathbf{K}$ , then we can order  $\widehat{K}$  by  $\widehat{M} \leq_{\widehat{K}} \widehat{N}$  if and only if  $M \leq_{\mathbf{K}} N$ . This gives an abstract class  $\widehat{K} := (\widehat{K}, \leq_{\widehat{K}}).$ 

The specific functorial expansion we will use is what we call the  $(<\aleph_0)$ -Galois-Morleyization (Definition 2.3.3. It consists in adding a relation for each Galois type over a "small" set (here small means finite, but in general for the  $(< \kappa)$ -Galois Morleyization small means "of size less than  $\kappa$ ").

DEFINITION 21.4.17. Let  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  be an AEC. Define an expansion  $\hat{\tau}$  of  $\tau(K)$  by adding a relation symbol  $R_p$  of arity  $\ell(p)$  for each  $p \in \mathrm{gS}^{<\omega}(\emptyset)$ . Expand each  $N \in K$  to a  $\hat{\tau}$ -structure  $\hat{N}$  by specifying that for each  $\bar{a} \in {}^{<\omega}|\hat{N}|, R_p^{\hat{N}}(\bar{a})$  (where  $R_p^{\hat{N}}$  is the interpretation of  $R_p$  inside  $\hat{N}$ ) holds exactly when  $\mathrm{gtp}(\bar{a}/\emptyset; N) = p$ . Let  $\hat{K}$  be the class of all such  $\hat{N}$ , ordered as in Remark 21.4.16. We call  $\hat{K}$  the  $(<\aleph_0)$ -Galois Morleyization of  $\mathbf{K}$ .

REMARK 21.4.18. Let **K** be an AEC and let  $\widehat{K}$  be the  $(\langle \aleph_0 \rangle)$ -Galois Morleyization of **K**. Then  $|\tau(\widehat{K})| = |gS^{\langle \omega}(\emptyset)| + |\tau(\mathbf{K})|$ .

The basic facts about the Galois Morleyization that we will use are below. The most important says that if Galois types are determined by their finite restrictions, then they will become equivalent to quantifier-free types in the Galois Morleyization.

FACT 21.4.19. Let **K** be an AEC and let  $\widehat{K} = (\widehat{K}, \leq_{\widehat{K}})$  be its  $(\langle \aleph_0 \rangle)$ -Galois Morleyization.

- (1) (Proposition 2.3.5)  $\hat{K}$  is a functorial expansion of **K**.
- (2) (Remark 2.3.4)  $\hat{K}$  is an AEC with  $LS(\hat{K}) = LS(\mathbf{K}) + |\tau(\hat{K})|$ .
- (3) (Theorem 2.3.15) If **K** is  $(<\aleph_0,\aleph_0)$ -tame for types of finite length, then for any  $M \in \widehat{K}_{\leq\aleph_0}$ , any two N, N' with  $M \leq_{\widehat{K}} N, M \leq_{\widehat{K}} N'$  and any  $\overline{a} \in {}^{<\omega}|N|, \overline{b} \in {}^{<\omega}|N'|$ , if the  $\tau(\widehat{K})$ -quantifier-free type of  $\overline{a}$  over M inside N is the same as the  $\tau(\widehat{K})$ -quantifier-free type of  $\overline{b}$  over M inside N', then  $\operatorname{gtp}(\overline{a}/M; N) = \operatorname{gtp}(\overline{b}/M; N')$ . This also holds if M is empty.

We have arrived to the definition of the correspondence between quasiminimal AECs and quasiminimal pregeometry classes, and the proof that it works:

DEFINITION 21.4.20. For **K**, a quasimiminal AEC let  $\mathcal{C}(\mathbf{K})$  be the class  $\{(M, \mathrm{cl}^M) \mid M \in \widehat{K}\}$ , where  $\widehat{K}$  is the  $(\langle \aleph_0 \rangle)$ -Galois Morleyization of **K**.

THEOREM 21.4.21. If **K** is a quasiminimal AEC, then  $\mathcal{C}(\mathbf{K})$  is a quasiminimal pregeometry class, which is unbounded if and only if **K** is. Moreover  $\mathbf{K}(\mathcal{C}(\mathbf{K}))$  is the  $(<\aleph_0)$ -Galois Morleyization of **K**.

PROOF. Let  $\mathcal{C} := \mathcal{C}(\mathbf{K})$ . It is clear that the elements of  $\mathcal{C}$  are of the right form. The moreover part is clear from the definition of  $\mathbf{K}(\mathcal{C}(\mathbf{K}))$ . We check all the conditions of Definition 21.4.5. We will use without comments that  $\mathbf{K}_{\leq\aleph_0}$  is has amalgamation and joint embedding (Lemma 21.4.9) and is stable in  $\aleph_0$  (Lemma 21.4.10).

0: (1) Since **K** has a prime model (axiom (2) in Definition 21.4.1),  $C \neq \emptyset$ . By Remark 21.4.18,  $|\tau(C)| = |\tau(\widehat{K})| \leq |gS^{<\omega}(\emptyset)| + |\tau(\mathbf{K})|$ . Since  $\mathrm{LS}(\mathbf{K}) = \aleph_0$ , we have that  $|\tau(\mathbf{K})| \leq \aleph_0$ . Using that  $\mathbf{K} \neq \emptyset$ , pick  $M \in \mathbf{K}_{\leq \aleph_0}$ . Since  $\mathbf{K}_{\leq \aleph_0}$  has amalgamation and joint embedding, there is an injection from  $gS^{<\omega}(\emptyset)$  into  $gS^{<\omega}(M)$ . By amalgamation and stability, there exists  $M' \in \mathbf{K}_{\aleph_0}$  universal over M. Therefore  $|gS^{<\omega}(M)| \leq \aleph_0$ . Thus  $|\tau(\widehat{K})| \leq \aleph_0$ , as desired.

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- (2) This is clear. In fact, if  $f: M \cong N$ , then by definition of  $\operatorname{cl}^M$  and  $\operatorname{cl}^N$ , f is automatically an isomorphism from  $(|M|, \operatorname{cl}^M)$  onto  $(|N|, \operatorname{cl}^N)$ .
- (3) Let  $(M, \mathrm{cl}^M), (N, \mathrm{cl}^N) \in \mathcal{C}$ . If  $M_0 \leq_{\widehat{K}} M$ , then  $M_0 \subseteq M$  so  $M_0$ and M satisfy the same quantifier-free sentences, so without loss of generality M and N are already countable. Now use that  $\mathbf{K}_{\leq\aleph_0}$  has joint embedding (by Lemma 21.4.9).
- I: (1) Let  $(M, cl^M) \in \mathcal{C}$ . By Theorem 21.4.11,  $(M, cl^M)$  is a pregeometry. Moreover if  $A \subseteq |M|$  is finite then  $|cl^M(A)| \leq LS(\mathbf{K}) = \aleph_0$ , as desired.
  - (2) Let  $(M, \text{cl}^M) \in \mathcal{C}$  and  $X \subseteq |M|$ . By definition of admitting intersections,  $\text{cl}^M(X) \leq_{\mathbf{K}} M$  and so the result follows.
  - (3) At that point it will be useful to prove a claim:

<u>Claim</u>: Let  $(H, cl_H), (H', cl_{H'}) \in C$ . Let  $G \subseteq H$  and  $G' \subseteq H'$  be countable closed subsets or empty and let  $g: G \to G'$  be an isomorphism. If  $g \cup f: H \to H'$  is a partial embedding where f has finite preimage X, then for any enumeration  $\bar{a}$  of X and  $\bar{G}$  of G,  $\operatorname{gtp}(\bar{a}\bar{G}/\emptyset; H) = \operatorname{gtp}(f(\bar{a})f(\bar{G})/\emptyset; H')$ .

<u>Proof of Claim</u>: By renaming, we may assume that G = G' so what we really have to prove is that  $gtp(\bar{a}/G; H) = gtp(f(\bar{a})/G; H')$ . Note that by assumption  $\bar{a}$  and  $f(\bar{a})$  satisfy the same  $\tau(\hat{K})$ -quantifierfree types over G. Therefore the result follows from Fact 21.4.19.(3) (recalling Lemma 21.4.14).  $\dagger_{\text{Claim 1}}$ 

(recalling Lemma 21.4.14).  $\dagger_{\text{Claim 1}}$ Now let  $(M, \text{cl}^M), (M', \text{cl}^{M'}) \in \mathcal{C}, X \subseteq |M|, y \in \text{cl}^M(X)$ , and  $f: M \to M'$  be a partial embedding with  $X \cup \{y\} \subseteq \text{preim}(f)$ . We want to see that  $f(y) \in \text{cl}^{M'}(f[X])$ . By finite character, we may assume without loss of generality that X is finite and therefore preim(f) is also finite. Let  $\bar{a}$  be an enumeration of X. By the Claim,  $\text{gtp}(\bar{a}y/\emptyset; M) = \text{gtp}(f(\bar{a})f(y)/\emptyset; M')$ . The result now follows from the definition of the closure operator.

- II: Let  $(H, cl_H), (H', cl_{H'}) \in \mathcal{C}$ . Let  $G \subseteq H$  and  $G' \subseteq H'$  be countable closed subsets or empty and let  $g: G \to G'$  be an isomorphism.
  - (1) Let  $x \in |H|$  and  $x' \in |H'|$  be independent from G and G' respectively. We show that  $g \cup \{(x, x')\}$  is a partial embedding. By renaming without loss of generality G = G'. By uniqueness of the generic type, gtp(x/G; H) = gtp(x'/G; H'). The result follows.
  - (2) Let  $g \cup f : H \to H'$  be a partial embedding, where f has finite preimage X, and  $y \in cl_H(X \cup G)$ . We want to find  $y' \in H'$  such that  $g \cup f \cup \{(y, y')\}$  is a partial embedding. Without loss of generality again, g is the identity. Let  $\bar{a}$  be an enumeration of X. By the Claim,  $gtp(\bar{a}/G; H) = gtp(f(\bar{a})/G; H')$ . By Fact 21.3.7, there exists  $h: cl^H(G\bar{a}) \cong_G cl^{H'}(G\bar{a}')$  such that  $h(\bar{a}) = f(\bar{a})$ . Let y' := h(y).
- IV: Assume that **K** is unbounded.
  - (1) Because **K** is an AEC and the closure operator has finite character.
  - (2) Since  $\mathbf{K}$  is unbounded.

As a corollary, all the work on structural properties of quasiminimal pregeometry classes automatically applies also to quasiminimal AECs: COROLLARY 21.4.22. Let  $\mathbf{K}$  be a quasiminimal AEC.

- (1) Let  $M, N \in \mathbf{K}$  and let  $B_M, B_N$  be bases for  $(|M|, \mathrm{cl}^M)$  and  $(|N|, \mathrm{cl}^N)$  respectively. If f is a bijection from  $B_M$  onto  $B_N$ , then there exists an isomorphism  $g: M \cong N$  with  $f \subseteq g$ .
- (2) If  $\mathbf{K}$  is unbounded, then:
  - (a) **K** has no maximal models.
  - (b) **K** has exactly  $\aleph_0$  non-isomorphic countable models and **K** is categorical in every uncountable cardinal.

PROOF. Let  $C := C(\mathbf{K})$ . By Theorem 21.4.21, C is a quasiminimal pregeometry class. By  $[\mathbf{BHH^+14}]$ , it also satisfies the excellence axiom. By Zilber's main result on these classes  $[\mathbf{Zil05a}]$  (or see  $[\mathbf{Kir10}, \text{Theorem 3.3}]$  for an exposition), (1) holds for C. Therefore it also holds for  $\mathbf{K}(C)$ , which is a just a functorial expansion of  $\mathbf{K}$ . Hence it also holds for  $\mathbf{K}$ . Similarly, (2) holds in unbounded quasimininal AECs (see  $[\mathbf{Kir10}, \S4]$ ).

#### 21.5. On a counterexample of Shelah

We give a (non-quasiminimal) example, due to Shelah, where the exchange property fails. We show that, in this example, there is a good frame which cannot be extended to be type-full, answering a question of Boney and the author (Question 18.1.4). We assume familiarity with good frames in this section, see [She09a, Chapter II]; we use the definition from [JS13, Definition 2.1.1].

The following definitions come from [She09b, Exercise VII.5.7]:

Definition 21.5.1.

- (1) Let  $\tau^*$  be the vocabulary consisting of only a single a unary function symbol F.
- (2) Let  $\psi_0$  be the following first-order  $\tau^*$ -sentence:

$$\forall x : F(F(x)) = F(x)$$

(3) Let  $\psi$  be the following first-order  $\tau^*$ -sentence:

$$\psi_0 \land \forall x \forall y : F(x) \neq x \land F(y) \neq y \to F(x) = F(y)$$

- (4) Let  $\phi(x)$  be the sentence  $\exists y : F(y) = x \land y \neq x$ .
- (5) Let  $K^*$  be the class of  $\tau^*$ -structures that satisfy  $\psi$ .
- (6) Say  $M \leq_{\mathbf{K}^*} N$  if  $M, N \in K^*$  and  $M \subseteq N$ .
- (7) Let  $\mathbf{K}^* := (K^*, \leq_{\mathbf{K}^*}).$

REMARK 21.5.2. If  $M \in \mathbf{K}^*$ , then  $|\phi(M)| \leq 1$ .

Recall (see [She87b]) that a class K of structures in a fixed vocabulary is a *universal class* if it is closed under isomorphisms, substructures, and unions of  $\subseteq$ -increasing chains. It is straightforward to check that  $K^*$  is a universal class, and it induces the AEC  $\mathbf{K}^*$  with Löwenheim-Skolem-Tarski number  $\aleph_0$ . Thus it admits intersections. Moreover:

FACT 21.5.3 ((\*)<sub>2</sub> in VII.5.7 of [She09b]).  $K^*$  has amalgamation.

LEMMA 21.5.4. For any  $M \in \mathbf{K}^*$ ,  $2 \le |gS^{na}(M)| \le 3$ .

PROOF. For  $M \leq_{\mathbf{K}^*} N$ , there are three kinds of types realized in N (not necessarily all non-algebraic):

(1) The type of  $a \in \phi(N)$ .

(2) The type of  $a \notin \phi(N)$  with  $F^N(a) \neq a$ .

(3) The type of  $a \notin \phi(N)$  with  $F^N(a) = a$ .

Taking a suitable N, an instance of the last two can be found that is nonalgebraic.  $\hfill \Box$ 

FACT 21.5.5 ((\*)<sub>8</sub> in VII.5.7 of [She09b]).  $\mathbf{K}^*$  fails disjoint amalgamation (in any infinite cardinal).

PROOF. Take  $M_0 \leq_{\mathbf{K}^*} M_1 = M_2$  with  $\phi(M_0) = \emptyset$ ,  $\phi(M_\ell) = \{b\}$ ,  $\ell = 1, 2$ . Then it is clear that  $M_1$  and  $M_2$  cannot be disjointly amalgamated over  $M_0$  (as any disjoint amalgam N would have to satisfy  $|\phi(N)| \geq 2$ ).

LEMMA 21.5.6. For any  $N \in \mathbf{K}^*$  with  $|\phi(N)| = 1$ ,  $(|N|, \mathrm{cl}^N)$  does not have exchange over  $\emptyset$ .

PROOF. First note that  $\operatorname{cl}^N(\emptyset) = \emptyset$ . Now pick  $b \in \phi(N)$  and let  $a \neq b$  be such that  $F^N(a) = b$ . Then  $b \in \operatorname{cl}^N(a)$ . However  $F^N(b) = F^N(F^N(a)) = F^N(a) = b$ , so  $a \notin \operatorname{cl}^N(b)$ .

For the rest of this section, we fix an infinite cardinal  $\lambda \geq \aleph_0$ . We define a good  $\lambda$ -frame  $\mathfrak{s}$  (whose definition appears already in [She09b, VII.5.7]) and an object  $\mathfrak{s}'$  satisfying all the axioms of good frames except existence. It turns out that  $\mathfrak{s}'$  would be the only type-full extension of  $\mathfrak{s}$ , so  $\mathfrak{s}$  cannot be extended to be type-full.

DEFINITION 21.5.7. Define pre- $\lambda$ -frames (see [She09a, Definition III.0.2])  $\mathfrak{s}, \mathfrak{s}'$  as follows:

- (1) The underlying class of  $\mathfrak{s}$  and  $\mathfrak{s}'$  is  $(\mathbf{K}^*)_{\lambda}$ .
- (2) The basic types of  $\mathfrak{s}'$  are all the nonalgebraic types. The basic types of  $\mathfrak{s}$  are all the nonalgebraic types of the form gtp(a/M; N) with  $a \notin \phi(N)$ .
- (3) In both frames, gtp(a/M; N) does not fork over  $M_0$  if and only if  $F^N(a) \notin |N| \setminus |M|$ .

Theorem 21.5.8.

- (1)  $\mathfrak{s}'$  is type-full but  $\mathfrak{s}$  is not.
- (2) The frames satisfy all the properties of good frames except perhaps existence.
- (3)  $\mathfrak{s}$  has existence. Therefore it is a good  $\lambda$ -frame.
- (4)  $\mathfrak{s}'$  fails existence.
- (5)  $\mathfrak{s}$  cannot be extended to be type-full.

Proof.

- (1) Clear from the definitions.
- (2) Straightforward. For example:
  - Density of basic types in  $\mathfrak{s}$ : let  $M <_{\mathbf{K}^*} N$ . We may assume that  $|\phi(M)| = 0$ ,  $|\phi(N)| = 1$ , so let  $b \in \phi(N)$ . Fix  $a \neq b$  such that  $F^N(a) = b$ . Now since  $b \notin |M|$ ,  $a \notin |M|$ , so gtp(a/M; N) is basic.
  - Uniqueness in  $\mathfrak{s}'$ : Use the description of the nonalgebraic types in the proof of Lemma 21.5.4.
  - Symmetry: Suppose that (in one of the two frames) we have  $a \downarrow_{M_0}^N M$ .

Let  $b \in |M|$  be such that  $gtp(b/M_0; M)$  is basic. Let  $M_a := cl^N(M_0a)$ .

Note that  $|M_a| = |M_0| \cup \{a, F^N(a)\}$ . We claim that  $b \bigcup_{M_0}^N M_a$ . Note that since  $\operatorname{gtp}(b/M_0; N)$  is basic, then also  $\operatorname{gtp}(b/M_a; N)$  is basic (check it for each frame). It remains to see that  $F^N(b) \notin |M_a| \setminus |M_0|$ . First note that as  $a \notin |M|$ ,  $F^N(b) = F^M(b) \neq a$ . Further, if  $F^N(a) \notin |M_0|$ , then  $F^N(a) \notin |M|$ , so  $F^N(a) \neq F^N(b) = F^M(b)$ , as desired.

- (3) Straightforward.
- (4) As in the proof of Fact 21.5.5.
- (5) Any type-full extension of  $\mathfrak{s}$  would have to be  $\mathfrak{s}'$  and we have shown that  $\mathfrak{s}'$  fails existence.

The following question seems much harder:

QUESTION 21.5.9. Is there an example of a good- $\lambda$ -frame which is categorical in  $\lambda$  and cannot be extended to be type-full?

Note that by Theorem 23.4.19, if  $\mathbf{K}_{\leq\aleph_0}$  is categorical, has amalgamation, joint embedding, no maximal models, and is stable in  $\aleph_0$ , then it has a type-full good  $\aleph_0$ -frame. Therefore by canonicity of categorical good frames (Theorem 6.9.7, the answer is negative when  $\lambda = \aleph_0$ .

## CHAPTER 22

# On the uniqueness property of forking in abstract elementary classes

This chapter is based on [Vasc].

## Abstract

In the setup of abstract elementary classes satisfying a local version of superstability, we prove the uniqueness property for a certain independence notion arising from splitting. This had been a longstanding technical difficulty when constructing forking-like notions in this setup. As an application, we show that the two versions of forking symmetry appearing in the literature (the one defined by Shelah for good frames and the one defined by VanDieren for splitting) are equivalent.

## 22.1. Introduction

In the study of classification theory for abstract elementary classes (AECs), the question of when a forking-like notion exists is central. The present chapter deals with this question.

To state our result more precisely, we first recall that there is a semantic notion of type in AECs: for the rest of this introduction we fix an AEC **K** with amalgamation, joint embedding, and arbitrarily large models. This allows us to fix a big universal model-homogeneous<sup>1</sup> monster model  $\mathfrak{C}$  and work inside it. For  $M \leq_{\mathbf{K}} \mathfrak{C}$ and  $a \in \mathfrak{C}$ , let gtp(a/M) (the Galois, or orbital, type of a over M) be the orbit of a under the automorphisms of  $\mathfrak{C}$  fixing M (Galois types can be defined without any assumptions on **K**, but then the definition becomes more technical). Write gS(M)for the set of all Galois types over M. The definitions of stability and saturation are as expected. Two important results of Shelah are:

- (1) [She09a, II.1.14] If M is saturated, then M is model-homogeneous.
- (2) [She09a, II.1.16] If **K** is stable in  $\mu$  and  $M \in \mathbf{K}_{\mu}$ , then there exists  $N \in \mathbf{K}_{\mu}$  universal over M.

To motivate the main result of this chapter, let us first consider the following consequence:

COROLLARY 22.1.1. Let **K** be an AEC with amalgamation, joint embedding, and arbitrarily large models. Let  $LS(\mathbf{K}) < \mu < \lambda$  be given. If **K** is categorical in  $\lambda$ , then there is a relation "p does not  $\mu$ -fork over M" defined for  $M \leq_{\mathbf{K}} N$  both saturated models in  $\mathbf{K}_{\mu}$  and  $p \in gS(N)$  satisfying:

(1) The usual invariance and monotonicity properties.

<sup>&</sup>lt;sup>1</sup>*M* is model-homogeneous if whenever  $M_0 \leq_{\mathbf{K}} N_0$  are such that  $M_0 \leq_{\mathbf{K}} M$  and  $||N_0|| < ||M||$ , then  $N_0$  embeds inside *M* over  $M_0$ .

- (2) Existence-extension: for  $M \leq_{\mathbf{K}} N$  both saturated in  $\mathbf{K}_{\mu}$ , any  $p \in gS(M)$  has a  $\mu$ -nonforking extension to gS(N).
- (3) Uniqueness<sup>2</sup>: for  $M \leq_{\mathbf{K}} N$  both saturated in  $\mathbf{K}_{\mu}$ , if  $p, q \in \mathrm{gS}(N)$  do not  $\mu$ -fork over M and  $p \upharpoonright M = q \upharpoonright M$ , then p = q.
- (4) Symmetry: for M saturated in  $\mathbf{K}_{\mu}$  and  $a, b \in \mathfrak{C}$ , the following are equivalent:
  - (a) There exists  $M_a$  saturated in  $\mathbf{K}_{\mu}$  containing a such that  $M \leq_{\mathbf{K}} M_a$ and  $\operatorname{tp}(b/M_a)$  does not  $\mu$ -fork over M.
  - (b) There exists  $M_b$  saturated in  $\mathbf{K}_{\mu}$  containing b such that  $M \leq_{\mathbf{K}} M_b$ and  $\operatorname{tp}(a/M_b)$  does not  $\mu$ -fork over M.
- (5) Local character for universal chains: if  $\delta < \mu^+$  is a limit ordinal,  $\langle M_i : i \leq \delta \rangle$  is an increasing continuous sequence of saturated models in  $\mathbf{K}_{\mu}$  with  $M_{i+1}$  universal over  $M_i$  for all  $i < \delta$ , then for any  $p \in \mathrm{gS}(M_{\delta})$  there exists  $i < \delta$  such that p does not  $\mu$ -fork over  $M_i$ .

We give a proof at the end of this introduction. Several remarks are in order.

First remark: we work only over models of a fixed cardinality, so we deal with a (potentially) different nonforking relation for each cardinal  $\mu$ . Note in particular that the uniqueness property is for types over models of the same size, so there are no obvious relationships between  $\mu_0$ -forking and  $\mu_1$ -forking (for LS(**K**)  $< \mu_0 < \mu_1 < \lambda$ ).

Second remark: we work only over *saturated* models. We do not know how to generalize our result to *all* models of cardinality  $\mu$ . It is worth mentioning that in the setup of Corollary 22.1.1 the  $\mu$ -saturated models are closed under unions (Corollary 17.5.7.(3). In fact they form an AEC with Löwenheim-Skolem-Tarski number  $\mu$ .

Third remark: it is known (using an argument of Morley, see [She99, I.1.7.(a)]) that in the setup of Corollary 22.1.1, **K** is stable in  $\mu$ . Moreover (5) can be seen as a version of superstability: it is a replacement for "every type does not fork over a finite set". In fact (5) is equivalent to superstability if **K** is first-order axiomatizable (see Chapter 9).

Fourth remark: if we strengthen condition (5) to:

(5+) Local character: if  $\delta < \mu^+$  is a limit ordinal,  $\langle M_i : i \leq \delta \rangle$  is an increasing continuous sequence of saturated models in  $\mathbf{K}_{\mu}$ , then for any  $p \in \mathrm{gS}(M_{\delta})$  there exists  $i < \delta$  such that p does not  $\mu$ -fork over  $M_i$ .

(note the difference with (5): we do *not* require that  $M_{i+1}$  be universal over  $M_i$ ) then we have arrived to Shelah's definition of a (type-full) good  $\mu$ -frame [**She09a**, Definition II.2.1]. Good frames are the main concept in Shelah's books [**She09a**, **She09b**] on classification theory for AECs. They have several applications, including the author's proof of the eventual categoricity conjecture for universal classes (Chapters 8 and 16). Thus the existence question for them is important.

Fifth remark: if we add to the assumptions of Corollary 22.1.1 that Galois types over saturated models of size  $\mu$  are determined by their restrictions to model of size  $\chi$ , for some  $\chi < \mu$  (this is called weak tameness in the literature), then the conclusion is known (see Theorem 10.6.4 and Corollary 17.5.7.(1)) and one can strengthen (5) to (5+), i.e. one gets a good  $\mu$ -frame. It is known how to

<sup>&</sup>lt;sup>2</sup>This can also be described as "types over saturated models are stationary".

derive eventual weak tameness from categoricity in a high-enough cardinal, thus the conclusion also holds if  $\mu$  is "high-enough" ( $\mu \geq \beth_{(2^{\text{LS}(\mathbf{K})})^+}$  suffices), see Corollary 17.5.7.(5). However we are interested in arbitrary, potentially small,  $\mu$ . In this case the conclusion of Corollary 22.1.1 is new.

Sixth remark: we actually prove a more local statement than Corollary 22.1.1: let us take a step back and explain how Corollary 22.1.1 is proven. As is customary, we first study an independence notion called  $\mu$ -splitting [**She99**, 3.2]: For  $M \leq_{\mathbf{K}} N$ both in  $\mathbf{K}_{\mu}$ ,  $p \in \mathrm{gS}(N)$   $\mu$ -splits over M if there exists  $N_1, N_2 \in \mathbf{K}_{\mu}$  with  $M \leq_{\mathbf{K}} N_{\ell} \leq_{\mathbf{K}} N$  for  $\ell = 1, 2$  and  $f : N_1 \cong_M N_2$  such that  $f(p \upharpoonright N_1) \neq p \upharpoonright N_2$ . In the context of Corollary 22.1.1, Shelah and Villaveces (see Fact 22.2.2) have shown that  $\mu$ -splitting satisfies (5).  $\mu$ -splitting also satisfies weak analogs of uniqueness and extension (see Fact 22.2.5).

The weak uniqueness statement is the following: if  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$  are all in  $\mathbf{K}_{\mu}$ , M is universal over  $M_0$ ,  $p, q \in \mathrm{gS}(N)$  both do not  $\mu$ -split over  $M_0$  and  $p \upharpoonright M = q \upharpoonright M$ , then p = q. Thus it is natural to define forking by "shifting" splitting by a universal model (this is already implicit in [She99] but is defined explicitly for the first time in Definition 4.3.8). Let us say that  $p \in \mathrm{gS}(N)$  does not  $\mu$ -fork over M if there exists  $M_0 \leq_{\mathbf{K}} M$  such that M is universal over  $M_0$ and p does not  $\mu$ -split over  $M_0$  (see Definition 22.2.4; it can be shown that any reasonable forking-like notion must be  $\mu$ -forking over saturated models (Theorem 6.9.7). In the setup of Corollary 22.1.1, it was known that  $\mu$ -forking satisfies all the conditions there except (3) (for symmetry, this is a recent result of the author, see Corollary 17.5.7.(1), relying on joint work with VanDieren (Chapter 10).

Let us describe the problem in proving uniqueness: let  $M \leq_{\mathbf{K}} N$  both be saturated in  $\mathbf{K}_{\mu}$  and  $p, q \in gS(N)$  be not  $\mu$ -forking over M with  $p \upharpoonright M = q \upharpoonright M$ . Thus we have witnesses  $M_p, M_q$  such that M is universal over both  $M_p$  and  $M_q$ , p does not  $\mu$ -split over  $M_p$  and q does not  $\mu$ -split over  $M_q$ . If we knew that  $M_p$ and  $M_q$  were the same (or at least had a common extension over which M is still universal), then we could use the weak uniqueness described in the previous paragraph. However we do not know how the witnesses fit together, so we are stuck. This causes several technical difficulties, forcing for example the witnesses to be carried over in the study of towers in [SV99, Van06, GVV16, Van16a] and Chapter 10. In this chapter, we prove the uniqueness property (this implies for example that the equivalence relation  $\approx$  defined in [SV99, Definition 3.2.1] is just equality).

More precisely, let us say that an AEC **K** is  $\mu$ -superstable if  $\mathbf{K}_{\mu}$  is nonempty, has amalgamation, joint embedding, no maximal models, is stable in  $\mu$ , and  $\mu$ -splitting satisfies (5) (see Definition 22.2.1). The main result of this chapter is:

THEOREM 22.2.16. If **K** is  $\mu$ -superstable, then  $\mu$ -forking has the uniqueness property over limit models in  $\mathbf{K}_{\mu}$ .

Recall that M is *limit* if it is the union of an increasing continuous chain in  $\mathbf{K}_{\mu}$  of the form  $\langle M_i : i \leq \delta \rangle$ ,  $\delta < \mu^+$  limit and  $M_{i+1}$  universal over  $M_i$  for all  $i < \delta$ . Limit models are a replacement for saturated models in a local context where we only know information about models of a single cardinality (see [**GVV16**] for an introduction to the theory of limit models). The proof of Theorem 22.2.16 proceeds by contradiction: if uniqueness fails, then we can build a tree of failures and this contradicts stability.

With Theorem 22.2.16 stated, we can now give a full proof of Corollary 22.1.1:

PROOF OF COROLLARY 22.1.1. By Fact 22.2.2, **K** is  $\mu$ -superstable. By Corollary 17.5.7, saturated models in  $\mathbf{K}_{\mu}$  are the same as limit models. Therefore Theorem 22.2.16 applies. We have that  $\mu$ -forking (from Definition 22.2.4) satisfies (5). By Fact 22.2.5, it also satisfies (2) and it is clear that it satisfies (1). By Theorem 22.2.16, it satisfies (2). Finally, by Corollary 17.5.7.(1) it satisfies (4).

As an application of Theorem 22.2.16, we can show that the symmetry property for splitting introduced by VanDieren in [Van16a] (which in essence is a symmetry property for  $\mu$ -forking with certain uniformity requirements on the witnesses) is the same as the symmetry property given in the statement of Corollary 22.1.1: see Corollary 22.2.18. Thus the "hierarchy of symmetry properties" described in Section 10.4 collapses: all the properties there are equivalent. We do not know whether symmetry follows from  $\mu$ -superstability. We also do not know whether in Theorem 22.2.16 we can assume only stability in  $\mu$  (and amalgamation, etc.) rather than superstability.

Another open problem would be to study the properties of the weak kind of good frames derived in Corollary 22.1.1. They are called H-almost good frames by Shelah (see [She09b, VII.5.9] and [Shee]). There has been some work on *almost* (not H-almost) good frames (see [She09b, VII.5], [JS]), where in addition to (5) a continuity property is required for all chains (i.e. given an increasing union of types where all the elements do not fork over a common model, the union of the chain does not fork over this model). In particular, conditions are given under which almost good frames are good frames. It would be interesting to know whether similar statements hold for H-almost good frames.

## 22.2. The main theorem

For the rest of this chapter, we assume that the reader has some basic familiarity with AECs ([**Bal09**, Chapters 4-12] should be more than enough). We work inside a fixed AEC  $\mathbf{K}$ .

The following definition is implicit already in [She99] and is studied in several papers including [SV99, Van06, GVV16, Van16a], and Chapter 10. It is given the name superstability for the first time in [Gro02, 7.12].

DEFINITION 22.2.1. **K** is  $\mu$ -superstable if:

- (1)  $\mu \geq \text{LS}(\mathbf{K})$  and  $\mathbf{K}_{\mu} \neq \emptyset$ .
- (2)  $\mathbf{K}_{\mu}$  has amalgamation, joint embedding, and no maximal models.
- (3) **K** is stable in  $\mu$ .
- (4) **K** has no long  $\mu$ -splitting chains: for any limit ordinal  $\delta < \mu^+$ , any increasing continuous chain  $\langle M_i : i \leq \delta \rangle$  with  $M_{i+1}$  universal over  $M_i \in \mathbf{K}_{\mu}$  for all  $i < \delta$ , and any  $p \in \mathrm{gS}(M_{\delta})$ , there exists  $i < \delta$  such that p does not  $\mu$ -split over  $M_i$ .

A justification for this rather technical definition is the fact that it follows from categoricity. This is proven (with slightly different hypotheses) in [SV99, 2.2.1]. For an exposition and complete proof, see Chapter 20.
FACT 22.2.2. Assume that **K** has amalgamation and no maximal models. Let  $LS(\mathbf{K}) \leq \mu < \lambda$ . If **K** is categorical in  $\lambda$ , then **K** is  $\mu$ -superstable.

From now on, we assume that **K** is  $\mu$ -superstable (we will repeat this hypothesis at the beginning of important statements). We fix a "monster model"  $\mathfrak{C} \in \mathbf{K}_{\mu^+}$  that is universal and model-homogeneous and work inside it.

REMARK 22.2.3. We could work in the more general setup of [SV99] (with only density of amalgamation bases, existence of universal extensions, limit models being amalgamation bases, and no long splitting chains), but we prefer to avoid technicalities.

The following is the main object of study of this chapter:

DEFINITION 22.2.4 (Definition 4.3.8). For  $M \leq_{\mathbf{K}} N$  both in  $\mathbf{K}_{\mu}$ ,  $p \in gS(N)$ does not  $\mu$ -fork over  $(M_0, M)$  if M is universal over  $M_0$  and p does not  $\mu$ -split over  $M_0$ . We say that p does not  $\mu$ -fork over M if it does not  $\mu$ -fork over  $(M_0, M)$  for some  $M_0$ .

Since  $\mu$  is always clear from context, we will omit it: we will say "p does not fork" and "p does not split" instead of "p does not  $\mu$ -fork" and "p does not  $\mu$ -split".

It is clear that forking has the basic invariance and monotonicity properties (see Lemma 4.3.9). The following are implicit in [She99] and stated explicitly in [Van06, I.4.10, I.4.12]. We will use them without much comments.

FACT 22.2.5. Let  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} N'$  all be in  $\mathbf{K}_{\mu}$ .

- (1) Extension: If  $p \in gS(N)$  does not fork over  $(M_0, M)$ , then there exists an extension  $q \in gS(N')$  of p that does not fork over  $(M_0, M)$ .
- (2) Weak uniqueness: If  $p, q \in gS(N)$  do not fork over  $(M_0, M)$  and  $p \upharpoonright M = q \upharpoonright M$ , then p = q.

We now state a weak version of the conjugation property that types enjoy in good frames [She09a, III.1.21]. This will be key in the proof of the main theorem.

DEFINITION 22.2.6. Let  $M, M' \in \mathbf{K}_{\mu}$ ,  $p \in gS(M)$ ,  $p' \in gS(M')$ . Let  $A \subseteq |M| \cap |M'|$ . We say that p and p' are *conjugate over* A if there exists  $f : M \cong_A M'$  such that p' = f(p). When  $A = \emptyset$ , we omit it.

FACT 22.2.7 (Conjugation property). Let  $\delta < \mu^+$  be a limit ordinal. Let  $M_0, M, N \in \mathbf{K}_{\mu}$ , with  $M_0 \leq_{\mathbf{K}} M \leq_{\mathbf{K}} N$ . Assume that M is  $(\mu, \delta)$ -limit over  $M_0$  and N is  $(\mu, \delta)$ -limit over M. If  $p \in \mathrm{gS}(N)$  does not fork over  $(M_0, M)$ , then p and  $p \upharpoonright M$  are conjugate over  $M_0$ .

PROOF. Since M is limit over  $M_0$ , there exists  $M_1 \in \mathbf{K}_{\mu}$  such that  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$ ,  $M_1$  is universal over  $M_0$ , and M is  $(\mu, \delta)$ -limit over  $M_1$ . Note that then also N is  $(\mu, \delta)$ -limit over  $M_1$ . Using uniqueness of limit models of the same length, pick  $f: N \cong_{M_1} M$ . Let q := f(p). We claim that  $q = p \upharpoonright M$ . Note that by invariance q does not fork over  $(M_0, f[M])$ , hence (by monotonicity) over  $(M_0, M_1)$ . By assumption and monotonicity, also  $p \upharpoonright M$  does not fork over  $(M_0, M_1)$ . Since f fixes  $M_1, p \upharpoonright M_1 = q \upharpoonright M_1$ , so using weak uniqueness  $q = p \upharpoonright M$ , as desired.  $\Box$ 

REMARK 22.2.8. We do *not* know here that limit models of different lengths are isomorphic.

The following result says that certain chains of types have least upper bounds. It is an easy use of extension, uniqueness, and local character.

FACT 22.2.9. Assume that **K** is  $\mu$ -superstable. Let  $\delta < \mu^+$  be a limit ordinal and let  $\langle M_i : i \leq \delta \rangle$  be increasing continuous in  $\mathbf{K}_{\mu}$  with  $M_{i+1}$  universal over  $M_i$ for all  $i < \delta$ . Suppose we are given an increasing chain of types  $\langle p_i : i < \delta \rangle$  such that  $p_i \in gS(M_i)$  for all  $i < \delta$ . Then there exists a unique  $p_{\delta} \in gS(M_{\delta})$  such that  $p_{\delta} \upharpoonright M_i = p_i$  for all  $i < \delta$ .

PROOF. Without loss of generality,  $\delta$  is regular. If  $\delta = \omega$ , the conclusion is given by a straightforward direct limit argument [**Bal09**, 11.1], so assume that  $\delta > \omega$ . Using no long splitting chains, for each limit  $i < \delta$  there exists  $j_i < i$ such that  $p_i$  does not split over  $M_{j_i}$ . By Fodor's lemma, there exists a stationary  $S \subseteq \delta$  and a  $j < \delta$  such that  $p_i$  does not split over  $M_j$  for all  $i \in S$ . Since S is unbounded and the  $p_i$ 's are increasing,  $p_i$  does not split over  $M_j$  for all  $i \in [j, \delta)$ . Let  $q \in gS(M_{\delta})$  be an extension of  $p_{j+1}$  that does not split over  $M_j$ . By weak uniqueness,  $q \upharpoonright M_i = p_i$  for all  $i \in [j+1, \delta)$ . This proves existence and uniqueness is similar: any  $q' \in gS(M_{\delta})$  extending all the  $p_i$ 's must be nonsplitting over  $M_j$ , so use weak uniqueness.

Recall that our goal is to prove uniqueness of nonforking extension. To this end, we define a type to be *bad* if it witnesses a failure of uniqueness. We then close this definition under nonforking extensions.

DEFINITION 22.2.10. Let  $M \in \mathbf{K}_{\mu}$  be limit. We define by induction on  $n < \omega$  what it means for a type  $p \in gS(M)$  to be *n*-bad:

- (1) p is 0-bad if there exists a limit model  $N \in \mathbf{K}_{\mu}$  with  $M \leq_{\mathbf{K}} N$  and  $q_1, q_2 \in \mathrm{gS}(N)$  such that:
  - (a) Both  $q_1$  and  $q_2$  extend p.
  - (b)  $q_1 \neq q_2$ .
  - (c) Both  $q_1$  and  $q_2$  do not fork over M.
- (2) For  $n < \omega$ , p is (n+1)-bad if there exists a limit model  $M_0 \in \mathbf{K}_{\mu}$  with  $M_0 \leq_{\mathbf{K}} M$  such that  $p \upharpoonright M_0$  is *n*-bad and *p* does not fork over  $M_0$ .
- (3) p is bad if p is n-bad for some  $n < \omega$ .

The following is an easy consequence of the definition (in fact the definition is tailored exactly to make this work):

REMARK 22.2.11. Let  $M \leq_{\mathbf{K}} N$  both be limit in  $\mathbf{K}_{\mu}$ . If  $p \in \mathrm{gS}(N)$  does not fork over M and  $p \upharpoonright M$  is bad, then p is bad.

We now proceed to develop some the theory of bad types. In the end, we will conclude that this contradicts stability in  $\mu$ , hence there cannot be any bad types. The next two lemmas are crucial: bad types are closed under unions of universal chains, and any bad type has two distinct bad extensions.

LEMMA 22.2.12. Assume that **K** is  $\mu$ -superstable. Let  $\delta < \mu^+$  be a limit ordinal. Let  $\langle M_i : i \leq \delta \rangle$  be an increasing continuous chain of limit models in  $\mathbf{K}_{\mu}$ with  $M_{i+1}$  limit over  $M_i$  for all  $i < \delta$ . Let  $\langle p_i : i \leq \delta \rangle$  be an increasing chain of types, with  $p_i \in gS(M_i)$  for all  $i < \delta$ . If  $p_i$  is bad for all  $i < \delta$ , then  $p_{\delta}$  is bad.

PROOF. Since there are no long splitting chains, there exists  $i < \delta$  such that  $p_{\delta}$  does not fork over  $M_i$ . By assumption,  $p \upharpoonright M_i$  is bad, so by Remark 22.2.11  $p_{\delta}$  is also bad, as desired.

LEMMA 22.2.13. Assume that **K** is  $\mu$ -superstable. Let  $M \in \mathbf{K}_{\mu}$  be a limit model. If  $p \in gS(M)$  is bad, then there exists a limit model N in  $\mathbf{K}_{\mu}$  with  $M \leq_{\mathbf{K}} N$  and  $q_1, q_2 \in gS(N)$  such that:

(1) Both  $q_1$  and  $q_2$  extend p.

(2)  $q_1 \neq q_2$ .

(3) Both  $q_1$  and  $q_2$  are bad.

PROOF. By definition, p is n-bad for some  $n < \omega$ . We proceed by induction on n.

- If n = 0, this is the definition of being 0-bad (note that  $q_1$  and  $q_2$  from Definition 22.2.10 are bad because they are nonforking extensions of the bad type p, see Remark 22.2.11)
- If n = m + 1, let  $M_0 \in \mathbf{K}_{\mu}$  be a limit model such that  $M_0 \leq_{\mathbf{K}} M$ , p does not fork over  $M_0$ , and  $p \upharpoonright M_0$  is m-bad. Pick  $M'_0$  such that p does not fork over  $(M'_0, M_0)$ . Let  $M'_1$  be  $(\mu, \omega)$ -limit over  $M'_0$  with  $M'_1 \leq_{\mathbf{K}} M_0$ . By monotonicity, p does not fork over  $(M'_0, M'_1)$ . Let  $M^*$  be  $(\mu, \omega)$ -limit over M (hence over  $M'_1$ ). Let  $q \in \mathrm{gS}(M^*)$  be an extension of p that does not fork over  $(M'_0, M)$ , hence over  $(M'_0, M'_1)$ . By Fact 22.2.7, q and  $p \upharpoonright M'_1$  are conjugate. Now by the induction hypothesis, there exists a limit model  $N^*$  extending  $M_0$  and two distinct bad extensions of  $p \upharpoonright M_0$  to  $N^*$ . These are also extensions of  $p \upharpoonright M'_1$ , so the result follows from the fact that qand  $p \upharpoonright M'_1$  are conjugate.

The following nominally stronger version of Lemma 22.2.13 (where N is fixed first) is the one that we will use to show that there are no bad types:

LEMMA 22.2.14. Assume that **K** is  $\mu$ -superstable. Let M be a limit model in  $\mathbf{K}_{\mu}$  and let N be limit over M. If  $p \in gS(M)$  is bad, then there exists  $q_1, q_2 \in gS(N)$  such that:

- (1) Both  $q_1$  and  $q_2$  extend p.
- (2)  $q_1 \neq q_2$ .
- (3) Both  $q_1$  and  $q_2$  are bad.

PROOF. By Lemma 22.2.13, there exists  $N' \in \mathbf{K}_{\mu}$  limit with  $M \leq_{\mathbf{K}} N'$  and  $q'_1, q'_2 \in \mathrm{gS}(N')$  distinct bad extensions of p. Use universality of N to pick  $f : N' \xrightarrow{M} N$ . For  $\ell = 1, 2$ , let  $q''_{\ell} := f(q'_{\ell})$ . Clearly,  $q''_1, q''_2$  are still distinct bad extensions of p. Now for  $\ell = 1, 2$ , let  $q_{\ell} \in \mathrm{gS}(N)$  be an extension of  $q''_{\ell}$  that does not fork over f[N'] (use no long splitting chains and extension). Then  $q_1$  and  $q_2$  are as desired (they are bad because they are nonforking extensions of the bad types  $q''_1, q''_2$ , see Remark 22.2.11).

LEMMA 22.2.15. If **K** is  $\mu$ -superstable, then there are no bad types.

PROOF. Suppose for a contradiction that there is a limit model M in  $\mathbf{K}_{\mu}$  and a bad type  $p \in \mathrm{gS}(M)$ . Fix an increasing continuous chain  $\langle M_i : i \leq \mu \rangle$  with  $M_0 = M$  and  $M_{i+1}$  limit over  $M_i$  for all  $i < \mu$ . We build a tree of types  $\langle p_\eta : \eta \in {}^{\leq \mu}2 \rangle$  satisfying:

- (1)  $p_{<>} = p$ .
- (2) For all  $\eta \in {}^{\leq \mu}2, p_{\eta} \in gS(M_{\ell(\eta)}).$

(3) For all  $\nu \leq \eta \in {}^{\leq \mu}2$ ,  $p_{\eta}$  is an extension of  $p_{\nu}$ .

- (4) For all  $\eta \in {}^{\leq \mu}2$ ,  $p_{\eta}$  is bad.
- (5) For all  $\eta \in {}^{<\mu}2$ ,  $p_{\eta \frown 0} \neq p_{\eta \frown 1}$ .

This is enough: Then for all  $\eta, \nu \in {}^{\mu}2, \ \eta \neq \nu$  implies  $p_{\eta} \neq p_{\nu}$ . Therefore  $|gS(\overline{M_{\mu}})| = 2^{\mu} > \mu$ , contradicting stability.

This is possible: We proceed by induction on  $\ell(\eta)$ . The base case has already been specified. At limits, we use Fact 22.2.9 and Lemma 22.2.12. At successors, we use Lemma 22.2.14.

THEOREM 22.2.16 (Uniqueness of forking). Assume that **K** is  $\mu$ -superstable. Let  $M \leq_{\mathbf{K}} N$  both be limits in  $\mathbf{K}_{\mu}$ . Let  $p, q \in \mathrm{gS}(N)$ . If  $p \upharpoonright M = q \upharpoonright M$  and both p and q do not fork over M, then p = q.

PROOF. Otherwise, this would mean that  $p \upharpoonright M$  is 0-bad, contradicting Lemma 22.2.15.

**22.2.1.** The hierarchy of symmetry properties collapses. In Section 10.4, VanDieren and the author defined several variations of the symmetry property (we have highlighted the differences between each, see the previously-cited paper for more motivation):

**DEFINITION 22.2.17.** 

- K has uniform μ-symmetry if for any limit models N, M<sub>0</sub>, M in K<sub>μ</sub> where M is limit over M<sub>0</sub> and M<sub>0</sub> is limit over N, if gtp(b/M) does not μ-split over M<sub>0</sub>, a ∈ |M|, and gtp(a/M<sub>0</sub>) does not μ-fork over (N, M<sub>0</sub>), there exists M<sub>b</sub> ∈ K<sub>μ</sub> containing b and limit over M<sub>0</sub> so that gtp(a/M<sub>b</sub>) does not μ-fork over (N, M<sub>0</sub>).
- (2) **K** has weak uniform  $\mu$ -symmetry if for any limit models  $N, M_0, M$  in  $\mathbf{K}_{\mu}$ where M is limit over  $M_0$  and  $M_0$  is limit over N, if  $\underline{\operatorname{gtp}}(b/M)$  does not  $\mu$ -fork over  $M_0$ ,  $a \in |M|$ , and  $\underline{\operatorname{gtp}}(a/M_0)$  does not  $\mu$ -fork over  $(N, \overline{M_0})$ , there exists  $\overline{M_b} \in$  $\mathbf{K}_{\mu}$  containing b and limit over  $M_0$  so that  $\underline{\operatorname{gtp}}(a/M_b)$  does not  $\mu$ -fork over  $(N, M_0)$ .
- (3) **K** has non-uniform  $\mu$ -symmetry if for any limit models  $M_0, M$  in  $\mathbf{K}_{\mu}$ where M is limit over  $M_0$ , if gtp(b/M) does not  $\mu$ -split over  $M_0$ ,  $a \in |M|$ , and  $gtp(a/M_0)$  does not  $\mu$ -fork over  $M_0$ , there exists  $M_b \in \mathbf{K}_{\mu}$  containing b and limit over  $M_0$  so that  $gtp(a/M_b)$  does not  $\mu$ -fork over  $M_0$ .
- (4) **K** has weak non-uniform  $\mu$ -symmetry if for any limit models  $M_0, M$  in  $\mathbf{K}_{\mu}$ where M is limit over  $M_0$ , if  $\underline{\operatorname{gtp}}(b/M)$  does not  $\mu$ -fork over  $M_0, a \in |M|$ , and  $\underline{\operatorname{gtp}}(a/M_0)$  does not  $\mu$ -fork over  $M_0$ , there exists  $M_b \in \mathbf{K}_{\mu}$  containing b and limit over  $M_0$  so that  $\underline{\operatorname{gtp}}(a/M_b)$  does not  $\mu$ -fork over  $M_0$ .

In Section 10.4, it was shown that the uniform variation corresponds to the symmetry property for splitting introduced by VanDieren in [Van16a] and the weak non-uniform variation corresponds to the symmetry property of good frames (over limit models). It was also proven that  $(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4)$ . Using Theorem 22.2.16, it is now easy to show that all these properties are equivalent.

COROLLARY 22.2.18. If **K** is  $\mu$ -superstable, then uniform  $\mu$ -symmetry is equivalent to weak non-uniform  $\mu$ -symmetry.

PROOF. We show that weak non-uniform  $\mu$ -symmetry implies weak uniform  $\mu$ -symmetry, which is known to be equivalent to uniform  $\mu$ -symmetry (Lemma 10.4.6).

So assume that we are given  $N, M_0, M, a, b$  as in the definition of weak uniform  $\mu$ -symmetry. Let  $M_b$  be as given by the definition of weak *non-uniform*  $\mu$ -symmetry. We know that  $gtp(a/M_b)$  does not  $\mu$ -fork over  $M_0$ , but we really want to conclude that it does not  $\mu$ -fork over  $(N, M_0)$ .

By assumption,  $gtp(a/M_0)$  does not  $\mu$ -fork over  $(N, M_0)$ . Therefore by extension there is a' such that  $gtp(a'/M_b)$  does not  $\mu$ -fork over  $(N, M_0)$ . We have that  $gtp(a/M_0) = gtp(a'/M_0)$ , and  $gtp(a/M_b)$ ,  $gtp(a'/M_b)$  both do not  $\mu$ -fork over  $M_0$ . Therefore by uniqueness (Theorem 22.2.16),  $gtp(a/M_b) = gtp(a'/M_b)$ . In particular,  $gtp(a/M_b)$  does not  $\mu$ -fork over  $(N, M_0)$ , as desired.

#### CHAPTER 23

## Abstract elementary classes stable in $\aleph_0$

This chapter is based on [SV] and is joint work with Saharon Shelah.

#### Abstract

We study abstract elementary classes (AECs) that, in  $\aleph_0$ , have amalgamation, joint embedding, no maximal models and are stable (in terms of the number of orbital types). We prove that such classes exhibit superstable-like behavior at  $\aleph_0$ . More precisely, there is a superlimit model of cardinality  $\aleph_0$  and the class generated by this superlimit has a type-full good  $\aleph_0$ -frame (a local notion of nonforking independence) and a superlimit model of cardinality  $\aleph_1$ . This extends Shelah's earlier study of PC $_{\aleph_0}$ -representable AECs and also improves results of Hyttinen-Kesälä and Baldwin-Kueker-VanDieren.

#### 23.1. Introduction

**23.1.1.** Motivation. In [She87a] (a revised version of which appears as [She09a, Chapter I], from which we cite), Shelah introduced *abstract elementary classes* (AECs): a semantic framework generalizing first-order model theory and also encompassing logics such as  $\mathbb{L}_{\omega_1,\omega}(Q)$ . Shelah studied  $\mathrm{PC}_{\aleph_0}$ -representable AECs (roughly, AECs which are reducts of a class of models of a first-order theory omitting a countable set of types) and generalized and improved some of his earlier results on  $\mathbb{L}_{\omega_1,\omega}$  [She83a, She83b] and  $\mathbb{L}_{\omega_1,\omega}(Q)$  [She75a].

For example, fix a  $PC_{\aleph_0}$ -representable AEC **K** and assume for simplicity that it is categorical in  $\aleph_0$ . Assuming  $2^{\aleph_0} < 2^{\aleph_1}$  and  $1 \leq I(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$ , Shelah shows [**She09a**, I.3.8] that **K** has amalgamation in  $\aleph_0$ . Further, [**She09a**, I.4, I.5], it has a lot of structure in  $\aleph_0$  and assuming more set-theoretic assumptions as well as few models in  $\aleph_2$ , **K** has a superlimit model in  $\aleph_1$  [**She09a**, I.5.34, I.5.40]. This means roughly (see [**She09a**, I.3.3]) that there is a saturated model in  $\aleph_1$  and that the union of an increasing chain of type  $\omega$  consisting of saturated models of cardinality  $\aleph_1$  is saturated.

**23.1.2.** Main result. The present chapter improves this result by removing the need for the extra set-theoretic and structure hypotheses on  $\aleph_2$ :

THEOREM 23.1.1. Assume  $2^{\aleph_0} < 2^{\aleph_1}$ . Let **K** be a PC<sub> $\aleph_0$ </sub>-representable AEC (with LS(**K**) =  $\aleph_0$  and countable vocabulary). If **K** is categorical in  $\aleph_0$  and  $1 \leq I(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$ , then **K** has a superlimit model of cardinality  $\aleph_1$ .

We give the proof of Theorem 23.1.1 at the end of this introduction. For now, notice that it implies the nontrivial fact that **K** has a model of size  $\aleph_2$ . However this consequence was known because under the hypotheses of Theorem 23.1.1, one can change the ordering on **K** to obtain a new class **K'** that has a good  $\aleph_0$ -frame [**She09a**, II.3.4] (a local axiomatic notion of nonforking independence. Its existence implies that there is a model of size  $\aleph_2$ ). Note also that the assumption of categoricity in  $\aleph_0$  is not really needed (see [**She09a**, I.3.10]) but then one has to change the class to obtain one that is categorical in  $\aleph_0$  and get a superlimit in the new class.

An additional difficulty in [**She09a**, I.5] is the lack of stability: one can only get that there are  $\aleph_1$ -many orbital types over countable models. A workaround is to redefine the ordering (but not the class of models) to get a stable class, see [**She09a**, I.5.29].

**23.1.3.** Outline of the chapter. In this chapter, we start with some of the consequences of [She09a, Chapter I]: amalgamation (plus joint embedding and no maximal models) in  $\aleph_0$  and stability in  $\aleph_0$ . We show that once we have them we can derive all the rest (e.g. existence of a superlimit in  $\aleph_0$  and existence of a good  $\aleph_0$ -frame) without assuming anything else (no need for  $2^{\aleph_0} < 2^{\aleph_1}$  or  $I(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$ ). In fact, we do not need to assume that  $\mathbf{K}$  is  $PC_{\aleph_0}$  (rather, we can prove that a certain subclass of  $\mathbf{K}$  is  $PC_{\aleph_0}$ , see Theorem 23.4.2 and Corollary 23.4.14). Moreover, we do not need to start with full amalgamation but can work in the slightly more general setup of [SV99].

One of the main tool is model-theoretic forcing in the style of Robinson, as used in [She09a, Chapter I]. When assuming amalgamation, the notion is wellbehaved. In particular, every formula is decided. We prove (Theorem 23.4.10) that one can characterize brimmed models (also called limit models in the literature) as those that are homogeneous for orbital types, or equivalently homogeneous for the syntactic types induced by the forcing notion (we call them generic types). This has as immediate consequence that the brimmed model of cardinality  $\aleph_0$  is superlimit (Corollary 23.4.11). This sheds light on an argument of Lessmann [Les05] and answers a question of Fred Drueck (see footnote 3 on [Dru13, p. 25].

We also deduce (Corollary 23.4.13) that orbital types over countable models are determined by their restrictions to finite sets (this is often called  $(\langle \aleph_0, \aleph_0 \rangle)$ tameness in the literature, we call it locality). This generalizes a result of Hyttinen and Kesälä, who proved it in the context of *finitary* AECs [**HK06**, 3.12].

One can then build a good frame (Theorem 23.4.19) as in the proof of [She09a, II.3.4] but a key new point given by the locality is that this frame will be good<sup>+</sup> (a technical condition characterized in Theorem 23.3.15). Using it, we can obtain the superlimit model in  $\aleph_1$ .

Another application of the construction of a good frame is that if the class has global amalgamation and all its orbital types are determined by their countable restrictions (this is called  $\aleph_0$ -tameness in other places in the literature), then  $\aleph_0$ -stability implies stability in all cardinals. This follows from e.g. the stability transfer in Theorem 4.5.6 and improves a result of Baldwin-Kueker-VanDieren [**BKV06**, 3.6] (by removing the hypothesis of  $\omega$ -locality there; in fact it follows from the rest by the existence of the good frame).

PROOF OF THEOREM 23.1.1. The global hypotheses of [She09a, I.5] are satisfied, and in particular we have amalgamation in  $\aleph_0$ . By [She09a, I.5.36], we can assume without loss of generality that **K** is stable in  $\aleph_0$ . Therefore the hypotheses of Theorem 23.4.19 hold, hence its conclusion.

**23.1.4.** Notes. Note that at the beginning of several sections, we make global hypotheses assumed throughout the section.

#### 23.2. Preliminaries

We assume familiarity with the basics of AECs, as presented for example in [Gro02, Bal09], or the first three sections of Chapter I together with the first section of Chapter II in [She09a]. We also assume familiarity with good frames (see [She09a, Chapter II]). This section mostly fixes the notation that we will use.

Given a  $\tau$ -structure M, we write |M| for its universe and ||M|| for its cardinality. We may abuse notation and write e.g  $a \in M$  instead of  $a \in |M|$ . We may even write  $\bar{a} \in M$  instead of  $\bar{a} \in {}^{<\omega}|M|$ .

We write  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  for an AEC. We may abuse notation and write  $M \in \mathbf{K}$ instead of  $M \in K$ . For a cardinal  $\lambda$ , we write  $\mathbf{K}_{\lambda}$  for the AEC restricted to its models of size  $\lambda$ . As shown in [She09a, II.1], any AEC is uniquely determined by its restriction  $\mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ .

When we say that  $M \in \mathbf{K}$  is an *amalgamation base*, we mean (as in [SV99]) that it is an amalgamation base in  $\mathbf{K}_{||M||}$ , i.e. we do not require that larger models can be amalgamated.

Given an AEC K, we may extend the relation  $\leq_{\mathbf{K}}$  to allow the empty set on the left hand side by requiring that  $\emptyset \leq_{\mathbf{K}} M$  for all  $M \in \mathbf{K}$ . This is useful when looking at universal models.

For  $M_0 \in \mathbf{K} \cup \{\emptyset\}$  we say that M is universal over  $M_0$  if  $M \leq_{\mathbf{K}} N$  and for any  $N \in \mathbf{K}$  with  $M_0 \leq_{\mathbf{K}} N$ , if  $||N|| \leq ||M_0|| + \mathrm{LS}(\mathbf{K})$ , there exists  $f: N \xrightarrow{M_0} M$ . We say that M is  $(\lambda, \delta)$ -brimmed over  $M_0$  (often also called  $(\lambda, \delta)$ -limit e.g. in **[SV99, GVV16]**) if  $\delta < \lambda^+$  is a limit ordinal,  $M_0 = \emptyset$  or  $M_0 \in \mathbf{K}_{\lambda}$ , and there exists an increasing continuous chain  $\langle N_i : i \leq \delta \rangle$  of members of  $\mathbf{K}_{\lambda}$  such that  $N_0$ is universal over  $M_0$ ,  $N_{\delta} = M$ , and  $N_{i+1}$  is universal over  $N_i$  for all  $i < \delta$ . We say that M is brimmed over  $M_0$  if it is  $(||M||, \delta)$ -brimmed over  $M_0$  for some limit  $\delta < \|M\|^+$ . We say that M is brimmed if it is brimmed over  $\emptyset$ .

The following notion of types already appears in [She87b]. It is called Galois types by many, but we prefer the term *orbital types* here. They are the same types that are defined in [She09a, II.1.9], except we also define them over sets. As pointed out in the preliminaries of Chapter 2, this causes no additional difficulties.

Definition 23.2.1. Fix an AEC  $\mathbf{K}$ .

- (1) We say  $(A, N_1, \bar{b}_1) E_{\text{at}}(A, N_2, \bar{b}_2)$  if:

  - (a) For  $\ell = 1, 2, N_{\ell} \in \mathbf{K}, A \subseteq |N_{\ell}|$ , and  $\bar{b}_{\ell} \in {}^{<\infty}|N_{\ell}|$ . (b) There exists  $N \in \mathbf{K}$  and  $f_{\ell} : N_{\ell} \xrightarrow{} N, \ell = 1, 2$ , such that  $f_1(\bar{b}_1) = \bar{b}_2$ .
- (2)  $E_{\rm at}$  is a reflexive and symmetric relation. Let E be its transitive closure.
- (3) Let  $gtp(\bar{b}, A, N)$  be the *E*-equivalence class of  $(\bar{b}, A, N)$ .
- (4) Define gS(A, N), gS(M),  $gS^{<\omega}(M)$ , etc. as expected. See for example the preliminaries of Chapter 2.

Let us say that an AEC **K** is stable in  $\lambda$  if for any  $M \in \mathbf{K}_{\lambda}$ ,  $|gS(M)| \leq \lambda$ . This makes sense in any AEC, and is quite well-behaved assuming amalgamation and no maximal models (since then it is known that one can build universal extensions). We will often work in the following axiomatic setup, a slight weakening where full amalgamation is not assumed. This comes from the context derived in [SV99]:

DEFINITION 23.2.2. Let **K** be an AEC and let  $\lambda$  be a cardinal. We say that **K** is *nicely stable in*  $\lambda$  (or *nicely*  $\lambda$ -*stable*) if:

- (1)  $LS(\mathbf{K}) \leq \lambda$ .
- (2)  $\mathbf{K}_{\lambda} \neq \emptyset$ .
- (3) **K** has joint embedding in  $\lambda$ .
- (4) Density of amalgamation bases: For any  $M \in \mathbf{K}_{\lambda}$ , there exists  $N \in \mathbf{K}_{\lambda}$  such that  $M \leq_{\mathbf{K}} N$  and N is an amalgamation base (in  $\mathbf{K}_{\lambda}$ ).
- (5) Existence of universal extensions: For any amalgamation base  $M \in \mathbf{K}_{\lambda}$ , there exists an amalgamation base  $N \in \mathbf{K}_{\lambda}$  such that  $M <_{\mathbf{K}} N$  and N is universal over M.
- (6) Any brimmed model in  $\mathbf{K}_{\lambda}$  is an amalgamation base.

We say that **K** is very nicely stable in  $\lambda$  if in addition it has amalgamation in  $\lambda$ .

REMARK 23.2.3. An AEC **K** is very nicely stable in  $\lambda$  if and only if  $LS(\mathbf{K}) \leq \lambda$ ,  $\mathbf{K}_{\lambda} \neq \emptyset$ , **K** is stable in  $\lambda$ , and  $\mathbf{K}_{\lambda}$  has amalgamation, joint embedding, and no maximal models.

We will make use of good frames for types of finite length (not just length one). Their definition is just like for types of length one, see Definition 5.3.8. We call them good ( $< \omega, \lambda$ )-frames. Note that any good  $\lambda$ -frame (i.e. for types of length one) extends to a good ( $< \omega, \lambda$ )-frame (using independent sequences, see [**She09a**, III.9.4]) or Corollary 5.5.8.

Given a good  $(\langle \omega, \lambda \rangle)$ -frame  $\mathfrak{s}$ , we write  $gS_{\mathfrak{s}}^{\mathrm{bs}}(M)$  for the basic types over Mand  $\mathbf{K}_{\mathfrak{s}}$  for the underlying class of the frames (so for some essentially unique AEC  $\mathbf{K}, \mathbf{K}_{\mathfrak{s}} = \mathbf{K}_{\lambda}$ ). We write  $M \leq_{\mathbf{K}^*} N$  to mean that  $M, N \in \mathbf{K}_{\mathfrak{s}}$  (so in particular both M and N have cardinality  $\lambda$ ) and  $M \leq_{\mathbf{K}} N$ .

#### 23.3. Weak nonforking amalgamation

In this section, we work in a good  $\lambda$ -frame and study a natural weak version of nonforking amalgamation, LWNF<sub>s</sub>. The main results are the existence property (Theorem 23.3.11) and how the symmetry property of LWNF<sub>s</sub> is connected to  $\mathfrak{s}$ being good<sup>+</sup> (Theorem 23.3.15). All throughout, we assume:

Hypothesis 23.3.1.

- (1)  $\mathfrak{s}$  is a good (<  $\omega, \lambda$ )-frame, except that it may not satisfy the symmetry axiom.
- (2)  $\mathbf{K}_{\mathfrak{s}}$  is categorical in  $\lambda$ . Write **K** for the AEC generated by  $\mathbf{K}_{\mathfrak{s}}$ .

REMARK 23.3.2. In this section,  $\lambda$  is allowed to be uncountable.

The reason for not assuming symmetry is that we will use some of the results of this section to *prove* that the symmetry axiom holds of a certain nonforking relation in Section 23.4.

We will use:

FACT 23.3.3 (II.4.3 in [She09a]). Let  $\delta < \lambda^+$  be a limit ordinal divisible by  $\lambda$ . Let  $\langle M_i : i \leq \delta \rangle$  be increasing continuous in  $\mathbf{K}_{\mathfrak{s}}$ . If for any  $i < \delta$  and any  $p \in \mathrm{gS}^{\mathrm{bs}}_{\mathfrak{s}}(M_i)$ , there exists  $\lambda$ -many  $j \in [i, \delta)$  such that the nonforking extension of p to  $M_j$  is realized in  $M_{j+1}$ , then  $M_{\delta}$  is brimmed over  $M_0$ .

DEFINITION 23.3.4. Define the following 4-ary relations on  $\mathbf{K}_{\mathfrak{s}}$ :

- (1) LWNF<sub>\$\$</sub> $(M_0, M_1, M_2, M_3)$  if and only if  $M_0 \leq_{\mathbf{K}^*} M_\ell \leq_{\mathbf{K}^*} M_3$  for  $\ell = 1, 2$ and for any  $\bar{b} \in {}^{<\omega}|M_1|$ , if  $\operatorname{gtp}(\bar{b}, M_2, M_3)$  and  $\operatorname{gtp}(\bar{b}, M_0, M_3)$  are basic then  $\operatorname{gtp}(\bar{b}, M_2, M_3)$  does not fork over  $M_0$ .
- (2) RWNF<sub>s</sub> $(M_0, M_1, M_2, M_3)$  if and only if LWNF<sub>s</sub> $(M_0, M_2, M_1, M_3)$ .
- (3) WNF<sub>s</sub> $(M_0, M_1, M_2, M_3)$  if and only if both LWNF<sub>s</sub> $(M_0, M_1, M_2, M_3)$  and RWNF<sub>s</sub> $(M_0, M_1, M_2, M_3)$ .

When  $\mathfrak{s}$  is clear from context, we write LWNF, RWNF, and WNF.

REMARK 23.3.5. WNF stands for weak nonforking amalgamation, and LWNF, RWNF stand for left (respectively right) weak nonforking amalgamation.

The following result often comes in handy.

LEMMA 23.3.6. Let  $\delta < \lambda^+$  be a limit ordinal. Let  $\langle M_i : i \leq \delta \rangle$ ,  $\langle N_i : i \leq \delta \rangle$ be increasing continuous in  $\mathbf{K}_{\mathfrak{s}}$ . Assume that for each  $i \leq j < \delta$ , we have that LWNF $(M_i, N_i, M_j, N_j)$ . If for each  $i < \delta$ ,  $N_i$  realizes all the basic types over  $M_i$ , then  $N_{\delta}$  realizes all the basic types over  $M_{\delta}$ .

PROOF. Let  $p \in \mathrm{gS}^{\mathrm{bs}}_{\mathfrak{s}}(M_{\delta})$ . By local character, there exists  $i < \delta$  such that p does not fork over  $M_i$ . By assumption, there exists  $a \in |N_i|$  such that  $p \upharpoonright M_i = \mathrm{gtp}(a, M_i, N_i)$ . Because for all  $j \in [i, \delta)$ , LWNF $(M_i, N_i, M_j, N_j)$ , we have by continuity that  $\mathrm{gtp}(a, M_{\delta}, N_{\delta})$  does not fork over  $M_i$ , hence by uniqueness it must be equal to p. Therefore a realizes p, as needed.

We will see that there seems to be a clear difference between LWNF and RWNF. The following ordering is defined similarly to  $\leq_{\lambda^+}^*$  from [She09a, II.7.2]:

DEFINITION 23.3.7. For  $R \in \{\text{LWNF}, \text{RWNF}, \text{WNF}\}$ , define a relation  $\leq_R$ on  $\mathbf{K}_{\lambda^+}$  as follows. For  $M^0, M^1 \in \mathbf{K}_{\lambda^+}, M^0 \leq_R M^1$  if and only if there exists increasing continuous resolutions  $\langle M_i^{\ell} \in \mathbf{K}_{\lambda} : i < \lambda^+ \rangle$  of  $M^{\ell}$  for  $\ell = 0, 1$  such that for all  $i < j < \lambda^+, R(M_i^0, M_i^1, M_j^0, M_i^1)$ .

The following is a straightforward "catching your tail argument", see the proof of Lemma 14.4.6.

FACT 23.3.8. Let  $M, N \in \mathbf{K}_{\lambda^+}$ . If  $M \leq_{\mathbf{K}} N$ , then  $M \leq_{\text{LWNF}} N$ .

Whether  $M \leq_{\text{RWNF}} N$  can be concluded as well seems to be a much more complicated question, and in fact is equivalent to  $\mathfrak{s}$  being good<sup>+</sup> (Theorem 23.3.15). Observe that an increasing union of a  $\leq_{\text{RWNF}}$ -increasing chain of saturated models is saturated:

LEMMA 23.3.9. Let  $\delta < \lambda^{++}$  be a limit ordinal. If  $\langle M_i : i < \delta \rangle$  is a  $\leq_{\text{RWNF}}$ increasing sequence of saturated models in  $\mathbf{K}_{\lambda^+}$ , then  $\bigcup_{i < \delta} M_i$  is saturated.

PROOF. Without loss of generality,  $\delta = \operatorname{cf} \delta < \lambda^+$ . Let  $M_{\delta} := \bigcup_{i < \delta} M_i$ . We build  $\langle M_{i,j} : i \leq \delta, j \leq \lambda^+ \rangle$  such that:

- (1) For any  $i \leq \delta$ ,  $M_{i,\lambda^+} = M_i$ .
- (2) For any  $i < \delta, j < \lambda^+, M_{i,j} \in \mathbf{K}_{\mathfrak{s}}$ .
- (3) For any  $i \leq \delta$ ,  $\langle M_{i,j} : j < \lambda^+ \rangle$  is increasing and continuous.
- (4) For any  $j \leq \lambda^+$ ,  $\langle M_{i,j} : i < \delta \rangle$  is increasing and  $M_{\delta,j} = \bigcup_{i < \delta} M_{i,j}$ .
- (5) For any  $i_1 < i_2 \leq \delta$ ,  $j_1 < j_2 \leq \lambda^+$ ,  $M_{i_2,j_2}$  realizes all the types in  $gS_{\mathfrak{s}}^{\mathrm{bs}}(M_{i_1,j_1})$ .

This is easy to do. Now for each  $i_1 < i_2 < \delta$ , we have by assumption that  $M_{i_1} \leq_{\text{RWNF}} M_{i_2}$ . Thus the set  $C_{i_1,i_2}$  of  $j < \lambda^+$  such that for all  $j' \in [j, \lambda^+)$ , RWNF $(M_{i_1,j}, M_{i_2,j}, M_{i_1,j'}, M_{i_2,j'})$  is a club. Therefore  $C := \bigcap_{i_1 < i_2 < \delta} C_{i_1,i_2}$  is also a club. Hence by renaming without loss of generality for all  $i_1 < i_2 < \delta$  and all  $j \leq j' < \lambda^+$ , RWNF $(M_{i_1,j}, M_{i_2,j}, M_{i_1,j'}, M_{i_2,j})$ .

Now let  $N \leq_{\mathbf{K}} M_{\delta}$  be such that  $N \in \mathbf{K}_{\lambda}$ . We want to see that any type over N is realized in  $M_{\delta}$ . By Fact 23.3.3, it is enough to show that any *basic* type over N is realized in  $M_{\delta}$ .

Let  $j < \lambda^+$  be big-enough such that  $N \leq_{\mathbf{K}} M_{\delta,j}$ . It is enough to see that any basic type over  $M_{\delta,j}$  is realized in  $M_{\delta,j+1}$ . To see this, use Lemma 23.3.6 with  $\langle M_i : i \leq \delta \rangle$ ,  $\langle N_i : i \leq \delta \rangle$  there standing for  $\langle M_{i,j} : i \leq \delta \rangle$ ,  $\langle M_{i,j+1} : i \leq \delta \rangle$  here. We know that for each  $i \leq i' < \delta$ , RWNF $(M_{i,j}, M_{i',j}, M_{i,j+1}, M_{i',j+1})$ and therefore LWNF $(M_{i,j}, M_{i,j+1}, M_{i',j}, M_{i',j+1})$ . Thus the hypotheses of Lemma 23.3.6 are satisfied.

The proof of the following fact is a direct limit argument similar to e.g. [**GVV16**, 5.3]. Note that the symmetry axiom is *not* needed.

FACT 23.3.10. Let  $\alpha < \lambda^+$ . Let  $\langle M_i : i \leq \alpha \rangle$  be  $\leq_{\mathbf{K}^*}$ -increasing continuous and let  $\langle \bar{a}_i : i < \alpha \rangle$  be given such that  $\bar{a}_i \in M_{i+1}$  for all  $i < \alpha$  and  $\operatorname{gtp}(\bar{a}_i, M_i, M_{i+1}) \in \operatorname{gS}_{\mathfrak{s}}^{\operatorname{bs}}(M_i)$ .

There exists  $\langle N_i : i \leq \alpha \rangle \leq_{\mathbf{K}^*}$ -increasing continuous such that:

- (1)  $M_i <_{\mathbf{K}^*} N_i$  for all  $i \leq \alpha$ .
- (2)  $gtp(\bar{a}_i, N_i, N_{i+1})$  does not fork over  $M_i$ .

We are now ready to list some basic properties of weak nonforking amalgamation.

THEOREM 23.3.11. Let  $R \in \{\text{LWNF}, \text{RWNF}, \text{WNF}\}$ .

- (1) Invariance: If  $R(M_0, M_1, M_2, M_3)$  and  $f: M_3 \cong M'_3$ , then  $R(f[M_0], f[M_1], f[M_2], M'_3)$ .
- (2) Monotonicity: If  $R(M_0, M_1, M_2, M_3)$  and  $M_0 \leq_{\mathbf{K}^*} M'_{\ell} \leq_{\mathbf{K}^*} M_{\ell}$  for  $\ell = 1, 2$ , then  $R(M_0, M'_1, M'_2, M_3)$ .
- (3) Ambiant monotonicity: If  $R(M_0, M_1, M_2, M_3)$  and  $M_3 \leq_{\mathbf{K}^*} M'_3$ , then  $R(M_0, M_1, M_2, M'_3)$ . If  $M''_3 \leq_{\mathbf{K}^*} M_3$  contains  $|M_1| \cup |M_2|$ , then  $R(M_0, M_1, M_2, M''_3)$ .
- (4) Continuity: If  $\delta < \lambda^+$  is a limit ordinal and  $\langle M_i^{\ell} : i \leq \delta \rangle$  are increasing continuous for  $\ell < 4$  with  $R(M_i^0, M_i^1, M_i^2, M_i^3)$  for each  $i < \delta$ , then  $R(M_{\delta}^0, M_{\delta}^1, M_{\delta}^2, M_{\delta}^3)$ .
- (5) Long transitivity: If  $\alpha < \lambda^+$  is an ordinal,  $\langle M_i : i \leq \alpha \rangle$ ,  $\langle N_i : i \leq \alpha \rangle$  are increasing continuous and LWNF $(M_i, N_i, M_{i+1}, N_{i+1})$  for all  $i < \alpha$ , then LWNF $(M_0, N_0, M_\alpha, N_\alpha)$ .
- (6) Existence: If  $R \neq \text{WNF}$ ,  $M_0 \leq_{\mathbf{K}^*} M_\ell$ ,  $\ell = 1, 2$ , then there exists  $M_3 \in \mathbf{K}_\lambda$ and  $f_\ell : M_\ell \xrightarrow[M_0]{} M_3$  such that  $R(M_0, f_1[M_1], f_2[M_2], M_3)$ .

PROOF. Invariance and the monotonicity properties are straightforward to prove. Continuity and long transitivity follow directly from the local character, continuity, and transitivity properties of good frames. We prove existence via the following claim:

<u>Claim</u>: There exists  $N_0, N_1, N_2, N_3 \in \mathbf{K}_{\mathfrak{s}}$  such that LWNF $(N_0, N_1, N_2, N_3)$  and  $N_{\ell}$  is brimmed over  $N_0$  for  $\ell = 1, 2$ .

Existence easily follows from the claim: given  $M_0 \leq_{\mathbf{K}^*} M_\ell$ ,  $\ell = 1, 2$ , there is (by categoricity in  $\lambda$ ) an isomorphism  $f : M_0 \cong N_0$  and (by universality of brimmed

models) embeddings  $f_{\ell}: M_{\ell} \to N_{\ell}$  extending f for  $\ell = 1, 2$ . After some renaming, we obtain the desired LWNF-amalgam. To obtain an RWNF-amalgam, reverse the role of  $M_1$  and  $M_2$ .

<u>Proof of Claim</u>: Let  $\delta := \lambda \cdot \lambda$ . We choose  $(\bar{M}^{\alpha}, \bar{a}^{\alpha})$  by induction on  $\alpha \leq \delta$  such that:

- (1)  $\overline{M}^{\alpha} = \langle M_i^{\alpha} : i \leq \alpha \rangle$  is  $\leq_{\mathbf{K}^*}$ -increasing continuous.
- (2)  $\bar{a}^{\alpha} = \langle \bar{a}_i : i < \alpha \rangle$ , and  $\bar{a}_i \in M_{i+1}^{\alpha}$  for all  $i < \alpha$ .
- (3) For all  $i < \alpha$ ,  $\operatorname{gtp}(\bar{a}_i^{\alpha}, M_i^{\alpha}, M_{i+1}^{\alpha}) \in \operatorname{gS}_{\mathfrak{s}}^{\operatorname{bs}}(M_i^{\alpha})$ .
- (4) For each  $i \leq \delta$ ,  $\langle M_i^{\alpha} : \alpha \in [i, \delta] \rangle$  is  $\langle_{\mathbf{K}^*}$ -increasing continuous.
- (5) For each  $i < \delta$  and each  $\alpha \in (i, \delta]$ ,  $gtp(\bar{a}_i, M_i^{\alpha}, M_{i+1}^{\alpha})$  does not fork over  $M_i^i$ .
- (6) If  $p \in \mathrm{gS}^{\mathrm{bs}}_{\mathfrak{s}}(M_i^{\alpha})$  for  $i \leq \alpha < \delta$ , then for  $\lambda$ -many  $\beta \in [\alpha, \delta)$ ,  $\mathrm{gtp}(\bar{a}_{\beta}, M_{\beta}^{\beta+1}, M_{\beta+1}^{\beta+1})$ is a nonforking extension of p.
- (7) If  $i < \alpha < \delta$  and  $gtp(\bar{a}, M_0^{\alpha}, M_{i+1}^{\alpha}) \in gS_{\mathfrak{s}}^{\mathrm{bs}}(M_0^{\alpha})$ , then for some  $\beta \in (\alpha, \delta)$ exactly one of the following occurs: (a)  $gtp(\bar{a}, M_0^{\beta+1}, M_{i+1}^{\beta+1})$  forks over  $M_0^{\alpha}$ . (b) There is no  $\langle M_j^* : j \leq i+1 \rangle \leq_{\mathbf{K}^*}$ -increasing continuous such that:

  - - (i)  $M_i^{\beta} \leq_{\mathbf{K}^*} M_i^*$  for all  $j \leq i+1$ .
    - (ii)  $gtp(\bar{a}_j, M_j^*, M_{j+1}^*)$  does not fork over  $M_j^\beta$  for all j < i+1.
    - (iii) gtp $(\bar{a}, M_0^*, M_{i+1}^*)$  forks over  $M_0^{\beta}$ .

This is possible: Along the construction, we also build an enumeration  $\langle (\bar{b}_{i}^{\gamma}, k_{i}^{\gamma}, i_{i}^{\gamma}, \alpha_{i}^{\gamma}) : j < \lambda, \gamma < \lambda \rangle$  such that for any  $\gamma \in (0, \lambda)$ , any  $\alpha < \lambda \cdot \gamma$ , any  $i < \alpha$ , any  $k \leq i$ , and any  $\bar{a} \in {}^{<\omega}M^{\alpha}_{i+1}$ , if  $gtp(\bar{a}, M^{\alpha}_k, M^{\alpha}_{i+1}) \in$  $\mathrm{gS}^{\mathrm{bs}}_{\mathfrak{s}}(M_k^{\alpha})$ , then there exists  $j < \lambda$  so that  $\bar{b}_j^{\gamma} = \bar{a}$ ,  $i_j^{\gamma} = i$ ,  $k_j^{\gamma} = k$ , and  $\alpha_j^{\gamma} = \alpha$ . We require that always  $k_j^{\gamma} \leq i_j^{\gamma} < \alpha_j^{\gamma} < \lambda \cdot \gamma$  and the triple  $(\bar{b}_{j}^{\gamma}, M_{k_{j}^{\gamma}}^{\alpha_{j}^{\gamma}}, M_{i_{j}^{\gamma}+1}^{\alpha_{j}^{\gamma}})$  represents a basic type. We make sure that at stage  $\lambda \cdot (\gamma + 1)$  of the construction below,  $\bar{b}_{i}^{\gamma'}, k_{i}^{\gamma'}, \dot{a}_{j}^{\gamma'}, \alpha_{i}^{\gamma'}$  are defined for all  $j < \lambda, \, \gamma' \leq \gamma.$ 

For  $\alpha = 0$ , take any  $M_0^0 \in \mathbf{K}_{\mathfrak{s}}$ . For  $\alpha$  limit, let  $M_i^{\alpha} := \bigcup_{\beta \in [i,\alpha)} M_i^{\beta}$ for  $i < \alpha$  and  $M^{\alpha}_{\alpha} := \bigcup_{i < \alpha} M^{\alpha}_i$ . Now assume that  $\overline{M}^{\alpha}$ ,  $\overline{a}^{\alpha}$  have been defined for  $\alpha < \delta$ . We define  $\overline{M}^{\alpha+1}$  and  $\overline{a}_{\alpha}$ . Fix  $\rho$  and  $j < \lambda$  such that  $\alpha = \lambda \cdot \rho + j$ . We consider two cases.

- Case 1:  $\rho$  is zero or a limit: Use Fact 23.3.10 to get  $\langle M_i^{\alpha+1} : i \leq \alpha \rangle$  $\frac{\partial \operatorname{dec} \Pi p \text{ is large of a minimizer construct 2 distributed get } (M_i) = 1 \neq \underline{\exists} \alpha_i / (M_i) = 1 \neq \underline{d} / (M_i) = 1 \neq \underline$
- Case 2:  $\rho$  is a successor: Say  $\rho = \gamma + 1$ . Let  $\bar{a} := \bar{b}_{j}^{\gamma}$ ,  $\alpha_{0} := \alpha_{j}^{\gamma}$ ,  $k_0 := k_i^{\gamma}, i_0 := i_i^{\gamma}$ . There are two subcases. It is possible that  $k_0 \neq 0$ or  $k_0 = 0$  and (7b) holds with  $i, \alpha, \beta$  there standing for  $i_0, \alpha_0, \alpha$  here. In this case, we proceed as in Case 1 to define  $\langle M_i^{\alpha+1} : i \leq \alpha \rangle$ . Then we pick  $\bar{a}_{\alpha}$ ,  $M_{\alpha+1}^{\alpha+1}$  such that  $\operatorname{gtp}(\bar{a}_{\alpha}, M_{\alpha}^{\alpha}, M_{\alpha+1}^{\alpha+1})$  is the nonforking extension of  $\operatorname{gtp}(\bar{a}, M_{i_0}^{\alpha_0}, M_{i_0+1}^{\alpha_0})$ .

On the other hand, it is possible that  $k_0 = 0$  and (7b) fails. In this case let  $\langle M_j^* : j \leq i_0 + 1 \rangle$  witness the failure and set  $M_j^{\alpha+1} := M_j^*$  for  $j \leq i_0 + 1$ . Then continue as in Case 1 and define  $\bar{a}_{\alpha}$ ,  $M_{\alpha+1}^{\alpha+1}$  as before.

This is enough: We choose  $\overline{M}^* = \langle M_i^* : i \leq \delta \rangle$  increasing continuous such that  $M_0^*$  is brimmed over  $M_0^{\delta}$ ,  $M_i^{\delta} \leq_{\mathbf{K}^*} M_i^*$  for all  $i \leq \delta$ , and  $\operatorname{gtp}(\overline{a}_i, M_i^*, M_{i+1}^*)$  does not fork over  $M_i^{\delta}$ . This is possible, see case 1 above. Now:

 $- M_0^*$  is brimmed over  $M_0^{\delta}$ .

[Why? By construction].

- If  $p \in gS_{\mathfrak{s}}^{\mathrm{bs}}(M_i^{\delta})$  for  $i < \delta$ , then for  $\lambda$ -many  $\beta \in [i, \delta)$ ,  $gtp(\bar{a}_{\beta}, M_{\beta}^{\delta}, M_{\beta+1}^{\delta})$  is a nonforking extension of p.

[Why? Pick  $i' \in (i, \delta)$  such that p does not fork over  $M_i^{i'}$ . By (6), we know that for  $\lambda$ -many  $\beta \in [i', \delta)$ , the nonforking extension of  $p \upharpoonright M_i^{i'}$  to  $M_{\beta}^{\beta+1}$  is realized in  $M_{\beta+1}^{\beta+1}$  by  $\bar{a}_{\beta}$ . But by (5) we also have that  $\operatorname{gtp}(\bar{a}_{\beta}, M_{\beta}^{\delta}, M_{\beta+1}^{\delta})$  does not fork over  $M_{\beta}^{\beta}$ . In particular by uniqueness  $\bar{a}_{\beta}$  also realizes p.]

 $-M_{\delta}^{\delta}$  is brimmed over  $M_{0}^{\delta}$ .

[Why? We apply Fact 23.3.3 to the chain  $\langle M_i^{\delta} : i \leq \delta \rangle$ , using the previous step.].

- LWNF $(M_0^{\delta}, M_{\delta}^{\delta}, M_0^*, M_{\delta}^*)$ .

[Why? Pick  $\bar{a} \in {}^{<\omega}M_{\delta}^{\delta}$  such that  $\operatorname{gtp}(\bar{a}, M_0^{\delta}, M_{\delta}^{\delta})$  is basic. By local character, there exists  $\alpha < \delta$  such that  $\operatorname{gtp}(\bar{a}, M_0^{\delta}, M_{\delta}^{\delta})$  does not fork over  $M_0^{\alpha}$ . Further, we can increase  $\alpha$  if necessary and pick  $i < \alpha$  such that  $\bar{a} \in {}^{<\omega}M_{i+1}^{\alpha}$ . We now apply Clause (7). We know that (7a) fails for all  $\beta \in (\alpha, \delta)$  by the choice of  $\alpha$ , therefore (7b) must hold for all  $\beta \in (\alpha, \delta)$ . Now if  $\operatorname{gtp}(\bar{a}, M_0^*, M_{\delta}^*)$  forks over  $M_0^{\delta}$ , then it must fork over  $M_0^{\beta}$  for all high-enough  $\beta$ , but then  $\langle M_j^* : j \leq i+1 \rangle$  would contradict Clause (7b). Therefore  $\operatorname{gtp}(\bar{a}, M_0^*, M_{\delta}^*)$  does not fork over  $M_0^{\delta}$ , as desired.]

Therefore we can take  $(M_0, M_1, M_2, M_3) := (M_0^{\delta}, M_{\delta}^{\delta}, M_0^*, M_{\delta}^*)$ .  $\dagger_{\text{Claim}}$ 

Definition 23.3.12. Let  $R \in \{\text{LWNF}, \text{RWNF}, \text{WNF}\}$ .

- (1) We say that R has the symmetry property if  $R(M_0, M_1, M_2, M_3)$  implies  $R(M_0, M_2, M_1, M_3)$ .
- (2) We say that R has the uniqueness property if whenever  $R(M_0, M_1, M_2, M_3)$ and  $R(M_0, M_1, M_2, M'_3)$ , there exists  $M''_3$  with  $M'_3 \leq_{\mathbf{K}^*} M''_3$  and  $f : M_3 \xrightarrow[|M_1|\cup|M_2]{} M''_3$ .

The following are trivial observations about the definitions:

Remark 23.3.13.

- (1) WNF has the symmetry property, and LWNF has the symmetry property if and only if RWNF has the symmetry property if and only if LWNF = RWNF = WNF.
- (2) LWNF has the uniqueness property if and only RWNF has it.

Recall from [She09a, III.1.3]:

DEFINITION 23.3.14.  $\mathfrak{s}$  is  $good^+$  when the following is *impossible*:

There exists an increasing continuous  $\langle M_i : i < \lambda^+ \rangle$ ,  $\langle N_i : i < \lambda^+ \rangle$ , a basic type  $p \in gS^{bs}_{\mathfrak{s}}(M_0)$ , and  $\langle \bar{a}_i : i < \lambda^+ \rangle$  such that for any  $i < \lambda^+$ :

- (1)  $M_i \leq_{\mathbf{K}^*} N_i$ .
- (2)  $\bar{a}_{i+1} \in |M_{i+2}|$  and  $\operatorname{gtp}(\bar{a}_{i+1}, M_{i+1}, M_{i+2})$  is a nonforking extension of p, but  $\operatorname{gtp}(\bar{a}_{i+1}, N_0, N_{i+2})$  forks over  $M_0$ .
- (3)  $\bigcup_{j < \lambda^+} M_j$  is saturated.

THEOREM 23.3.15.  $(1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$ , where:

- (1) LWNF has the symmetry property.
- (2)  $\mathfrak{s}$  is good<sup>+</sup>.
- (3) For  $M, N \in \mathbf{K}_{\lambda^+}$  both saturated,  $M \leq_{\mathbf{K}} N$  implies  $M \leq_{\text{WNF}} N$ .
- (4) There is a superlimit model in  $\mathbf{K}_{\lambda^+}$ .

#### Proof.

- (3) implies (4): This follows from Lemma 23.3.9.
- $\neg(2)$  implies  $\neg(3)$ : Fix a witness  $\langle M_i : i < \lambda^+ \rangle$ ,  $\langle N_i : i < \lambda^+ \rangle$ ,  $\langle \bar{a}_i : i < \lambda^+ \rangle$ , p to the failure of being good<sup>+</sup>. Write  $M_{\lambda^+} := \bigcup_{i < \lambda^+} M_i$ ,  $N_{\lambda^+} := \bigcup_{i < \lambda^+} N_i$ . By assumption,  $M_{\lambda^+}$  is saturated. Clearly, increasing the  $N_i$ 's will not change that we have a witness so without loss of generality  $N_{\lambda^+}$  is also saturated. We claim that  $M_{\lambda^+} \not\leq_{\text{RWNF}} N_{\lambda^+}$ . We show this by proving that for any  $i < \lambda^+$  and any  $j \le i + 1$ ,  $\neg \text{RWNF}(M_j, N_j, M_{i+2}, N_{i+2})$ . Indeed,  $\text{gtp}(\bar{a}_{i+1}, N_j, N_{i+2})$  forks over  $M_j$ : if not, then by transitivity  $\text{gtp}(\bar{a}_{i+1}, N_j, N_{i+2})$  does not fork over  $M_0$ , and hence  $\text{gtp}(\bar{a}_{i+1}, N_0, N_{i+2})$  does not fork over  $M_0$ , and we know that this is not the case of the witness we selected.
- $\neg(3)$  implies  $\neg(2)$ : Fix M, N saturated in  $\mathbf{K}_{\lambda^+}$  such that  $M \leq_{\mathbf{K}} N$  but  $M \not\leq_{\mathrm{RWNF}} N$ .

<u>Claim</u>: For any  $A \subseteq |M|$  of size  $\lambda$ , there exists  $M_0 \leq_{\mathbf{K}^*} M_1 \leq_{\mathbf{K}} M$ and  $N_0 \leq_{\mathbf{K}^*} N_1 \leq_{\mathbf{K}} N$  such that  $M_0 \leq_{\mathbf{K}^*} N_0, M_1 \leq_{\mathbf{K}^*} N_1, A \subseteq |M_0|$ , but  $\neg \operatorname{RWNF}(M_0, N_0, M_1, N_1)$ .

<u>Proof of Claim</u>: If not, we can use failure of the claim and continuity of RWNF to build increasing continuous resolution  $\langle M_i : i \leq \lambda^+ \rangle$ ,  $\langle N_i : i \leq \lambda^+ \rangle$  of M and N respectively such that RWNF $(M_i, N_i, M_j, N_j)$  for all  $i < j < \lambda^+$ . Thus  $M \leq_{\text{RWNF}} N$ , contradicting the assumption.  $\dagger_{\text{Claim}}$ 

Build  $\langle M_i^* : i \leq \lambda^+ \rangle$ ,  $\langle N_i^* : i \leq \lambda^+ \rangle$  increasing continuous resolutions of M, N respectively such that for all  $i < \lambda^+$ ,  $M_i^* \leq_{\mathbf{K}^*} N_i^*$  and  $\neg \operatorname{RWNF}(M_{i+1}^*, N_{i+1}^*, M_{i+2}^*, N_{i+2}^*)$ . This is possible by the claim. Let  $\bar{a}_{i+1}^* \in |M_{i+2}^*|$  witness the RWNF-forking, i.e.  $\operatorname{gtp}(\bar{a}_{i+1}^*, N_{i+1}^*, N_{i+2}^*)$  forks over  $M_{i+1}^*$ . By Fodor's lemma, local character, and stability, there exists a stationary set S,  $i_0 < \lambda^+$  and  $p \in \operatorname{gS}_{\mathfrak{s}}^{\operatorname{bs}}(M_{i_0}^*)$  such that for all  $i \in S$ ,  $\operatorname{gtp}(\bar{a}_{i+1}^*, M_i^*, M_{i+2}^*)$  is the nonforking extension of p. Without loss of generality,  $i_0$  is limit and all elements of S are also limit ordinals.

Now build an increasing continuous sequence of ordinals  $\langle j_i : i < \lambda^+ \rangle$  as follows. Let  $j_0 := i_0$ . For *i* limit, let  $j_i := \sup_{k < i} j_k$ . For *i* successor, pick any  $j_i \in S$  with  $j_i > j_{i-1}$ .

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Now for *i* not the successor of a limit, let  $M_i := M_{j_i}^*$ ,  $N_i := N_{j_i}^*$ ,  $\bar{a}_i := \bar{a}_{j_i}^*$ . For i = k + 1 with k a limit, set  $M_i := M_{j_k}^*$ ,  $N_i := N_{j_k}^*$ ,  $\bar{a}_i := \bar{a}_{i_i}^*$ . This gives a witness to the failure of being good<sup>+</sup>.

• (1) implies (3): If LWNF has the symmetry property, then by Remark 23.3.13, LWNF = RWNF = WNF. By Fact 23.3.8, it follows that  $M \leq_{\mathbf{K}} N$  implies  $M \leq_{\text{WNF}} N$  for any  $M, N \in \mathbf{K}_{\lambda^+}$ , so (3) holds.

QUESTION 23.3.16. Are the conditions in Theorem 23.3.15 all equivalent?

QUESTION 23.3.17. Is there a good  $\lambda$ -frame  $\mathfrak{s}$  such that LWNF $\mathfrak{s}$  does *not* have the symmetry property?

The next result shows that the uniqueness property has strong consequences. Shelah has given conditions under which when  $\lambda = \aleph_0$ , failure of uniqueness implies nonstructure [She09b, VII.4.16].

THEOREM 23.3.18. Assume that  $\mathfrak{s}$  is a good ( $\langle \omega, \lambda \rangle$ )-frame (so it satisfies symmetry). If LWNF has the uniqueness property, then LWNF has the symmetry property and  $\mathfrak{s}$  is successful good<sup>+</sup> (see [She09a, III.1.1]).

PROOF. By Corollary 14.3.11 (used with the pre- $(\leq \lambda, \lambda)$ -frame induced by LWNF, recalling Fact 23.3.8)  $\mathfrak{s}$  is weakly successful. This implies that there is a relation NF = NF $\mathfrak{s}$  that is a nonforking relation respecting  $\mathfrak{s}$  (see [She09a, II.6.1], in particular it has all the properties listed in Theorem 23.3.11, as well as uniqueness and symmetry). Now as NF respects  $\mathfrak{s}$ , we must have that NF $(M_0, M_1, M_2, M_3)$ implies LWNF $(M_0, M_1, M_2, M_3)$ . Since LWNF has the uniqueness property and NF has the existence property, it follows from Lemma 3.4.1 that LWNF = NF. In particular, LWNF has the symmetry property.

To see that  $\mathfrak{s}$  is successful good<sup>+</sup>, it is enough to show that for  $M, N \in \mathbf{K}_{\lambda^+}$ ,  $M \leq_{\mathbf{K}} N$  implies  $M \leq_{\mathrm{NF}} N$  (where  $\leq_{\mathrm{NF}}$  is defined as in Definition 23.3.7). This is immediate from Fact 23.3.8 and LWNF = NF.

To prepare for the proof of symmetry in the  $\lambda = \aleph_0$  case, we introduce yet another notion of nonforking amalgamation (VWNF stands for "very weak nonforking amalgamation").

DEFINITION 23.3.19.

- (1) For  $M \leq_{\mathbf{K}^*} N$ ,  $B \subseteq |N|$ ,  $\bar{a} \in {}^{<\omega}N$ , we say that  $\operatorname{gtp}(\bar{a}, B, N)$  does not fork over M if there exists M', N' with  $N \leq_{\mathbf{K}^*} N'$ ,  $M \leq_{\mathbf{K}^*} M' \leq_{\mathbf{K}^*} N'$ , and  $B \subseteq |M'|$  such that  $\operatorname{gtp}(\bar{a}, M', N')$  does not fork over  $M_0$ .
- (2) We define a 4-ary relation VWNF<sub>s</sub> = VWNF on  $\mathbf{K}_{s}$  by VWNF $(M_{0}, M_{1}, M_{2}, M_{3})$ if and only if  $M_{0} \leq_{\mathbf{K}^{*}} M_{\ell} \leq_{\mathbf{K}^{*}} M_{3}$ ,  $\ell = 1, 2$  and for any  $\bar{a} \in {}^{<\omega}M_{1}$  and any finite  $B \subseteq |M_{2}|$ , if  $\operatorname{gtp}(\bar{a}, M_{0}, M_{3})$  and  $\operatorname{gtp}(\bar{a}, M_{2}, M_{3})$  are both basic, then  $\operatorname{gtp}(\bar{a}, B, M_{3})$  does not fork over  $M_{0}$ .

THEOREM 23.3.20. Assume that  $\mathfrak{s}$  is a type-full good ( $\langle \omega, \lambda \rangle$ -frame.

- (1) VWNF has the symmetry property:  $VWNF(M_0, M_1, M_2, M_3)$  if and only if  $VWNF(M_0, M_2, M_1, M_3)$ .
- (2) If for any  $M \in \mathbf{K}_{\mathfrak{s}}$  and any  $p \neq q \in \mathrm{gS}_{\mathfrak{s}}^{\mathrm{bs}}(M)$  there exists  $B \subseteq |M|$  finite such that  $p \upharpoonright B \neq q \upharpoonright B$ , then VWNF = WNF. In particular, LWNF has the symmetry property.

Proof.

- (1) By the symmetry axiom of good frames.
- (2) This is observed in Lemma 6.4.5. In details, it suffices to show that for  $M \leq_{\mathbf{K}^*} N, p \in \mathrm{gS}^{\mathrm{bs}}_{\mathfrak{s}}(N)$  does not fork over M if and only if  $p \upharpoonright B$  does not fork over M for all finite  $B \subseteq |N|$ . Let  $q \in \mathrm{gS}^{\mathrm{bs}}_{\mathfrak{s}}(N)$  be the nonforking extension of  $p \upharpoonright M$ . For any finite  $B \subseteq |N|$ , we have that  $q \upharpoonright B = p \upharpoonright B$ , by the uniqueness property for (the extended notion of) forking, see Lemma 3.5.4. Therefore by the assumption we must have p = q, as desired.

#### **23.4.** Building a good $\aleph_0$ -frame

In this section, we work in  $\aleph_0$  and aim to build a good  $\aleph_0$ -frame from stability and amalgamation.

HYPOTHESIS 23.4.1.  $\mathbf{K} = (K, \leq_{\mathbf{K}})$  is an AEC with  $LS(\mathbf{K}) = \aleph_0$  (and countable vocabulary).

First note that if  $\mathbf{K}$  is stable and has few models, we can say something about its definability:

THEOREM 23.4.2. Assume that  $I(\mathbf{K}, \aleph_0) \leq \aleph_0$ .

- (1) The set  $\{M \in K_{\aleph_0} : |M| \subseteq \omega\}$  is Borel.
- (2) If **K** has amalgamation in  $\aleph_0$  and is stable in  $\aleph_0$ , then the set  $\{(M, N) : M \leq_{\mathbf{K}} N \text{ and } |N| \subseteq \omega\}$  is  $\Sigma_1^1$ .

In particular if **K** has amalgamation in  $\aleph_0$  and is stable in  $\aleph_0$ , then **K** is a  $PC_{\aleph_0}$ -representable AEC.

Proof.

(1) Fix  $M \in \mathbf{K}_{\aleph_0}$ . By Scott's isomorphism theorem, there exists a formula  $\phi_M$  of  $\mathbb{L}_{\aleph_1,\aleph_0}(\tau_{\mathbf{K}})$  such that  $N \models \phi_M$  if and only if  $M \cong N$ . Now observe that the set

 $\{N : N \text{ is a } \tau_{\mathbf{K}} \text{-structure with } |N| \subseteq \omega \text{ and } N \models \phi_M \}$ 

is Borel and use that  $I(\mathbf{K}, \aleph_0) \leq \aleph_0$ .

(2) For  $M, N \in \mathbf{K}_{\aleph_0}$  with  $M \leq_{\mathbf{K}} N$ , let us say that N is almost brimmed over M if either N is brimmed over M, or N is  $\leq_{\mathbf{K}}$ -maximal. Using amalgamation, it is easy to check that if N, N' are both almost brimmed over M, then  $N \cong_M N'$ . Moreover there always exists an almost brimmed model over any  $M \in \mathbf{K}_{\aleph_0}$ .

Fix  $\langle M_n^* : n < \omega \rangle$  such that for any  $M \in \mathbf{K}_{\aleph_0}$  there exists  $n < \omega$  such that  $M \cong M_n^*$  (possible as  $I(\mathbf{K}, \aleph_0) \leq \aleph_0$ ). For each  $n < \omega$ , fix  $N_n^* \in \mathbf{K}_{\aleph_0}$  almost brimmed over  $M_n^*$ . We have:

- $\circledast_1$  For  $M, N \in \mathbf{K}_{\aleph_0}$ :
  - (a) There is  $n < \omega$  and an isomorphism  $f : M_n^* \cong M$ .
  - (b) If N is almost brimmed over M, then any such f extends to  $g: N_n^* \cong N$ .
- $\circledast_2$  For  $M_1, M_2 \in \mathbf{K}_{\aleph_0}, M_1 \leq_{\mathbf{K}} M_2$  if and only if  $M_1 \subseteq M_2$  and for some  $n < \omega$ , for some  $(N, f_1, f_2)$  we have:  $M_1 \subseteq M_2 \subseteq N$  and  $f_\ell$  is an isomorphism from  $(M_n^*, N_n^*)$  onto  $(M_\ell, N)$ .

[Why? The implication "if" holds by the coherence axiom of AECs. The implication "only if" holds as there is  $N \in \mathbf{K}_{\aleph_0}$  which is almost brimmed over  $M_2$  (and so  $M_2 \leq_{\mathbf{K}} N$ ) hence N is almost brimmed over  $M_1$  and use  $\circledast_1$  above.]

The result now follows from  $\circledast_2$ .

By [**BL16**, 3.3], it follows that **K** is  $PC_{\aleph_0}$ .

The following appears already in [She09a, I.4.3]:

DEFINITION 23.4.3. Let  $\phi(\bar{x})$  be a formula in  $\mathbb{L}_{\infty,\aleph_0}(\tau_{\mathbf{K}})$  and let  $M \in \mathbf{K}_{\aleph_0}$ ,  $\bar{a} \in {}^{<\omega}M$ . We define  $M \Vdash_{\mathbf{K}} \phi[\bar{a}]$  (we will just write  $M \Vdash \phi[\bar{a}]$  as  $\mathbf{K}$  is fixed) by induction on  $\phi$  as follows:

- If  $\phi$  is atomic,  $M \Vdash \phi[\bar{a}]$  if and only if  $M \models \phi[\bar{a}]$ .
- If  $\phi(\bar{x}) = \wedge_{i < \alpha} \phi_i[\bar{x}]$ , then  $M \Vdash \phi[\bar{a}]$  if and only if  $M \Vdash \phi_i[\bar{a}]$  for all  $i < \alpha$ .
- If  $\phi(\bar{x}) = \exists \bar{y}\psi(\bar{y},\bar{x})$ , then  $M \Vdash \phi[\bar{a}]$  if and only if for every  $N \in \mathbf{K}_{\aleph_0}$  with  $M \leq_{\mathbf{K}} N$ , there exists  $N' \in \mathbf{K}_{\aleph_0}$  with  $N \leq_{\mathbf{K}} N'$  and  $\bar{b} \in {}^{<\omega}N'$  such that  $N' \Vdash \psi[\bar{b},\bar{a}]$ .
- If  $\phi(\bar{x}) = \neg \psi(\bar{x})$ , then  $M \Vdash \phi[\bar{a}]$  if and only if for every  $N \in \mathbf{K}_{\aleph_0}$  with  $M \leq_{\mathbf{K}} N, N \not\models \psi[\bar{a}].$
- If  $\phi(\bar{x}) = \forall \bar{y}\psi(\bar{y},\bar{x})$ , then  $M \Vdash \phi[\bar{a}]$  if and only if  $M \Vdash \neg \exists \bar{y} \neg \psi(\bar{y},\bar{a})$ .
- If  $\phi(\bar{x}) = \bigvee_{i < \alpha} \phi_i(\bar{x})$ , then  $M \Vdash \phi[\bar{a}]$  if and only if  $M \Vdash \neg \wedge_{i < \alpha} \neg \phi_i[\bar{a}]$ .

We now state some basic facts about forcing. In particular, forcing is very well-behaved on amalgamation bases.

LEMMA 23.4.4. Let  $M, N \in \mathbf{K}_{\aleph_0}$  with  $M \leq_{\mathbf{K}} N$ ,  $\bar{a} \in {}^{<\omega}M$ , and  $\phi(\bar{x})$  be an  $\mathbb{L}_{\infty,\aleph_0}(\tau_{\mathbf{K}})$ -formula.

Then:

- (1) If  $M \Vdash \phi[\bar{a}]$ , then  $N \Vdash \phi[\bar{a}]$ .
- (2) If  $M \Vdash \phi[\bar{a}]$ , then  $M \nvDash \neg \phi[\bar{a}]$ . If  $M \nvDash \neg \phi[\bar{a}]$ , then there exists  $N \in \mathbf{K}_{\aleph_0}$ with  $M \leq_{\mathbf{K}} N$  such that  $N \Vdash \phi[\bar{a}]$ .
- (3)  $M \Vdash \phi[\bar{a}]$  if and only if for every  $N \in \mathbf{K}_{\aleph_0}$  with  $M \leq_{\mathbf{K}} N$  there exists  $N' \in \mathbf{K}_{\aleph_0}$  such that  $N \leq_{\mathbf{K}} N'$  and  $N' \Vdash \phi[\bar{a}]$ .
- (4) If  $M \not\models \phi[\bar{a}]$ , then there exists  $N \in \mathbf{K}_{\aleph_0}$  such that  $M \leq_{\mathbf{K}} N$  and  $N \Vdash \neg \phi[\bar{a}]$ .
- (5) If M is an amalgamation base, then either  $M \Vdash \phi[\bar{a}]$  or  $M \Vdash \neg \phi[\bar{a}]$ .
- (6) If M is an amalgamation base, then  $M \Vdash \phi[\bar{a}]$  if and only if  $N \Vdash \phi[\bar{a}]$ .
- (7) If  $M_0 \in \mathbf{K}_{\aleph_0} \cup \{\emptyset\}$ , M is brimmed over  $M_0$ , and  $N \Vdash \psi[\bar{a}, \bar{b}]$  (with  $\bar{b} \in N$ ), then there exists  $\bar{b}' \in M$  such that  $M \Vdash \psi[\bar{a}, \bar{b}']$ .
- (8) If M is a brimmed amalgamation base, then  $M \Vdash \phi[\bar{a}]$  if and only if  $M \models \phi[\bar{a}]$ .

Proof.

- (1) Straightforward induction on  $\phi$ .
- (2) By definition of  $M \Vdash \neg \phi[\bar{a}]$ .
- (3) Straightforward induction on  $\phi$ .
- (4) By the previous part and definition of forcing a negation.
- (5) If  $M \not\models \phi[\bar{a}]$  and  $M \not\models \neg \phi[\bar{a}]$ , then by the previous parts there exists extensions  $M_1, M_2 \in \mathbf{K}_{\aleph_0}$  of M which force  $\phi$  and  $\neg \phi$  respectively. Use amalgamation to get a contradiction.

- (6) We have already shown the left to right direction. For the right to left direction, suppose that  $M \not\models \phi[\bar{a}]$ . Then by the previous part  $M \Vdash \neg \phi[\bar{a}]$  so  $N \Vdash \neg \phi[\bar{a}]$  so  $N \not\models \phi[\bar{a}]$ , as desired.
- (7) Since M is brimmed over  $M_0$ , there exists  $M_1 \in \mathbf{K}_{\aleph_0}$  such that  $\bar{a} \in M_1$ ,  $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M$ , and M is universal over  $M_1$ . Let  $f : N \xrightarrow[M_1]{} M$ . Then

 $f[N] \Vdash \psi[f(\bar{b}), \bar{a}]$ , so  $M \Vdash \psi[f(\bar{b}), ba]$ , so  $\bar{b}' := f(\bar{b})$  is as desired.

(8) Straightforward induction on  $\phi$ , using the previous part for the existential case.

DEFINITION 23.4.5. For  $M \in \mathbf{K}_{\aleph_0}$ ,  $B \subseteq |M|$ , and  $\bar{a} \in {}^{<\omega}M$ , let gentp $(\bar{a}, B, M)$ (the generic type of  $\bar{a}$  over B in M) be the following set:

$$\{\phi(\bar{x},\bar{b}) \mid \phi(\bar{x},\bar{y}) \in \mathbb{L}_{\aleph_1,\aleph_0}(\tau_{\mathbf{K}}), \bar{b} \in {}^{<\omega}B, M \Vdash \phi[\bar{a},\bar{b}]\}$$

Note that generic types are always rougher than orbital types. See Corollary 23.4.12 for a converse.

LEMMA 23.4.6. Let  $M_1, M_2 \in \mathbf{K}_{\aleph_0}$  be amalgamation bases,  $B \subseteq |M_1| \cap |M_2|$ and  $\bar{a}_{\ell} \in {}^{<\omega}M_{\ell}$ . If  $\operatorname{gtp}(\bar{a}_1, B, M_1) = \operatorname{gtp}(\bar{a}_2, B, M_2)$ , then  $\operatorname{gentp}(\bar{a}_1, B, M_1) = \operatorname{gentp}(\bar{a}_2, B, M_2)$ .

**PROOF.** By the definition of orbital types and Lemma 23.4.4(6).  $\Box$ 

Assuming there is a universal extension over  $M_0$ , the set of generic types over  $M_0$  will be the set of generic types realized in the universal extension. In particular, it will be countable:

LEMMA 23.4.7. For any  $M_0 \in \mathbf{K}_{\aleph_0} \cup \{\emptyset\}$  and any  $M \in \mathbf{K}_{\aleph_0}$  universal over  $M_0$ , we have:

 $\{\operatorname{gentp}(\bar{a}, M_0, M) \mid \bar{a} \in {}^{<\omega}M\} = \{\operatorname{gentp}(\bar{a}, M_0, N) \mid N \in \mathbf{K}_{\aleph_0}, M_0 \leq_{\mathbf{K}} N, \bar{a} \in {}^{<\omega}N\}$ (where by convention we set  $\emptyset \leq_{\mathbf{K}} N$  for every  $N \in \mathbf{K}$ )

**PROOF.** Use universality of M and Lemma 23.4.4(6).

The following technical lemma shows how to code a generic type inside a single formula.

LEMMA 23.4.8. Assume that **K** is nicely stable in  $\aleph_0$  (recall Definition 23.2.2). Fix an amalgamation base  $M_0 \in \mathbf{K}_{\aleph_0} \cup \{\emptyset\}$ . There exists a sequence  $\langle \phi_m^{M_0} : m < \omega \rangle$  such that:

- (1) For each  $m < \omega$ ,  $\phi_m^{M_0}$  is an  $\mathbb{L}_{\aleph_1,\aleph_0}(\tau_{\mathbf{K}})$ -formula with parameters from  $M_0$ .
- (2) For any  $M \in \mathbf{K}_{\aleph_0}$  extending  $M_0$  and any  $\bar{a} \in {}^{<\omega}M$ , there is a unique  $m = m(\bar{a}, M_0, M) < \omega$  such that  $M \Vdash \phi_m^{M_0}[\bar{a}]$  and  $M \Vdash \neg \phi_{m'}^{M_0}[\bar{a}]$  for all  $m' \neq m$ .
- (3) For any  $M \in \mathbf{K}_{\aleph_0}$  extending  $M_0$  and any  $\bar{a}, \bar{b} \in {}^{<\omega}M$ , gentp $(\bar{a}, M_0, M) =$  gentp $(\bar{b}, M_0, M)$  if and only if  $m(\bar{a}, M_0, M) = m(\bar{b}, M_0, M)$ .

PROOF. Say  $\{p_i : i < \omega\} = \{\text{gentp}(\bar{a}, M_0, N) : N \in \mathbf{K}_{\aleph_0}, M_0 \leq_{\mathbf{K}} N, \bar{a} \in {}^{<\omega}N \}$ (this set is countable by Lemma 23.4.7). For each  $i \neq j$  in  $\omega$ , there exists  $\psi_{i,j}$  such that  $\psi_{i,j} \in p_i$  and  $\neg \psi_{i,j} \in p_j$ . For  $m < \omega$ , set  $\phi_m^{M_0} := \wedge_{m \neq j} \psi_{m,j}$ . It is straightforward to see that this works. We have all the tools available to study homogeneous models and show that they coincide with brimmed models.

DEFINITION 23.4.9. Let D be a set of orbital types and let  $M \in \mathbf{K}$ . We say that M is  $(D, \aleph_0)$ -homogeneous if whenever  $p \in D$  is the type of an (n + m)-elements sequence and  $\bar{a} \in {}^n M$  realizes  $p^n$  (the restriction of p to its first n "variables"), there exists a sequence  $\bar{b} \in {}^m M$  such that  $\bar{a}\bar{b}$  realizes p. When  $D = \mathrm{gS}^{<\omega}(\emptyset, M)$ , we omit it.

THEOREM 23.4.10. Assume that **K** is nicely stable in  $\aleph_0$ . Let  $M_0 \in \mathbf{K}_{\aleph_0} \cup \{\emptyset\}$  be an amalgamation base, and let  $M \in \mathbf{K}_{\aleph_0}$  be such that  $M_0 \leq_{\mathbf{K}} M$ . The following are equivalent:

(1) M is brimmed over  $M_0$ 

(2) M is  $(gS^{<\omega}(M_0), \aleph_0)$ -homogeneous.

**PROOF.** First we show:

<u>Claim 1</u>: If M is brimmed over  $M_0$ , then M is  $\aleph_0$ -homogeneous over  $M_0$  in the sense of generic types. That is, if  $\bar{a}_1, \bar{b}_1, \bar{a}_2 \in {}^{<\omega}M$  and  $\operatorname{gentp}(\bar{a}_1, M_0, M) = \operatorname{gentp}(\bar{a}_2, M_0, M)$ , then there exists  $\bar{b}_2 \in {}^{<\omega}M$  such that  $\operatorname{gentp}(\bar{a}_1\bar{b}_1, M_0, M) = \operatorname{gentp}(\bar{a}_2\bar{b}_2, M_0, M)$ .

<u>Proof of Claim 1</u>: Let  $m := m(\bar{a}_1, M_0, M)$  and  $n := m(\bar{a}_1 \bar{b}_1, M_0, M)$  (see Lemma 23.4.8). Since the generic types are equal, we must have that  $m = m(\bar{a}_2, M_0, M)$ . Consider the formula

$$\psi(\bar{x}) := \phi_m(\bar{x}) \land \exists \bar{y} \phi_n(\bar{x}, \bar{y})$$

where  $\ell(\bar{x}) = \ell(\bar{a}_1)$  and  $\ell(\bar{y}) = \ell(\bar{b}_1)$ . We have that  $M \Vdash \psi[\bar{a}_1]$  (the existential part is witnessed by  $\bar{b}_1$ ) so also  $M \Vdash \psi[\bar{a}_2]$  by equality of the generic types. By definition of forcing this means that there exists  $N \in \mathbf{K}_{\aleph_0}$  and  $\bar{b}_2^* \in {}^{<\omega}N$  such that  $M \leq_{\mathbf{K}} N$  and  $N \Vdash \phi_n[\bar{a}_2, \bar{b}_2^*]$ . Now by Lemma 23.4.4(7 (using that M is brimmed over  $M_0$ ), there exists  $\bar{b}_2 \in M$  such that  $M \Vdash \phi_n[\bar{a}_2, \bar{b}_2]$ , as desired.  $\dagger_{\text{Claim 1}}$ 

<u>Claim 2</u>: If M is brimmed over  $M_0$  and  $\bar{a}, \bar{b} \in {}^{<\omega}M$ , then gentp $(\bar{a}, M_0, M) =$  gentp $(\bar{b}, M_0, M)$  if and only if there is an automorphism of M sending  $\bar{a}$  to  $\bar{b}$  and fixing  $M_0$  pointwise.

<u>Proof of Claim 2</u>: The right to left direction is clear and the left to right direction is a direct back and forth argument using Claim 1.  $\dagger_{\text{Claim 2}}$ 

From Claim 2, it follows directly that if M is brimmed over  $M_0$  then it is  $(gS^{<\omega}(M_0), \aleph_0)$ -homogeneous. Conversely, the countable  $(gS^{<\omega}(M_0), \aleph_0)$ -homogeneous model is unique and so it must also be brimmed over  $M_0$ .

COROLLARY 23.4.11. If **K** is nicely stable in  $\aleph_0$ , then there is a superlimit model of cardinality  $\aleph_0$ .

PROOF. The  $\aleph_0$ -homogeneous model works (Theorem 23.4.10 with  $M_0 := \emptyset$  implies its existence).

We deduce the following characterization of types:

COROLLARY 23.4.12. Assume that **K** is nicely stable in  $\aleph_0$ . Let  $M_0 \in \mathbf{K}_{\aleph_0} \cup \{\emptyset\}$  be an amalgamation base. Let  $M \in \mathbf{K}_{\aleph_0}$  be a brimmed model extending  $M_0$  (but *not* necessarily brimmed *over*  $M_0$ ). Let  $\bar{a}_1, \bar{a}_2 \in {}^{<\omega}M$ . The following are equivalent:

(1)  $gtp(\bar{a}_1, M_0, M) = gtp(\bar{a}_2, M_0, M).$ 

(2) gentp( $\bar{a}_1, M_0, M$ ) = gentp( $\bar{a}_2, M_0, M$ ).

- (3)  $\operatorname{tp}_{\mathbb{L}_{\infty,\aleph_0}(\tau_{\mathbf{K}})}(\bar{a}_1, M_0, M) = \operatorname{tp}_{\mathbb{L}_{\infty,\aleph_0}}(\bar{a}_2, M_0, M).$
- (4)  $\operatorname{tp}_{\mathbb{L}_{\aleph_1,\aleph_0}(\tau_{\mathbf{K}})}(\bar{a}_1, M_0, M) = \operatorname{tp}_{\mathbb{L}_{\aleph_1,\aleph_0}}(\bar{a}_2, M_0, M).$

**PROOF.** Let N be brimmed over M (hence over  $M_0$ ). First we prove:

Claim: For any  $\bar{a} \in M$ , gentp $(\bar{a}, M_0, M)$  = gentp $(\bar{a}, M_0, N)$  and tp $_{\mathbb{L}_{\infty,\aleph_0}(\tau_{\mathbf{K}})}(\bar{a}, M_0, M)$  = tp $_{\mathbb{L}_{\infty,\aleph_0}(\tau_{\mathbf{K}})}(\bar{a}, M_0, N)$ 

<u>Proof of Claim</u>: This follows from Lemmas 23.4.4(6),(8).  $\dagger_{\text{Claim}}$ Now consider the following statement:

(1)' There is an automorphism of N fixing  $M_0$  sending  $\bar{a}_1$  to  $\bar{a}_2$ .

Using it, we complete the proof of the theorem as follows:

- (1) is equivalent to (1)' (by a back and forth argument).
- (1)' implies (3) (straightforward using the Claim).
- (3) implies (4) (trivial).
- (4) is equivalent to (2) by Lemmas 23.4.4(6),(8), recalling that generic types are defined using L<sub>ℵ1,ℵ0</sub>(τ)-formulas.
- (4) implies (1)' by the Claim, the equivalence of (2) with (4), and Claim 2 in the proof of Theorem 23.4.10.

COROLLARY 23.4.13 (Locality). Assume that **K** is nicely stable in  $\aleph_0$ . Let  $M \in \mathbf{K}_{\aleph_0}$  be an amalgamation base. Let  $p, q \in \mathrm{gS}^{<\omega}(M)$ . If  $p \neq q$ , then there exists  $A \subseteq |M|$  finite such that  $p \upharpoonright A \neq q \upharpoonright A$ .

PROOF. Suppose that  $p \neq q$ . Say  $p = \operatorname{gtp}(\overline{a}, M, N)$ ,  $q = \operatorname{gtp}(\overline{b}, M, N)$ , with N brimmed over M. By Corollary 23.4.12,  $\operatorname{gentp}(\overline{b}, M, N) \neq \operatorname{gentp}(\overline{b}, M, N)$ , so there exists  $A \subseteq |M|$  finite such that  $\operatorname{gentp}(\overline{a}, A, N) \neq \operatorname{gentp}(\overline{b}, A, N)$ . By Lemma 23.4.6, this implies that  $\operatorname{gtp}(\overline{a}, A, N) \neq \operatorname{gtp}(\overline{b}, A, N)$ , as desired.  $\Box$ 

We have also justified assuming amalgamation in the following sense:

COROLLARY 23.4.14. If **K** is nicely stable in  $\aleph_0$ , then there exists an AEC  $\mathbf{K}' = (K', \leq_{\mathbf{K}'})$  such that:

- (1)  $LS(\mathbf{K}') = \aleph_0$ .
- (2)  $\mathbf{K}'_{\langle \aleph_0} = \emptyset.$
- (3)  $\tau_{\mathbf{K}'} = \tau_{\mathbf{K}}.$
- (4)  $K' \subseteq K$  and for  $M, N \in K', M \leq_{\mathbf{K}'} N$  if and only if  $M \leq_{\mathbf{K}} N$ .
- (5) For any  $M \in \mathbf{K}$  there exists  $M' \in \mathbf{K}'$  with  $M \leq_{\mathbf{K}} M'$ .
- (6)  $\mathbf{K}'$  is categorical in  $\aleph_0$ .
- (7)  $\mathbf{K}'$  is very nicely stable in  $\aleph_0$ . In particular it has amalgamation in  $\aleph_0$ .
- (8) For  $M, N \in \mathbf{K}'_{\aleph_0}, M \leq_{\mathbf{K}'} N$  implies  $M \preceq_{\mathbb{L}_{\infty, \aleph_0}(\tau_{\mathbf{K}'})} N$ .
- (9)  $\mathbf{K}'$  is  $\mathrm{PC}_{\aleph_0}$ .

PROOF. Let  $M \in \mathbf{K}_{\aleph_0}$  be superlimit (exists by Corollary 23.4.11). Let  $K'_{\aleph_0} := \{N \in K : N \cong M\}$ . Now let  $\mathbf{K}'$  be the AEC generated by  $(K'_{\aleph_0}, \leq_{\mathbf{K}})$ . One can easily check that  $\mathbf{K}'$  is nicely stable in  $\aleph_0$  and from categoricity in  $\aleph_0$  we get amalgamation in  $\aleph_0$ , hence (7) holds. To see (8), use Corollary 23.4.12. As for (9), it follows from Theorem 23.4.2.

We can now construct the promised good  $\aleph_0$ -frame. Its nonforking relation will be define terms of splitting. We will work in the class generated by the superlimit so the reader may assume that all the models are brimmed.

DEFINITION 23.4.15. For  $M \in \mathbf{K}_{\aleph_0}$  brimmed and  $A \subseteq |M|, p \in \mathrm{gS}^{<\omega}(M)$  splits over A if there exists an automorphism f of M such that  $f(p) \neq p$ .

REMARK 23.4.16. Using Corollary 23.4.12, one can check that (assuming that **K** is nicely stable in  $\aleph_0$ ) this is equivalent to the syntactic definition using  $\mathbb{L}_{\aleph_1,\aleph_0}(\tau_{\mathbf{K}})$ -formulas.

The following is proven in [She09a, I.5.6].

FACT 23.4.17. Assume that **K** is nicely stable in  $\aleph_0$  and categorical in  $\aleph_0$ . If  $M \in \mathbf{K}_{\aleph_0}$  and  $p \in \mathrm{gS}^{<\omega}(M)$ , then there exists  $A \subseteq |M|$  finite such that p does not split over A.

DEFINITION 23.4.18. Assume that **K** is nicely stable in  $\aleph_0$ . We define a pre-( $< \omega, \aleph_0$ )-frame  $\mathfrak{s} = (\mathbf{K}_{\mathfrak{s}}, \mathrm{gS}_{\mathfrak{s}}^{\mathrm{bs}}, \downarrow)$  by:

- (1)  $\mathbf{K}_{\mathfrak{s}} = \mathbf{K}'_{\aleph_0}$ , where  $\mathbf{K}'$  is as given by Corollary 23.4.14.
- (2)  $gS_{\mathfrak{s}}^{\mathrm{bs}}(M)$  is the set of all nonalgebraic types of finite sequences over M.
- (3) For  $M \leq_{\mathbf{K}^*} N$ ,  $p \in \mathrm{gS}^{\mathrm{bs}}_{\mathfrak{s}}(N)$  does not fork over M if and only if there exists a finite  $A \subseteq |M|$  so that p does not split over A.

THEOREM 23.4.19. If **K** is nicely stable in  $\aleph_0$ , then  $\mathfrak{s}$  is a categorical typefull good ( $\langle \omega, \aleph_0 \rangle$ )-frame. Moreover LWNF<sub> $\mathfrak{s}$ </sub> has the symmetry property (recall Definitions 23.3.4 and 23.3.12). In particular,  $\mathfrak{s}$  is good<sup>+</sup> and **K** has a superlimit of cardinality  $\aleph_1$ .

PROOF. Without loss of generality assume to simplify the notation that the class has already been changed, i.e.  $\mathbf{K} = \mathbf{K}'$  where  $\mathbf{K}'$  is from Corollary 23.4.14. Equivalently,  $\mathbf{K}$  is categorical in  $\aleph_0$ . Once we have shown that  $\mathfrak{s}$  is a type-full good frame, the moreover part follows from Corollary 23.4.13 and Theorem 23.3.20. The last sentence is by Theorem 23.3.15 (it is easy to check that if  $\mathbf{K}'$  has a superlimit in  $\aleph_1$  then  $\mathbf{K}$  also has one).

Except for symmetry, the axioms of good frames are easy to check (see the proof of [She09a, II.3.4]). For example:

- Local character: Let  $\langle M_i : i \leq \delta \rangle$  be increasing continuous in  $\mathbf{K}_s$ . Let  $p \in \mathrm{gS}^{\mathrm{bs}}_s(M_\delta)$ . By Fact 23.4.17, there exists a finite  $A \subseteq |M_\delta|$  such that p does not split over A. Pick  $i < \delta$  such that  $A \subseteq |M_i|$ . Then p does not fork over  $M_i$ .
- Uniqueness: standard, see for example Lemma 7.4.8 (and Remark 23.4.16).
- Extension: follows on general grounds, see Lemma 19.3.5.

Symmetry is the hardest to prove, and is done as in [She09a, I.5.30]. We give a full proof for the convenience of the reader.

Suppose that  $\operatorname{gtp}(\overline{b}, N_2, N_3)$  does not fork over  $N_0$  and let  $\overline{c} \in {}^{<\omega}N_2 \setminus N_1$ . We want to find  $N_1, N'_3$  such that  $N_0 \leq_{\mathbf{K}^*} N_1 \leq_{\mathbf{K}^*} N'_3$ ,  $N_3 \leq_{\mathbf{K}^*} N'_3$ ,  $\overline{b} \in {}^{<\omega}N_1$  and  $\operatorname{gtp}(\overline{c}, N_1, N'_3)$  does not fork over  $N_0$ . Assume for a contradiction that there is no such  $N_1$ . Using existence for LWNF<sub>s</sub> = LWNF (see Theorem 23.3.11), as well as the extension property for nonforking, we can increase  $N_2$  and  $N_3$  if necessary and find  $N_1$  such that LWNF $(N_0, N_1, N_2, N_3)$ ,  $N_\ell$  is brimmed over  $N_0$ , and  $N_3$  is brimmed over  $N_\ell$  for  $\ell = 1, 2$ . By assumption,  $p := \operatorname{gtp}(\overline{c}, N_1, N_3)$  forks over  $N_0$ .

<u>Claim 1</u>: Let *I* be the linear order  $[0, \infty) \cap \mathbb{Q}$ . There exists an increasing chain  $\langle M_s : s \in I \rangle$  such that for any s < t in *I*,  $M_s, M_t$  are in  $\mathbf{K}_{\aleph_0}$  and  $M_t$  is brimmed over  $M_s$ .

<u>Proof of Claim 1</u>: Fix  $\langle M_i^* : i < \omega_1 \rangle$  increasing continuous in  $\mathbf{K}_{\aleph_0}$  such that  $M_{i+1}^*$  is brimmed over  $M_i^*$  for all  $i < \omega_1$ . Using undefinability of well-ordering, pick a countable ill-founded model of set theory  $\mathfrak{B} = (A, E, \langle M_s : s \in I^* \rangle)$  elementarily equivalent to  $(H(\aleph_2), \in, \langle M_i^* : i < \omega_1 \rangle)$ . Now  $I^*$  contains a copy of the rationals by a general argument on ill-founded models of set theory, see [**Fri73**, Section 3]). Recalling that **K** is PC<sub>\aleph\_0</sub> (see Corollary 23.4.14) and the syntactic characterization of brimmed models (Theorem 23.4.10), the result follows.  $\dagger_{\text{Claim 1}}$ 

Fix I,  $\langle M_s : s \in I \rangle$  as in Claim 1. Fix  $N'_0$  such that  $N_0$  is brimmed over  $N'_0$  and  $p \upharpoonright N_0$  does not fork over  $N'_0$ .

For any fixed infinite  $J \subseteq I$ , write  $M_J := \bigcup_{s \in J} M_s$ . Assume now that  $M_I$  is brimmed over  $M_J$ . Let  $N_0^J := M_J$ ,  $N_1^J := M_I$ . Let  $N_3^J$  be brimmed over  $N_1^J$ . By categoricity and uniqueness of brimmed models, there exists  $f_0 : N_0' \cong M_0$ ,  $f_0^J : N_0 \cong N_0^J$ ,  $f_1^J : N_1 \cong N_1^J$ , and  $f_3^J : N_3 \cong N_3^J$  such that  $f_0 \subseteq f_0^J \subseteq f_1^J \subseteq f_3^J$ . Let  $f_2^J := f_3^J \upharpoonright N_2$  and let  $N_2^J := f_2^J[N_2]$ . Note that LWNF $(N_0^J, N_1^J, N_2^J, N_3^J)$ holds.

Let  $p_J := \operatorname{gtp}(f_3^J(\bar{c}), f_3^J[N_1], f_3^J[N_3]) = \operatorname{gtp}(f_3^J(\bar{c}), M_I, N_3^J)$ . Since we are assuming that  $\operatorname{gtp}(\bar{c}, N_1, N_3)$  forks over  $N_0$ , we have that  $p_J$  forks over  $N_0^J$ . Moreover  $p_J \upharpoonright N_0^J$  does not fork over  $M_0$ .

<u>Claim 2</u>: If J has no last elements,  $I \setminus J$  has no first elements, and  $t \in I \setminus J$ , then  $p_J \upharpoonright M_t$  forks over  $N_0^J$ .

<u>Proof of Claim 2</u>: Suppose that  $p_J \upharpoonright M_t$  does not fork over  $N_0^J$ . Note that  $M_t$  is brimmed over  $M_J$ . Find  $N'_1$  such that  $N_0 \leq_{\mathbf{K}^*} N'_1 \leq_{\mathbf{K}^*} N_1, N'_1$  is brimmed over  $N_1$ , and  $f_1^J : N'_1 \cong M_t$ . Let  $\bar{b}' \in {}^{<\omega}N'_1$  be such that  $\operatorname{gtp}(\bar{b}', N_0, N'_1) = \operatorname{gtp}(\bar{b}, N_0, N_1)$ . Since LWNF $(N_0, N_1, N_2, N_3)$ , we know that  $\operatorname{gtp}(\bar{b}', N_2, N_3)$  does not fork over  $N_0$ , hence by uniqueness  $\operatorname{gtp}(\bar{b}, N_2, N_3) = \operatorname{gtp}(\bar{b}', N_2, N_3)$ . But we have assumed shown that  $\operatorname{gtp}(\bar{c}, N'_1, N_3)$  does not fork over  $N_0$  and  $\bar{b}' \in {}^{<\omega_1}N'_1$ , hence by a simple renaming we obtain a contradiction to our hypothesis that symmetry failed.  $\dagger_{\operatorname{Claim 2}}$ 

<u>Claim 3</u>: If  $J_1 \subsetneq J_2$  are both proper initial segments of I with no last elements and  $J_2 \setminus J_1$  has no first elements, then  $p_{J_1} \neq p_{J_2}$ .

<u>Proof of Claim 3</u>: Fix  $t \in J_2 \setminus J_1$ . By Claim 2,  $p_{J_1} \upharpoonright M_t$  forks over  $N_0^{J_1}$ . We claim that  $p_{J_2} \upharpoonright M_t$  does not fork over  $N_0^{J_1}$ . Indeed recall that  $N_0^{J_2} = M_{J_2}$  and by assumption  $p_{J_2} \upharpoonright N_0^{J_2}$  does not fork over  $M_0$ . Therefore by monotonicity also  $p_{J_2} \upharpoonright M_t$  does not fork over  $M_{J_1} = N_0^{J_1}$ .  $\dagger_{\text{Claim 3}}$ 

To finish, observe that there are  $2^{\aleph_0}$  cuts of I as in Claim 3. Therefore stability fails, a contradiction.

#### CHAPTER 24

# Indiscernible extraction and Morley sequences

This chapter is based on [Vas17b]. I thank John Baldwin, José Iovino, Itay Kaplan, Alexei Kolesnikov, Anand Pillay, and Akito Tsuboi for valuable comments on earlier versions of this chapter.

#### Abstract

We present a new proof of the existence of Morley sequences in simple theories. We avoid using the Erdős-Rado theorem and instead use only Ramsey's theorem and compactness. The proof shows that the basic theory of forking in simple theories can be developed using only principles from "ordinary mathematics", answering a question of Grossberg, Iovino and Lessmann, as well as a question of Baldwin.

#### 24.1. Introduction

Shelah [**She80**, Lemma 9.3] has shown that, in a simple first-order theory T, Morley sequences exist for every type. The proof proceeds by first building an independent sequence of length  $\beth_{(2^{|T|})^+}$  for the given type and then using the Erdős-Rado theorem together with Morley's method to extract the desired indiscernibles.

After slightly improving on the length of the original independent sequence [GIL02, Appendix A], Grossberg, Iovino and Lessmann observed that, in contrast, most of the theory of forking in a stable first-order theory T does not need the existence of such "big" cardinals. The authors then asked whether the same could be said about simple theories, and so in particular whether there was another way to build Morley sequences there.

Baldwin (see [**Bal10**] and [**Bal13**, Question 3.1.9]) similarly asked<sup>1</sup> whether the equivalence between forking and dividing in simple theories had an alternative proof.

We answer those questions in the affirmative by showing how to extract a Morley sequence from any infinite independent sequence. Our construction relies on a property of forking we call *dual finite character*. We show it holds in simple theories, and that the converse is also true (the latter was noticed by Itay Kaplan).

#### 24.2. Preliminaries

For the rest of this chapter, fix a complete first-order theory T in a language L(T) and work inside its monster model  $\mathfrak{C}$ . We write |T| for  $|L(T)| + \aleph_0$ . We denote by  $\operatorname{Fml}(L(T))$  the set of first-order formulas in the language L(T). If A is a set, we say a formula is *over* A if it has parameters from A. For a tuple  $\bar{a}$  in  $\mathfrak{C}$  and  $\phi$  a formula, we write  $\models \phi[\bar{a}]$  instead of  $\mathfrak{C} \models \phi[\bar{a}]$ .

<sup>&</sup>lt;sup>1</sup>Akito Tsuboi [**Tsu14**] has independently answered this question.

When I is a linearly ordered set,  $(\bar{a}_i)_{i \in I}$  are tuples, and  $i \in I$ , we write  $\bar{a}_{< i}$  for  $(\bar{a}_j)_{j < i}$ . It is often assumed without comments that all the  $\bar{a}_i$ s have the same (finite) arity.

We assume the reader is familiar with forking. We will use the combinatorial definition stated e.g. in [She80, Definition 1.2]. It turns out that our construction of Morley sequences does not rely on this exact definition, but only on abstract properties of forking such as invariance, extension, and symmetry.

Recall also the definition of a Morley sequence:

DEFINITION 24.2.1. Let I be a linearly ordered set. Let  $\mathbf{I} := \langle \bar{a}_i \mid i \in I \rangle$  be a sequence of finite tuples of the same arity. Let  $A \subseteq B$  be sets, and let  $p \in S(B)$  be a type that does not fork over A.

**I** is said to be an *independent sequence for* p *over* A if:

(1) For all  $i \in I$ ,  $\bar{a}_i \models p$ .

(2) For all  $i \in I$ ,  $\operatorname{tp}(\bar{a}_i/B\bar{a}_{\leq i})$  does not fork over A.

**I** is said to be a Morley sequence for p over A if:

(1) **I** is an independent sequence for p over A.

(2)  $\mathbf{I}$  is indiscernible over B.

#### 24.3. Morley sequences in simple theories

It is well known that independent sequences can be built by repeated use of the extension property of forking. If the theory is stable, the existence of Morley sequences follows, because in such theories any sufficiently long sequence contains indiscernibles. The latter fact is no longer true in general, and in fact there are counterexamples among both simple [She85b, p. 209] and dependent [KS14] theories. Thus a different approach is needed in the unstable case. Recall from the introduction that we do not want to use big cardinals, so Morley's method cannot be used. We can however make use of the following variation of the Ehrenfeucht-Mostowski theorem:

FACT 24.3.1 ([**TZ12**], Lemma 5.1.3). Let A be a set, and let I be a linearly ordered set. Let  $\mathbf{J} := \langle \bar{a}_j \mid j < \omega \rangle$  be a sequence of finite tuples of the same arity. Then there exists a sequence  $\mathbf{I} := \langle \bar{b}_i \mid i \in I \rangle$ , indiscernible over A such that:

For any  $i_0 < \ldots < i_{n-1}$  in I, for all finite  $q \subseteq \operatorname{tp}(\overline{b}_{i_0} \ldots \overline{b}_{i_{n-1}}/A)$ , there exists  $j_0 < \ldots < j_{n-1} < \omega$  so that  $\overline{a}_{j_0} \ldots \overline{a}_{j_{n-1}} \models q$ .

Do we get a Morley sequence if we apply Fact 24.3.1 to an independent sequence? In general, we see no reason why it should be true. However, we will see that it *is* true if we assume the following local definability property of forking:

DEFINITION 24.3.2 (Dual finite character). Forking is said to have *dual finite* character (DFC) if whenever  $tp(\bar{c}/A\bar{b})$  forks over A, there is a formula  $\phi(\bar{x}, \bar{y})$  over A such that:

- $\models \phi[\bar{c}, \bar{b}]$ , and:
- $\models \phi[\bar{c}, \bar{b}']$  implies  $\operatorname{tp}(\bar{c}/A\bar{b}')$  forks over A.

A variation of DFC appears as property A.7' in [Mak84], but we haven't found any other explicit occurrence in the literature. Notice that DFC immediately implies something stronger:

PROPOSITION 24.3.3. Assume forking has DFC. Assume  $p := \operatorname{tp}(\bar{c}/A\bar{b})$  forks over A, and  $\phi(\bar{x}, \bar{y})$  is as given by Definition 24.3.2. Then  $\operatorname{tp}(\bar{c}'/A) = \operatorname{tp}(\bar{c}/A)$  and  $\models \phi[\bar{c}', \bar{b}']$  imply  $\operatorname{tp}(\bar{c}'/A\bar{b}')$  forks over A.

PROOF. Assume  $\operatorname{tp}(\overline{c}'/A) = \operatorname{tp}(\overline{c}/A)$ . Let f be an automorphism of  $\mathfrak{C}$  fixing A such that  $f(\overline{c}') = \overline{c}$ . Assume  $\models \phi[\overline{c}', \overline{b}']$ . Applying f,  $\models \phi[\overline{c}, f(\overline{b}')]$ . Since  $\phi$  witnesses DFC,  $\operatorname{tp}(\overline{c}/Af(\overline{b}'))$  forks over A. Applying  $f^{-1}$  and using invariance of forking,  $\operatorname{tp}(\overline{c}'/A\overline{b}')$  forks over A.

THEOREM 24.3.4. Assume forking has DFC. Let  $A \subseteq B$  be sets. Let  $p \in S(B)$  be a type that does not fork over A. Let I be a linearly ordered set. Then there is a Morley sequence  $\mathbf{I} := \langle \bar{b}_i \mid i \in I \rangle$  for p over A.

PROOF. By repeated use of the extension property of forking, build an independent sequence  $\mathbf{J} := \langle \bar{a}_i \mid j < \omega \rangle$  for p over A.

Let  $\mathbf{I} := \langle \bar{b}_i | i \in I \rangle$  be indiscernible over B as described by Fact 24.3.1. We claim  $\mathbf{I}$  is as required.

It is indiscernible over B, and for every  $i \in I$ , every  $\bar{b}_i$  realizes p: If  $\bar{b}_i \not\models p$ , fix a formula  $\phi(\bar{x}, \bar{b}) \in p$  so that  $\models \neg \phi[\bar{b}_i, \bar{b}]$ . By the defining property of **I**, there exists  $j < \omega$  so that  $\models \neg \phi[\bar{a}_j, \bar{b}]$ , so  $\bar{a}_j \not\models p$ , a contradiction.

It remains to see that for every  $i \in I$ ,  $p_i := \operatorname{tp}(\bar{b}_i/B\bar{b}_{<i})$  does not fork over A. Assume not, and fix  $i \in I$  so that  $p_i$  forks over A. Fix  $\bar{b} \in B$  and  $i_0 < \ldots < i_{n-1} < i$ such that  $p'_i := \operatorname{tp}(\bar{b}_i/A\bar{b}_{i_0}\ldots\bar{b}_{i_{n-1}}\bar{b})$  forks over A. Fix  $\phi(\bar{x},\bar{b}_{i_0}\ldots\bar{b}_{i_{n-1}}\bar{b}) \in p'_i$  a formula over A witnessing DFC.

Find  $j_0 < \ldots < j_n < \omega$  such that  $\models \phi[\bar{a}_{j_n}, \bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}}\bar{b}]$ . Since it has already been observed that  $\operatorname{tp}(\bar{a}_{j_n}/A) = \operatorname{tp}(\bar{b}_i/A) = p \upharpoonright A$ , Proposition 24.3.3 implies that  $\operatorname{tp}(\bar{a}_{j_n}/A\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}}\bar{b})$  forks over A, contradicting the independence of  $\mathbf{J}$ .  $\Box$ 

We now show that a simple theory has DFC (this was essentially already observed by Makkai). Recall [**Kim01**, Theorem 2.4] that T is simple exactly when forking has the symmetry property. Moreover, the methods of [**Adl09b**] show that the equivalence can be proven without using Morley sequences. The key is [**Adl09b**, Theorem 3.6], which shows (without using Morley sequences) that if the D-rank is bounded, then symmetry holds.

LEMMA 24.3.5. Assume T is simple. Then forking has DFC.

PROOF. Assume  $p := \operatorname{tp}(\bar{c}/A\bar{b})$  fork over A. By symmetry,  $q := \operatorname{tp}(\bar{b}/A\bar{c})$  forks over A. Fix  $\psi(\bar{y}, \bar{x})$  over A such that  $\psi(\bar{y}, \bar{c}) \in q$  witnesses forking, i.e. if  $\models \psi[\bar{b}', \bar{c}]$  then  $\operatorname{tp}(\bar{b}'/A\bar{c})$  forks over A.

Let  $\phi(\bar{x}, \bar{y}) := \psi(\bar{y}, \bar{x})$ . Then  $\phi(\bar{x}, \bar{b}) \in p$ , and if  $\models \phi[\bar{c}, \bar{b}']$ , then  $\models \psi[\bar{b}', \bar{c}]$ , so  $\operatorname{tp}(\bar{b}'/A\bar{c})$  forks over A, so by symmetry,  $\operatorname{tp}(\bar{c}/A\bar{b}')$  forks over A. This shows  $\phi(\bar{x}, \bar{y})$  witnesses DFC.

COROLLARY 24.3.6 (Existence of Morley sequences in simple theories). Assume T is simple. Let  $A \subseteq B$  be sets. Let  $p \in S(B)$  be a type that does not fork over A. Let I be a linearly ordered set. Then there is a Morley sequence  $\mathbf{I} := \langle \bar{b}_i \mid i \in I \rangle$  for p over A.

PROOF. Combine Lemma 24.3.5 and Theorem 24.3.4.  $\hfill \Box$ 

We end by closing the loop on our study of DFC: Lemma 24.3.5 shows that simplicity implies DFC, but it turns out that they are equivalent! This was pointed out by Itay Kaplan in a personal communication. Definition 24.3.8 and (2) implies (3) implies (1) in Theorem 24.3.9 below are due to Kaplan, and I am grateful to him for allowing me to include them here.

The key is to observe that symmetry fails very badly when the theory is not simple:

FACT 24.3.7 ([Che14], Lemma 6.16). Assume T is not simple. Then there is a model M and tuples  $\bar{b}, \bar{c}$  such that  $\operatorname{tp}(\bar{b}/M\bar{c})$  is finitely satisfiable in M, but  $\operatorname{tp}(\bar{c}/M\bar{b})$  divides over M.

We are now ready to prove that forking has DFC exactly when the theory is simple. In fact, we only need the following version of DFC:

DEFINITION 24.3.8. Forking is said to have weak dual finite character (weak DFC) if whenever M is a model and  $tp(\bar{c}/M\bar{b})$  divides over M, there is a formula  $\phi(\bar{x},\bar{y})$  over M such that:

- $\models \phi[\bar{c}, \bar{b}]$ , and:
- $\models \phi[\bar{c}, \bar{b}']$  implies  $\operatorname{tp}(\bar{c}/M\bar{b}')$  is not finitely satisfiable in M.

THEOREM 24.3.9. The following are equivalent:

- (1) T is simple.
- (2) Forking has DFC.
- (3) Forking has weak DFC.

PROOF. (1) implies (2) is Lemma 24.3.5, and (2) implies (3) is because finite satisfiability implies nonforking. We show (3) implies (1). Assume T is not simple. Fix M and  $\bar{b}$ ,  $\bar{c}$  as given by Fact 24.3.7. In particular,  $p := \operatorname{tp}(\bar{c}/M\bar{b})$  divides over M. Let  $\phi(\bar{x}, \bar{y})$  be a formula over M such that  $\models \phi[\bar{c}, \bar{b}]$ . By assumption,  $\operatorname{tp}(\bar{b}/M\bar{c})$  is finitely satisfiable in M, so in particular there is  $\bar{b}' \in M$  such that  $\models \phi[\bar{c}, \bar{b}']$ . Thus  $\operatorname{tp}(\bar{c}/M\bar{b}') = \operatorname{tp}(\bar{c}/M)$  must be finitely satisfiable over M, hence  $\phi(\bar{x}, \bar{y})$  cannot witness weak DFC for p. Since  $\phi$  was arbitrary, this shows weak DFC fails.  $\Box$ 

We end by pointing out that all the results of this chapter could be formalized in a weak fragment of ZFC, such as ZFC - Replacement - Power set + "For any set X of size  $\leq |T|$ ,  $\mathcal{P}(\mathcal{P}(X))$  exists"<sup>2</sup>. Going further, it would be interesting to extend Harnik's work on the reverse mathematics of stability theory [Har85, Har87] by finding the exact proof-theoretic strength of the existence of Morley sequences.

<sup>&</sup>lt;sup>2</sup>Formally, we have to work in work in a language containing a constant symbol standing for |T|.

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