GALOIS-STABILITY FOR TAME ABSTRACT ELEMENTARY CLASSES

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ABSTRACT. We introduce *tame abstract elementary classes* as a generalization of all cases of abstract elementary classes that are known to permit development of stability-like theory. In this paper we explore stability results in this new context. We assume that $\mathcal{K}$ is a tame abstract elementary class satisfying the amalgamation property with no maximal model. The main results include:

Theorem 0.1. *Suppose that $\mathcal{K}$ is not only tame, but $\mathrm{LS}(\mathcal{K})$-tame. If $\mu \geq \mathrm{Hanf}(\mathcal{K})$ and $\mathcal{K}$ is Galois stable in μ, then $\kappa_\mu(\mathcal{K}) < \beth_{(2^{\mathrm{Hanf}(\mathcal{K})})^+}$, where $\kappa_\mu(\mathcal{K})$ is a relative of $\kappa(T)$ from first order logic.*

$\mathrm{Hanf}(\mathcal{K})$ is the Hanf number of the class $\mathcal{K}$. It is known that $\mathrm{Hanf}(\mathcal{K}) \leq \beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$

The theorem generalizes a result from [Sh3]. It is used to prove both the existence of Morley sequences for non-splitting (improving Claim 4.15 of [Sh 394] and a result from [GrLe1]) and the following initial step towards a stability spectrum theorem for tame classes:

Theorem 0.2. *If $\mathcal{K}$ is Galois-stable in some $\mu > \beth_{(2^{\mathrm{Hanf}(\mathcal{K})})^+}$, then $\mathcal{K}$ is stable in every κ with $\kappa^\mu = \kappa$. E.g. under GCH we have that $\mathcal{K}$ Galois-stable in μ implies that $\mathcal{K}$ is Galois-stable in μ^{+n} for all $n < \omega$.*

INTRODUCTION

In the last twenty years most of Shelah's effort in model theory was in developing classification theory for abstract elementary classes (Some important mile stones are: [Sh 87a], [Sh 87b],[Sh 88], [Sh 300], [Sh 394], [Sh 576], [Sh 600] and [Sh 705]). An abstract elementary class (AEC) is a class of structures of the same similarity type endowed with a morphism satisfying natural properties such as closure under directed limits (see [Gr1] for an introduction). The notion of an AEC (introduced by Shelah in [Sh 88]) is

broad enough to capture classes axiomatized by non-first-order logics including $L_{\omega_1,\omega}(\mathbf{Q})$. To date, there is no known stability theory or even a categoricity theorem for abstract elementary classes without some additional strong assumptions.

The most general of Shelah's attempts to develop a stability theory is [Sh 600] where Shelah works in the context of AECs with the amalgamation and joint embedding properties and a very nicely behaved forking-like relation. He introduces the notion of good-frame which is a generalization of a first-order superstable theory (the definition alone needs a couple of pages). Most of his 100-page paper is dedicated to the derivation of structural results under the assumption that the AEC has a forking-like relation.

In our paper we deal with a much wider class of AECs than Shelah in [Sh 600] as we do not assume as part of the setting built-in forking and existence of regular types. We deal with stable but tame classes (see Definitions 1.7 and 3.2), in other words, stable classes where inequality of types behaves locally. Tameness is a general property capturing finite diagrams (homogeneous model theory). While it is not difficult to see that excellent classes of atomic models of a first-order theory (from [Sh 87a] and [Sh 87b]) are tame, Grossberg and Alexei Kolesnikov show that every abstract excellent class is also tame (see [GrKo]). Very recently using ideas from [GrKo], Andrés Villaveces and Pedro Zambrano discovered that the class of structures obtained from Hrushovski's fusion of strongly minimal first-order theories is another source of examples of tame AECs. See [ViZa] and [Ba].

Despite the generality of our context, recent progress on a Morley's Theorem for tame abstract elementary classes provides evidence that a stability-theory can be developed. A short time after submitting this paper for publication we realized that using the splitting machinery developed here, we can prove in ZFC an instance of Shelah's categoricity conjecture for tame classes (see [GrVa1] and [GrVa2]).

Theorem 0.3 (Grossberg and VanDieren 2003). *Let $\mathcal{K}$ be an AEC, $\kappa := \beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$. Denote by $\mu_0 := \beth_{(2^\kappa)^+}$. Suppose that $\mathcal{K}_{>\kappa}$ has the amalgamation property and is κ-tame. If $\mathcal{K}$ is categorical in some $\lambda^+ > \mu_0$ then $\mathcal{K}$ is categorical in every $\mu \geq \mu_0$.*

This work removes the set-theoretic assumption of the existence of a strongly compact cardinal from the work of Makkai and Shelah [MaSh] replacing it with a weaker model theoretic notion.

In addition to examining splitting in this paper, we prove the existence of Morley sequences from the assumption of stability in tame AECs. We plan to use Morley sequences for non-splitting to define a dividing-like concept for these classes and to prove a stability spectrum theorem for AECs. More work on the stability spectrum theorem (extension of our results under the additional assumption of $\mathrm{LS}(\mathcal{K}) = \aleph_0$) appears in [BaKuVa].

We are grateful to John Baldwin for reading several preliminary versions and suggesting us to consider the stability spectrum problem for AECs.

The referee's comments and questions led to significant improvement of the presentation.

1. BACKGROUND

Much of the necessary background for this paper can be found in [Gr1]. We will use $\alpha, \beta, \gamma, i, j$ to denote ordinals and $\kappa, \lambda, \mu, \chi$ will be used for cardinals. We will use $(\mathcal{K}, \prec_\mathcal{K})$ to denote an abstract elementary class and $\mathcal{K}_\mu$ is the subclass of models in $\mathcal{K}$ of cardinality μ. Similarly we define $\mathcal{K}_{<\mu}$ and $\mathcal{K}_{\geq\mu}$. Models are denoted by M, N and may be decorated with superscripts and subscripts. Sequences of elements from M are written as $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ and we write ${}^\beta M$ to denote the collection of all sequences of length β made up of elements from M. The letters e, f, g, h are reserved for $\mathcal{K}$-mappings and id is the identity mapping.

For the remainder of this paper we will fix $(\mathcal{K}, \prec_\mathcal{K})$ to be an abstract elementary class.

We begin by reviewing the definition of Galois-type, since we will be considering variations of the underlying equivalence relation E later in this paper.

Definition 1.1. Let $\beta > 0$ be an ordinal. For triples $(\bar{a}_\ell, M, N_l)$ where $\bar{a}_\ell \in {}^\beta N_\ell$ and $M_\ell \prec_\mathcal{K} N_\ell \in \mathcal{K}$ for $\ell = 0, 1$, we define a binary relation E as follows: $(\bar{a}_0, M, N_0)E(\bar{a}_1, M, N_1)$ iff and there exists $N \in \mathcal{K}$ and $\mathcal{K}$-mappings f_0, f_1 such that $f_\ell : N_\ell \to N$, $f_\ell \upharpoonright M = \mathrm{id}_M$ for $\ell = 0, 1$ and $f_0(\bar{a}_0) = f_1(\bar{a}_1)$:

$$
\begin{array}{ccc}
N_1 & \xrightarrow{\ f_1\ } & N \\
{\scriptstyle\mathrm{id}}\Big\uparrow & & \Big\uparrow{\scriptstyle f_0} \\
M & \xrightarrow[\ \mathrm{id}\ ]{} & N_0
\end{array}
$$

The relation E is used to define Galois-types in classes that satisfy the amalgamation property.

Definition 1.2.   (1) Let $\mu \geq \mathrm{LS}(\mathcal{K})$. We say that $\mathcal{K}$ has the μ-amalgamation property iff for any $M_\ell \in \mathcal{K}_\mu$ (for $\ell \in \{0, 1, 2\}$) such that $M_0 \prec_\mathcal{K} M_1$ and $M_0 \prec_\mathcal{K} M_2$ there are $N \in \mathcal{K}_\mu$ and $\mathcal{K}$-embeddings $f_\ell : M_\ell \to N$ such that $f_\ell \upharpoonright M_0 = \mathrm{id}_{M_0}$ for $\ell = 1, 2$.
   (2) A model $M_0 \in \mathcal{K}_\mu$ satisfying the above requirement is called an *amalgamation base*.
   (3) $\mathcal{K}$ has the *amalgamation property* iff $\mathcal{K}$ has the μ-amalgamation property for all $\mu \geq \mathrm{LS}(\mathcal{K})$.

Remark 1.3. E is an equivalence relation on the class of triples of the form $(\bar{a}, M, N)$ where $M \prec_\mathcal{K} N$, $\bar{a} \in N$ and both M and N are amalgamation bases. When N is not an amalgamation base, E may fail to be transitive, but the transitive closure of E could be used instead.

Remark 1.4. While, the focus of this paper is on classes with the amalgamation property, several of the proofs in this paper can be adjusted to the context of abstract elementary classes with density of amalgamation bases as in [ShVi] and [Va].

We will make the following assumption for the remainder of the paper:

Assumption 1.5. $\mathcal{K}$ satisfies the amalgamation property.

Under the assumption of the amalgamation property, there exists a large model-homogeneous model, which we will denote by $\mathfrak{C}$ and call the monster model. A μ-*model homogeneous model* is a model M in which for every $N \prec_{\mathcal{K}} M$ of cardinality μ and every extension N_1 of cardinality μ, there exists a $\mathcal{K}$-embedding $f : N_1 \to M$ with $f \restriction N = \mathrm{id}_N$. M is model homogeneous when it is μ-model homogeneous for every $\mu < \|\mathfrak{C}\|$. All models and sequences of elements will be assumed to come from $\mathfrak{C}$. Thus, E is an equivalence relation and we can now define types as equivalence classes of E.

In order to avoid confusing this non-elementary notion of "type" with the conventional first-order one (i.e. set of formulas) we will follow the notation of [Gr1] and [Gr2] and introduce it below under the name of Galois-type. This concept was introduced by Shelah in [Sh 394] using the usual notation of first-order logic.

Definition 1.6. Let β be a positive ordinal (can be one).
  (1) For $M, N \in \mathcal{K}$ and $\bar{a} \in {}^{\beta}\mathrm{N}$. The *Galois-type of* $\bar{a}$ *in* N *over* M, written ga-tp$(\bar{a}/M, N)$, is defined to be $(\bar{a}, M, N)/E$.
  (2) We abbreviate ga-tp$(\bar{a}/M, N)$ by ga-tp$(\bar{a}/M)$.
  (3) For $M \in \mathcal{K}$,

   ga-S$^{\beta}(M) := \{$ga-tp$(\bar{a}/M, N) \mid M \prec N \in \mathcal{K}_{\|M\|}, \bar{a} \in {}^{\beta}\mathrm{N}\}$.

   We write ga-S(M) for ga-S$^1(M)$.
  (4) Let $p := $ ga-tp$(\bar{a}/M', N)$ for $M \prec_{\mathcal{K}} M'$ we denote by $p \restriction M$ the type ga-tp$(\bar{a}/M, N)$. The *domain of* p is denoted by dom(p) and it is by definition M'.
  (5) Let $p = $ ga-tp$(\bar{a}/M, N)$, suppose that $M \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} N$ and let $\bar{b} \in {}^{\beta}\mathrm{N}'$ we say that $\bar{b}$ *realizes* p iff ga-tp$(\bar{b}/M, N') = p \restriction M$.
  (6) For types p and q, we write $p \leq q$ if dom$(p) \subseteq$ dom(q) and there exists $\bar{a}$ realizing p in some N extending dom(p) such that $(\bar{a}, $ dom$(p), N) = q \restriction$ dom(p).

Definition 1.7. We say that $\mathcal{K}$ is β-*Galois-stable in* μ if for every $M \in \mathcal{K}_{\mu}$, $|$ga-S$^{\beta}(M)| = \mu$. The class $\mathcal{K}$ is *Galois stable in* μ iff $\mathcal{K}$ is 1-stable in μ.

Remark 1.8. Let T be a stable, countable, first-order, complete theory. Set $\mathcal{K} := \mathrm{Mod}(T)$ and $\prec_{\mathcal{K}}$ the usual elementary submodel relation. Take $\mu = 2^{\aleph_0}$. While $\mathcal{K}$ is 1-Galois-stable in μ (in fact it is n-Galois-stable in μ

for all $n < \omega$), it can be shown that $\mathcal{K}$ is ω-Galois-stable in μ iff $\mathcal{K}$ has the ndop and notop.

While there is a nice relationship between β-stability and β'-stability for first order theories, the corresponding relationship between β-stability and β'-stability for AECs in general is unknown and probably fails.

Notation 1.9. Although not standard in AECs with the amalgamation property, we will use the notation ga-tp$(\bar{a}/\emptyset)$ to denote the orbit of $\bar{a}$ under Aut$(\mathfrak{C})$. This notation is only used as a device in the proof of Theorem 4.13.

2. Saturated and Limit Models

In this section we recall that limit models (see Definition 2.6) can be used as a substitute for saturated models. Later, we will see that limit models will be more convenient in situations where the first order case would call for saturated models.

Definition 2.1. We say that $M \in \mathcal{K}_{>\mathrm{LS}(\mathcal{K})}$ is *Galois-saturated* if for every $N \prec_{\mathcal{K}} M$ of cardinality $< \|M\|$, and every $p \in$ ga-S(N), we have that M realizes p.

Remark 2.2. When $\mathcal{K} = \mathrm{Mod}(T)$ for a first-order T, using the compactness theorem one can show (Theorem 2.2.3 of [Gr1]) that for $M \in \mathcal{K}$, the model M is Galois-saturated iff M is saturated in the first-order sense.

We begin with universal extensions which are used to build limit models. A universal extension captures some properties of saturated models without referring explicitly to types. The notion of universality over countable models was first analyzed by Shelah in Theorem 1.4(3) of [Sh 87a].

Definition 2.3. (1) Let κ be a cardinal $\geq \mathrm{LS}(\mathcal{K})$. We say N is κ-*universal over* M iff for every $M' \in \mathcal{K}_\kappa$ with $M \prec_{\mathcal{K}} M'$ there exists a $\mathcal{K}$-embedding $g : M' \to N$ such that $g \upharpoonright M = \mathrm{id}_M$:

$$
\begin{array}{ccc}
M' & & \\
\mathrm{id} \uparrow & \searrow^{g} & \\
M & \xrightarrow{\mathrm{id}} & N
\end{array}
$$

(2) We say N is *universal over* M or N is *a universal extension of* M whenever N is $\|M\|$-universal over M.

Remark 2.4. Notice that the definition of N *universal over* M requires all extensions of M of cardinality $\|M\|$ to be embeddable into N. Variants of this definition in the literature often involve $\|M\| < \|N\|$. We would like to emphasize that in this paper we are concentrating on the case when $\|M\| = \|N\|$.

Examples 2.5.   (1) Suppose that T is a first-order complete theory that
is stable in some regular μ. Then every model M of T of cardinality
μ has an elementary extension N of cardinality μ which is universal
over M. To see this define an elementary-increasing and continuous
chain of models of T of cardinality μ, $\langle N_i \mid i < \mu \rangle$ such that N_{i+1}
realizes all types over N_i. Let $N = \bigcup_{i<\mu} N_i$. By a back-and-forth
construction, one can show that N is universal over M.

(2) Let $\mathcal{K}$ be the class of atomic models of a complete first-order the-
ory in a countable language and $\prec_{\mathcal{K}}$ the usual notion of first-order
elementary submodel. If $1 \leq I(\aleph_1, \mathcal{K}) < 2^{\aleph_1}$ and $2^{\aleph_0} < 2^{\aleph_1}$ then for
every $M \in \mathcal{K}_{\aleph_0}$ there exists a proper extension $N \in \mathcal{K}_{\aleph_0}$ universal
over M, by Theorem 1.4(3) of [Sh 87a].

Definition 2.6. For $M', M \in \mathcal{K}_\mu$ and σ a limit ordinal with $\sigma < \mu^+$, we
say that M' is a (μ, σ)-*limit over* M iff there exists a $\prec_{\mathcal{K}}$-increasing and
continuous sequence of models $\langle M_i \in \mathcal{K}_\mu \mid i < \sigma \rangle$ such that

(1) $M = M_0$,
(2) $M' = \bigcup_{i<\sigma} M_i$ and
(3) M_{i+1} is universal over M_i.

Remark 2.7. Notice that in Definition 2.6, for $i < \sigma$ and i a limit ordinal,
M_i is a (μ, i)-limit model.

Definition 2.8. We say that M' is a (μ, σ)-*limit* iff there is some $M \in \mathcal{K}$
such that M' is a (μ, σ)-limit over M.

When $\mathcal{K} = \mathrm{Mod}(T)$ for a first-order and stable T then automatically (by
Theorem III.3.12 of [Shc]):

$$M \in \mathcal{K}_\mu \text{ is saturated} \implies \begin{array}{l} M \text{ is } (\mu, \sigma)\text{-limit for all } \sigma < \mu^+ \\ \text{of cofinality } \geq \kappa(T). \end{array}$$

The uniqueness of (μ, σ)-limit models is an interesting problem: The
statement is

(∗)   If N_ℓ are (μ, σ_ℓ)-limit models over M then $N_1 \cong_M N_2$.

A standard back and forth construction gives a uniqueness proof when
$\mathrm{cf}(\sigma_1) = \mathrm{cf}(\sigma_2)$.

Problem (∗) for $\sigma_1 \neq \sigma_2$ is important. Let's make some basic observations:

1. Suppose $\mathcal{K} = \mathrm{Mod}(T)$ and T is a complete first-order theory. If μ is
regular and $\sigma = \mu$ then the (μ, σ)-limit model (if it exists) is the saturated
model of T of cardinality μ.

2. Showing that two (μ, σ)-limit models are isomorphic for different σs
is a central (dichotomy) property even for first-order theories: E.g. Let
T be a countable (for simplicity), stable, non-superstable first-order theory.
Suppose T is stable in μ. By Harnik's theorem [Ha] T has a saturated model
of cardinality μ, in fact for $\sigma < \mu^+$ the theory has (μ, σ)-limit models. Since
T is not superstable one can modify the construction in Thm 13.1 of [AlGr]

to construct a (μ, ω)-limit model which is not saturated. there exists a union of an ω-sequence of saturated models which is not saturated (see [AlGr]). The union of such as sequence is not isomorphic to a (μ, ω_1)-limit model (that was obtained by taking a union of saturated models, see Theorem III.3.12 of [Shc]). Thus, when T is countable and stable but not superstable, the saturated model of cardinality μ is a (μ, ω_1)-limit but not a (μ, ω)-limit model.

We will see that under the amalgamation property, the existence of limit models follows from stability. Limit models have been applied in several contexts including [Sh 394],[Sh 576] and [Sh 600]. Even without the amalgamation property, the existence [ShVi] and uniqueness [Va] have been shown for categorical AECs under some mild set-theoretic assumptions.

Lately Grossberg, VanDieren and Villaveces proved uniqueness of limit models under mild model-theoretic assumptions on the AEC without assuming categoricity or working outside of ZFC (see [GVV]).

We now describe how one proves the existence of universal extensions and limit models in stable AECs. It turns out that for first-order theories stable in μ any model of cardinality μ has a universal extension of cardinality μ. The result for AECs is claimed in [Sh600] without a proof.

Claim 2.9 (Claim 1.15.1 from [Sh 600]). *Suppose $\mathcal{K}$ is an abstract elementary class with the amalgamation property. If $\mathcal{K}$ is Galois-stable in μ, then for every $M \in \mathcal{K}_\mu$, there exists $M' \in \mathcal{K}_\mu$ such that M' is universal over M. Moreover M' can be chosen to be a (μ, σ)-limit over M for any $\sigma < \mu^+$.*

We provide a proof of this claim here for completeness.
As it is easy to prove:

Proposition 2.10. *Suppose $\kappa \geq \mathrm{LS}(\mathcal{K})$, $\mathcal{K}$ has the κ-amalgamation property and $M, N \in \mathcal{K}_\kappa$. If $M \precneqq_\mathcal{K} N$ and N is maximal (i.e. there is no $N' \succneqq_\mathcal{K} N$) then N is universal over M.*

For the rest of this paper we assume:

Assumption 2.11. $\mathcal{K}$ *does not have maximal models.*

This is a reasonable assumption, since (in the presence of the disjoint amalgamation property) it holds automatically for categorical classes or classes with the joint embedding property.

Theorem 2.12. *Let $\mathcal{K}$ be an AEC which is 1-Galois-stable in μ. If $\mathcal{K}$ has the amalgamation property then for every $M \in \mathcal{K}_\mu$ there exists $M^* \succneqq_\mathcal{K} M$, universal over M of cardinality μ.*

Proof. Define an increasing continuous chain $\langle M_i \mid i < \mu \rangle \subseteq \mathcal{K}_\mu$ such that $M_0 := M$ and every $p \in \mathrm{ga\text{-}S}(M_i)$ is realized in M_{i+1}. Let $M^* := \bigcup_{i<\mu} M_i$.

We now show that M^* is universal over M. Let $N \succ_\mathcal{K} M$ be a given model of cardinality μ, we'll construct an embedding $f : N \to M^*$ s.t. $f \upharpoonright M = \mathrm{id}_M$.

Fix $\{a_i \mid i < \mu\}$ an enumeration of $|N| - |M|$.

By induction on $i < \mu$ define $\prec_\mathcal{K}$-increasing and continuous chains $\{N^i_\ell \mid i < \mu, \ell = 0, 1\} \subseteq \mathcal{K}_\mu$ and $\{f_i \mid i < \mu\}$ such that

(1) $N^i_0 \prec_\mathcal{K} N^i_1$,
(2) $N^0_0 = M$, $N^0_1 = N$
(3) $a_i \in N^{i+1}_0$
(4) $f_i : N^i_0 \to M_i$.

Clearly this is sufficient since $(\bigcup_{i < \mu} f_i) \restriction N$ is as required. By continuity it is enough to describe the construction at successor stages.

If $a_j \in N^j_0$ let $N^{j+1}_\ell := N^j_\ell$.

Otherwise, denote by $M^j_0 \prec_\mathcal{K} M_j$ the image of N^j_0 under f_j, let $g \supseteq f_j$ and $\overline{M}^j_1 \succ_\mathcal{K} M^j_0$ be such that $g : N^j_1 \cong \overline{M}^j_1$.

By the amalgamation property there are M^j_1 and $\overline{g} : \overline{M}^j_1 \to M^j_1$ whose restriction is the identity over M^j_1 such that the diagram

$$
\begin{array}{ccccc}
N^j_1 & \xrightarrow{\ g\ } & \overline{M}^j_1 & \xrightarrow{\ \overline{g}\ } & M^j_1 \\
{\scriptstyle\mathrm{id}}\big\uparrow & & {\scriptstyle\mathrm{id}}\big\uparrow & & {\scriptstyle\mathrm{id}}\big\uparrow \\
N^j_0 & \xrightarrow[\ f_j\ ]{} & M^j_0 & \xrightarrow[\ \mathrm{id}\ ]{} & M_j & \xrightarrow[\ \mathrm{id}\ ]{} & M
\end{array}
$$

commutes.

Since $a_j \in N^j_1 - N^j_0$, $g(a_j) \in \overline{M}^j_1 - M^j_0$. We can choose $\overline{g}$ such that $\overline{g}(g(a_j)) \notin M_j$. Let $p := \mathrm{ga\text{-}tp}(\overline{g}(g(a_j))/M_j, M^j_1)$. By the construction of $\langle M_i \rangle_{i < \mu}$ there exists $b \in M_{j+1}$ realizing the type p. By definition of Galois-types we have $(M_j, M^j_1, \overline{g}(g(a_j)))E(M_j, M_{j+1}, b)$. Using the definition of E (Definition 1.1), there are $N^{**} \in \mathcal{K}_\mu$ and mappings h_1, h_2 such that the diagram

$$
\begin{array}{ccc}
M^j_1 & \xrightarrow{\ h_2\ } & N^{**} \\
{\scriptstyle\mathrm{id}}\big\uparrow & & \big\uparrow {\scriptstyle h_1} \\
M_j & \xrightarrow[\ \mathrm{id}\ ]{} & M_{j+1}
\end{array}
$$

commutes and

$$(\ast) \quad h_2(\overline{g}(g_j(a_j))) = h_1(b).$$

We may assume that $h_2 = \mathrm{id}_{M^j_1}$. Thus by gluing the last two diagrams together we get that the diagram

$$
\begin{array}{ccccc}
N^j_1 & \xrightarrow{\ g \circ \overline{g}\ } & M^j_1 & \xrightarrow{\ \mathrm{id}\ } & N^{**} \\
{\scriptstyle\mathrm{id}}\big\uparrow & & {\scriptstyle\mathrm{id}}\big\uparrow & & \big\uparrow {\scriptstyle h_1} \\
N^j_0 & \xrightarrow[\ f_j\ ]{} & M_j & \xrightarrow[\ \mathrm{id}\ ]{} & M_{j+1} & \xrightarrow[\ \mathrm{id}\ ]{} & M
\end{array}
$$

commutes. Now pick $N_1^{j+1} \succ_{\mathcal{K}} N_1^j$ and $h \supseteq g \circ \bar{g}$ such that $h : N_1^i \cong N^{**}$.
So we have that

$$
\begin{array}{ccc}
N_1^{i+1} & & \\
\text{\scriptsize id} \uparrow & \searrow^{h} & \\
N_1^j & \xrightarrow{\ g \circ \bar{g}\ } & N^{**} \\
\text{\scriptsize id} \uparrow & & \uparrow h_1 \\
N_0^j & \xrightarrow{\ f_j\ } & M_{j+1}
\end{array}
$$

commutes. Let $N_0^{j+1} := h^{-1}[h_1[M_{j+1}]]$.

Notice that by the above diagram, we get that $N_0^{j+1} \supseteq N_0^j$. Now we are ready to define the mapping f_i, let $f_{j+1} := h_1^{-1} \circ (h \upharpoonright N_0^j)$. It is a $\mathcal{K}$-embedding that extends f_j as required.

One can easily verify that $\bigcup_{i<\mu} f_i$ is an embedding of N_2 over N_1 into M. $\dashv$

The following is now immediate

Corollary 2.13. *Let $\mathcal{K}$ be an AEC without maximal models and suppose it is Galois-stable in μ. If $\mathcal{K}$ has the amalgamation property and arbitrarily large models then for every $M \in \mathcal{K}_\mu$ and every $\sigma < \mu^+$ there exists $N \in \mathcal{K}_\mu$ a (μ, σ)-limit model over M.*

Remark 2.14. In a preliminary draft of this paper we included the μ-Disjoint Amalgamation Property as one of the assumptions of Theorem 2.12. We thank John Baldwin and Alexei Kolesnikov for analyzing our proof and bringing to our attention that we actually didn't use disjoint amalgamation.

3. Tame Abstract Elementary Classes

In AECs we view types as equivalences classes of triples (Definition 1.6) instead of sets of formulas. However, a localized equivalence relation (denoted E_μ) is eventually utilized in various partial solutions to Shelah's Categoricity Conjecture (see [Sh 394] and [Sh 576]).

Shelah identified E_μ in [Sh 394]. Here we recall the definition.

Definition 3.1. Triples $(\bar{a}_1, M, N_1)$ and $(\bar{a}_2, M, N_2)$ are said to be E_μ-*related* provided that for every $M' \prec_{\mathcal{K}} M$ with $M' \in \mathcal{K}_{\leq\mu}$,

$$(\bar{a}_1, M', N_1) E (\bar{a}_2, M', N_2).$$

Notice that in first order logic, the finite character of consistency implies that two types are equal if and only if they are $E_{<\omega}$-related.

Main Claim 2.3 of part II on page 288 [Sh 394] Shelah ultimately proves that, under categoricity in some $\lambda > \beth_{(2^{\mathrm{Hanf}(\mathcal{K})})^+}$ and under the assumption

that $\mathcal{K}$ has the amalgamation property, E-equivalence is the same as E_μ equivalence for 1-types over saturated models, for some $\mu < \lambda$.

We now define a context for abstract elementary classes where consistency has small character.

Definition 3.2. Let χ be a cardinal number. For $\beta > 0$, we say $\mathcal{K}$ is χ-*tame for β-types* provided that for every $M \in \mathcal{K}_{>\chi}$, $p \neq q \in$ ga-$S^\beta(M)$ implies existence of $N \prec_\mathcal{K} M$ of cardinality χ such that $p \restriction N \neq q \restriction N$. We say $\mathcal{K}$ is χ-*tame* if it is χ-tame for 1-types.

Notice that in [Sh 394] Shelah defines a relative to tameness which considers only those types over saturated models. Here we are interested in types over arbitrary models. The property of tameness seems to be much stronger than the one considered in [Sh 394].

Definition 3.3. $\mathcal{K}$ *is tame* iff there exists a $\chi < \text{Hanf}(\mathcal{K})$ such that $\mathcal{K}$ is χ-tame.

The relationship between tameness and χ-tameness is not completely clear. It is plausible that the following holds:

Conjecture 3.4. *Let $\mathcal{K}$ be an AEC. If $\mathcal{K}$ is χ-tame for some χ then it is tame.*

Remark 3.5. In this paper, we will actually only use that E-equivalence is the same as E_χ-equivalence for types over limit models.

Notice that if $\mathcal{K}$ is a finite diagram in the sense of [Sh3] (i.e. we have amalgamation not only over all models but also over all subsets of every model), then $\mathcal{K}$ is a tame AEC.

There are tame AECs with amalgamation which are not finite diagrams. In fact Leo Marcus in [Ma] constructed an $L_{\omega_1,\omega}$-sentence which is categorical in every cardinality but does not have an uncountable sequentially homogeneous model. Recently Boris Zilber found a mathematically more natural example [Zi] motivated by Schanuel's Conjecture. His example is not homogeneous nor $L_{\omega_1,\omega}$-axiomatizable.

Shelah proves excellence (from **V**=**L**) for countable $L_{\omega_1,\omega}$-theories satisfying $I(\aleph_{n+1}, \mathcal{K}) < 2^{\aleph_{n+1}}$ for all $n < \omega$ ([Sh 87a] and [Sh 87b]).

In [GrKo] it is shown that excellence implies that $\mathcal{K}$ is $\aleph_0$-tame. In fact, a certain two dimensional property weaker than $(< \chi, \mathcal{P}^-(3))$-AP already implies χ-tameness. Even more examples of tame classes which are not finite diagrams can be found from the work of Villaveces and Zambrano (see [ViZa] or [Ba]).

Let $\mathcal{K}$ be an AEC, suppose $\kappa \geq \text{LS}(\mathcal{K})$ is a strongly compact cardinal. If $\mathcal{K}$ is categorical in some $\lambda^+ > \beth_{(2^\kappa)^+}$ then by Proposition 1.13 of [MaSh] $\mathcal{K}_{\geq\kappa}$ has the amalgamation property and since Galois types can be identified with sets of formulas in $L_{\kappa,\kappa}$ (see [MaSh]). Thus classes satisfying the above assumptions (categoricity above the Hanf number of a strongly compact

cardinal), by Makkai and Shelah's results fit into our framework of κ-tame classes.

While we are convinced that there are examples of arbitrary levels of tameness at the time of submission of this paper we did not have any.

Question 3.6. *For $\mu_1 < \mu_2 < \beth_{\omega_1}$, find an AEC which is μ_2-tame but not μ_1-tame.*

Since the submission of our paper, some progress on this question has been made. Back in a discussion with Baldwin in Bogotá in 2001 we suggested that perhaps a non-free but almost free group of cardinality χ could be used to find an example of a class which is not χ-tame. In August 2005 Baldwin informed us that he and Shelah have a work in progress for class which is not $\aleph_1$-tame they obtain this with groups and short exact sequences, by essentially coding the functor $\text{Ext}(G, \mathbf{Z})$ and using an $\aleph_1$-free group of cardinality $\aleph_1$ which is not free and not Whitehead (see [BaSh]).

Historical Comment 3.7. We introduce tameness as a sufficient assumption to prove Theorem 4.13. Later we realized that a related and a weaker property appears in [Sh 394] where it is proved that categoricity implies that galois types over *saturated* models are determined by smaller substructures. A preliminary version of this paper (our first article containing tameness) was circulated by us in July 2001 (under the title of Morley sequences in AEC). Several authors among them Baldwin, Hyttinen, Kesalla and Lessmann recently published papers using tameness (usually $\aleph_0$-tameness). Unfortunately some of these blur the distinction between tameness and Shelah's notion of local character from [Sh 394].

4. BOUND ON κ_μ^β

In this section we study a non-elementary relative to the first order notion of splitting and we derive bounds for an invariant that corresponds to $\kappa(T)$ under the assumption of stability and tameness.

First let us recall the notion of μ-splitting introduced by Shelah in [Sh 394].

Definition 4.1. A type $p \in \text{ga-S}^\beta(N)$ μ-*splits* over $M \prec_{\mathcal{K}} N$ if and only if $\|M\| \leq \mu$, there exist $N_1, N_2 \in \mathcal{K}_{\leq \mu}$ and h, a $\mathcal{K}$-embedding such that $M \prec_{\mathcal{K}} N_l \prec_{\mathcal{K}} N$ for $l = 1, 2$ and $h : N_1 \to N_2$ such that $h \restriction M = \text{id}_M$ and $p \restriction N_2 \neq h(p \restriction N_1)$.

Remark 4.2. If T is a first-order theory stable in μ and M is saturated, then for all $N \prec M$ of cardinality μ, the first order type, $\text{tp}(a/M)$, does not split (in the first order sense) over N iff $\text{ga-tp}(a/M)$ does not μ-split over N.

Notice that non-splitting is monotonic: I.e. If $p \in \text{ga-S}(N)$ does not μ-split over M (for some $M \prec_{\mathcal{K}} N$) then p does not μ-split over M' for every $M \prec_{\mathcal{K}} M' \prec_{\mathcal{K}} N$ of cardinality μ.

Similar to $\kappa(T)$ when T is first-order the following is a natural cardinal invariant of $\mathcal{K}$:

Definition 4.3. Let $\beta > 0$. We define $\kappa_\mu^\beta(\mathcal{K})$ to be the minimal $\kappa < \mu^+$ such that for every $\langle M_i \in \mathcal{K}_\mu \mid i \leq \kappa \rangle$ which satisfies

(1) $\kappa = \mathrm{cf}(\kappa) < \mu^+$,
(2) $\langle M_i \mid i \leq \kappa \rangle$ is $\prec_{\mathcal{K}}$-increasing and continuous and
(3) for every $i < \kappa$, M_{i+1} is a (μ, θ)-limit over M_i for some $\theta < \mu^+$,

and for every $p \in \mathrm{ga\text{-}S}^\beta(M_\kappa)$, there exists $i < \kappa$ such that p does not μ-split over M_i. If no such κ exists, we say $\kappa_\mu^\beta(\mathcal{K}) = \infty$.

Another relative of $\kappa(T)$ is the following

Definition 4.4. For $\beta > 0$, $\bar{\kappa}_\mu^\beta(\mathcal{K})$ is the minimal cardinal $\bar{\kappa}$ such that for every $N \in \mathcal{K}_\mu$ and every $p \in \mathrm{ga\text{-}S}^\beta(N)$ there are $\lambda < \bar{\kappa}$ and $M \in \mathcal{K}_\lambda$ such that p does not λ-split over M.

It is not difficult to verify that

Proposition 4.5. *For μ with $\mathrm{cf}(\mu) > \bar{\kappa}_\mu^\beta(\mathcal{K})$, we have $\kappa_\mu^\beta(\mathcal{K}) \leq \bar{\kappa}_\mu^\beta(\mathcal{K})$.*

While for stable first order theories (when $\beta < \omega$) both invariants are equal, the situation for non-elementary classes is more complicated. Already in [Sh 394] it was observed that $\kappa_\mu^\beta(\mathcal{K})$ is better behaved than $\bar{\kappa}_\mu^\beta(\mathcal{K})$ when a bound for $\kappa_\mu^\beta(\mathcal{K})$ was found. A corresponding bound for $\bar{\kappa}_\mu^\beta(\mathcal{K})$ is unknown. We will defer dealing with the invariant $\bar{\kappa}_\mu^\beta(\mathcal{K})$ to a future paper.

Notice that Theorem 2.2.1 of [ShVi] states that certain categorical abstract elementary classes satisfy $\kappa_\mu^1(\mathcal{K}) = \omega$, for various μ. A slight modification of the argument of Claim 3.3 from [Sh 394] can be used to prove a related result using the weaker assumption of Galois-stability only:

Fact 4.6. *Let $\beta > 0$. Suppose that $\mathcal{K}$ is β-Galois-stable in μ. For every $p \in \mathrm{ga\text{-}S}^\beta(N)$ there exists $M \prec_{\mathcal{K}} N$ of cardinality μ such that p does not μ-split over M. Thus $\bar{\kappa}_\mu^\beta(\mathcal{K}) \leq \mu$.*

For the sake of completeness an argument for Fact 4.6 is included:

Proof. Suppose $p = \mathrm{ga\text{-}tp}(\bar{a}/N, \mathfrak{C})$ with $\bar{a} \in {}^\beta|\mathfrak{C}|$ and p μ-splits over M, for every $M \prec_{\mathcal{K}} N$ of cardinality μ. We may assume that $\bar{a} \notin N$, since it is routine to check that an algebraic type never splits over its domain.

Let $\chi := \min\{\chi \mid 2^\chi > \mu\}$. Notice that $\chi \leq \mu$ and $2^{<\chi} \leq \mu$.

First we will define $\langle M_\alpha, N_{\alpha,\ell} \prec N \mid \alpha < \chi, \ \ell = 1, 2 \rangle \subseteq \mathcal{K}_\mu$ which will then be used to construct another sequence $\langle M_\alpha^* \in \mathcal{K}_\mu \mid \alpha \leq \chi \rangle$ so that

$$|\mathrm{ga\text{-}S}^\beta(M_\chi^*)| \geq 2^\chi > \mu \text{ will witness Galois-instability in } \mu.$$

The definition of $\langle M_\alpha, N_{\alpha,1}, N_{\alpha,2} \mid \alpha < \chi \rangle$ follows by induction on α and our choice of p. Let $M_0 \prec M$ be any model of cardinality μ.

For $\alpha = \gamma + 1$, by our choice of p, we know p must μ-split over M_γ. Then, there are $N_{\gamma,\ell} \prec_{\mathcal{K}} M$ of cardinality μ for $\ell = 1, 2$ and there is $F_\gamma : N_{\gamma,1} \cong_{M_\gamma}$

$N_{\gamma,2}$ such that $F_\gamma(p \restriction N_{\gamma,1}) \neq p \restriction N_{\gamma,2}$. Pick $M_\alpha \prec_\mathcal{K} M$ of cardinality μ containing the set $|N_{\gamma,1}| \cup |N_{\gamma,2}|$.

When α is a limit ordinal we define $M_\alpha := \bigcup_{\gamma<\alpha} M_\gamma$.

Now we use our sequence $\langle M_\alpha \mid \alpha < \chi \rangle$ to define for $\alpha \leq \chi$ another sequence of models $M_\alpha^* \in \mathcal{K}_\mu$ and a tree of $\mathcal{K}$-embeddings $\langle h_\eta \mid \eta \in {}^\alpha 2 \rangle$ satisfying

  (1) $\gamma < \alpha \implies M_\gamma^* \prec_\mathcal{K} M_\alpha^*$,
  (2) for α limit let $M_\alpha^* = \bigcup_{\gamma<\alpha} M_\gamma^*$,
  (3) $\gamma < \alpha \wedge \eta \in {}^\alpha 2 \implies h_{\eta \restriction \gamma} \subseteq h_\eta$,
  (4) $\eta \in {}^\alpha 2 \implies h_\eta : M_\alpha \to M_\alpha^*$ and
  (5) $\alpha = \gamma + 1 \wedge \eta \in {}^\alpha 2 \implies h_{\eta \hat{} 0}(N_{\gamma,1}) = h_{\eta \hat{} 1}(N_{\gamma,2})$

The construction is possible by using the μ-amalgamation property several times. For $\alpha = 0$, define $M_0^* = M$ and $h_{\langle\rangle} := \mathrm{id}_{M_0^*}$. When α is a limit ordinal we define $M_\alpha^* := \bigcup_{\gamma<\alpha} M_\gamma$ and for $\eta \in {}^\alpha 2$, $h_\eta := \bigcup_{\gamma<\alpha} h_{\eta \restriction \gamma}$.

Suppose we have completed the construction for γ and $\alpha = \gamma + 1$. Given $\eta \in {}^\gamma 2$, let $\bar{h}_\eta$ be an extension of h_η to an automorphism of $\mathfrak{C}$. Notice that $\bar{h}_\eta \circ F_\gamma(N_{\gamma,1}) = \bar{h}_\eta(N_{\gamma,2})$ by our choice of F_γ, $N_{\gamma,1}$ and $N_{\gamma,2}$. Let $h_{\eta \hat{} \langle 0 \rangle}$ be some extension of $(\bar{h}_\eta \restriction N_{\gamma,2}) \circ F_\gamma$ to $M_{\gamma+1}$ and let $h_{\eta \hat{} \langle 1 \rangle}$ be $\bar{h}_\eta \restriction M_{\gamma+1}$. Let $M_{\gamma+1}^*$ be a model of cardinality μ extending M_γ^* and containing $h_{\eta \hat{} \langle 0 \rangle}(M_{\gamma+1})$ and $h_{\eta \hat{} \langle 1 \rangle}(M_{\gamma+1})$ for each $\eta \in {}^\gamma 2$.

This completes the construction and for each $\eta \in {}^\chi 2$ we can fix $H_\eta \in \mathrm{Aut}(\mathfrak{C})$ extending h_η.

Claim 4.7. $\eta \neq \nu \in {}^\chi 2 \implies \mathrm{ga\text{-}tp}(H_\eta(\bar{a})/M_\chi^*) \neq \mathrm{ga\text{-}tp}(H_\nu(\bar{a})/M_\chi^*)$.

Proof. Let $\rho := \eta \wedge \nu$ and suppose that $\rho \hat{} \langle 0 \rangle < \eta$ and $\rho \hat{} \langle 1 \rangle < \nu$. Let γ be the length of ρ. Suppose for the sake of contradiction that

$$\mathrm{ga\text{-}tp}(H_\eta(\bar{a})/M_\chi^*) = \mathrm{ga\text{-}tp}(H_\nu(\bar{a})/M_\chi^*).$$

In particular,

$$\mathrm{ga\text{-}tp}(H_\eta(\bar{a})/H_{\rho \hat{} \langle 0 \rangle}(N_{\gamma,1})) = \mathrm{ga\text{-}tp}(H_\nu(\bar{a})/H_{\rho \hat{} \langle 1 \rangle}(N_{\gamma,2})).$$

Referring back to our choice of $\bar{h}_\rho$ and $h_{\rho \hat{} \langle 0 \rangle}$, we can apply H_ν^{-1} to obtain

$$\mathrm{ga\text{-}tp}(H_\nu^{-1} \circ H_\eta(\bar{a})/F_\gamma(N_{\gamma,1}) = \mathrm{ga\text{-}tp}(\bar{a}/N_{\gamma,2}).$$

All we need to verify in order to contradict our original choices of $N_{\gamma,1}$, $N_{\gamma,2}$ and F_γ is that for some extension of F_γ to an automorphism of $\mathfrak{C}$ we have that

$$(*) \quad \mathrm{ga\text{-}tp}(F_\gamma(\bar{a})/N_{\gamma,2}) = \mathrm{ga\text{-}tp}(H_\nu^{-1} \circ H_\eta(\bar{a})/N_{\gamma,2}).$$

Notice that by our choice of H_ν and H_η we have that $H_\nu^{-1} \circ H_\eta \restriction N_{\gamma,1} = F_\gamma$. Thus $H_\nu^{-1} \circ H_\eta \supseteq F_\gamma$, so by definition of the image of a type, $F_\gamma(p \restriction N_{\gamma,1}) = \mathrm{ga\text{-}tp}(H_\nu^{-1} \circ H_\eta(\bar{a})/N_{\gamma,2})$. Combining this with $(*)$ we obtain a contradiction to our choice of $N_{\gamma,1}$, $N_{\gamma,2}$ and F_γ which witness the splitting of p over M_γ. $\dashv$

Thus Claim 4.7 gives us that $|\text{ga-S}(M_\chi^*)| \geq 2^\chi > \mu$ and $\mathcal{K}$ is not β-Galois-stable in μ. ⊣

In Theorem 4.13 below we present an improvement of Fact 4.6 for tame AECs: In case $\mathcal{K}$ is β-stable in μ for some μ above its second Hanf number then $\kappa_\mu^\beta(\mathcal{K})$ is bounded by the second Hanf number. Notice that the bound does not depend on μ.

We will need to relate Galois-stability with the failure of a certain infinitary order property in order to find a bound for $\kappa_\mu^\beta(\mathcal{K})$. The order property, defined next, is an analog of the first order definition using formulas. This order property for non-elementary classes was introduced by Shelah in [Sh 394].

Definition 4.8. $\mathcal{K}$ is said to have the χ-*order property* provided that for every α, there exists $\langle \bar{d}_i \mid i < \alpha \rangle$ and where $\bar{d}_i \in {}^\chi\mathfrak{C}$ such that if $i_0 < j_0 < \alpha$ and $i_1 < j_1 < \alpha$,

$$(*) \text{ then for no } f \in Aut(\mathfrak{C}) \text{ do we have } f(\bar{d}_{i_0}{}^\frown\bar{d}_{j_0}) = \bar{d}_{j_1}{}^\frown\bar{d}_{i_1}.$$

Example 4.9. Let T be a first order theory in a countable language. Set $\mathcal{K} := \text{Mod}(T)$ and $\prec_\mathcal{K}$ the usual elementary submodel relation. $\mathcal{K}$ has the $\aleph_0$-order property iff T is unstable of T has dop or otop.

Remark 4.10 (Trivial monotonicity). Notice that for $\kappa_1 < \kappa_2$ if a class has the κ_1-order property then it has the κ_2-order property.

Fact 4.11 (Claim 4.6.3 of [Sh 394]). *We may replace the phrase* every α *in Definition 4.8 with* every $\alpha < \beth_{(2^\chi + \text{LS}(\mathcal{K}))^+}$ *and get an equivalent definition.*

Fact 4.12 (Claim 4.8.2 of [Sh 394]). *If $\mathcal{K}$ has the χ-order property and $\mu \geq \chi$, then for some $M \in \mathcal{K}_\mu$ we have that $|\text{ga-S}^\chi(M)/E_\chi| \geq \mu^+$. Moreover, we can conclude that $\mathcal{K}$ is not χ-Galois-stable in μ.*

The following is a generalization of an old theorem (Theorem 4.17 in [GrLe2]) of Shelah from [Sh3]. A special case of this theorem is Theorem 0.1 in the abstract.

Theorem 4.13. *Let $\beta > 0$. Suppose that $\mathcal{K}$ is χ-tame for β-types. If $\mathcal{K}$ is β-stable in μ with $\beth_{(2^\chi + \text{LS}(\mathcal{K}))^+} \leq \mu$, then $\kappa_\mu^\beta(\mathcal{K}) < \beth_{(2^\chi + \text{LS}(\mathcal{K}))^+}$.*

Proof. Let $\chi' := \beth_{(2^\chi + \text{LS}(\mathcal{K}))^+}$. Suppose that the conclusion of the theorem does not hold. Let $\langle M_i \in \mathcal{K}_\mu \mid i \leq \chi' \rangle$ and $p \in \text{ga-S}^\beta(M_{\chi'})$ witness the failure. Namely, the following hold:

(1) $\langle M_i \mid i \leq \chi' \rangle$ is $\prec_\mathcal{K}$-increasing and continuous,
(2) for every $i < \chi'$, M_{i+1} is a (μ, θ)-limit over M_i for some $\theta < \mu^+$ and
(3) for every $i < \mu^+$, p μ-splits over M_i.

For every $i < \chi'$ let f_i, N_i^1 and N_i^2 witness that p μ-splits over M_i. Namely,

$$M_i \prec_\mathcal{K} N_i^1, N_i^2 \prec_\mathcal{K} M,$$

$$f_i : N_i^1 \cong N_i^2 \text{ with } f_i \upharpoonright M_i = \text{id}_{M_i}$$

$$\text{and } f_i(p \upharpoonright N_i^1) \neq p \upharpoonright N_i^2.$$

By χ-tameness, there exist models B_i and $A_i := f_i^{-1}(B_i)$ of size $< \chi$ such that

$$f_i(p \upharpoonright A_i) \neq p \upharpoonright B_i.$$

By renumbering our chain of models, we may assume that

(4) $A_i, B_i \subset M_{i+1}$.

Since M_{i+1} is a limit model over M_i, we can additionally fix

(5) $\bar{c}_i \in M_{i+1}$ realizing $p \upharpoonright M_i$.

For each $i < \mu$, let $\bar{d}_i := A_i\hat{\,}B_i\hat{\,}\bar{c}_i$.

Claim 4.14. $\langle \bar{d}_i \mid i < \chi' \rangle$ *witnesses the χ-order property.*

Proof. Suppose for the sake of contradiction that there exist $g \in Aut(\mathfrak{C})$, $i_0 < j_0 < \chi'$ and $i_1 < j_1 < \chi'$ such that

$$g(\bar{d}_{i_0}\hat{\,}\bar{d}_{j_0}) = \bar{d}_{j_1}\hat{\,}\bar{d}_{i_1}.$$

Notice that since $i_0 < j_0 < \alpha$ we have that $\bar{c}_{i_0} \in M_{j_0}$. So $f_{j_0}(\bar{c}_{i_0}) = \bar{c}_{i_0}$. Recall that $f_{j_0}(A_{j_0}) = B_{j_0}$. Thus, f_{j_0} witnesses that

$$(*) \quad \text{ga-tp}(\bar{c}_{i_0}\hat{\,}A_{j_0}/\emptyset) = \text{ga-tp}(\bar{c}_{i_0}\hat{\,}B_{j_0}/\emptyset).$$

Applying g to $(*)$ we get

$$(**) \quad \text{ga-tp}(\bar{c}_{j_1}\hat{\,}A_{i_1}/\emptyset) = \text{ga-tp}(\bar{c}_{j_1}\hat{\,}B_{i_1}/\emptyset).$$

Applying f_{i_1} to the RHS of $(**)$, we notice that

$$(\sharp) \quad \text{ga-tp}(f_{i_1}(\bar{c}_{j_1})\hat{\,}B_{i_1}/\emptyset) = \text{ga-tp}(\bar{c}_{j_1}\hat{\,}B_{i_1}/\emptyset).$$

Because $i_1 < j_1$, we have that $\bar{c}_{j_1}$ realizes $p \upharpoonright M_{i_1}$. Thus, $(\sharp)$ implies

$$f_{i_1}(p \upharpoonright A_{i_1}) = p \upharpoonright B_{i_1},$$

which contradicts our choice of f_{i_1}, A_{i_1} and B_{i_1}.

$\dashv$

By Claim 4.11 and Fact 4.12, we have that $\mathcal{K}$ is β-unstable in μ, contradicting our hypothesis.

$\dashv$

5. Morley sequences

We have now developed enough splitting machinery to derive the existence of Morley sequences for tame, Galois-stable AECs.

The following is a new Galois-theoretic notion of indiscernible sequence.

Definition 5.1. (1) $\langle \bar{a}_i \mid i < i^* \rangle$ is a *Galois-indiscernible sequence over* M iff for every $i_1 < \cdots < i_n < i^*$ and every $j_1 < \cdots < j_n < i^*$, $\mathrm{ga\text{-}tp}(\bar{a}_{i_1} \ldots \bar{a}_{i_n}/M) = \mathrm{ga\text{-}tp}(\bar{a}_{j_1} \ldots \bar{a}_{j_n}/M)$.

(2) $\langle \bar{a}_i \mid i < i^* \rangle$ is a *Galois-indiscernible sequence over* A iff for every $i_1 < \cdots < i_n < i^*$ and every $j_1 < \cdots < j_n < i^*$, there exists $M_i, M_j, M^* \in \mathcal{K}$ and $\prec_{\mathcal{K}}$-mappings f_i, f_j such that
   (a) $A \subseteq M_i, M_j$;
   (b) $f_\ell : M_\ell \to M^*$, for $\ell \in \{i, j\}$;
   (c) $f_i(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}) = f_j(\bar{a}_{j_0}, \ldots, \bar{a}_{j_n})$ and
   (d) and $f_i \restriction A = f_j \restriction A = \mathrm{id}_A$.

Remark 5.2. This is on the surface a weaker notion of indiscernible sequence than is presented in [Sh 394]. However, under the amalgamation property, this definition and the definition in [Sh 394] are equivalent.

Lemma 5.3 provides us with sufficient conditions to find an indiscernible sequence.

Lemma 5.3. *Let* $\mu \geq \mathrm{LS}(\mathcal{K})$, κ, λ *be ordinals and* β *a positive ordinal. Suppose that* $\langle M_i \mid i < \lambda \rangle$ *and* $\langle \bar{a}_i \mid i < \lambda \rangle$ *satisfy*

   (1) $\langle M_i \in \mathcal{K}_\mu \mid i < \lambda \rangle$ *are* $\prec_{\mathcal{K}}$*-increasing;*
   (2) M_{i+1} *is a* (μ, κ)*-limit over* M_i;
   (3) $\bar{a}_i \in {}^\beta M_{i+1}$;
   (4) $p_i := \mathrm{ga\text{-}tp}(\bar{a}_i/M_i, M_{i+1})$ *does not* μ*-split over* M_0 *and*
   (5) *for* $i < j < \lambda$, $p_i \leq p_j$.

Then, $\langle \bar{a}_i \mid i < \lambda \rangle$ *is a Galois-indiscernible sequence over* M_0.

Definition 5.4. A sequence $\langle \bar{a}_i, M_i \mid i < \lambda \rangle$ satisfying conditions $(1) - (5)$ of Lemma 5.3 is called a *Morley sequence*. It is a *Morley sequence for* $p \in \mathrm{ga\text{-}S}(\bigcup_{i<\lambda} M_i)$ if p extends each $\mathrm{ga\text{-}tp}(\bar{a}_i/M_i)$. For $A \subseteq M_0$, such a sequence is a *Morley sequence over* A.

Remark 5.5. Notice that our definition of Morley sequence varies from some literature. An alternative name for our sequences (suggested by John Baldwin) is a *coherent non-splitting sequence*.

Remark 5.6. While the statement of the lemma is similar to Shelah's first order Lemma I.2.5 in [Shc], the proof differs, since types are not sets of formulas.

Proof. We prove that for $i_0 < \cdots < i_n < \lambda$ and $j_0 < \cdots < j_n < \lambda$, $\mathrm{ga\text{-}tp}(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/M_0, M_{i_{n+1}}) = \mathrm{ga\text{-}tp}(\bar{a}_{j_0}, \ldots, \bar{a}_{j_n}/M_0, M_{j_{n+1}})$ by induction on $n < \omega$.

$n = 0$: Let $i_0, j_0 < \lambda$ be given. Condition 5, gives us

$$\text{ga-tp}(\bar{a}_{i_0}/M_0, M_{i_0+1}) = \text{ga-tp}(\bar{a}_{j_0}/M_0, M_{j_0+1}).$$

$n > 0$: Suppose that the claim holds for all increasing sequences $\bar{i}$ and $\bar{j} \in \lambda$ of length n. Let $i_0 < \cdots < i_n < \lambda$ and $j_0 < \cdots < j_n < \lambda$ be given. Without loss of generality, $i_n \leq j_n$. Define $M^* := M_1$. From condition 2 and uniqueness of (μ, ω)-limits, we can find a $\prec_{\mathcal{K}}$-isomorphism, $g : M_{j_n} \to M_{i_n}$ such that $g \upharpoonright M_0 = \text{id}_{M_0}$. Moreover we can extend g to $g : M_{j_n+1} \to M_{i_n+1}$. Denote by $\bar{b}_{j_l} := g(\bar{a}_{j_l})$ for $l = 0, \ldots, n$. Notice that $b_{j_l} \in M_{i_n}$ for $l < n$. Since $\text{ga-tp}(\bar{b}_{j_0}, \ldots, \bar{b}_{j_n}/M_0, M_{i_n+1}) = \text{ga-tp}(\bar{a}_{j_0}, \ldots, \bar{a}_{j_n}/M_0, M_{j_n+1})$ it suffices to prove that $\text{ga-tp}(\bar{b}_{j_0}, \ldots, \bar{b}_{j_n}/M_0, M_{i_n+1}) = \text{ga-tp}(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/M_0, M_{i_n+1})$.

Also notice that our $\prec_{\mathcal{K}}$-mapping g preserves some properties of p_j. Namely, since p_j does not μ-split over M_0, $g(p_j \upharpoonright M_{j_n}) = p_j \upharpoonright M_{i_n}$. Thus, $\text{ga-tp}(\bar{b}_{j_n}/M_{i_n}, M_{i_n+1}) = \text{ga-tp}(\bar{a}_{j_n}/M_{i_n}, M_{i_n+1})$. In particular we have that $\text{ga-tp}(\bar{b}_{j_n}/M_{i_n}, M_{i_n+1})$ does not μ-split over M_0.

By the induction hypothesis

$$\text{ga-tp}(\bar{b}_{j_0}, \ldots, \bar{b}_{j_{n-1}}/M_0, M_{i_n}) = \text{ga-tp}(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}/M_0, M_{i_n}).$$

Thus we can find $h_i : M_{i_n+1} \to M^*$ and $h_j : M_{i_n+1} \to M^*$ such that $h_i(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}) = h_j(\bar{b}_{j_0}, \ldots, \bar{b}_{j_{n-1}})$. Let us abbreviate $\bar{b}_{j_0}, \ldots, \bar{b}_{j_{n-1}}$ by $\bar{b}_{\bar{j}}$. Similarly we will write $\bar{a}_{\bar{i}}$ for $\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}$.

By appealing to condition 4, we derive several equalities that will be useful in the latter portion of the proof. Since p_j does not μ-split over M_0, we have that $p_j \upharpoonright h_j(M_{i_n}) = h_j(p_j \upharpoonright M_{i_n})$, rewritten as

$$(*) \quad \text{ga-tp}(\bar{b}_{j_n}/h_j(M_{i_n}), M_{i_n+1}) = \text{ga-tp}(h_j(\bar{b}_{j_n})/h_j(M_{i_n}), M^*).$$

Similarly as p_i does not μ-split over M_0, we get $p_i \upharpoonright h_j(M_{i_n}) = h_j(p_i \upharpoonright M_{i_n})$ and $p_i \upharpoonright h_i(M_{i_n}) = h_i(p_i \upharpoonright M_{i_n})$. These equalities translate to

$$(**)_j \quad \text{ga-tp}(\bar{a}_{i_n}/h_j(M_{i_n}), M_{i_n+1}) = \text{ga-tp}(h_j(\bar{a}_{i_n})/h_j(M_{i_n}), M^*) \text{ and}$$

$$(**)_i \quad \text{ga-tp}(\bar{a}_{i_n}/h_i(M_{i_n}), M_{i_n+1}) = \text{ga-tp}(h_i(\bar{a}_{i_n})/h_i(M_{i_n}), M^*), \text{ respectively.}$$

Finally, from condition 5, notice that

$$(***) \quad \text{ga-tp}(\bar{a}_{i_n}/M_{i_n}, M_{i_n+1}) = \text{ga-tp}(\bar{b}_{j_n}/M_{i_n}, M_{i_n+1}).$$

Applying h_j to $(***)$ yields

$$(\dagger) \quad \text{ga-tp}(h_j(\bar{b}_{j_n})/h_j(M_{i_n}), M^*) = \text{ga-tp}(h_j(\bar{a}_{i_n})/h_j(M_{i_n}), M^*).$$

Since $h_i(\bar{a}_{\bar{i}}) = h_j(\bar{b}_{\bar{j}}) \in h_j(M_{i_n})$, we can draw from $(\dagger)$ the following:

$$(1) \quad \text{ga-tp}(h_j(\bar{b}_{j_n})\char94 h_j(\bar{b}_{\bar{j}})/M_0, M^*) = \text{ga-tp}(h_j(\bar{a}_{j_n})\char94 h_i(\bar{a}_{\bar{i}})/M_0, M^*).$$

Similarly, equality $(**)_i$ allows us to see

$$(2) \quad \text{ga-tp}(\bar{a}_{i_n}\char94 h_i(\bar{a}_{\bar{i}})/M_0, M^*) = \text{ga-tp}(h_i(\bar{a}_{i_n})\char94 h_i(\bar{a}_{\bar{i}})/M_0, M^*).$$

Since ga-tp$(h_j(\bar{a}_{i_n})/h_j(M_{i_n}), M^*) = $ ga-tp$(\bar{a}_{i_n}/h_j(M_{i_n}), M_{i_n+1})$ (equality $(**)_j)$) and $h_i(\bar{a}_{\bar{i}}) = h_j(\bar{b}_{\bar{j}}) \in h_j(M_{i_n})$, we get that

(3)   ga-tp$(h_j(\bar{a}_{i_n})\hat{}h_i(\bar{a}_{\bar{i}})/M_0, M^*) = $ ga-tp$(\bar{a}_{i_n}\hat{}h_i(\bar{a}_{\bar{i}})/M_0, M^*)$.

Combining equalities (1), (2) and (3), we get

(††)   ga-tp$(h_i(\bar{a}_{\bar{i}})\hat{}h_i(\bar{a}_{i_n})/M_0, M^*) = $ ga-tp$(h_j(\bar{b}_{\bar{j}})\hat{}h_j(\bar{b}_{j_n})/M_0, M^*)$.

Recall that $h_i \restriction M_0 = h_j \restriction M_0 = \text{id}_{M_0}$. Thus (††), witnesses that

ga-tp$(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/M_0, M_{i_n+1}) = $ ga-tp$(\bar{b}_{j_0}, \ldots, \bar{b}_{j_n}/M_0, M_{i_n+1})$.

$\dashv$

Theorem 5.7. *Fix $\beta > 0$. Suppose $\mathcal{K}$ is χ-tame for β-types with $\chi <$ $\beth_{(2^{\text{Hanf}(\mathcal{K})})^+}$. Suppose $\mu \geq \beth_{(2^{Hanf(\mathcal{K})})^+}$. Let $M \in \mathcal{K}_{>\mu}$, $A \subset M$ and $I \subseteq {}^\beta M$ be given such that $|I| \geq \mu^+ > |A|$. If $\mathcal{K}$ is Galois β-stable in μ, then there exists $J \subset I$ of cardinality μ^+, Galois indiscernible over A. Moreover J can be chosen to be a Morley sequence over A.*

Proof. Fix $\kappa = \kappa_\mu^\beta$. By Theorem 4.13, $\kappa < \beth_{(2^{\text{Hanf}(\mathcal{K})})^+}$. Let $\{\bar{a}_i \in I \mid i < \mu^+\}$ be given. Define $\langle M_i \in K_\mu \mid i < \mu^+\rangle$, $\prec_{\mathcal{K}}$-increasing and continuous, satisfying

(1) $A \subseteq |M_0|$
(2) M_{i+1} is a (μ, κ)-limit over M_i
(3) $\bar{a}_i \in M_{i+1}$

Let $p_i := $ ga-tp$(\bar{a}_i/M_i, M_{i+1})$ for every $i < \mu^+$. Define $f : S_\kappa^{\mu^+} \to \mu^+$ by

$$f(i) := \min\{j < \mu^+ \mid p_i \text{ does not } \mu\text{-split over } M_j\}.$$

By our choice of κ, f is regressive. Thus by Fodor's Lemma, there are a stationary set $S \subseteq S_\kappa^{\mu^+}$ and $j_0 < \mu^+$ such that for every $i \in S$,

(†) p_i does not μ-split over M_{j_0}.

By stability and the pigeon-hole principle there exists $p^* \in $ ga-S(M_{j_0}) and $S^* \subseteq S$ of cardinality μ^+ such that for every $i \in S^*$, $p^* = p_i \restriction M_{j_0}$. Let $M^* := M_{j_0}$. Enumerate and $\langle M_{j_i} \mid i \in S^*\rangle$. Let $M^{**} := M_{j_1}$ in this enumeration. Again, by stability we can find $S^{**} \subset S^*$ of cardinality μ^+ such that for every $i \in S^{**}$, $p^{**} = p_i \restriction M^*$. Notice that $M^*, M^{**} \prec_{\mathcal{K}} M_i$ for every $i \in S^{**}$.

Subclaim 5.8. *For $i < j \in S^{**}$, $p_i = p_j \restriction M_i$.*

Proof. Let $0 < i < j \in S^{**}$ be given. Since M_{i+1} and M_{j+1} are (μ, κ)-limits over M_i, there exists an isomorphism $g : M_{j+1} \to M_{i+1}$ such that $g \restriction M_i = \text{id}_{M_i}$. Let $\bar{b}_j := g(\bar{a}_j)$. Since the type p_j does not μ-split over M^*, g cannot witness the splitting. Therefore, it must be the case that ga-tp$(\bar{b}_j/M_i, M_{i+1}) = p_j \restriction M_i$. Then, it suffices to show that $p_i = $ ga-tp$(\bar{b}_j/M_i, M_{i+1})$.

Since $p_i \restriction M^* = p_j \restriction M^*$, we can find $\prec_{\mathcal{K}}$-mappings witnessing the equality. Furthermore since M^{**} is universal over M^*, we can find $h_l : M_{l+1} \to M^{**}$ such that $h_l \restriction M^* = \mathrm{id}_{M^*}$ for $l = i, j$ and $h_i(\bar{a}_i) = h_j(\bar{b}_j)$.

We will use (†) to derive several inequalities. Consider the following possible witness to splitting. Let $N_1 := M_i$ and $N_2 := h_i(M_i)$. Since p_i does not μ-split over M^*, we have that $p_i \restriction N_2 = h_i(p_i \restriction N_1)$, rewritten as

$$(*) \quad \text{ga-tp}(\bar{a}_i/h_i(M_i), M_{i+1}) = \text{ga-tp}(h_i(\bar{a}_i)/h_i(M_i), M^{**}).$$

Similarly we can conclude that

$$(**) \quad \text{ga-tp}(\bar{b}_j/h_j(M_i), M_{i+1}) = \text{ga-tp}(h_j(\bar{b}_j)/h_j(M_i), M^{**}).$$

By choice of S^{**}, we know that

$$(***) \quad \text{ga-tp}(\bar{b}_j/M^{**}) = \text{ga-tp}(\bar{a}_i/M^{**}).$$

Now let us consider another potential witness of splitting. $N_1^* := h_i(M_i)$ and $N_2^* := h_j(M_i)$ with $H^* := h_j \circ h_i^{-1} : N_1^* \to N_2^*$. Since $p_j \restriction M_i$ does not μ-split over M_0, $p_j \restriction N_2^* = H^*(p_j \restriction N_1^*)$. Thus by $(**)$ we have

$$(\sharp) \quad H^*(p_j \restriction N_1^*) = \text{ga-tp}(h_j(\bar{b}_j)/h_j(M_i), M^{**}).$$

Now let us translate $H^*(p_j \restriction N_1^*)$. By monotonicity and $(***)$, we have that $p_j \restriction N_1^* = \text{ga-tp}(\bar{b}_j/h_i(M_i), M_{i+1}) = \text{ga-tp}(\bar{a}_i/h_i(M_i), M_{i+1})$. We can then conclude by $(*)$ that $p_j \restriction N_1^* = \text{ga-tp}(h_i(\bar{a}_i)/h_i(M_i), M_{i+1})$. Applying H^* to this equality yields

$$(\sharp\sharp) \quad H^*(p_j \restriction N_1^*) = \text{ga-tp}(h_j(\bar{a}_i)/h_j(M_i), M^{**)}.$$

By combining the equalities from $(\sharp)$ and $(\sharp\sharp)$ and applying h_j^{-1} we get that

$$\text{ga-tp}(\bar{b}_j/M_i, M_{i+1}) = \text{ga-tp}(\bar{a}_i/M_i, M_{i+1}).$$

$$\dashv$$

Notice that by Subclaim 5.8 and our choice of S^{**}, $\langle M_i \mid i \in S^{**} \rangle$ and $J := \langle \bar{a}_i \mid i \in S^{**} \rangle$ satisfy the conditions of Lemma 5.3. Applying Lemma 5.3, we get that $\langle \bar{a}_i \mid i \in S^{**} \rangle$ is a Morley sequence over M^*. In particular, since $A \subset M^*$, we have that $\langle \bar{a}_i \mid i \in S^{**} \rangle$ is a Morley sequence over A.

$$\dashv$$

Remark 5.9. Our effort to prove the results of this section in summer 2001 lead us to the notion of tameness as defined in this paper. A preliminary version of this paper containing the results of this section was posted by us on the web already in July 27th, 2001 with the title *Morley Sequences in AECs*.

6. Towards a Stability Spectrum Theorem

In this section we begin work towards a stability spectrum theorem for tame classes. Other partial results in this direction appear in [BaKuVa].

Theorem 6.1 (Extension property for non-μ-splitting types). *Let $\mathcal{K}$ be an AEC. Suppose $N \prec_{\mathcal{K}} M' \prec_{\mathcal{K}} M$ are such that M' is universal over N and all three models have cardinality μ. Then for every $p' \in \text{ga-S}(M')$ which does not μ-split over N, there exists $p \in \text{ga-S}(M)$ extending p' such that p does not μ-split over N.*

Proof. Let N, M', M be as in the statement of the theorem. Suppose $p = \text{ga-tp}(a'/M')$ does not μ-split over N. Since M' is universal over N, there exists $f : M' \to M$ with $f \restriction N = \text{id}_N$. Notice that by monotonicity of non-μ-splitting, we have that $\text{ga-tp}(a'/f(M))$ does not μ-split over N. Then, by invariance we have that also

$$(*) \quad \text{ga-tp}(f(a')/M) \text{ does not } \mu\text{-split over } N.$$

By one more appeal to non-splitting, we can conclude that $\text{ga-tp}(f(a')/M') = \text{ga-tp}(a'/M')$. Otherwise these two types would contradict $(*)$. $\dashv$

Theorem 6.2 (Uniqueness of non-splitting extensions). *Let $\mathcal{K}$ be χ-tame abstract elementary class and μ a cardinal with $\mu \geq \chi$. Let $M \in \mathcal{K}_{>\mu}$ and $M', N \in \mathcal{K}_\mu$ with $N \prec_{\mathcal{K}} M' \prec_{\mathcal{K}} M$. If M' is universal over N, then for every $p' \in \text{ga-S}(M')$ which does not μ-split over N, if there are $q, p \in \text{ga-S}(M)$ such that q and p both extend p' and do not μ-split over N, then $p = q$.*

Proof. Suppose for the sake of contradiction that there exists $q \neq p \in \text{ga-S}(M)$ extending $p' \in \text{ga-S}(M')$ such that both p and q do not μ-split over N. Let a, b be such that $p = \text{ga-tp}(a/M)$ and $q = \text{ga-tp}(b/M)$. By tameness there exists $M^* \in \mathcal{K}_\mu$ such that $M' \prec_{\mathcal{K}} M^* \prec_{\mathcal{K}} M$ and $p \restriction M^* \neq q \restriction M^*$.

Since M' is universal over N, there exists a $\prec_{\mathcal{K}}$-mapping $f : M^* \to M'$ with $f \restriction N = \text{id}_N$. Since p and q do not μ-split over N we have

$$(*)_a \quad \text{ga-tp}(a/f(M^*)) = \text{ga-tp}(f(a)/f(M^*)) \text{ and}$$

$$(*)_b \quad \text{ga-tp}(b/f(M^*)) = \text{ga-tp}(f(b)/f(M^*)).$$

On the other hand, since $p \restriction M^* \neq q \restriction M^*$, we have that their images are also not equal

$$(*) \quad \text{ga-tp}(f(a)/f(M^*)) \neq \text{ga-tp}(f(b)/f(M^*)).$$

Combining $(*)_a$, $(*)_b$ and $(*)$, we get

$$\text{ga-tp}(a/f(M^*)) \neq \text{ga-tp}(b/f(M^*)).$$

Since $f(M^*) \prec_{\mathcal{K}} M'$, this inequality witnesses that

$$\text{ga-tp}(a/M') \neq \text{ga-tp}(b/M'),$$

contradicting our choice of p and q both extending p'.

     $\dashv$

Corollary 6.3. *Let $\mathcal{K}$ be χ-tame abstract elementary class and let μ be a cardinal with $\mu > \chi$. Let $M \in \mathcal{K}_{>\mu}$, $N \prec_{\mathcal{K}} M$ with M universal over N. If $\mathcal{K}$ is Galois-stable in μ then*

$$|\{p \in \text{ga-S}(M) \ : \ p \text{ does not } \mu\text{-split over } N\}| \leq \mu.$$

Proof. Since M is universal over N, there is a $M' \in \mathcal{K}_{\mu}$ with $N \prec_{\mathcal{K}} M' \prec_{\mathcal{K}} M$ and M' universal over N. By μ-stability there are only μ-many types over M' and by Theorem 6.2 each type over N has at most one non-μ-splitting extension to M. $\dashv$

Corollary 6.4 (Partial stability spectrum)**.** *Suppose that $\mathcal{K}$ is a tame abstract elementary class satisfying the amalgamation property. If $\mathcal{K}$ is Galois-stable in some $\mu > \text{Hanf}(\mathcal{K})$, then $\mathcal{K}$ is stable in every κ with $\kappa^{\mu} = \kappa$.*

Proof. Let $\kappa = \kappa^{\mu}$. Consider $M \in \mathcal{K}_{\kappa}$ and let M' be a μ-saturated extension of M. Such an extension exists by stability in μ and our cardinal arithmetic assumption. Suppose that $\text{ga-tp}(a/M, \mathfrak{C}) \neq \text{ga-tp}(b/M, \mathfrak{C})$, then surely $\text{ga-tp}(a/M', \mathfrak{C}) \neq \text{ga-tp}(b/M', \mathfrak{C})$. So the number of Galois-types over M is $\leq$ the number of Galois-types over M'.

Thus we may assume that M is μ-saturated. Again by our assumption that $\kappa^{\mu} = \kappa$, we can enumerate $\langle N_i \mid i < \kappa \rangle$ all the $\prec_{\mathcal{K}}$-submodels of M of cardinality μ. Since M is μ-saturated, for each one of the N_i there is M'_i of cardinality μ such that $N_i \prec_{\mathcal{K}} M'_i \prec_{\mathcal{K}} M$ with M'_i universal over N_i. By Fact 4.6 for every $p \in \text{ga-S}(M)$, there exists $N_i \prec_{\mathcal{K}} M$ of cardinality μ such that p does not μ-split over N_i. Thus

$$|\text{ga-S}(M)| = |\bigcup_{i<\kappa} \{p \in \text{ga-S}(M) \mid p \text{ does not } \mu\text{-split over } N_i\}|.$$

By monotonicity of non-splitting we have that for each $p \in \text{ga-S}(M)$ that does not μ-split over N_i, also $p \upharpoonright M'_i$ does not μ-split over N_i. By Corollary 6.3, for each $i < \kappa$,

$$|\{p \in \text{ga-S}(M) \mid p \text{ does not } \mu\text{-split over } N_i\}|$$

$$\leq |\{p \in \text{ga-S}(M'_i) \mid p \text{ does not } \mu\text{-split over } N_i\}|.$$

Now by stability in μ and our assumption that $\kappa^{\mu} = \kappa$, we can conclude that $|\text{ga-S}(M)| \leq \kappa$.

$\dashv$

7. Exercise on Dividing

With the existence of Morley sequences a natural extension is to study the following dependence relation to determine whether or not it satisfies properties such as transitivity, symmetry or extension. Here we derive the existence property for types over saturated models.

Definition 7.1. Let $p \in \text{ga-S}(M)$ and $N \prec_{\mathcal{K}} M$. We say that p *divides over* N iff there are $a \in |M| \setminus |N|$ and a Morley sequence, $\{a_n \mid n < \omega\}$ for the ga-tp$(a/N, M)$ such that for some collection $\{f_n \in \text{Aut}_M \mathfrak{C} \mid n < \omega\}$ with $f_n(a) = a_n$ we have

$$\{f_n(p) \mid n < \omega\} \text{ is inconsistent.}$$

I.e. there is no common realization to the above mentioned ω-many types.

Theorem 7.2 (Existence)**.** *Suppose that $\mathcal{K}$ is Galois-stable in μ and χ-tame for some $\chi < \mu$. For every μ-model homogeneous model $M \in \mathcal{K}_{\geq \mu}$ and ever $p \in \text{ga-S}(M)$, there exists $N \prec_{\mathcal{K}} M$ of cardinality μ such that p does not divide over N.*

Proof. Suppose that p and M form a counter-example. We will find N_i, N_i^1, N_i^2 and h_i for $i < \mu$ satisfying:

(1) $\langle N_i \in \mathcal{K}_\mu \mid i \leq \mu \rangle$ is a $\prec_{\mathcal{K}}$-increasing and continuous sequence of models;
(2) $N_i \prec_{\mathcal{K}} N_i^l \prec_{\mathcal{K}} N_{i+1}$ for $i < \mu$ and $l = 1, 2$;
(3) for $i < \mu$, $h_i : N_i^1 \cong N_i^2$ and $h_i \restriction N_i = \text{id}_{N_i}$ and
(4) $p \restriction N_i^2 \neq h_i(p \restriction N_i^1)$.

Suppose that N_i has been defined. Since p divides over every substructure of cardinality μ, we may find $\bar{a}, \{\bar{a}_n \mid n < \omega\}$ and $\{f_n \mid n < \omega\}$ witnessing that p divides over N_i. Namely, we have that $\{f_n(p) \mid n < \omega\}$ is inconsistent. Let $n < \omega$ be such that $f_0(p) \neq f_n(p)$. Then $p \neq f_0^{-1} \circ f_n(p)$. By χ-tameness, we can find $N^* \prec_{\mathcal{K}} M$ of cardinality μ containing N such that $p \restriction N^* \neq (f_0^{-1} \circ f_n(p)) \restriction N^*$. WLOG $f_0^{-1} \circ f_n \in \text{Aut}_N N^*$.

Let $h_i := f_0^{-1} \circ f_n$, $N_i^1 := N^*$ and $N_i^2 := N^*$. Choose $N_{i+1} \prec_{\mathcal{K}} M$ to be an extension of N^* of cardinality μ.

Now we can use $\langle N_i, N_i^1, N_i^2, h_i \mid i < \mu \rangle$ to contradict Galois-stability in μ by repeating the argument from the proof of Fact 4.6, (starting at the second paragraph) we construct an increasing chain of models and a tree of $\mathcal{K}$-embedding which by Claim 4.7 gives many Galois-types contradicting stability in μ. $\dashv$

Remark 7.3. In a previous version of this paper, we stated for the existence property for arbitrary models $M \in \mathcal{K}_{\geq \mu}$. We do not know how to remove the assumption of μ-model homogeneity and we suspect that this is not possible with only the assumption of μ-stability and tameness.

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