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Uniqueness of limit models in classes with amalgamation

Rami Grossberg[1], **Monica VanDieren**[2]*, and **Andrés Villaveces**[3]

[1] Department of Mathematical Sciences, Carnegie Mellon University, Wean Hall 6113, Pittsburgh PA 15213, United States of America
[2] Department of Mathematics & University Honors Program, Robert Morris University, 6001 University Boulevard, Moon Township PA 15108, United States of America
[3] Departamento de Matemáticas, Universidad Nacional de Colombia, AK 30 # 45–03–111321, Bogotá, Colombia

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We prove the following main theorem: Let $\mathcal{K}$ be an abstract elementary class satisfying the joint embedding and the amalgamation properties with no maximal models of cardinality μ. Let μ be a cardinal above the the Löwenheim-Skolem number of the class. If $\mathcal{K}$ is μ-Galois-stable, has no μ-Vaughtian Pairs, does not have long splitting chains, and satisfies locality of splitting, then any two (μ, σ_ℓ)-limits over M, for $\ell \in \{1, 2\}$, are isomorphic over M.

1 Introduction

We work in the general context of abstract elementary classes (AECs) with the amalgamation property (AP) and Galois-stability at one fixed cardinality μ above the Löwenheim-Skolem number. We assume there is a model of cardinality μ^+. We prove the uniqueness of limit models under a unidimensionality-like assumption of no μ-Vaughtian pairs and superstability-like assumptions of the μ-splitting dependence relation.

The basic model theory of abstract elementary classes (definitions, the role of the AP and the joint embedding property, JEP, the existence of a "monster model" $\mathfrak{C}$, Galois types and the foundational development of stability theory in that context) can be checked in the monograph [5] and the books [1, 16]. For the sake of completeness, we include some of the notation and fundamentals of this context here. We fix an abstract elementary class $\mathcal{K}$ with ordering $\prec_{\mathcal{K}}$. For a cardinal μ, we use the notation $\mathcal{K}_\mu$ for the class of models of $\mathcal{K}$ of cardinality μ.

In practice, abstract elementary classes were not as approachable as one would hope and much work in non-elementary model theory takes place in contexts which additionally satisfy the amalgamation property so that a monster model can be utilized. The following fact can be traced back to Jónsson's 1960 paper [11]; the present formulation is from [4]:

Theorem 1.1 *Let $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ be an AEC with no maximal models and suppose that there is $\lambda \geq \kappa > \mathrm{LS}(\mathcal{K})$ such that $K_{<\lambda}$ has the AP and the JEP. Suppose $M \in \mathcal{K}$. If $\lambda^{<\kappa} = \lambda \geq \|M\|$ then there exists $N \succ M$ of cardinality λ which is κ-model-homogeneous.*

Thus if an AEC $\mathcal{K}$ has the AP and the JEP, then like in first-order stability theory we may assume that there is a large model-homogeneous $\mathfrak{C} \in \mathcal{K}$ that acts like a monster model. We will refer to the model $\mathfrak{C}$ as the *monster model*. All models considered will be of size less than $\|\mathfrak{C}\|$, and we will find realizations of types we construct inside this monster model. From now on, we assume that the monster model $\mathfrak{C}$ has been fixed. We use the notation $\mathrm{Aut}_M(\mathfrak{C})$ to denote the set of automorphisms of $\mathfrak{C}$ fixing M pointwise.

The notion of type as a set of formulas, even when the class is described in some infinitary logic, does not behave as nicely as in first-order logic. A replacement was introduced by Shelah in [19]. In order to avoid confusion between this and the classical, syntactic notion, we will use the terminology in [5] and call this alternative notion the *Galois type*.

* Corresponding author; e-mail: vandieren@rmu.edu

Since in this paper we deal only with AECs with the AP, the notion of Galois type has a simpler definition than in the general case.

Definition 1.2 (Galois types) Suppose that $\mathcal{K}$ has the AP.

(1) Given $M \in \mathcal{K}$ consider the action of $\mathrm{Aut}_M(\mathfrak{C})$ on $\mathfrak{C}$, for an element $a \in |\mathfrak{C}|$ let ga-tp(a/M) denote the *Galois type of a over M* which is defined as the orbit of a under $\mathrm{Aut}_M(\mathfrak{C})$.
(2) For $M \in \mathcal{K}$, we let ga-S$(M) = \{$ ga-tp(a/M) ; $a \in |\mathfrak{C}|\}$.
(3) $\mathcal{K}$ is λ-*Galois-stable* if and only if $N \in \mathcal{K}_\lambda$ implies $|$ ga-S$(N)| \leq \lambda$.
(4) Given $p \in$ ga-S(M) and $N \in \mathcal{K}$ such that $N \succ_{\mathcal{K}} M$, we say that p is *realized* by $a \in N$ if and only if ga-tp$(a/M) = p$. Just as in the first-order case we will write $a \models p$ when a is a realization of p.
(5) For $h \in \mathrm{Aut}(\mathfrak{C})$ and $p =$ ga-tp(a/M), then the notation $h(p)$ refers to ga-tp$(h(a)/h(M))$.

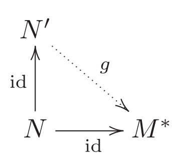
For a more detailed discussion of Galois types, their extensions, restrictions, equivalent forms and generalizations, the reader may consult [5].

The next notion to consider is that of a saturated model. In homogeneous abstract elementary classes (cf., e.g., [6]) where one may study classes of models omitting given sets of types, the existence of a saturated model presents some problems. One solution is to consider models which realize as many types as possible. Such models are called Galois-saturated. More formally, a model M of size $\kappa > \mathrm{LS}(\mathcal{K})$ is *Galois-saturated* if it realizes all Galois types over submodels $N \prec_{\mathcal{K}} M$ of cardinality $< \kappa$. When stability theory has been ported to contexts more general than first order logic, many situations have appeared when Galois-saturated models do not fulfill the main roles that saturated models play in elementary classes.

The main concept of this paper is Shelah's *limit model* which (among other things) serves as a substitute for the role of saturation in stability theory (cf. [5, 16, 24], etc.) or at least serves as a stepping stone to prove the properties of Galois-saturated models. For example, under the assumption of categoricity with reasonable stability conditions, the existence of Galois-saturated models in singular cardinals is not straightforward and is proved by first considering limit models [20]. In some contexts limit models have been successfully used as "tools" towards finding Galois-saturated models [12, 21]. Furthermore, the notion of limit model refines the notion of saturation; more detailed information is given on the particular way one model is embedded inside another.

Limit models appear in [12, 22] under the name (μ, α)-*saturated models*. In [23], Shelah calls this notion *brimmed*. Later papers, beginning with Shelah-Villaveces [24], adopt the name *limit models*. We use the more recent terminology. Before defining limit models, we must introduce their building blocks, universal extensions.

Definition 1.3 (1) Let κ be a cardinal $\geq \mathrm{LS}(\mathcal{K})$. We say $M^* \succ_{\mathcal{K}} N$ is κ-*universal over N* if and only if for every $N' \in \mathcal{K}_\kappa$ with $N \prec_{\mathcal{K}} N'$ there exists a $\mathcal{K}$-embedding $g : N' \underset{N}{\rightarrow} M^*$ such that the following diagram commutes:



(2) We say M^* is *universal over N* or M^* is *a universal extension of N* if and only if M^* is $\|N\|$-universal over N.

Definition 1.4 (Limit models) Consider $\mu \geq \mathrm{LS}(\mathcal{K})$ and $\alpha < \mu^+$ a limit ordinal and $N \in \mathcal{K}_\mu$. We say that M is (μ, α)-*limit model over N* if and only if there exists an increasing and continuous chain $\langle M_i \in \mathcal{K}_\mu \,;\, i < \alpha \rangle$ such that $M_0 = N$; $M = \bigcup_{i<\alpha} M_i$; M_i is a proper $\mathcal{K}$-submodel of M_{i+1}; and M_{i+1} is universal over M_i for all $i < \alpha$.

From Theorem 1.5 we get that for $\alpha \leq \mu^+$ there always exists a (μ, α)-limit model provided $\mathcal{K}$ has the AP, has no maximal models and is μ-Galois-stable. This theorem was stated without proof as [23, Claim 1.16]; for a proof, cf. [7] or [4].

Theorem 1.5 (Existence) *Let $\mathcal{K}$ be an AEC without maximal models and suppose it is Galois-stable in μ. If $\mathcal{K}$ has the amalgamation property then for every $N \in \mathcal{K}_\mu$ there exists $M^* \succneqq_{\mathcal{K}} N$, universal over N of cardinality μ.*

The following theorem partially clarifies the analogy with saturated models:

Theorem 1.6 *Let T be a stable, complete, first-order theory and let $\mathcal{K}$ be the elementary class of models of T with the usual notion of elementary submodel. If M is a (μ, δ)-limit model for δ a limit ordinal with* $\mathrm{cf}(\delta) \geq \kappa(T)$, *then M is saturated.*

Proof. Use an argument similar to the proof of [15, Theorem III 3.11]. □

Thus in elementary classes superstability implies that limit models are saturated, in particular are unique. This raises the following natural question for AECs about the uniqueness of limit models:

> Let $\mathcal{K}$ be an AEC, $\mu \geq \mathrm{LS}(\mathcal{K})$, $M \in \mathcal{K}_\mu$ and σ_1, σ_2 limit ordinals $< \mu^+$, and suppose that for $\ell = 1, 2$, N_ℓ is a (μ, σ_ℓ)-limit model over M. What "reasonable" assumptions on $\mathcal{K}$ will imply that there exists $f : N_1 \cong_M N_2$?

This question is non-trivial only for the case where $\mathrm{cf}(\sigma_1) \neq \mathrm{cf}(\sigma_2)$. Using a back and forth argument one can show that when $\mathrm{cf}(\sigma_1) = \mathrm{cf}(\sigma_2)$, we get uniqueness without any assumptions on $\mathcal{K}$. More precisely:

Theorem 1.7 *Let $\mu \geq \mathrm{LS}(\mathcal{K})$ and $\sigma < \mu^+$. If M_1 and M_2 are (μ, σ)-limits over M, then there exists an isomorphism $g : M_1 \to M_2$ such that $g \restriction M = \mathrm{id}_M$. Moreover if M_1 is a (μ, σ)-limit over M_0, if N_1 is a (μ, σ)-limit over N_0 and if $g : M_0 \cong N_0$, then there exists a $\mathcal{K}$-embedding, $\hat{g}$, extending g such that $\hat{g} : M_1 \cong N_1$.*

Theorem 1.8 *Let μ be a cardinal and σ a limit ordinal with $\sigma < \mu^+$. If M is a (μ, σ)-limit model, then M is a $(\mu, \mathrm{cf}(\sigma))$-limit model.*

The main result of this paper provides an answer to the question of uniqueness of limit models:

Theorem 1.9 (Main Theorem) *Let $\mathcal{K}$ be an AEC and $\mu > \mathrm{LS}(\mathcal{K})$. Suppose $\mathcal{K}$ has both the AP and the JEP and has no maximal models of cardinality μ. If $\mathcal{K}$ is μ-Galois-stable, does not have long splitting chains, has no μ-Vaughtian pairs and satisfies locality of splitting*[1]*, then any two (μ, σ_ℓ)-limits over M, for $\ell \in \{1, 2\}$, are isomorphic over M.*

This theorem extends results of Shelah from [20,22,23], Kolman and Shelah in [12] and Shelah and Villaveces from [24]. A preliminary version of our uniqueness theorem, which was circulated in 2006, was used by Grossberg and VanDieren to prove a case of Shelah's categoricity conjecture for tame abstract elementary classes in [8]. Preprints of this paper have also influenced the Ph.D. theses of Drueck [3] and Zambrano [31].

In a preprint of this paper, the assumption of disjoint amalgamation was made in Theorem 1.9. After reading a preprint of this paper, Fred Drueck in his Ph.D. thesis [3] pointed out that the disjoint amalgamation property is not necessary to carry out the arguments here. In particular, it is not needed in Theorem 4.4.

This paper also serves the expository role of presenting together the arguments in [25, 26] in a more natural context in which the amalgamation property holds and this work provides an approach to the uniqueness of limit models that does not rely on Ehrenfeucht-Mostowski constructions.

The last section of this paper describes different approaches to the question of the uniqueness of limit models.

2 The setting

In what follows, $\mathcal{K}$ is assumed to be an AEC, and μ is a cardinal $\geq \mathrm{LS}(\mathcal{K})$. In this section we summarize all of the assumptions that will be made on the class $\mathcal{K}$, and in the subsequent sections we introduce two of the main components of the proof of the uniqueness of limit models: strong types and towers.

We will prove the uniqueness of limit models in μ-Galois stable AECs that are essentially unidimensional and are equipped with a moderately well-behaved dependence relation. We will use μ-splitting as the dependence relation, but any dependence relation which is local and has existence, uniqueness and extension properties suffices.

[1] Cf. Assumption 2.8 for the precise description of long splitting chains and locality.

Definition 2.1 A type $p \in \text{ga-S}(M)$ *μ-splits* over $N \in \mathcal{K}_{\leq\mu}$ if and only if N is a $\prec_{\mathcal{K}}$-submodel of M and there exist $N_1, N_2 \in \mathcal{K}_\mu$ and a $\mathcal{K}$-mapping h such that $N \prec_{\mathcal{K}} N_l \prec_{\mathcal{K}} M$ for $l = 1, 2$ and $h : N_1 \to N_2$ with $h \restriction N = \mathrm{id}_N$ and $p \restriction N_2 \neq h(p \restriction N_1)$.

The existence property for non-μ-splitting types follows from Galois stability in μ:

Theorem 2.2 (Existence; [20, Claim 3.3]) *Assume $\mathcal{K}$ has the AP and is Galois-stable in μ. For every $M \in \mathcal{K}_{\geq\mu}$ and $p \in$ ga-S(M), there exists $N \in \mathcal{K}_\mu$ such that p does not μ-split over N.*

The uniqueness and extension properties of non-μ-splitting types hold for types over limit models:

Theorem 2.3 (Uniqueness; [25, Theorem I.4.15]) *Let $N \prec_{\mathcal{K}} M \prec_{\mathcal{K}} M'$ be models in $\mathcal{K}_\mu$ such that M' is universal over M and M is universal over N. If $p \in$ ga-S(M) does not μ-split over N, then there is a unique $p' \in$ ga-S(M') such that p' extends p and p' does not μ-split over N.*

A variation of this fact is later used in an induction construction in the proof of Theorem 5.7. We state it explicitly here:

Theorem 2.4 (VanDieren; [25, Theorem I.4.10]) *Let M, N, M^* be models in $\mathcal{K}_\mu$. Suppose that M is universal over N and that M^* is an extension of M. If a type $p =$ ga-tp(a/M) does not μ-split over N then there exists an automorphism g of $\mathfrak{C}$ fixing M such that* ga-tp$(g(a)/M^*)$ *does not μ-split over N and* ga-tp$(g(a)/M) = p$.

The other concepts that show up in the assumptions of the main theorem of this paper are minimal types [20] and μ-Vaughtian Pairs [8].

Definition 2.5 (1) For M a model of cardinality μ, $p \in$ ga-S(M) is *minimal* if it is non-algebraic and for each N extending M of cardinality μ if there are non-algebraic extensions p_1 and p_2 of p to N, then $p_1 = p_2$.

(2) For M a limit model of cardinality μ a *μ-Vaughtian Pair* is a pair of limit models M' and N' of cardinality μ so that there exist $M \preceq_{\mathcal{K}} M' \prec_{\mathcal{K}} N'$ and $p \in$ ga-S(M) a minimal type for which N' contains no new realizations of p, in other words, $p(M') = p(N')$.

Theorem 2.6 (Existence of minimal types; [20, Fact $(*)_5$ in Theorem 9.8]) *Let $\mu >$ LS$(\mathcal{K})$. If $\mathcal{K}$ is Galois-stable in μ, then for every $M \in \mathcal{K}_\mu$ and every $q \in$ ga-S(M), there are $N \in \mathcal{K}_\mu$ and $p \in$ ga-S(N) such that $M \preceq_{\mathcal{K}} N$, $q \leq p$ and p is minimal.*

Theorem 2.7 (Shelah; [20, Claim $(*)_8$ of Theorem 9.8]) *If $\mathcal{K}$ is categorical in some successor cardinal $\lambda^+ >$ LS$(\mathcal{K})^+$, then for every μ satisfying* LS$(\mathcal{K}) \leq \mu \leq \lambda$*, there are no μ-Vaughtian Pairs.*

It is worth mentioning that our "no μ-Vaughtian pairs" assumption is weaker, in general, than assuming categoricity (as in [26]). Consider the first order case. There are first order theories such as the theory of real closed fields which are quite far from being categorical but also have no Vaughtian pairs. Of course, under ω-stability, no Vaughtian pairs and categoricity are equivalent (in first order). But our stability assumptions are of a "superstable" nature and not ω-stable, and in abstract elementary classes, the connections between superstability, no Vaughtian pairs, and categoricity are far from being completely understood.

Here are the assumptions of the paper:

Assumption 2.8 The class $\mathcal{K}$ is an AEC with the AP and the JEP with no maximal models of cardinality μ, and $\mathcal{K}$ satisfies the following properties:

(1) All models are submodels of a fixed monster model $\mathfrak{C}$.
(2) The class $\mathcal{K}$ is stable in μ.
(3) There are no μ-Vaughtian Pairs.
(4) The relation of μ-splitting in $\mathcal{K}$ satisfies the following locality (sometimes called continuity) and *no long splitting chains* properties. For all infinite α, for every sequence $\langle M_i \,;\, i < \alpha\rangle$ of limit models of cardinality μ with M_{i+1} universal over M_i and for every $p \in$ ga-S(M_α), where $M_\alpha = \bigcup_{i<\alpha} M_i$, we have that
  (a) If for every $i < \alpha$, the type $p \restriction M_i$ does not μ-split over M_0, then p does not μ-split over M_0.
  (b) There exists $i < \alpha$ such that p does not μ-split over M_i.

In the context of an AEC with the full amalgamation property and the JEP, categoricity in a cardinal $\lambda > \mu$ implies all parts of Assumption 2.8. For a proof of Assumption 2.8.2 from categoricity, cf. [20, Claim 1.7] or [1]. [20, Claim $(*)_8$ of Theorem 9.8] is Assumption 2.8.3 when λ is a successor cardinal. The observation that Assumption 2.8(4a) follows from categoricity is a consequence of [20, Observation 6.2 and Main Lemma 9.4]. [20, Lemma 6.3] is the statement that assumption 2.8(4b) follows from categoricity when the cofinality of the categoricity cardinal is larger than μ.

The amalgamation and joint embedding properties hold in homogeneous classes (cf. [17] or [14]), in excellent classes (cf. [18]) and are axioms in the definition of finitary classes (cf. [10]). They also hold for cats consisting of existentially closed models of positive Robinson theories [31]. In each of these contexts dependence relations satisfying Assumption 2.8 have been developed. Finally, the locality and existence of non-μ-splitting extensions are akin to consequences of superstability in first order logic.

Theorem 2.9 ("No long splitting chains" follows from stability in FO) *Suppose that T is first order complete. If T is stable then Assumption* 2.8*(4b) holds for α such that* $\mathrm{cf}(\alpha) \geq |T|^+$.

P r o o f. Let $\langle M_i | i < \alpha \rangle$ be an increasing sequence of saturated models. Let $M_\alpha := \bigcup_{i<\alpha} M_i$. Suppose $p \in S(M_\alpha)$ is such that $\forall i < \alpha$, p μ-splits over M_i. Because for every $i < \alpha$, we know there exists $i < j(i) < \alpha$ such that $p \restriction M_{j(i)}$ splits over M_i, we may assume that for all $i < \alpha$, $p \restriction M_{i+1}$ splits over M_i. Let $\varphi_i(\bar{x}, \bar{y})$ be a formula witnessing the splitting of $p \restriction M_{i+1}$ over M_i. As $\mathrm{cf}(\alpha) \geq |T|^+$, there exists $S \subset \alpha$ infinite such that $i, j \in S$ implies $\varphi_i = \varphi_j$.

Without loss of generality, suppose that $\langle M_n | n \leq \omega \rangle$ is an increasing sequence of saturated models, and $p \in S_\varphi(M_\omega)$ is such that $\bar{a}_i, \bar{b}_i \in M_{i+1}$ witness that $p \restriction M_{i+1}$ splits over M_i. Then $p(x_1, \bar{y}_1, \bar{z}_1, x_2, \bar{y}_2, \bar{z}_2)$ and $\{\bar{d}_i | i < \omega\}$ witness that p has the order property, where $\bar{d}_i = \bar{a}_i \hat{\ } \bar{b}_i \hat{\ } c_i$, $c_i \in M_{i+2}$ and $c_i \models p \restriction \{\bar{a}_k, \bar{b}_k | k \leq i\} \cup \{d_k | k < i\}$. Now use [4, Lemma VII, 2.12]. □

Note that Assumption 2.8.3 is used only to show that reduced towers are continuous (cf. Theorem 5.7). It is conjectured that this assumption may be eliminated or replaced with a weaker assumption related to superstability in first order logic.

3 Strong types

Under the assumption of μ-stability, we can define *strong types* as in [24]. These strong types will allow us to achieve a better control of extensions of towers of models than what we obtain using just Galois types. Specifically, consider the situation in which $M \preceq_{\mathcal{K}} M'$ and $p, q \in \text{ga-S}(M')$ with $p \restriction M = q \restriction M$. If p does not μ-split over some $N_1 \prec_{\mathcal{K}} M$ and q does not μ-split over $N_2 \prec_{\mathcal{K}} M$, then we cannot conclude that $p = q$, even if we assume M is universal over N_1 and N_2.

Definition 3.1 (Shelah & Villaveces; [24, Definition 3.2.1]) For a (μ, ϑ)-limit model M, let

$$\mathfrak{St}(M) := \{(p, N)\,;\, N \prec_{\mathcal{K}} M \text{ is a } (\mu, \vartheta)\text{-limit model such that } M \text{ is universal over } N$$
$$\text{and } p \in \text{ga-S}(M) \text{ is non-algebraic and does not } \mu\text{-split over } N\}$$

Elements of $\mathfrak{St}(M)$ are called *strong types.* Two strong types $(p_1, N_1) \in \mathfrak{St}(M_1)$ and $(p_2, N_2) \in \mathfrak{St}(M_2)$ are *parallel* if and only if for every M' of cardinality μ extending M_1 and M_2 there exists $q \in \text{ga-S}(M')$ such that q extends both p_1 and p_2 and q does not μ-split over N_1 nor over N_2.

Under the assumption of the existence of universal extensions, it is equivalent to say two strong types $(p_1, N_1) \in \mathfrak{St}(M_1)$ and $(p_2, N_2) \in \mathfrak{St}(M_2)$ are parallel if and only if for some M' of cardinality μ universal over some common extension of M_1 and M_2 there exists $q \in \text{ga-S}(M')$ such that q extends both p_1 and p_2 and q does not μ-split over N_1 and N_2.

Lemma 3.2 (Monotonicity of parallel types) *Suppose $M_0, M_1 \in \mathcal{K}_\mu$ and $M_0 \prec_{\mathcal{K}} M_1$ and $(p, N) \in \mathfrak{St}(M_1)$. If M_0 is universal over N, then we have $(p \restriction M_0, N)$ is parallel to (p, N).*

P r o o f. Straightforward using the uniqueness of non-μ-splitting extensions. □

Let $M, M' \in \mathcal{K}_\mu$ and suppose that $M \prec_{\mathcal{K}} M'$. For $(p, N) \in \mathfrak{St}(M')$, if M is universal over N, we define the restriction $(p, N)\restriction M \in \mathfrak{St}(M)$ to be $(p\restriction M, N)$. If we write $(p, N)\restriction M$, we mean that p does not μ-split over N and M is universal over N. We denote by $\sim$ the parallelism relation between strong types in $\mathfrak{St}(M)$, for fixed M.

Notice that $\sim$ is an equivalence relation on $\mathfrak{St}(M)$ (cf. [25]). Stability in μ implies that there are few strong types over any model of cardinality μ:

Theorem 3.3 (Shelah & Villaveces; [24, Claim 3.2.2 (3)]) *If $\mathcal{K}$ is Galois-stable in μ, then for any $M \in \mathcal{K}_\mu$, $|\mathfrak{St}(M)/\sim| \le \mu$.*

The referee has pointed out that several of our uses of parallel types fit into the more simplified situation described in the remark below. In particular, parallel types can be replaced by equal restrictions in Theorem 4.4.

Remark 3.4 Let (p_1, N_1) and (p_2, N_2) be parallel strong types with $p_l \in$ ga-S(M_l). If $M_1 \prec_{\mathcal{K}} M_2$, then by uniqueness of non-splitting extensions $p_1 = p_2\restriction M_1$.

However, we cannot replace parallelism with equality of types everywhere. In particular parallelism shows its necessity in the proof that the union of a $<$-chain of relatively full towers in $\mathcal{K}^*_{\mu,\alpha}$ is relatively full. The strength of parallel types can be seen in the following situation which arises in the proof of Claim 5.10. Suppose that $M \prec_{\mathcal{K}} M'$ and that there are types p and p' over M and M', respectively. Suppose p does not μ-split over N, $p' =$ ga-tp(a'/M') does not μ-split over both N and N', and $p'\restriction M = p$. Without having any understanding of the relationship between N and N' (and this is the case in the definition of towers: the N_i's of Definition 4.1 are bases for non-splitting but are not in principle related to one another) or the stronger condition that the strong types are parallel, it is not possible to predict which M^* extending M will have the property that ga-tp(a'/M^*) does not μ-split over N. Under the assumption that (p, N) and (p, N') are parallel, we only need to be able to extend M^* to a model M^{**} for which ga-tp(a'/M^{**}) does not μ-split over N' to be able to conclude that ga-tp(a'/M^*) also does not μ-split over N.

The way we use parallel types in the proof of Claim 5.10 is summarized in the following lemma:

Lemma 3.5 *Suppose $M_1, M_1', M_2, M_2' \in \mathcal{K}_\mu$ with $M_1 \prec_{\mathcal{K}} M_1'$, $M_2 \prec_{\mathcal{K}} M_2'$, $M_1 \prec_{\mathcal{K}} M_2$, and $M_1' \prec_{\mathcal{K}} M_2'$. Suppose (p_1, N_1) and (p_2, N_2) are parallel types in $\mathfrak{St}(M_1)$ and $\mathfrak{St}(M_2)$, respectively. If $p_l' \in$ ga-S(M_l') extends p_l and does not μ-split over N_l for $l \in \{1, 2\}$, then (p_1', N_1) and (p_2', N_2) are parallel.*

Proof. By uniqueness of non-splitting extensions, first notice that $p_1'\restriction M_1 = p_1$ and $p_2'\restriction M_2 = p_2$.

To see that (p_1', N_1) is parallel to (p_2', N_2), let M' be an extension of M_2' of cardinality μ. Since $(p_2', N_2)\restriction M_2 = (p_2, N_2)$ is parallel to (p_1, N_1) there is $q \in$ ga-S(M') so that q extends both p_2 and p_1, and q does not μ-split over N_1 and N_2. By the uniqueness of non-splitting extensions, we have the $q\restriction M_2'$ must agree with p_2'. But, we also can conclude by the uniqueness of non-splitting extensions that $q\restriction M_1'$ must agree with p_1' since they are both non-splitting extensions of p_1. Therefore q witnesses that (p_1', N_1) and (p_2', N_2) are parallel. □

4 Towers

We use the technology of towers in our proof. Towers have been used before by Shelah and Villaveces [24] and VanDieren [25,26,28]. Towers enable us to control in multidimensional arrays notions of "relative saturation" apt to our aim: obtaining limits that can be approached through chains of two different cofinalities requires controlling the way in which we gradually "saturate" the models with realizations of Galois types. Properties of towers are "filtered" analogues of properties of Galois types (extension and other properties of independence).

To each (μ, ϑ)-limit model M we can naturally associate a $\prec_{\mathcal{K}}$-increasing chain $\bar{M} = \langle M_i \in \mathcal{K}_\mu\,;\, i < \vartheta\rangle$ witnessing that M is a (μ, ϑ)-limit model (that is, $\bigcup_{i<\vartheta} M_i = M$ and M_{i+1} is universal over M_i). Furthermore, by Theorems 1.7 and 1.8 we can require that this chain satisfies additional requirements such as M_{i+1} is a limit model over M_i. In this section we will be considering a related chain of models which we will refer to as a tower (cf. Definition 4.1). But first, we will describe how towers will be used to prove the main theorem of this paper.

To prove the uniqueness of limit models we will construct a model which is simultaneously a (μ, ϑ_1)-limit model over some fixed model M and a (μ, ϑ_2)-limit model over M. Notice that, by Theorem 1.7, it is enough to construct a model M^* that is simultaneously a (μ, ω)-limit model and a (μ, ϑ)-limit model for arbitrary ordinal $\vartheta < \mu^+$. By Theorem 1.8 we may assume that ϑ is a limit ordinal $< \mu^+$ such that $\vartheta = \mu \cdot \vartheta$.

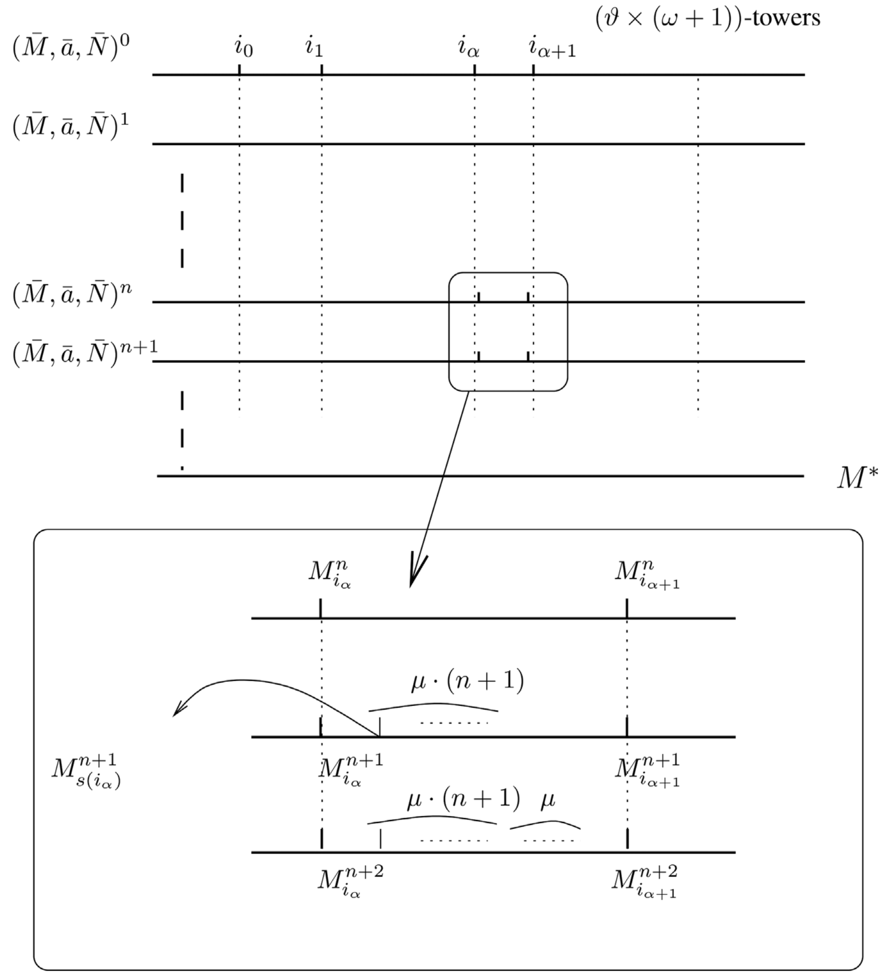


Fig. 1

So, we actually construct an array of models with $\omega + 1$ rows and the number of columns of this array will have the same cofinality as $\vartheta + 1$. Cf. Figure 1 for the big picture of the construction. We intend to carry out the construction *down* and *to the right* in that picture. In the array, the bottom right hand corner (M^*) will be a (μ, ω)-limit model witnessed by the right-most column of the array. We will see that M^* is a (μ, ϑ)-limit model by examining the last (the ωth) row of the array. This last row will be an $\prec_{\mathcal{K}}$-increasing sequence of models, $\bar{M}^*$ whose length will have the same cofinality as ϑ. However we will not be able to guarantee that M^*_{i+1} is universal over M^*_i in this last row. Thus we need another method to conclude that M^* is a (μ, ϑ)-limit model. This involves attaching more information to our sequence $\bar{M}^*$. We call this accessorized sequence of models a tower (cf. Definition 4.1 below). Each row in our construction of the array of models will be such a tower.

Under the assumption of Galois-superstability, given any sequence $\langle a_i \,;\, i < \vartheta \rangle$ of elements with $a_i \in M_{i+1} \backslash M_i$, we can identify some $N_i \prec_{\mathcal{K}} M_i$ such that ga-tp(a_i/M_i) does not μ-split over N_i. Furthermore, by Assumption 2.8, we may choose this N_i such that M_i is a limit model over N_i. We abbreviate this situation by the triple $(\bar{M}, \bar{a}, \bar{N})$.

Definition 4.1 (Towers) Let $(I, <)$ be a well ordering of cardinality $< \mu^+$. For cleaner notation, we will identify I with ϑ, its order-type, and we will denote the successor of i in the ordering I by $i + 1$ when it is clear. Then, we define a *tower* to be a triple $(\bar{M}, \bar{a}, \bar{N})$ where $\bar{M} = \langle M_i \,;\, i < \vartheta \rangle$ is a $\prec_{\mathcal{K}}$-increasing sequence of limit models of cardinality μ; $\bar{a} = \langle a_i\,;\, i+1 < \vartheta\rangle$ and $\bar{N} = \langle N_i\,;\, i+1 < \vartheta\rangle$[2] satisfy $a_i \in M_{i+1}\backslash M_i$; ga-tp$(a_i/M_i)$ does not μ-split over N_i; and M_i is a (μ, σ)-limit model over N_i.

We denote by $\mathcal{K}^*_{\mu,I}$ the set of towers of the form $(\bar{M}, \bar{a}, \bar{N})$ where the sequences $\bar{M}$, $\bar{a}$ and $\bar{N}$ are indexed by I. Occasionally, I will be an ordinal ϑ with the usual ordering, and we write $\mathcal{K}^*_{\mu,\vartheta}$ for this set of towers. At times, we will be considering towers based on different well orderings I and I' simultaneously. In these contexts if $i \in I \bigcap I'$, the notation $i+1$ is not necessarily well-defined so we will use the notation $\mathrm{succ}_I(i)$ for the successor of i in the ordering I. Finally when I is a sub-order of I' for any $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I'}$ we write $(\bar{M}, \bar{a}, \bar{N})\restriction I$ for the tower in $\mathcal{K}^*_{\mu,I}$ given by the subsequences $\langle M_i\,;\, i \in I\rangle$, $\langle N_i\,;\, i+1 \in I\rangle$ and $\langle a_i\,;\, i+1 \in I\rangle$.

In addition to having control over the last row of the array, we also need to be able to guarantee that the last column of the tower witnesses that M^* is a (μ, ω)-limit model. This will be done by prescribing the following ordering on rows of the array and working with towers indexed by I having cofinality $\vartheta + 1$ for some $\vartheta < \mu^+$.

Definition 4.2 For towers $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ and $(\bar{M}', \bar{a}', \bar{N}') \in \mathcal{K}^*_{\mu,I'}$ with $I \subseteq I'$, we write $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}', \bar{a}', \bar{N}')$ if and only if for every $i \in I$, $a_i = a'_i$, $N_i = N'_i$ and M'_i is a universal extension of M_i.

The ordering $<$ on towers is identical to the ordering $<^c_\mu$ defined in [24]. The superscript was used by Shelah and Villaveces to distinguish this ordering from others. We only use one ordering on towers, so we omit the superscripts and subscripts here.

Once we have established an ordering on towers, we can define a specific tower which will be called a *union of an increasing sequence of towers*. Suppose that $\langle (\bar{M}, \bar{a}, \bar{N})^\gamma \in \mathcal{K}^*_{\mu,I_\gamma}\,;\, \gamma < \beta\rangle$ is an increasing sequence of towers such that the index set I_γ of $(\bar{M}, \bar{a}, \bar{N})^\gamma$ is a sub-ordering of the index set $I_{\gamma'}$ for $(\bar{M}, \bar{a}, \bar{N})^{\gamma'}$ whenever $\gamma < \gamma'$. Let $I_\beta := \bigcup_{\gamma<\beta} I_\gamma$. Then denote by $(\bar{M}, \bar{a}, \bar{N})^\beta \in \mathcal{K}^*_{\mu,I_\beta}$ the "union" of the sequence of towers where

$$a^\beta_i = a^{\min\{\gamma\,;\, i\in I_\gamma\}}_i,$$

$$N^\beta_i = N^{\min\{\gamma\,;\, i\in I_\gamma\}}_i, \text{ and}$$

$$\bar{M}^\beta = \langle M^\beta_i\,;\, i \in I_\beta\rangle \text{ with } M^\beta_i = \bigcup_{\gamma<\beta}\bigcup_{i\in I_\gamma} M^\gamma_i.$$

By Assumption 2.8.4a, $(\bar{M}, \bar{a}, \bar{N})^\beta$ is indeed a tower. In particular, this assumption guarantees that for $i \in I_\beta$, ga-tp(a_i/M^β_i) does not μ-split over N_i.

Notice that we do not assume an individual tower to be continuous. Nor do we assume that inside of a tower M_{i+1} is universal over M_i. If one considers the approach of defining an array of models row by row, then generally (even in the first order case) even if all rows are continuous and satisfy the universality property mentioned in this paragraph, it is not necessarily true that the union of these rows will be a tower in which every model is universal over its predecessors.

For a tower $(\bar{M}, \bar{a}, \bar{N})$, it was shown in [24], that even if M_{i+1} is not universal over M_i, one can conclude that $\bigcup_{i<\vartheta} M_i$ is a (μ, ϑ)-limit model provided that all types over each of the M_i are realized by a sufficient number of a_js in the tower. Unfortunately constructing such a tower meeting these along with all of our other requirements is beyond reach. However, in [25], VanDieren showed that slightly less was needed (cf. Definition 4.3). In [25], the amalgamation property is not assumed resulting in noise that can be avoided in our context. Thus because we have at our disposal the AP, we provide a complete, undistracted proof here.

Definition 4.3 (Relatively Full Towers) Suppose that I is a well-ordered set. Let $(\bar{M}, \bar{a}, \bar{N})$ be a tower indexed by I such that each M_i is a (μ, σ)-limit model. For each i, let $\langle M^\gamma_i\,;\, \gamma < \sigma\rangle$ witness that M_i is a (μ, σ)-limit model.

The tower $(\bar{M}, \bar{a}, \bar{N})$ is *full relative to* $(M^\gamma_i)_{\gamma<\sigma, i\in I}$ iff

(1) there exists a cofinal sequence $\langle i_\alpha\,;\, \alpha < \vartheta\rangle$ of I of order type ϑ such that there are $\mu \cdot \omega$ many elements between i_α and $i_{\alpha+1}$ and

[2] Since $a_i \notin M_i$, if the sequence $\bar{M}$ has order type $\alpha + 1$ (with M_α the final model in the sequence), it does not make sense to define a_α which would lie outside of the top model in the tower. Therefore in the situation that the sequence $\bar{M}$ has order type $\alpha + 1$, the sequences $\bar{a}$ and $\bar{N}$ will have order type α.

(2) for every $\gamma < \sigma$ and every $(p, M_i^\gamma) \in \mathfrak{St}(M_i)$ with $i_\alpha \leq i < i_{\alpha+1}$, there exists $j \in I$ with $i \leq j < i_{\alpha+1}$ such that $(\text{ga-tp}(a_j/M_j), N_j)$ and (p, M_i^γ) are parallel.

The following theorem is an improvement of a result in a previous version of this paper in which we made the additional assumption of disjoint amalgamation over limit models. This improvement is [3, Proposition 4.1.5] and uses special models. We provide the proof here for completeness.

Theorem 4.4 (Relatively full towers provide limit models) *Let ϑ be a limit ordinal $< \mu^+$ satisfying $\vartheta = \mu \cdot \vartheta$. Suppose that I is a well-ordered set as in Definition* 4.3.

Let $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ be a continuous tower made up of (μ, σ)-limit models, for some fixed $\sigma < \mu^+$. If $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ is full relative to $(M_i^\gamma)_{i \in I, \gamma < \sigma}$, then $M := \bigcup_{i \in I} M_i$ is a (μ, ϑ)-limit model over M_{i_0}.

P r o o f. Because the sequence $\langle i_\alpha \,;\, \alpha < \vartheta \rangle$ is cofinal in I and $\vartheta = \mu\vartheta$, we can rewrite $M := \bigcup_{i \in I} M_i = \bigcup_{\beta < \vartheta} M_{i_\beta} = \bigcup_{\alpha < \vartheta} \bigcup_{\delta < \mu} M_{i_{\mu\alpha+\delta}}$.

For $\alpha < \vartheta$ and $\delta < \mu$, notice

$$M_{i_{\mu\alpha+\delta+1}} \text{ realizes every type over } M_{i_{\mu\alpha+\delta}}. \tag{1}$$

To see this take $p \in \text{ga-S}(M_{i_{\mu\alpha+\delta}})$. By our superstability assumptions, p does not μ-split over $M^\gamma_{i_{\mu\alpha+\delta}}$ for some $\gamma < \sigma$. Therefore $(p, M^\gamma_{i_{\mu\alpha+\delta}}) \in \mathfrak{St}(M_{i_{\mu\alpha+\delta}})$. By definition of relatively full towers, there is an a_k with $i_{\mu\alpha+\delta} \leq k < i_{\mu\alpha+\delta+1}$ so that $(\text{ga-tp}(a_k/M_k), N_k)$ and $(p, M^\gamma_{i_{\mu\alpha+\delta}})$ are parallel. Because $M_{i_{\mu\alpha+\delta}} \prec_{\mathcal{K}} M_k$ and by the definition of parallel strong types, it must be the case that $a_k \models p$.

By a back and forth argument we can conclude from (1) that $M_{i_{\mu\alpha+\mu}}$ is universal over $M_{i_{\mu\alpha}}$. Thus M is a (μ, ϑ)-limit model.

To see the details of the back-and-forth argument mentioned in the previous paragraph, first translate (1) to the terminology of [1]: (1) witnesses that $M_{i_{\mu\alpha+\mu}}$ is 1-special over $M_{i_{\mu\alpha}}$. Then, refer to the proof of [1, Lemma 10.5]. □

Remark 4.5 The referee has pointed out that our proof of Theorem 4.4 gives a slightly stronger result. In particular, the hypothesis of Theorem 4.4 can be weakened by replacing the relatively full tower with a tower that has the property that

$$\begin{aligned} &\text{for every } i < \vartheta \text{ and every } p \in \text{ga-S}(M_i) \text{ with } i_\alpha \leq i < i_{\alpha+1}, \\ &\quad \text{there exists } j \in I \text{ with } i \leq j < i_{\alpha+1} \text{ such that } a_j \models p. \end{aligned} \tag{2}$$

Constructing a $<$-increasing chain of towers satisfying this weaker condition becomes problematic at limit stages, so we will ultimately need to work with relatively full towers. The problem arises when we take the union of a $<$-increasing chain of towers $\langle (\bar{M}^i, \bar{a}, \bar{N}) \,;\, i < \omega \rangle$. If we only had the weaker condition (2) holding for each $(\bar{M}^i, \bar{a}, \bar{N})$, we would not be able to conclude that (2) also held for the tower, $(\bar{M}^\omega, \bar{a}, \bar{N})$, made up of the union of the towers. To see this, just consider towers of length 2 with models M_i and M_i' satisfying the weaker condition (2), and suppose there are ω many of these towers. In other words, we have models $\langle M_i \prec M_i' \,;\, i < \omega \rangle$ with every type over M_i realized in M_i'. We would want to see if the weaker condition (2) would hold at the tower formed by $M_\omega := \bigcup_{i<\omega} M_i$ and $M' := \bigcup_{i<\omega} M_i'$. Take $p \in \text{ga-S}(M_\omega)$. If $p \restriction M_i$ is realized in M_i' for all $i < \omega$, it is not clear that p would be realized in M'_ω. Without compactness or structure beyond what we have assumed, it would be difficult to conclude that condition (2) held for the tower composed of M_ω and M'_ω. On the other hand, Lemma 3.5 allows use to pass the second bulleted condition from Definition 4.3 through the unions of towers, as we will see later in Claim 5.10.

5 Uniqueness of limit models

We now begin the construction of our array of models and M^*. Let ϑ be an ordinal as in the previous section. The goal is to build an array of models with $\omega + 1$ rows so that the bottom row of the array is a relatively full tower indexed by a set of cofinality ϑ. To do this, we will be adding elements to the index set of towers row by row so that at stage n of our construction the tower that we build is indexed by I_n described here:

The index sets I_n will be defined inductively so that $\langle I_n \,;\, n < \omega + 1\rangle$ is an increasing and continuous chain of well-ordered sets. We fix I_0 to be an index set of order type $\vartheta + 1$ and will denote it by $\langle i_\alpha \,;\, \alpha \leq \vartheta\rangle$. We will refer to the members of I_0 by name in many stages of the construction. These indices serve as anchors for the members of the remaining index sets in the array. Next we demand that for each $n < \omega$, $\{j \in I_n \,;\, i_\alpha < j < i_{\alpha+1}\}$ has order type $\mu \cdot n$ such that each I_n has supremum i_ϑ. An example of such $\langle I_n \,;\, n \leq \omega\rangle$ is $I_n = \vartheta \times (\mu \cdot n) \bigcup \{i_\vartheta\}$ ordered lexicographically, where i_ϑ is an element $\geq$ each $i \in \bigcup_{n<\omega} I_n$. Also, let $I = \bigcup_{n<\omega} I_n$.

To prove the main theorem of the paper, we need to prove that for a fixed $M \in \mathcal{K}$ of cardinality μ any (μ, ϑ)-limit and (μ, ω)-limit model over M are isomorphic over M. Let us begin by fixing a limit model $M \in \mathcal{K}_\mu$ and ϑ such that $\mu \cdot \vartheta = \vartheta$. We define by induction on $n \leq \omega$ a $<$-increasing and continuous sequence of towers $(\bar{M}, \bar{a}, \bar{N})^n$ such that

(1) $(\bar{M}, \bar{a}, \bar{N})^0$ is a tower with $M^0_0 = M$.
(2) $(\bar{M}, \bar{a}, \bar{N})^n \in \mathcal{K}^*_{\mu, I_n}$.
(3) For every $(p, N) \in \mathfrak{St}(M^n_i)$ with $i_\alpha \leq i < i_{\alpha+1}$ there is $j \in I_{n+1}$ with $i < j < i_{\alpha+1}$ so that $(\text{ga-tp}(a_j/M^{n+1}_j), N^{n+1}_j)$ and (p, N) are parallel.

Given M, we can find a tower $(\bar{M}, \bar{a}, \bar{N})^0 \in \mathcal{K}^*_{\mu, I_0}$ with $M^0_0 = M$ because of the existence of universal extensions and because of Assumption 2.8.4b. The last pages of this section provide a picture of this construction of an array of models, explanations for carrying out the final stage of the construction and a proof that this is sufficient to prove the main theorem. We spend most of the remainder of this section verifying that it is possible to carry out the induction step of the construction. This is a particular case of [25, Theorem II.7.1]. But since our context is somewhat easier, we do not encounter so many obstacles as in [25] and we provide a different, more direct proof here:

Theorem 5.1 (Dense $<$-extension property) *Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu, I_n}$ there exists $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu, I_{n+1}}$ such that $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}', \bar{a}, \bar{N})$ and for each $(p, N) \in \mathfrak{St}(M_i)$ with $i_\alpha \leq i < i_{\alpha+1}$, there exists $j \in I_{n+1}$ with $i < j < i_{\alpha+1}$ such that $(\text{ga-tp}(a_j/M'_j), N_j)$ and (p, N) are parallel. Here, the M_i's are defined for $i \in I_n$ and the M'_j are defined for $j \in I_{n+1}$.*

Before we prove Theorem 5.1, we prove a slightly weaker extension property, one in which we can find an extension of the tower $(\bar{M}, \bar{a}, \bar{N})$ of the same index set. Variations of this lemma appear in various places for instance [25, Theorem II.8.2].

Lemma 5.2 ($<$-extension property) *Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu, I}$, there exists a (discontinuous) $<$-extension $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu, I}$ of $(\bar{M}, \bar{a}, \bar{N})$ such that for each i, M'_i is a (μ, μ)-limit model over $\bigcup_{j<i} M'_j$.*

P r o o f . Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu, I}$ we will define a $<$-extension $(\bar{M}', \bar{a}, \bar{N})$ by induction on $i \in I$. Notice that a straightforward induction proof is not sufficient here for if we have defined $\langle M_j \,;\, j \leq i\rangle$ as a tower extending $(\bar{M}, \bar{a}, \bar{N})$ restricted to $\langle j \,;\, j \leq i\rangle$ and are at the stage of defining M'_{i+1}, we may be faced with an impossible task: during our construction we may have inadvertently placed inside M'_i witnesses for the splitting of the type of a_{i+1} over N_{i+1}; this would prevent us from extending M'_i to M'_{i+1} so that ga-tp(a_{i+1}/M'_{i+1}) does not μ-split over N_{i+1}. Therefore, we will instead define approximations, M^+_i, for M'_i by induction on $i \in I$ and at each stage i of the induction we will make adjustments of the previously defined approximation M^+_j for $j < i$. This leads us into defining M^+_i and a directed system of $\prec_{\mathcal{K}}$-embeddings $\langle f_{j,i} \,;\, j < i \in I\rangle$ such that for $i \in I$, $M_i \prec_{\mathcal{K}} M^+_i$ for $j \leq i$, $f_{j,i} : M^+_j \to M^+_i$ and $f_{j,i} \restriction M_j = \text{id}_{M_j}$. We further require that M^+_{i+1} is a limit model over $f_{i,i+1}(M^+_i)$ and ga-tp$(a_i/f_{i,i+1}(M^+_i))$ does not μ-split over N_i. When i is a limit, we choose M^+_i to be a (μ, μ)-limit model over $\bigcup_{j<i} f_{j,i}(M^+_j)$.

This construction is done by induction on $i \in I$ using the existence of non-μ-splitting extensions. Suppose that $\langle M^+_k \,;\, k \leq i\rangle$ and $\langle f_{k,l} \,;\, k \leq l \leq i\rangle$ have been defined. We explain how to define M^+_{i+1} and $f_{i,i+1}$. The rest of the definitions required for the $i + 1$st stage are dictated by the requirement that we are forming a directed system. Let M^*_{i+1} be a limit model over both M^+_i and M_{i+1}. Since ga-tp(a_{i+1}/M_{i+1}) does not μ-split over N_{i+1}, by Theorem 2.4 there exists $f \in \text{Aut}_{M_{i+1}}(\mathfrak{C})$ so that ga-tp$(a_{i+1}/f(M^*_{i+1}))$ does not μ-split over N_{i+1}. Take $M^+_{i+1} := f(M^*_{i+1})$ and $f_{i,i+1} := f \restriction M^+_i$.

At limit stages we take direct limits so that $f_{j,i}\restriction M_j = \mathrm{id}_{M_j}$. This is possible by [25, Subclaims II.7.10 and II.7.11] or [8, Claim 2.17]. Notice that the direct limit may not be universal over M_i. Take an extension of the direct limit that is both universal over M_i and is a (μ, μ)-limit over $\bigcup_{j<i} f_{j,i}(M_j)$ and call this M_i^+. Notice that at this point we do not obtain a continuous tower; continuity will be recovered later using reduced towers.

Let $f_{j,\sup\{I\}}$ and $M'_{\sup\{I\}}$ be the direct limit of this system such that $f_{j,\sup\{I\}}\restriction M_j = \mathrm{id}_{M_j}$. We can now define $M'_j := f_{j,\sup\{I\}}(M_j^+)$ for each $j \in I$. By construction, we have that ga-tp$(a_i/f_{i,i+1}(M_i^+))$ does not μ-split over N_i. Mapping into $M_{\sup(I)}$ by $f_{i+1,\sup(I)}$, and noting that both a_i and N_i are fixed by $f_{i+1,\sup(I)}$, we conclude that ga-tp(a_i/M'_i) does not μ-split over N_i as required. □

We can now use the extension property for towers of the same index set from Lemma 5.2 to prove the dense extension property which allows us to grow the index set as we add elements to the models in the extension.

P r o o f o f T h e o r e m 5.1. Given $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I_n}$, let $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I_n}$ be an extension of $(\bar{M}, \bar{a}, \bar{N})$ as in Lemma 5.2 so that each $M'_{i_{\alpha+1}}$ is a (μ, μ)-limit model over $\bigcup_{j<i_{\alpha+1}} M'_j$.

For each i_α, let $\langle M'_l\,;\, l \in I_{n+1},\ i_\alpha + \mu\cdot n < l < i_{\alpha+1}\rangle$ witness that $M'_{i_{\alpha+1}}$ is a (μ, μ)-limit model over $\bigcup_{j<i_{\alpha+1}} M'_j$. Without loss of generality we may assume that each of these M'_l is a limit model over its predecessor.

Fix $\{(p, N)^k_{i_\alpha}\,;\, 0 < k < \mu\}$ an enumeration of $\bigcup\{\mathfrak{S}\mathrm{t}(M_i) : i \in I_n, i_\alpha \le i < i_{\alpha+1}\}$. By our choice of I_{n+1}, stability in μ, and Theorem 3.3, such an enumeration is possible. For each $k < \mu$, we will consider the model indexed by $l = i_\alpha + \mu n + k$. These are the models in the sequence $\bar{M}'$ which do not appear indexed in $\bar{M}$. Therefore $p^k_{i_\alpha}$ will be a type over some submodel of M'_l. Since $M'_{\mathrm{succ}_{I_{n+1}}(l)}$ is universal over M'_l, there exists a realization in $M'_{\mathrm{succ}_{I_{n+1}}(l)}$ of the non-μ-splitting extension of $p^k_{i_\alpha}$ to M'_l. Let a_l be such a realization and take $N_l := N^l_{i_\alpha}$.

Notice that $(\langle M'_j\,;\, j \in I_{n+1}\rangle, \langle a_j\,;\, j \in I_{n+1}\rangle, \langle N_j\,;\, j \in I_{n+1}\rangle)$ provide the desired extension of $(\bar{M}, \bar{a}, \bar{N})$ in $\mathcal{K}^*_{\mu,I_{n+1}}$. □

We are almost ready to carry out the complete construction. However, notice that Lemma 5.2 and Theorem 5.1 do not provide us with a continuous extension. In particular in Lemma 5.2, at the limit stage of the construction we need to take an extension of a direct limit to find a universal extension and this precludes continuity.

Therefore the bottom (i.e., the $\omega + 1$st) row of our array may not be continuous which would prevent us from applying Theorem 4.4 to conclude that M^* is a (μ, ϑ)-limit model. So we will further require that the towers that occur in the rows of our array are all continuous. This can be guaranteed by restricting ourselves to reduced towers as in [24, 25].

Definition 5.3 A tower $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ is said to be *reduced* provided that for every $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ with $(\bar{M}, \bar{a}, \bar{N}) \le (\bar{M}', \bar{a}, \bar{N})$ we have that for every $i \in I$,

$$(*)_i \quad M'_i \cap \bigcup_{j\in I} M_j = M_i.$$

If we take a $<$-increasing chain of reduced towers, the union will be reduced. The following fact appears as [24, Theorem 3.1.14]. We provide the proof for completeness.

Theorem 5.4 *Let $\langle(\bar{M}, \bar{a}, \bar{N})^\gamma \in \mathcal{K}^*_{\mu,I_\gamma}\,;\, \gamma < \beta\rangle$ be a $<$-increasing and continuous sequence of reduced towers such that the sequence is continuous in the sense that for a limit $\gamma < \beta$, the tower $(\bar{M}, \bar{a}, \bar{N})^\gamma$ is the union of the towers $(\bar{M}, \bar{a}, \bar{N})^\zeta$ for $\zeta < \gamma$. Then the union of the sequence of towers $\langle(\bar{M}, \bar{a}, \bar{N})^\gamma \in \mathcal{K}^*_{\mu,I_\gamma}\,;\, \gamma < \beta\rangle$ is itself a reduced tower.*

P r o o f . Suppose that $(\bar{M}, \bar{a}, \bar{N})^\beta$ is not reduced. Let $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I_\beta}$ witness this. Then there exists an $i \in I_\beta$ and an element b such that $b \in (M'_i \cap \bigcup_{j\in I_\beta} M^\beta_j)\setminus M^\beta_i$. There exists $\gamma < \beta$ such that $b \in \bigcup_{j\in I_\gamma} M^\gamma_j \setminus M^\gamma_i$. Notice that $(\bar{M}', \bar{a}, \bar{N})\restriction I_\gamma$ witnesses that $(\bar{M}, \bar{a}, \bar{N})^\gamma$ is not reduced. □

The following appears in [24, Theorem 3.1.13].

Theorem 5.5 (Density of reduced towers) *There exists a reduced $<$-extension of every tower in $\mathcal{K}^*_{\mu,I}$.*

P r o o f. Assume for the sake of contradiction that no $<$-extension of $(\bar{M}, \bar{a}, \bar{N})$ is reduced. This allows us to construct a $\leq$-increasing and continuous sequence of towers $\langle (\bar{M}, \bar{a}, \bar{N})^\zeta \in \mathcal{K}^*_{\mu,I} ; \zeta < \mu^+ \rangle$ such that $(\bar{M}, \bar{a}, \bar{N})^{\zeta+1}$ witnesses that $(\bar{M}, \bar{a}, \bar{N})^\zeta$ is not reduced. The construction is done inductively in the obvious way.

For each $b \in \bigcup_{\zeta<\mu^+, i\in I} M^\zeta_i$ define

$$i(b) := \min\big\{i \in I \,;\, b \in \bigcup_{\zeta<\mu^+} \bigcup_{j\leq i} M^\zeta_j\big\} \text{ and}$$

$$\zeta(b) := \min\big\{\zeta < \mu^+ \,;\, b \in M^\zeta_{i(b)}\big\}.$$

The function $\zeta(\cdot)$ can be viewed as a function from μ^+ to μ^+. Since $|I| = \mu$ and each M^ζ_i has cardinality μ, there exists a club $E = \{\delta < \mu^+ \,;\, \forall b \in \bigcup_{i\in I} M^\delta_i,\ \zeta(b) < \delta\}$. Actually, all we need is that E is non-empty.

Fix $\delta \in E$. By construction $(\bar{M}, \bar{a}, \bar{N})^{\delta+1}$ witnesses the fact that $(\bar{M}, \bar{a}, \bar{N})^\delta$ is not reduced. So we may fix $i \in I$ and $b \in M^{\delta+1}_i \cap \bigcup_{j\in I} M^\delta_j$ such that $b \notin M^\delta_i$. Since $b \in M^{\delta+1}_i$, we have that $i(b) \leq i$. Since $\delta \in E$, we know that there exists $\zeta < \delta$ such that $b \in M^\zeta_{i(b)}$. Because $\zeta < \delta$ and $i(b) \leq i$, this implies that $b \in M^\delta_i$ as well. This provides a contradiction since on the one hand $b \in M^\delta_i$ and on the other hand, it is not. □

By revising the proof of Lemma 5.2, we can conclude:

Lemma 5.6 *Suppose that $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ is reduced. If I_0 is an initial segment of I, then $(\bar{M}, \bar{a}, \bar{N})\restriction I_0$ is reduced.*

P r o o f. Suppose that $(\bar{M}, \bar{a}, \bar{N})\restriction I_0$ is not reduced. Let $(\bar{M}', \bar{a}\restriction I_0, \bar{N}\restriction I_0)$ and $\delta < j \in I_0$ with $b \in (M'_\delta \cap M_j)\backslash M_\delta$ witness this. We can apply the inductive step of Lemma 5.2 (replacing an initial segment of the construction there with $\bar{M}'$), to find $(\bar{M}'', \bar{a}, \bar{N})$ an extension of $(\bar{M}, \bar{a}, \bar{N})$ such that there is a $\prec_\mathcal{K}$-mapping f from the models of $\bar{M}'$ into the models of $\bar{M}''$ with $f\restriction M_j = \mathrm{id}_{M_j}$. Notice that $(\bar{M}'', \bar{a}, \bar{N})$ and b, δ, j will witness that $(\bar{M}, \bar{a}, \bar{N})$ is not reduced. □

The following theorem makes use of the unidimensionality assumption. This generalizes a special case of the uniqueness of limit models result in the series of papers [25, 26] by replacing the assumption of categoricity in μ^+ with the weaker unidimensionality assumption. Further work of VanDieren in [28] weakens this assumption.

Theorem 5.7 (Reduced towers are continuous) *If $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ is reduced, then it is continuous, namely for each limit i in I, $M_i = \bigcup_{j<i} M_j$.*

P r o o f o f T h e o r e m 5.7 Suppose the theorem fails for μ. Let δ be the minimal limit ordinal such that there exists an index set I and $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ a reduced tower which is discontinuous at the δth element of I. We can apply Lemma 5.6 to assume without loss of generality that $I = \delta + 1$. Fix $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\delta+1}$ reduced and discontinuous at δ with $b \in M_\delta\backslash \bigcup_{i<\delta} M_i$. By Theorem 2.6, there exists a minimal type p over M_0. So by our unidimensionality Assumption 3, we know that the Galois type of p must be realized in $M_\delta\backslash \bigcup_{i<\delta} M_i$. Therefore, we may assume that $b \models p$.

Claim 5.8 *There exists a $<$-extension of $(\bar{M}, \bar{a}, \bar{N})\restriction\delta$, containing b. We will refer to such a tower in $\mathcal{K}^*_{\mu,\delta}$ as $(\bar{M}', \bar{a}\restriction\delta, \bar{N}\restriction\delta)$. Furthermore, b may be assumed to be an element of M'_0.*

P r o o f o f C l a i m 5.8 We use the minimality of δ and the $<$-extension property to find a tower of length δ, $(\bar{M}^*, \bar{a}\restriction\delta, \bar{N}\restriction\delta)$, that is a proper extension of $(\bar{M}, \bar{a}, \bar{N})\restriction\delta$. By the definition of $<$-extension, M^*_0 is universal over M_0; so we can find $b^* \in M^*_0\backslash M_0$ realizing p.

Notice that by Lemma 5.6, $(\bar{M}, \bar{a}, \bar{N})\restriction\delta$ is reduced. Thus we can conclude that $b^* \in M^*_0\backslash \bigcup_{i<\delta} M_i$ and ga-tp$(b^*/\bigcup_{i<\delta} M_i)$ is non-algebraic. Since p is minimal, it must be the case that ga-tp$(b^*/\bigcup_{i<\delta} M_i) =$ ga-tp$(b/\bigcup_{i<\delta} M_i)$. Let $f \in \mathrm{Aut}_{\bigcup_{i<\delta} M_i}\mathfrak{C}$ take b^* to b.

Consider the image of $(\bar{M}^*, \bar{a}, \bar{N})$ under f; denote this tower by $(\bar{M}', \bar{a}, \bar{N})$. Because f fixes $(\bar{M}, \bar{a}, \bar{N})\restriction\delta$, $(\bar{M}', \bar{a}, \bar{N})$ is an extension of $(\bar{M}, \bar{a}, \bar{N})\restriction\delta$ as required. □

Using $(\bar{M}', \bar{a}, \bar{N})$ from Claim 5.8, define M'_δ to be a limit model of cardinality μ containing $\bigcup_{i<\delta} M'_i$ so that it is universal over M_δ. Notice that the tower $(\bar{M}'^\frown\langle M'_\delta\rangle, \bar{a}, \bar{N})$ extends $(\bar{M}, \bar{a}, \bar{N})$ with $b \in (M'_0\backslash \bigcup_{i<\delta} M_i) \bigcap M_\delta$. This contradicts our assumption that $(\bar{M}, \bar{a}, \bar{N})$ is reduced and completes the proof of Theorem 5.7. □

Corollary 5.9 *In Theorem* 5.1*, we can choose* $(\bar{M}, \bar{a}, \bar{N})$ *to be reduced, and hence continuous.*

Proof. As in the proof of Theorem 5.1, first take $(\bar{M}'', \bar{a}', \bar{N}')$ extending $(\bar{M}, \bar{a}, \bar{N})$ to realize the required strong types. By Theorem 5.5 we can find a reduced extension $(\bar{M}', \bar{a}', \bar{N}')$ of $(\bar{M}'', \bar{a}', \bar{N}')$, which realizes the same required strong types. By Theorem 5.7, $(\bar{M}', \bar{a}', \bar{N}')$ is continuous. □

Now we return to the construction in the proof of the Main Theorem.

Corollary 5.9 tells us that the construction of our array of models as an increasing sequence of towers is possible in successor cases. In the limit case, let $I_\omega = \bigcup_{m<\omega} I_m$, and simply define $(\bar{M}, \bar{a}, \bar{N})^\omega \in \mathcal{K}^*_{\mu, I_\omega}$ to be the union of the towers $(\bar{M}, \bar{a}, \bar{N})^n$.

To see that the construction satisfies our requirements, first notice that the last column of the array, $\langle M^n_{i_\vartheta}\ ;\ n < \omega\rangle$, witnesses that M^* is a (μ, ω)-limit model. Recall that the towers that form the rows of the array are indexed by I_n which have cofinality $\vartheta + 1$ (and not ϑ) and are continuous. Therefore by the definition of $<$-extensions of towers, we know that each $M^{n+1}_{i_\vartheta}$ is universal over $M^n_{i_\vartheta}$, which demonstrates that M^* is indeed a (μ, ω)-limit model. If we had constructed our towers to only have cofinality ϑ, we could not arrive at this conclusion concerning M^*.

In light of Theorem 4.4 we need only verify that the last row of the array is a continuous relatively full tower of cofinality $\vartheta + 1$.

Claim 5.10 *The tower* $(\bar{M}, \bar{a}, \bar{N})^\omega$ *is full relative to* $(M^n_i)_{n<\omega, i\in I_\omega}$.

Proof. Given i with $i_\alpha \le i < i_{\alpha+1}$, let (p, M^n_i) be some strong type in $\mathfrak{St}(M^\omega_i)$. Notice that by monotonicity of non-splitting $(p \restriction M^{n+1}_i, M^n_i) \in \mathfrak{St}(M^{n+1}_i)$. By construction there is a $j \in I_{n+1}$ with $i < j < i_{\alpha+1}$ such that $(\text{ga-tp}(a_j/M^{n+2}_j), N^{n+2}_j)$ is parallel to $(p \restriction M^{n+1}_i, M^n_i)$. We will show that $(\text{ga-tp}(a_j/M^\omega_j), N^\omega_j)$ is parallel to (p, M^n_i).

First notice that $\text{ga-tp}(a_j/M^\omega_j)$ does not μ-split over $N^\omega_j = N^{n+2}_j$ because $(\bar{M}, \bar{a}, \bar{N})^\omega$ is a tower. By Lemma 3.5, $(\text{ga-tp}(a_j/M^\omega_j), N^{n+2}_j)$ is parallel to (p, M^n_i). □

This completes the proof of Theorem 1.9.

6 Concluding remarks

In this section we discuss other results related to the question of the uniqueness of limit models. First to understand the boundaries of the question of the uniqueness of limit models, consider the elementary case. Limit models are not necessarily unique even for first order complete stable theories.

Theorem 6.1 *Suppose T is a complete, stable theory. Let* $\mu \ge 2^{|T|}$ *such that* $\mu^{|T|} = \mu$*. If T is not superstable, then no* (μ, ω)*-limit model is isomorphic to any* (μ, κ)*-limit model for any* κ *with* $\text{cf}(\kappa) \ge \kappa(T)$.

Proof. Let T be a stable, but not superstable, complete theory, and fix κ and μ as in the statement of the theorem. As T is not superstable, by [15, Lemma VII, 3.5 (2)], for $\lambda := (2^\mu)^+$, there are $\langle \bar{a}_\eta | \eta \in {}^{\omega\,\ge}\lambda\rangle$ and $\langle \varphi_n(\bar{x}, \bar{y}_n) | n < \omega\rangle$ such that for every $n < \omega$, $\nu \in {}^n\lambda$, and all $\eta \in {}^\omega\lambda$,

$$(\mathfrak{C} \models \varphi_n[\bar{a}_\eta, \bar{a}_\nu]) \iff \nu = \eta \restriction n.$$

By induction on $n < \omega$ define $\langle M_n | n < \omega\rangle$ all of cardinality μ and $\langle \eta_n, \nu_n | n < \omega\rangle$ such that

(1) The model M_{n+1} is universal over M_n and saturated of cardinality μ,
(2) $\eta_{n+1} > \eta_n$, $\nu_{n+1} > \eta_n$, and $\eta_{n+1} \ne \nu_{n+1}$,
(3) $\bar{a}_{\eta_{n+1}}, \bar{a}_{\nu_{n+1}} \in M_{n+1}$ and
(4) $\text{tp}(\bar{a}_{\eta_{n+1}}/M_n) = \text{tp}(\bar{a}_{\nu_{n+1}}/M_n)$.

This construction is enough. Let $N' \models T$ be a (μ, κ)-limit over M_0. By Theorem 1.6, N' must be saturated. Let $N = \bigcup_{n<\omega} M_n$. Clearly N is a (μ, ω)-limit over M_0. To conclude that N and N' are non-isomorphic, it is enough to show that N is not saturated. Consider $p := \{\varphi_{n+1}(\bar{x}; \bar{a}_{\eta_{n+1}}) \wedge \neg\varphi_{n+1}(\bar{x}; \bar{a}_{\nu_{n+1}}) | n < \omega\}$. The set of formulas p is a type since it is realized in $\mathfrak{C}$ by $\bar{a}_\eta$ where $\eta := \bigcup_{n<\omega} \eta_n$. Notice that N cannot satisfy p. If $\bar{a} \in N$ would satisfy p, then M_n realizes p for some $n < \omega$. Thus by condition (4), we would have

$$\mathfrak{C} \models \varphi_{n+1}[\bar{a}, \bar{a}_{\eta_{n+1}}] \Longleftrightarrow \mathfrak{C} \models \varphi_{n+1}[\bar{a}, \bar{a}_{\nu_{n+1}}]$$

which would contradict the assumption that $\bar{a}$ satisfies p.

This is possible. By stability and $\mu^{|T|} = \mu$, using the proof of [15, Th. III 3.12], every model of cardinality μ has a saturated proper elementary extension. Let M_0 be a saturated model of cardinality μ and take $\eta_0 = \nu_0 := \langle\rangle$. Given η_n, ν_n, M_n, using Theorem 1.5 let M^* be universal over M_n of cardinality μ. Let $M^{**} \succ M^*$ of cardinality μ containing $\bar{a}_{\eta_n}$ and $\bar{a}_{\nu_n}$. By [15, Th. III 3.12], we can take $M_{n+1} \succ M^{**}$ saturated of cardinality μ. Clearly it is universal over M_n. For $n < \omega$, consider $F_n(\alpha) := \mathrm{tp}(\bar{a}_{\eta_n\hat{\ }\alpha}/M_n)$. As λ is regular and $\lambda > |S(M_n)|$, there is $S \subset \lambda$ of cardinality λ such that $\alpha \neq \beta \in S$ implies$F_n(\alpha) = F_n(\beta)$. Pick $\alpha \neq \beta \in S$ and define $\eta_{n+1} := \eta_n\hat{\ }\alpha$ and $\nu_{n+1} := \eta_n\hat{\ }\beta$. □

In the non-elementary setting, many authors have considered approximations to Theorem 1.9. Several authors have proved and used the uniqueness of limit models in AECs under the assumption of categoricity: [1, 12, 20, 22, 24–26]. Also, Shelah's [16] examines (as an aside) the uniqueness of limit models in good frames. Below we briefly describe the results and techniques of these papers and distinguish them from our context.

In [20, Theorem 6.5], Shelah claims uniqueness of limit models of cardinality μ for classes with the amalgamation property under little more than categoricity in some $\lambda > \mu > \mathrm{LS}(\mathcal{K})$ together with existence of arbitrarily large models. Shelah's claim in [20, Theorem 6.5] (isomorphism over the base) seems too strong for the proof that he suggests. Instead, he proves that (μ, κ)-limit models are Galois saturated, which implies uniqueness only over models of size $< \mu$. The argument in [20] depends in a crucial way on an analysis of Ehrenfeucht-Mostowski models. For an exposition of this result cf. [1]. Our result differs in two ways from Shelah's [20, Theorem 6.5]. First, we do not explicitly use Ehrenfeucht-Mostowski machinery; although, Ehrenfeucht-Mostowski machinery is used to derive the superstability assumptions in this paper from categoricity. Secondly, our result produces an isomorphism between the two limit models that fixes a submodel of cardinality μ.

Kolman and Shelah in [12] prove the uniqueness of limit models of cardinality μ in λ-categorical AECs that are axiomatized by a $L_{\kappa,\omega}$-sentence where $\lambda > \mu$ and κ is a measurable cardinal. Then Kolman and Shelah use this uniqueness result to prove that amalgamation occurs below the categoricity cardinal in $L_{\kappa,\omega}$-theories with κ measurable. Both the measurability of κ and the categoricity are used integrally in their proof of uniqueness.

Shelah in [22, Claim 7.8] proved a special case of the uniqueness of limit models under the assumption of the μ-AP, categoricity in μ and in μ^+ as well as assuming $K_{\mu^{++}} \neq \emptyset$. In that paper Shelah needs to produce *reduced types* and use some of their special properties.

In [24], Shelah and Villaveces attempted to prove a uniqueness theorem without assuming any form of amalgamation; however, they assumed that $\mathcal{K}$ is categorical in some sufficiently large λ, that every model in $\mathcal{K}$ has a proper extension and that $2^\lambda < 2^{\lambda^+}$. VanDieren in [25, 26] managed to prove the uniqueness statement under the assumptions of [24] together with the additional assumptions that the class is categorical in μ^+ and $\mathcal{K}^{am} := \{M \in \mathcal{K}_\mu\ ;\ M \text{is an amalgamation base}\}$ is closed under unions of increasing $\prec_{\mathcal{K}}$ chains.

In [16] the most important new concept is that of a λ-*good frame*, which is an axiomatization of the notion of superstability, *with hypothesis on just one cardinal* λ. Its full definition is more than a page long. Shelah's assumptions on the AEC include, among other things, the amalgamation property, the existence of a forking like dependence relation and of a family of types playing a role akin to that of regular types in first order superstable theories – Shelah calls them *bs*-types. One of the axioms of a good frame is the existence of a non-maximal super-limit model. This axiom along with μ-stability implies the uniqueness of limit models of cardinality μ. In [16, Lemma II.4.8] he states that in a good frame, limit models are unique. Boney and Vasey have isolated this result and use both symmetry and tameness to derive the uniqueness of limit models [2].

We are particularly interested in Theorem 1.9 not only for the sake of generalizing Shelah's result from [22] but due to the fact that the first and second author originally used an earlier draft of this uniqueness theorem (which did not assume unidimensionality) along with tools from [20] in a crucial step to prove:[3]

[3] Some time after Grossberg and VanDieren announced Theorem 6.2, Baldwin circulated an alternative proof of Theorem 6.2 that eventually appeared in [1]. Lessmann in [13] proved the result for $\mathcal{K}$ with $\mathrm{LS}(\mathcal{K}) = \aleph_0$ beginning with categoricity in $\aleph_1$.

Theorem 6.2 (Upward categoricity theorem; [8]) *Suppose that $\mathcal{K}$ has arbitrarily large models, is χ-tame and satisfies the amalgamation and joint embedding properties. Let λ be such that $\lambda > \mathrm{LS}(\mathcal{K})$ and $\lambda \geq \chi$. If $\mathcal{K}$ is categorical in λ^+ then $\mathcal{K}$ is categorical in all $\mu \geq \lambda^+$.*

After the addition of the unidimensionality assumption in 2014 to resolve an error found in 2012 in the proof of Theorem 5.7, Grossberg and VanDieren have revisited the proof of Theorem 6.2 to insure that the upward categoricity transfer still holds [9]. Grossberg and VanDieren's initial use of the uniqueness of limit models in this theorem hints at a connection between classical definitions of superstability in first order logic and the uniqueness of limit models. This link is explored in further work of VanDieren [27, 28].

It is worth mentioning that the links between classical notions of superstability from first order logic and the uniqueness of limit models have also produced interesting results in the connections between "continuous model theory" and so-called "metric AECs". Villaveces and Zambrano [29] have adapted our proofs and notions of independence used here to the metric AEC context, under the stronger hypothesis of categoricity ([30] but for the wider ambit of metric AECs) and at the same time explored various consequences of assuming forms of uniqueness of limit models in that metric (continuous) context.

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