

A Course in Model Theory I:

Introduction¹

Rami Grossberg

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213

¹This **preliminary draft** is dated from January 17, 2015. The book will be published by Cambridge University Press. The book is approximately 93.82% complete, I expect the final version to have about 745 pages, many sections of the current version will be revised and few will be added. I hope to have a stable version of this volume soon.

This version is made only for students studying model theory with me and not for distribution outside CMU. If you have a copy not received directly from me, it is an illegal copy and I request that you will not share with others.

Exercise #=701.

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