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Dependence relations in nonelementary classes
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ABSTRACT. We study the class $\mathcal{K}$ of models of a first order theory T omitting a prescribed set of types, under the assumption that $\mathcal{K}$ contains a model with a high level of sequential-homogeneity. The stability theory of such classes was initiated by Shelah in 1969. We introduce a rank which is bounded when $\mathcal{K}$ is $\aleph_0$-stable. The main difficulties are the failure of the compactness theorem for $\mathcal{K}$ and the fact that T may be unstable, even not simple. The rank induces a dependence relation on the subsets of the models in $\mathcal{K}$ which shares many of the formal properties of forking. We obtain pregeometries with respect to this dependence relation; the pregeometries exist on the set of realization of types of minimal rank. We prove the existence of prime models. We develop the parallel to orthogonality calculus and unidimensionality. Finally we generalize many of the classical results obtained for models of a first order totally transcendental theory. The global picture is similar (but the proofs different): This is illustrated with positive results (E.g. Chang Conjecture, Categoricity with a geometric proof) and negative results (construction of nonisomorphic models), as well as a proof of the Main Gap. The structure part of the Main Gap is done axiomatically, so that the proof covers the known first order NDOP cases, as well as a known nonelementary case: when the class is excellent.

In order to generalize ideas from first order stability theory to contexts where the compactness theorem fails, we also work in an abstract pregeometry satisfying some logical axioms. The main result is a proof of a group configuration theorem.

We also look at the consequences of the failure of the λ-order property inside the set of realizations of a fixed type, in nonelementary contexts. We are able to generalize many of the results known in the first order case.

Contents

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Introduction

The goal of this introduction is to draw attention to one of the main concerns of this thesis; the use of *dependence relations* in several *nonelementary* contexts.

The word nonelementary refers to the failure of the compactness theorem; it will be used as a synonym for *non first order*. One motivation for studying nonelementary classes of models is that many natural classes of structures, for example in algebra, do not have a first order axiomatization: Archimedean fields, periodic groups, p-groups, FC-groups, locally finite structures, to name only a few. There are several families of nonelementary classes of models. Historically, the study of nonelementary classes of models started in the late 1940s, when Erdős, Hanf, and Tarski among others obtained basic existence results for infinitary logics ($L_{\lambda^+,\omega}$, for infinite cardinals λ). In the 1950s, Mostowski introduced several new quantifiers, for example cardinality quantifiers, like 'there exists uncountably many'. In the 1960s and early 1970s, Barwise, Fuhrken, Keisler, Makkai, and Shelah proved fundamental existence results for logics with Mostowski quantifiers. Around the mid 1970s, motivated by a question of Baldwin, Shelah discovered connections between *existence* of models for $L_{\omega_1,\omega}(Q)$ and *categoricity* for this class. This analysis was done by working in an equivalent class of models: class of models omitting a prescribed set of types. Classes of models omitting a prescribed set of types under some additional assumption are going to be the main object of study of this thesis. It should be pointed out that it follows from the work of Chang and Shelah that many nonelementary contexts are essentially equivalent and that one can expect methods developed for one context to be useful in another. (See the introduction to each chapter for more details.)

A dependence relation is a relation among triples of subsets of a model, satisfying some basic requirements: invariance under automorphism, finite character, and so on. A dependence relation attempts to capture, in a reasonable way, the notion that a set A is *free* from a set C over another set B.

In the first order case, the main dependence relation is *forking*. Forking is a dependence relation discovered by Shelah, tailored to the compactness theorem, which extends the notion of linear dependence in linear algebra or algebraic dependence in field theory. It satisfies several additional properties (see the introduction to Chapter V, for example) and has become the crucial tool of classification theory for the first order case. In fact, it is so central that various kinds of contexts studied in classification theory for first order classes are characterized in terms of what properties forking satisfies.

In the nonelementary case, forking is useless; not only do several key properties of forking use the compactness theorem, but the first order theory of the models of a given nonelementary class may not be simple, even if the class is categorical (we give an example of this below). While a dependence relation as nicely behaved as forking has not been found, several dependence relations satisfying fewer properties than forking can be used. This thesis makes use of them extensively. In Chapter I and II, two dependence relations, *splitting* and *strong splitting* are used. In Chapter III, a new dependence relation is defined, via the introduction of a rank. This new dependence relation satisfies many of the properties of forking. We also prove the existence of *pregeometries*, which are sets inside which the dependence relation gives rise to a good notion of dimension. In Chapter IV and V, a more abstract point of view is developed and a dependence relation is considered axiomatically.

This thesis fits generally in the classification theory for nonelementary classes. It is organized as follows. There are five chapters and each starts with an introduction which explains the goals and how the results fit with the general theory.

Chapter I is an exposition of Shelah's stability spectrum theorem, homogeneity spectrum theorem, and equivalence between order and instability in finite diagrams. The framework of finite diagrams generalizes the first order case. It is one of the main frameworks studied in this thesis, and will appear (sometimes implicitly) in all the chapters. This exposition is done from a modern point of view, incorporating recent improvements, both in the proofs and in the presentation. This chapter provides the necessary background to Chapter III.

Chapter II is an extension of the first chapter in several ways. We study local versions of the order property in several frameworks: (1) Inside a fixed model, (2) for nonelementary classes of models (generalizing finite diagrams), (3) for finite diagrams, and (4) for the first order case. Appropriate localizations of the order property, the independence property, and the strict order property are introduced. We are able to generalize some of the results that were known in the case of local stability for first order theories, and for stability for nonelementary classes (existence of indiscernibles, existence of averages, stability spectrum, equivalence between order and instability). In particular, all the results of Chapter I can be localized in (3). In the first order case, we also prove the local version of Shelah's Trichotomy Theorem. Finally, as an application, we give a characterization of stable types when the ambient first order theory is simple.

Chapter III continues the work of Chapter I in a different direction. We work in the framework of finite diagrams and introduce a natural dependence relation on the subsets of the models for the $\aleph_0$-stable case, which share many of the formal properties of forking. This is achieved by considering a rank for this framework which is bounded when the diagram is $\aleph_0$-stable. We also obtain pregeometries with respect to this dependence relation. The dependence relation is the natural one induced by the rank, and the pregeometries exist on the set of realizations of types of minimal rank. Finally, these concepts are used to generalize many of the classical results for models of a totally transcendental first order theory. In fact, strong analogies arise: models are determined by their pregeometries or their relationship with their pregeometries. However the proofs are different, as we do not have the compactness theorem. This is illustrated with positive results (categoricity) as well as negative results (construction of nonisomorphic models). We also give a proof of Chang's Conjecture for this context.

In Chapter IV, we develop a more abstract framework which generalizes the framework of Chapter III and that of several other cases (first order, excellent Scott sentences). We show that if a given class of models admits a dependence relation on the subsets of its models, as well as prime models satisfying a prescribed set of axioms, then under the parallel of NDOP every model of this class is prime and minimal over an independent tree of models. This axiomatization is different from that of Shelah in the Main Gap in two crucial ways: the dependence relation satisfies weaker axioms than those of forking and the axiomatization does not depend on the compactness theorem. In fact the theorem is in the vein of combinatorial geometry; the logical ingredient is hidden in checking the particular axioms for the classes of models under consideration. As a new application, we show that the class of models isolated in Chapter III satisfies all the axioms, and thus admits a decomposition theorem. We also develop more orthogonality calculus for this class and prove the main gap.

Finally in Chapter V, we lay a foundation for separating the geometry from the logic in geometric model theory. Our feeling is that this separation is an elegant way of lifting results from geometric model theory to non first order logic. We introduce a relation between subsets of a pregeometry and show that it satisfies all the formal properties that forking satisfies in simple first order theories. This is important when one is trying to lift forking to nonelementary classes, in contexts where there exists pregeometries but not necessarily a well-behaved dependence relation, for example the one of Chapter III. This is used to reproduce S. Buechler's characterization of local modularity in general. We also present an axiomatic approach to the Hrushovski-Zilber group configuration theorem.

In this thesis, many terms like stable, totally transcendental, and stationary will be employed for nonelementary classes of models; typically classes of models of a first order theory T omitting a set of types Γ. We would like to alert the reader that these terms do not refer to the first order theory, but to the entire class. For example, such a class may be stable while the first order theory T is not. It may be helpful to keep in mind the following simple-minded example. The example given here merely illustrates that such natural situations exist. It is not the motivation for developing the theory.

Consider the first order theory T asserting that the unary predicate R is the domain of an ordered field of characteristic zero and that each model is an R-vector space (in the obvious language). Then, T has the strict order property, which implies that it is not stable and not even simple. Furthermore, T is not o-minimal. This roughly means that T is complicated from the point of view of first order model theory. Now consider the class $\mathcal{K}$ consisting of those models of T omitting the type $\{\neg R(x) \vee (n * x < 1) \mid n < \omega\}$. Then, $\mathcal{K}$ is the class of vector spaces over Archimedean fields of characteristic zero. ($\mathcal{K}$ does not have a first order axiomatization.) Notice that no model of $\mathcal{K}$ can encode a linear order of length more than the continuum ($\mathcal{K}$ does not have the $(2^{\aleph_0})^+$-order property). This implies immediately from results of Chapter II that, as a class, $\mathcal{K}$ must be *stable*. In addition, for each cardinal λ, this class contains λ-homogeneous models (take any sufficiently large real vector space). Hence, $\mathcal{K}$ can be made into a *finite diagram* in the sense of Chapter I: Let $V \in \mathcal{K}$ be a real vector space of large cardinality. Let D be the set of types in finitely many variables that V realizes over the empty set. Together with the previous observation, this shows that D is stable (in the sense of Chapter I). In fact, one can show directly that D is stable in every cardinal at least the continuum. Moreover, D is *totally transcendental* in the sense of Chapter III. Now consider the class $\mathcal{K}_1$ of $(D, \aleph_0)$-homogeneous models of $\mathcal{K}$. This class is studied for general D in Chapter III. It is easy to see that $\mathcal{K}_1$ is the class of real vector spaces, as every rational cut must be realized. This shows that $\mathcal{K}_1$ contains exactly one model up to isomorphism for each cardinal λ above the continuum. Thus, there exists nonelementary classes of models which are categorical in some uncountable cardinal, stable in all large enough cardinals, for which there exists a rank, a nice dependence relation, prime models and so forth, but can have a complicated first order theory.

CHAPTER I

Shelah's stability spectrum and homogeneity spectrum in finite diagrams

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Saharon Shelah's Finite Diagrams Stable in Power [**Sh3**], published in 1970, is one of the seminal articles in model theory. It contains a large number of key ideas which have shaped the development of classification theory. The model-theoretic framework of the paper is more general than the first order case, but while all the particular cases of the results in the first order case can be found in several more recent publications, the non first order content of [**Sh3**] is still not available in a concise from.

The primary purpose of this chapter is to present, in this more general framework, most of the stability results of [**Sh3**], together with the order/stability dichotomy from [**Sh16**], and the homogeneity spectrum appearing in The Lazy Model Theorist's Guide To Stability [**Sh54**]. A secondary purpose is to present in a compact form the necessary background to Chapter III. This is done in a contemporary and self-contained form, and includes improvements and techniques from [**Sh b**], [**Sh300**], and [**Gr1**]. Finally, the results are presented in such a way that with very little additional work, the theorems of this chapter can be localized. Local versions of the Stability Spectrum Theorem and the Homogeneity Spectrum Theorem will be proved in Chapter II, devoted to local stability.

The framework introduced by S. Shelah in [**Sh3**] is the study of classes of models of a finite diagram. These classes are described in more detail below. Such classes are examples of *nonelementary* classes and the results presented in this paper belong to what Shelah calls the *classification theory for nonelementary classes*. The word nonelementary refers to the fact that the compactness theorem fails. While many of the questions of classification theory for first order theories have been solved [**Sh b**], classification theory for nonelementary classes is still under-developed. This is not to say that the subject is small or not interesting. Thousands of pages have been devoted to its questions: See for example [**BaSh1**],[**BaSh2**], [**BaSh3**], [**Gr1**], [**Gr2**], [**GrHa**], [**GrSh1**], [**GrSh2**], [**GrSh3**], [**HaSh**], [**HySh1**], [**HySh2**], [**Ke**], [**Ki**], [**KoSh**], [**MaSh**], [**Sh3**], [**Sh48**], [**Sh87a**], [**Sh87b**], [**Sh88**], [**Sh tape**], [**Sh299**], [**Sh300**], [**Sh394**], [**Sh472**], [**Sh576**] and Shelah's forthcoming book [**Sh h**]. The techniques used are usually set-theoretic and combinatorial in nature, although more recently, new ideas coming from geometric stability theory have been imported (see Chapter III). The failure of the compactness theorem for a class of models makes their model theory delicate and sometimes sensitive to the axioms of set theory. This is one of the reasons why some additional assumptions are often required; a "monster model", set-theoretic assumptions, amalgamation properties and so on.

Let us describe briefly what is meant by the class of models of a finite diagram. Two perspectives are given below.

Given a first order theory T and a model M of T, the *finite diagram* of the model M is the set of complete types over the empty set realized in M. Fix a set D of complete T-types (over finitely many variables) and consider the class of models whose finite diagram is a subset of D. Such models are called D-*models* for convenience. In another language, we study the class of models omitting all the types over the empty set which do not belong to D. Both in [**Sh3**] and [**Sh54**], S. Shelah studied these classes under an additional assumption. Let us say a few words about exactly what this additional assumption is (it takes two equivalent forms in [**Sh3**] and [**Sh54**], and yet another equivalent formulation is given here). Since the compactness theorem fails for this class of models, it is crucial to have a good understanding of what the *meaningful* types are, that is which types can be realized by D-models. A corollary of the compactness theorem is that given a model M and a type p over a subset A of M, it is possible to find an elementary extension N of M in which p is realized. This fails, in general, for the class just described. There is a natural obstacle why this cannot work in general: Suppose p is a complete type over a set of parameters A, where A is a subset of a D-model M. Suppose there is a D-model N containing M in which p is realized, say by the sequence $\bar{c}$. Then, since $A \cup \bar{c} \subseteq N$ and N is a D-model, necessarily, all the subsequences of the set $A \cup \bar{c}$ realize (over the empty set) types that belong to D. The assumption that Shelah made (although not in those terms) is that this is the only restriction. This class of models, with the additional assumption on types, is the framework that S. Shelah calls *finite diagrams*. Note that when D is the set of all complete T-types over the empty set, then this is the first order case.

An alternative way of looking at this framework is as follows. Given a theory T, fix a large *homogeneous* model $\mathfrak{C}$ of T. In general, $\mathfrak{C}$ is not saturated. Let D be the diagram of $\mathfrak{C}$. Then, the class of D-models can be assumed to be the class of elementary submodels of $\mathfrak{C}$ and above meaningful types are the ones realized in $\mathfrak{C}$. Note that when $\mathfrak{C}$ is saturated, then this is the first order case.

Using the first order case as a guide, there are four important results in stability theory all due to S. Shelah [**Sh b**].

- A theory T is stable if and only if it does not have the order property.
- If a theory T is stable in λ, then given any set of finite sequences I of cardinality λ^+ and a set A of cardinality λ there exists a subset $J \subseteq I$ of cardinality λ^+ indiscernible over A.

- (The Stability Spectrum) For a theory T, either T is not stable or T is stable and there exist cardinals $\kappa(T)$ and $\lambda(T)$ satisfying $\kappa(T) \leq |T|^+$ and $\kappa(T) \leq \lambda(T) \leq 2^{|T|}$ such that T is stable in μ if and only if $\mu \geq \lambda(T)$ and $\mu^{<\kappa(T)} = \mu$.
- (The Saturation Spectrum) A theory T has a λ-saturated model of cardinality λ if and only if $\lambda \geq |D(T)|$ and either $\lambda^{<\lambda} = \lambda$ or T is stable in λ.

This chapter contains Shelah's generalizations of above theorems to the class of models of a finite diagram. In the first order case, the optimal versions of these results (at least the first three and the existence part of the last) are proved using *forking*. However, forking does not work in this more general context and generally in contexts where the compactness fails. Originally, however, the first two results were proved in this context using the notion of *splitting* and the third result using the notion of *strong splitting* [**Sh3**]. As to the last result, in this context the proof uses a combination of combinatorial methods based on splitting and strong splitting [**Sh54**]. Since *strong splitting* does not satisfy all the properties of forking, the proofs are more intricate and combinatorial in flavor. The first order notion of forking was invented by Saharon Shelah later and can be considered an improvement of strong splitting. The modern proofs have gained in conceptual structure over the original ones and we have attempted to integrate these improvements in the presentation by treating strong splitting more like forking (for example, we use the corresponding notion to $\kappa(T)$).

Classes of models of a finite diagram are important also because they provide a natural test-case to generalize ideas from first order logic to more general nonelementary classes. On the one hand, many of the technical difficulties arising from the failure of the compactness theorem are present. On the other hand, the model theory is more manageable as we have a good understanding of types. Note also that, in contrast to other nonelementary contexts, this work is completely done within ZFC. We added a discussion on the strength of the main assumption of Finite Diagrams after Hypothesis .1.5.

I.1. The framework of finite diagrams

The notation is standard. Abbreviations like AB stands for $A \cup B$, and $A\bar{b}$ for $A \cup \{\mathrm{ran}(\bar{b})\}$. When M is a model, $\|M\|$ stands for the cardinality of M. The notation $A \subseteq M$ means that A is a subset of the universe of M.

Let T be a first order complete theory in a language L. Denote by $L(T)$ the set of first order formulas in L. Let $\bar{M}$ be the a very saturated model of T. For $\Delta \subseteq L$, $A \subseteq M$, and a (not necessarily finite) sequence $\bar{a} \in M$, define the Δ-*type of* $\bar{a}$ *over* A *in* M by

$$\mathrm{tp}_\Delta(\bar{a}/A, M) = \{\phi(\bar{x}, \bar{b}) \mid \bar{b} \in A, \phi(\bar{x}, \bar{y}) \text{ or } \neg\phi(\bar{x}, \bar{y}) \in \Delta, \text{ and } M \models \phi[\bar{a}, \bar{b}]\}.$$

When Δ is $L(T)$ it is omitted and when M is $\bar{M}$, it is omitted also.

DEFINITION I.1.1.

(1) The *finite diagram* of A is
$$D(A) = \{\mathrm{tp}(\bar{a}/\emptyset) \mid \bar{a} \in A, \bar{a} \text{ finite }\}.$$
Such sets will be denoted by D and called *finite diagrams*.
(2) The set A is a D-*set* if $D(A) \subseteq D$. The model M is a D-*model* if $D(M) \subseteq D$.
(3) We let $S^n_\Delta(A) = \{\mathrm{tp}_\Delta(\bar{c}/A) \mid \bar{c} \in \bar{M}, \ell(\bar{c}) = n\}$, for $\Delta \subseteq L(T)$. When $\Delta = L(T)$ it is omitted. A type $p \in S^n(A)$ is called a D-*type* if and only if $A \cup \bar{c}$ is a D-set, for every $\bar{c}$ realizing p.
$S^n_D(A)$ will denote the set of D-types over A in n variables. We write $S_D(A)$ for $S^1_D(A)$.

When $D = D(T)$, then $S_D(A) = S(A)$.

DEFINITION I.1.2. The D-model M is a (D, λ)-*homogeneous* model if M realizes every $p \in S_D(A)$ for $A \subseteq M$ with $|A| < \lambda$.

When $D = D(T)$, then a model is (D, λ)-homogeneous if and only if it is λ-saturated.

The next lemma shows that if M is (D, λ)-homogeneous, then it is λ-universal for the class of D-models.

LEMMA I.1.3. *Let M be (D, λ)-homogeneous and A be a D-set of cardinality λ. Let $B \subseteq A$ such that $|B| < \lambda$. Then for every elementary mapping $f\colon B \to M$, there is an elementary mapping $g\colon A \to M$ extending f.*

PROOF. Write $A = B \cup \{a_i : i < \alpha \leq \lambda\}$. Construct an increasing sequence of elementary mappings $\langle f_i \mid i < \lambda\rangle$ by induction on $i < \alpha$, such that $f_0 = f$,
$$B \cup \{a_j : j < i\} \subseteq \mathrm{dom}(f_i) \quad \text{and} \quad \mathrm{ran}(f_i) \subseteq M.$$
In case $i = 0$ or i a limit, it is obvious. Assume f_i is constructed. Define $q_i = f_i(\mathrm{tp}(a_i/B \cup \{a_j : j < i\}))$. By induction hypothesis $q_i \in S_D(f_i(B \cup \{a_j : j < i\}))$. Hence, since M is (D, λ)-homogeneous, q_i is realized by some $b_i \in M$. Let $f_{i+1} = f_i \cup \langle a_i, b_i\rangle$. The elementary mapping $g = \bigcup_{i<\alpha} f_i$ is as required. $\square$

Recall from the first order case that a model is λ-*homogeneous*, if for any partial elementary mapping f from M into M with $|\mathrm{dom}(f)| < \lambda$ and $c \in M$, there is an elementary mapping g from M into M extending f such that $\mathrm{dom}(g) \supseteq \mathrm{dom}(f) \cup c$. The next lemma is an extension of the familiar first order result that a model M is λ-saturated if and only if M is λ-homogeneous and $< \aleph_0$-universal.

LEMMA I.1.4. *M is a (D, λ)-homogeneous model if and only if $D(M) = D$ and M is λ-homogeneous.*

PROOF. The only if part follows from the previous lemma. To see the converse, we show that M is (D, μ)-homogeneous for every $\mu \leq \lambda$ by induction on μ.

For the base case, assume that $\mu < \aleph_0$. Let $p \in S_D(\bar{c})$, where $\bar{c} \in M$ is finite. Let a be any element realizing p. By assumption $\mathrm{tp}(a\hat{}\bar{c}/\emptyset) \in D$. Since $D(M) = D$, there exist a' and $\bar{c}' \in M$ realizing $\mathrm{tp}(a\hat{}\bar{c}/\emptyset)$. Let f be a partial elementary mapping such that $f(\bar{c}) = \bar{c}'$ and $f(a) = a'$. Then, by λ-homogeneity of M, there is a partial elementary mapping g from M to M, extending $f^{-1} \upharpoonright \bar{c}'$, with $\mathrm{dom}(g) \supseteq \bar{c}' \cup a'$. Then we have that a' realizes $f(p)$, and so $g(a')$ realizes $g(f(p)) = p$. Hence, p is realized in M.

By induction, let $C \subseteq M$ of cardinality $\mu < \lambda$ and assume that we have already shown that M is (D, μ)-homogeneous. Let $p \in S_D(C)$ and a be any element realizing p. Then $C \cup a$ is a D-set of cardinality μ, so by (D, μ)-homogeneity of M, using the previous lemma, there exists an elementary mapping f sending $C \cup a$ into M. Hence, by λ-homogeneity of M, there is g, an elementary mapping from M into M, extending $f^{-1} \upharpoonright C$ with $\mathrm{dom}(g) \supseteq f(C) \cup f(a)$. To conclude, notice that since a realizes p, $f(a)$ realizes $f(p)$ and $g(f(a))$ realizes $g(f(p)) = p$. This shows that M realizes p, since $g(f(a)) \in M$, and completes the proof. □

The following hypothesis is made throughout the chapter. It is equivalent to Shelah's original assumption in [**Sh3**] and [**Sh54**]. Also, the same assumption was made by H. Jerome Keisler in his categoricity theorem [**Ke**].

HYPOTHESIS I.1.5. There exists a $(D, \bar{\kappa})$-homogeneous model $\mathfrak{C}$, with $\bar{\kappa}$ larger than any cardinality mentioned in this chapter.

In view of the preceding lemma, we may assume that any D-set lies in $\mathfrak{C}$. Also, satisfaction is with respect to $\mathfrak{C}$. Notice also that for any D-set A

$$S^n_D(A) = \{\mathrm{tp}(\bar{a}/A, \mathfrak{C}) \mid \bar{a} \in \mathfrak{C}\}.$$

The study of a *finite diagram* D is thus the study of the class of D-models under the additional assumption that there exists a $(D, \bar{\kappa})$-homogeneous model $\mathfrak{C}$, with $\bar{\kappa}$ very large.

Hypothesis .1.5 is a natural assumption to make. Let us say a few words about why we feel this is so. The most outstanding test question in the classification theory for nonelementary classes is a conjecture of S. Shelah, made in the mid-1970s:

CONJECTURE I.1.6 (Shelah). Let T be a countable $L_{\omega_1\omega}$ theory. If there exists a cardinal $\lambda \geq \beth_{\omega_1}$ such that T is categorical in λ, then T is categorical in every $\mu \geq \beth_{\omega_1}$.

Results of C.C.Chang and S. Shelah show that it is equivalent to solve this conjecture for the class of D-models of a countable first order theory, where D

is the set of isolated types over the empty set (whence the relevance of this discussion here). Most experts agree that the full conjecture seems currently out of reach. However, several attempts to solve the conjecture since the late 1970s have indicated that categoricity (sometimes in several cardinals and sometimes under additional set-theoretic axioms) implies the existence of various kinds of *amalgamation properties* and the existence of *monster models* (see for example [**Sh48**], [**Sh87a**], [**Sh87b**], [**Sh88**], [**KoSh**], or [**BaSh3**]). By monster model, we mean a large model with universal or homogeneous properties. By amalgamation properties we mean that the class of models of T satisfies the μ-amalgamation property for a class of cardinals μ. Recall that a class of models $\mathcal{K}$ has the μ-amalgamation property if for every triple of models $M_0, M_1, M_2 \in \mathcal{K}$ of cardinality μ such that $M_0 \prec M_1$, $M_0 \prec M_2$, and $M_0 \subseteq M_1 \cap M_2$, there exist a model $N \in \mathcal{K}$ and embeddings $f_i \colon M_i \to N$ for $i = 1, 2$ such that $f_1 \upharpoonright M_0 = f_2 \upharpoonright M_0$. For example, by Robinson's Consistency Lemma, the class of models of a first order theory T has the μ-amalgamation property, for every cardinal $\mu \geq |T|$.

While Shelah observed from work of Leo Marcus [**Mr**], that a monster model exactly as in Hypothesis .1.5 does not follow from the assumption of Shelah's conjecture, it is certainly reasonable to conjecture that it implies the existence of a monster model with a similar flavor. Thus, experience gained in this framework can shed light on the more general framework. These results are additional motivations to develop classification theory either inside a homogeneous model [**Sh3**], [**Sh54**], [**Gr1**], [**Gr2**], [**HySh1**], [**HySh2**] or for nonelementary classes with amalgamation properties [**Sh48**], [**Sh87a**], [**Sh87b**], [**GrHa**], [**Sh394**]. In fact, under *monster model* or *amalgamation properties* several approximations of Shelah conjecture are known: for example [**Ke**], [**KoSh**] [**Sh87a**], [**Sh87b**]. A Categoricity result for finite diagrams using geometric techniques appears in Chapter III. For a more detailed discussion of the categoricity problem, see the introduction to Chapter III.

In this vein, the two following conjectures were made by Rami Grossberg in 1989, in an email communication with John T. Baldwin:

CONJECTURE I.1.7. Let T be a countable $L_{\omega_1\omega}$ theory. If T is categorical is some large enough λ, then there exists a μ_0 such that the class of models of T has the μ-amalgamation property for every μ greater than μ_0.

Amalgamation properties are closely related to monster model hypotheses: When T is a Scott sentence, the conclusion of the previous conjecture implies the existence of arbitrarily large model-homogeneous models.

CONJECTURE I.1.8. Let T be a countable $L_{\omega_1\omega}$ theory such that there exists a μ_0 such that the class of models of T has the μ-amalgamation property for every μ greater than μ_0. If T is categorical in some $\lambda \geq \beth_{\omega_1}$, then T is categorical in every cardinal $\mu \geq \beth_{\omega_1}$.

Before finishing this discussion, we can ask the following related question:

QUESTION I.1.9. *Let T be a countable theory in $L_{\omega_1\omega}$. Is there a cardinal $\mu(T)$ such that if the class of models of T has the $\mu(T)$-amalgamation property then it has the λ-amalgamation property for arbitrarily large λ?*

I.2. Stability and order in finite diagrams

In this section, we present the equivalence between stability and the failure of the order property in the context of finite diagrams (Corollary .2.12).

DEFINITION I.2.1. Let D be a finite diagram.

(1) The diagram D is said to be *stable in* λ if for every $A \subseteq \mathfrak{C}$ of cardinality at most λ and for every $n < \omega$ we have $|S^n_D(A)| \leq \lambda$.
(2) We say that D is *stable* if there is a λ such that D is stable in λ.

By the pigeonhole principle, it is enough to consider $n = 1$, i.e. D is stable in λ if and only if for all $A \subseteq \mathfrak{C}$ of cardinality at most λ, we have $|S_D(A)| \leq \lambda$.

DEFINITION I.2.2. Let D be a finite diagram.

(1) D has the λ-*order property* if there exist a D-set $\{\bar{a}_i \mid i < \lambda\}$, and a formula $\phi(\bar{x}, \bar{y}) \in L(T)$ such that
$$\models \phi[\bar{a}_i, \bar{a}_j] \quad \text{if and only if} \quad i < j < \lambda.$$
(2) D has the *order property* if D has the λ-order property for every cardinal λ.

Notice that the order property is formulated differently from the order property used by Shelah in [**Sh b**]. The formulation given here is equivalent to the usual order property in the first order case, and is more natural in nonelementary cases; when it holds there are many nonisomorphic models (see [**Sh16**], [**GrSh1**], and [**GrSh3**]).

Recall some standard definitions. A set of finite sequences $\{\bar{a}_i \mid i < \alpha\}$ is said to be a sequence of n-*indiscernibles over* A, for $n < \omega$ if $\mathrm{tp}(\bar{a}_0, \ldots, \bar{a}_{n-1}/A) = \mathrm{tp}(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}/A)$. for every $i_0 < \cdots < i_{n-1} < \alpha$. Then $\{\bar{a}_i \mid i < \alpha\}$ is an *indiscernible sequence over* A, if it is an n-indiscernible sequence over A for every $n < \omega$. It is said to be an *indiscernible set*, if in addition, the ordering does not matter. We will not have to distinguish between the two, as in the presence of stability, every indiscernible sequence is, in fact, an indiscernible set (Remark .2.4 and Corollary .2.12). Hence, we will often say indiscernible for indiscernible sequence, or set when they coincide or when it does not matter.

REMARK I.2.3. If there exists a D-set $\{\bar{a}_i \mid i < \omega\}$, which is an indiscernible sequence, and a formula $\phi(\bar{x}, \bar{y})$ such that
$$\models \phi[\bar{a}_i, \bar{a}_j] \quad \text{if and only if} \quad i < j < \omega,$$
then D has the order property.

PROOF. Let λ be an infinite cardinal. Let $\{\bar{c}_i \mid i < \lambda\}$ be new constants. Consider the union of the following sentences:

- $\phi(\bar{c}_i, \bar{c}_j)$, if $i < j < \lambda$;
- $\neg\phi(\bar{c}_i, \bar{c}_j)$, if $i \geq j$, $i, j < \lambda$;
- $\psi(\bar{c}_{i_0}, \dots, \bar{c}_{i_n})$, for each $\psi(\bar{x}_0, \dots \bar{x}_n) \in \mathrm{tp}(\bar{a}_0, \dots, \bar{a}_n/\emptyset)$, and each $n < \omega$, and each $i_0 < \dots < i_n < \lambda$.

The above set of sentences is consistent (use $\{\bar{a}_i \mid i < \omega\}$). Let $\bar{b}_i$ be the interpretation of $\bar{c}_i$ in $\bar{M}$, the monster model for T. The last clause implies that $\{\bar{b}_i \mid i < \lambda\}$ is a D-set. By the first two clauses, we have

$$\models \phi[\bar{b}_i, \bar{b}_j] \quad \text{if and only if} \quad i < j < \lambda.$$

Hence, D has the λ-order property. We are done since λ was arbitrary. □

The next remark is a fact that goes back to Morley and Ehrenfeucht.

REMARK I.2.4. Suppose D does not have the order property. Let $\{\bar{a}_i \mid i < \alpha\}$ be an infinite indiscernible sequence over A. Then $\{\bar{a}_i \mid i < \alpha\}$ is an indiscernible set over A.

PROOF. Suppose that the conclusion fails. Then, there exist an integer $n < \omega$, a permutation $\sigma \in S_n$, and indices $i_0 < \dots < i_n < \alpha$ such that

$$\mathrm{tp}(\bar{a}_0, \dots, \bar{a}_n/A) \neq \mathrm{tp}(\bar{a}_{i_{\sigma(0)}}, \dots, \bar{a}_{i_{\sigma(n)}}/A).$$

Since $\{\bar{a}_i \mid i < \alpha\}$ is an indiscernible sequence over A, we have $\mathrm{tp}(\bar{a}_0, \dots, \bar{a}_n/A) \neq \mathrm{tp}(\bar{a}_{\sigma(0)}, \dots, \bar{a}_{\sigma(n)}/A)$. Since any permutation is a product of transpositions, we may assume that there exist $k_0 < k_1 \leq n$ such that $\sigma(k_0) = k_1$, $\sigma(k_1) = k_0$ and $\sigma(i) = i$, otherwise. Hence, there exists $\phi(\bar{x}, \bar{y}, \bar{b})$, where $\bar{b} \in A \cup \{\bar{a}_i \mid i \leq n, i \neq k_0, k_1\}$ such that $\models \phi[\bar{a}_{k_0}, \bar{a}_{k_1}, \bar{b}]$ and $\models \neg\phi[\bar{a}_{k_1}, \bar{a}_{k_0}, \bar{b}]$. Then, the D-set $\{\bar{a}_i\hat{\ }\bar{b} \mid n < i < \alpha\}$ is an infinite indiscernible sequence (over $\emptyset$). Hence $\models \phi[\bar{a}_i, \bar{a}_j, \bar{b}]$ if and only if $n < i < j < \alpha$. This implies that D has the order property by the previous remark. □

The main tool to prove that the failure of the order property implies stability (Theorem .2.9) is *splitting*. Recall the definition.

DEFINITION I.2.5. Let Δ_1 and Δ_2 be sets of formulas. Let A be a set and $B \subseteq A$. For $p \in S^n(A)$, we say that p (Δ_1, Δ_2)-*splits over* B if there are $\bar{b}, \bar{c} \in A$ and $\phi(\bar{x}, \bar{y}) \in \Delta_2$ such that $\mathrm{tp}_{\Delta_1}(\bar{b}/B) = \mathrm{tp}_{\Delta_1}(\bar{c}/B)$ with $\phi(\bar{x}, \bar{b}) \in p$ and $\neg\phi(\bar{x}, \bar{c}) \in p$.

When $\Delta_1 = \Delta_2 = L(T)$, we just say that p *splits over* B. When $\Delta_1 = \{\phi(\bar{x}, \bar{y})\}$ and $\Delta_2 = \{\psi(\bar{x}, \bar{y})\}$, we write $(\phi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}))$-*splits*, omitting the parentheses.

For a statement $\mathbf{t}$ and a formula ϕ, the following convention is made: $\phi^{\mathbf{t}} = \neg\phi$ if the statement $\mathbf{t}$ is false and $\phi^{\mathbf{t}} = \phi$, if the statement $\mathbf{t}$ is true. The same notation is used when $\mathbf{t} \in \{0, 1\}$, where 0 stands for falsehood and 1 stands for truth.

The next two lemmas give sufficient conditions guaranteeing the existence and uniqueness of nonsplitting extensions.

LEMMA I.2.6. *Let $A \subseteq B \subseteq C$ be sets such that B realizes all the Δ_1-types over A that are realized in C. Assume p_1,$p_2 \in S_{\Delta_2}(C)$ and $p_1 \restriction B = p_2 \restriction B$. If p_1, p_2 do not (Δ_1, Δ_2)-split over A, then $p_1 = p_2$.*

PROOF. By symmetry, it is enough to show that $p_1 \subseteq p_2$. Let $\phi(\bar{x}, \bar{b}) \in p_1$. By assumption $\mathrm{tp}_{\Delta_1}(\bar{b}/A)$ is realized by some $\bar{c} \in B$. Hence $\phi(\bar{x}, \bar{c}) \in p_1$ since p_1 does not (Δ_1, Δ_2)-split over A, and $\phi(\bar{x}, \bar{y})^t \in \Delta_2$ for $t = 0$ or 1. Thus $\phi(\bar{x}, \bar{c}) \in p_2$ and so $\phi(\bar{x}, \bar{b}) \in p_2$ also since p_2 does not (Δ_1, Δ_2)-split over A. $\square$

LEMMA I.2.7. *Let $A \subseteq B \subseteq C$ be D-sets, such that B realizes every D-type over A, which is realized in C. Suppose $p \in S_D(B)$ does not split over A. Then, there is a unique type $q \in S_D(C)$ extending p that does not split over A.*

PROOF. Uniqueness was proved in the previous lemma. Hence, it is enough to show existence. Define q explicitly by setting:

$$q := \{\phi(x, \bar{c}) \mid \text{There exists } \bar{b} \in B \text{ realizing } \mathrm{tp}(\bar{c}/A) \text{ and } \phi(x, \bar{b}) \in p\}.$$

This is well-defined. By assumption p does not split over A and so the definition does not depend on the choice of $\bar{b} \in B$.

First notice that q is complete. Suppose $\bar{c} \in C$ and $\phi(x, \bar{y}) \in L(T)$. Suppose $\phi(x, \bar{c}) \notin q$. Let $\bar{b} \in B$ realize $\mathrm{tp}(\bar{c}/A)$. By definition, we have $\phi(x, \bar{b}) \notin p$. Hence, $\neg\phi(x, \bar{b}) \in p$, since $p \restriction B$ is complete. Thus, $\neg\phi(x, \bar{c}) \in q$, by definition of q. Also, q is consistent. Let $\phi_1(x, \bar{c}_1), \dots, \phi_n(x, \bar{c}_n) \in q$. Then $\phi_i(x, \bar{b}_i) \in p$, for $\bar{b}_1\hat{}\dots\hat{}\bar{b}_n \in B$ realizing $\mathrm{tp}(\bar{c}_1\hat{}\dots\hat{}\bar{c}_n/A)$. Since p is consistent, we have

$$\models \exists x[\phi_1(x, \bar{b}_1) \wedge \cdots \wedge \phi_n(x, \bar{b}_n)].$$

Then, by an elementary mapping sending each $\bar{b}_i$ to $\bar{c}_i$ fixing A we conclude that

$$\models \exists x[\phi_1(x, \bar{c}_1) \wedge \ldots \wedge \phi_n(x, \bar{c}_n)].$$

Hence, the set $\{\phi_1(x, \bar{c}_1), \dots, \phi_n(x, \bar{c}_n)\}$ is consistent.

Now let us see that q does not split over A. Otherwise, there are $\bar{c}_1$, $\bar{c}_2 \in C$, and $\phi(x, \bar{y})$ such that $\mathrm{tp}(\bar{c}_1/A) = \mathrm{tp}(\bar{c}_2/A)$ and $\phi(x, \bar{c}_1)$, $\neg\phi(x, \bar{c}_2) \in q$. Choose $\bar{b}_1$, $\bar{b}_2 \in B$, such that $\mathrm{tp}(\bar{b}_1/A) = \mathrm{tp}(\bar{b}_2/A) = \mathrm{tp}(\bar{c}_1/A)$. We have $\phi(x, \bar{b}_1)$, $\neg\phi(x, \bar{b}_2) \in p$, by definition of q. Hence p splits over A, contradiction.

Finally, let us show that q is a D-type. Suppose not. Then, there is a realizing q and $\bar{c} \in C$ such that $\mathrm{tp}(a\hat{}\bar{c}/\emptyset) \notin D$. Let $\bar{b} \in B$ realize $\mathrm{tp}(\bar{c}/A)$. Since

a realizes p, we have $\mathrm{tp}(a\bar{b}/\emptyset) \in D$. Hence, in particular

$$\mathrm{tp}(a\bar{b}/\emptyset) \neq \mathrm{tp}(a\bar{c}/\emptyset).$$

Hence there is $\phi(x, \bar{y})$, with $\models \phi[a, \bar{b}]$, and $\models \neg\phi[a, \bar{c}]$. This implies that $\phi(x, \bar{b})$, and $\neg\phi(x, \bar{c}) \in q$. This shows that q splits over A,a contradiction. $\square$

We will use the following notational convention: For Δ a set of formulas, we write

$$S_{D,\Delta}(B) = \{\mathrm{tp}_\Delta(c/B, \mathfrak{C}) \mid c \in \mathfrak{C}\}.$$

When $\Delta = \{\phi(\bar{x}, \bar{y})\}$, we write $S_{D,\phi}(B)$ instead of $S_{D,\{\phi\}}(B)$.

COROLLARY I.2.8. *Let $A \subseteq B$ be D-sets. Then*

$$|\{p \in S_{D,\Delta_2}(B) : p \text{ does not } (\Delta_1, \Delta_2)\text{-split over } A\}| \leq 2^{2^{|L(T)|+|A|}}.$$

PROOF. Since $|S^n_D(A)| \leq 2^{|L(T)|+|A|}$, for each $n < \omega$, we can find C, with $|C| \leq 2^{|L(T)|+|A|}$ such that C realizes all the types in $S^n_{D,\Delta_1}(A)$, for each $n < \omega$. Then, by Lemma .2.6, we have

$$|\{p \in S_{D,\Delta_2}(B) : \ p \text{ does not } (\Delta_1, \Delta_2)\text{-split over } A\}| \leq$$
$$\leq |\{p : p \in S_{D,\Delta_2}(C)\}| \leq 2^{|C|} \leq 2^{2^{|L(T)|+|A|}}.$$

$\square$

The proof of the next theorem follows [**Gr1**].

THEOREM I.2.9. *Let $\lambda \geq |L(T)|$. If D is not stable in 2^{2^λ}, then D has the λ^+-order property.*

PROOF. We first claim that there exist a D-set A of cardinality 2^{2^λ} and a formula $\phi(x, \bar{y})$ such that

$$|S_{D,\phi}(A)| > |A|.$$

Since D is not stable in 2^{2^λ}, there is a D-set A of cardinality 2^{2^λ} such that $|S_D(A)| > |A|$. Define

$$f\colon S_D(A) \to \Pi_{\phi(x,\bar{y}) \in L} S_{D,\phi}(A), \quad \text{by} \quad f(p) = (p \restriction \phi)_{\phi \in L}.$$

Then, f is injective and since $\lambda \geq |L(T)|$, by the pigeonhole principle, there must be $\phi(x, \bar{y}) \in L$ such that $|S_{D,\phi}(A)| > |A|$. This proves the claim.

Let A and ϕ be as in the claim, we will show that

$$\psi(x_0, \bar{x}_1, \bar{x}_2, y_0, \bar{y}_1, \bar{y}_2) := \phi(x_0, \bar{y}_1) \leftrightarrow \phi(x_0, \bar{y}_2)$$

demonstrates the order property. For convenience, let $\mu = 2^{2^\lambda}$. Let $\{a_i : i < \mu^+\} \subseteq \mathfrak{C}$ be such that $i \neq j < \mu^+$ implies $\mathrm{tp}_\phi(a_i/A) \neq \mathrm{tp}_\phi(a_j/A)$. This is possible since $|S_{D,\phi}(A)| > |A|$. Let $\chi(\bar{y}, x) = \phi(x, \bar{y})$ and $n = \ell(\bar{y})$. Define an increasing continuous chain of sets $\langle A_i : i < \mu \rangle$ such that:

(1) $A_0 = \emptyset$ and $|A_i| \leq \mu$, $i < \mu$.
(2) For every $B \subseteq A_i$ of cardinality at most λ and every type $p \in S_{D,\phi}(A_i) \cup S^n_{D,\chi}(A_i)$, $p \restriction B$ is realized in A_{i+1}.

This is possible since there are at most $\mu^\lambda = \mu$ subsets of A_i of cardinality λ and at most $|S_D(B)| \leq 2^{|L(T)|+|B|} \leq (2^\lambda)^\lambda < \mu$ possible types for each set B.

CLAIM. For every $j < \mu^+$, there is i with $j < i < \mu^+$ such that for all $l < \lambda^+$ the type $q_i = \mathrm{tp}(a_i, A_l)$ (χ, ϕ)-splits over each $B \subseteq A_l$ of cardinality at most λ.

PROOF. Otherwise, there is $j < \mu^+$ such that for every i with $j < i < \mu^+$, there is $l < \lambda$ and $B^i \subseteq A_l$ of cardinality at most λ such that q_i does not (χ, ϕ)-split over B^i. Since $\mu^+ > \lambda$, by the pigeonhole principle, we can find $l < \lambda$ such that μ^+ many q_i's do not (χ, ϕ)-split over a subset of A_l. By a second application of the pigeonhole principle, since $\mu^+ > \mu \geq |A_l|^\lambda = |\{B \subseteq A_l : |B| \leq \lambda\}|$, we can find $\mu^+ > (2^{2^\lambda})$ many types that do not (χ, ϕ)-split over a set of cardinality at most λ. This contradicts Corollary .2.8. Hence, the claim is true. □

Among the i's satisfying the claim, pick one such that $a_i \notin \bigcup_{l<\lambda} A_l$. This is possible since $|\bigcup_{l<\lambda} A_l| \leq \mu$. Then, by construction, for every $l < \lambda^+$, the type $\mathrm{tp}_\phi(a_i/A_l)$ (χ, ϕ)-splits over every $B \subseteq A_l$ of cardinality at most λ. Define $\bar{a}_l$, $\bar{b}_l$ and c_l in A_{2l+2}, as well as $B_l = \cup\{\bar{a}_k, \bar{b}_k, c_k : k < l\}$ by induction on $l < \lambda^+$ such that

(1) $B_l \subseteq A_{2l}$ and $|B_l| \leq \lambda$;
(2) $\mathrm{tp}_\chi(\bar{a}_l/B_l) = \mathrm{tp}_\chi(\bar{b}_l/B_l)$;
(3) Both $\phi(x, \bar{a}_l)$ and $\neg\phi(x, \bar{b}_l)$ belong to $\mathrm{tp}(a_i/A_{2l})$;
(4) $c_l \in A_{2l+1}$ realizes $\phi(x, \bar{a}_l) \wedge \neg\phi(x, \bar{b}_l)$.

This is possible: Set $B_0 = \emptyset$. If B_l is constructed, since $B_l \subseteq A_{2l}$ of cardinality at most λ, $\mathrm{tp}_\phi(a_i/A_{2l})$ (χ, ϕ)-splits over B_l, hence we can find $\bar{a}_l$ and $\bar{b}_l$ in A_{2l} such that $\mathrm{tp}_\chi(\bar{a}_l/B_l) = \mathrm{tp}_\chi(\bar{b}_l/B_l)$ and both $\phi(x, \bar{a}_l)$ and $\neg\phi(x, \bar{b}_j)$ belong to $\mathrm{tp}(a_i/A_{2j})$. Then, by construction of A_{2l+1}, we can find $c_l \in A_{2l+1}$, realizing $\mathrm{tp}_\phi(a_i/A_2) \restriction \{\bar{a}_l, \bar{b}_l\}$ and hence realizing $\phi(x, \bar{a}_l) \wedge \neg\phi(x, \bar{b}_l)$. When l is a limit ordinal, we define B_l by continuity.

Now, let $\bar{d}_l = c_l\hat{\ }\bar{a}_l\hat{\ }\bar{b}_l$. It is easy to see from (2), (3) and (4) that $\{\bar{d}_l : l < \lambda^+\}$ and $\psi(x_0, \bar{x}_1, \bar{x}_2, y_0, \bar{y}_1, \bar{y}_2) = \phi(x_0, \bar{y}_1) \leftrightarrow \phi(x_0, \bar{y}_2)$ together demonstrate the (D, λ^+)-order property. □

The next theorem is a converse of Theorem .2.9. The proof uses Hanf number techniques. For a first order theory T and Γ a set of T-types over the empty set, the class $\mathrm{EC}(T, \Gamma)$ is the class of models of T omitting every type in Γ. For cardinals λ and κ, the *Hanf-Morley number* $\mu(\lambda, \kappa)$ is defined to be the smallest cardinal μ with the property that for every $\mathrm{EC}(T,\Gamma)$ with $|T| \leq \lambda$ and $|\Gamma| \leq \kappa$, if $\mathrm{EC}(T,\Gamma)$ contains a model of cardinality μ then $\mathrm{EC}(T,\Gamma)$ contains models of arbitrarily large cardinality. Clearly, when $\kappa = 0$, $\mu(\lambda,\kappa) = \aleph_0$; this is the first order case. When $\kappa \geq 1$, the notion of *wellordering number* $\delta(\lambda,\kappa)$ needs to be introduced. For cardinals λ,κ, the number $\delta(\lambda,\kappa)$ is the smallest ordinal δ with the property that for every $\mathrm{EC}(T,\Gamma)$ with $|T| \leq \lambda$ and $|\Gamma| \leq \kappa$, if $\mathrm{EC}(T,\Gamma)$ contains a model with a predicate of order type δ, then $\mathrm{EC}(T,\Gamma)$ contains a model where this predicate is not wellordered. If $\kappa \geq 1$, it is a standard result that $\mu(\lambda,\kappa) = \beth_{\delta(\lambda,\kappa)}$. (Note that the methods of the proof below show $\mu(\lambda,\kappa) \leq \beth_{\delta(\lambda,\kappa)}$.) A standard result on wellordering numbers states that $\delta(\lambda,\kappa) \leq (2^\lambda)^+$. This will be used in the proof and explains the cardinal $\beth_{(2^{|T|})^+}$ appearing in the statement.

THEOREM I.2.10. *If D has the λ-order property for every $\lambda < \beth_{(2^{|T|})^+}$, then D is not stable and D has the ω-order property witnessed by an indiscernible sequence.*

PROOF. We will show first that D has the ω-order property witnessed by an indiscernible sequence. By assumption, for each $\alpha < (2^{|T|})^+$, we can find a D-set

$$P_\alpha = \{\bar{a}_{\alpha,j} \mid j < (\beth_\alpha)^+\}$$

and a formula ϕ_α witnessing the order property. Hence, by the pigeonhole principle, we may assume that $\phi_\alpha = \phi$ is fixed for all α.

Notice that M is a D-model of T if and only if $M \in \mathrm{EC}(T,\Gamma)$, with $\Gamma = D(T) \setminus D$. But $|D(T) \setminus D| \leq 2^{|T|}$, and so the wellordering number for this class is at most $\delta(|T|, 2^{|T|}) = (2^{|T|})^+$.

For $\alpha < (2^{|T|})^+$, define $M_\alpha \prec \mathfrak{C}$ containing $\{\bar{a}_{\alpha,j} : j < (\beth_\alpha)^+\}$ of cardinality $(\beth_\alpha)^+$. This is possible by the downward Löwenheim-Skolem Theorem. Each M_α belongs to $EC(T,\Gamma)$. Define $F\colon (2^{|T|})^+ \to \{M_\alpha : \alpha < (2^{|T|})^+\}$, by $F(\alpha) = M_\alpha$.

Consider the following model

$$M = \langle H(\bar{\chi}), \in, F, (2^{|T|})^+, T, P, \models, \psi\rangle_{\psi \in L},$$

where $H(\bar{\chi})$ is the set of sets of hereditary power less than $\bar{\chi}$, and $\bar{\chi}$ is a regular cardinal chosen so that $H(\bar{\chi})$ contains everything that has been mentioned so far in this proof. The predicates $(2^{|T|})^+$ and T are unary predicates whose interpretations are the corresponding sets. The meaning of the binary predicates $\models$ and $\in$ and of the constants ψ, for each $\psi \in L$ is their true meaning in $H(\chi)$. Also F is a unary function and the interpretation of F is the one we just defined. P is a unary predicate, whose interpretation in each M_α is the D-set P_α witnessing the order property. More precisely, we have that

$$M \models \forall \alpha \in (2^{|T|})^+[\bar{a}_{\alpha,i} \in M_\alpha \wedge \bar{a}_{\alpha,j} \in M_\alpha \wedge P\bar{a}_{\alpha,i} \wedge P\bar{a}_{\alpha,j}] \\ \to (M_\alpha \models \phi[\bar{a}_{\alpha,i}, \bar{a}_{\alpha,j}] \leftrightarrow i \in j).$$

Let $N \prec M$ such that $(2^{|T|})^+ \subseteq N$ of cardinality $(2^{|T|})^+$. Therefore, we can fix a bijection $G\colon |N| \to (2^{|T|})^+$. Define $a < b$ if and only if $G(a) \in G(b)$.

Form $N' = \langle N, <, G\rangle$ an expansion of N. Let $T' = Th(N')$ and for each $\psi(\bar{x}) \in L$ define $\psi'(\bar{x}, y)$ by $\exists \alpha \in (2^{|T|})^+(y = M_\alpha \wedge \bar{x} \in M_\alpha \wedge M_\alpha \models \psi[\bar{x}])$. Let $\Gamma' = \{\{\psi'(\bar{x}, y) : \psi(\bar{x}) \in p\} : p \in \Gamma\}$. Then, we have that $|T'| = |T|$ and $|\Gamma'| = |\Gamma|$, so $\delta(|T'|, 2^{|T'|}) = (2^{|T|})^+$.

We first claim that N' omits every type in Γ'.

Suppose not. There is $p' \in \Gamma'$ such that for some $\bar{c}\hat{\ }a \in N'$ we have that $\models \psi'[\bar{c}, a]$, for all $\psi' \in p'$. But then, by definition $\bar{c}$ is in some M_α and $\bar{c}$ realizes every $\psi(\bar{x})$ in p. But $p \in \Gamma$ so this contradicts the fact that $M_\alpha \in \mathrm{EC}(T, \Gamma)$. Hence, we have a model $N' \in \mathrm{EC}(T', \Gamma')$ wellordered by $<$ and of order-type $(2^{|T|})^+$. Thus, we can find a model $N'' \in \mathrm{EC}(T', \Gamma')$, whose universe is not wellordered by $<$. Therefore, by taking away elements if necessary, there exists elements $b_n \in N''$ such that $N'' \models b_{n+1} + n + 1 < b_n$ and $N'' \models b_n \in (2^{|T|})^+$ for $n < \omega$.

Define a sequence of sets $\langle X_n \mid n < \omega\rangle$ such that

(1) $N'' \models$ "X_n is a sequence of n-indiscernibles in M_{b_0} of cardinality $\beth_{b_n}$".
(2) $N'' \models$ "X_n has the D-order property"

This is possible. Construct the X_n by induction on $n < \omega$. For $n = 0$, let $X_0 = \{\bar{a}_{b_0, j} : j < \beth_{b_0}\}$, i.e. the interpretation in N'' of the interpretation of the predicate P in M_α. Then the first requirement is satisfied since X_0 has the right cardinality and there is nothing to check for 0-indiscernibility. The second requirement is also satisfied since M and so N'' knows that they witness the order property.

Assume X_n has already been constructed. Define

$$f : [X_n]^{n+1} \to S^{n+1}_{L(T)}(\emptyset), \quad \text{by} \quad (c_1, \dots, c_{n+1}) \mapsto \mathrm{tp}(c_1, \dots, c_{n+1}/\emptyset).$$

We know by Erdős-Rado that

$$\beth_n^+(\beth_{b_{n+1}}) \to (\beth^+_{b_{n+1}})^{n+1}_{\beth_{b_{n+1}}}$$

and we have $\beth_{b_n} \geq \beth_{b_{n+n+1}} \geq \beth_n^+(\beth_{b_{n+1}})$, so we can find a subset X_{n+1} of X_n of cardinality $\beth_{b_{n+1}}$ such that every increasing $(n+1)$-tuple from it has the same type. This implies that X_{n+1} is an $(n+1)$-indiscernible sequence with the right cardinality. Since the second requirement is preserved by renumbering if needed, we are done.

This is enough. Let $\{\bar{c}_i : i < \omega\}$ be a new set of constants. Define T_1 to be the union of the following set of sentences:

- T;
- $\bar{c}_i \neq \bar{c}_j$, whenever $i \neq j$;

- $\phi(\bar{c}_i, \bar{c}_j)^{i<j}$, for every $i, j < \omega$;
- $\chi(\bar{c}_{i_1}, \dots, \bar{c}_{i_n})$, for every $\chi \in \mathrm{tp}(\bar{a}_1, \dots, \bar{a}_n/\emptyset)$, $i_1 < \dots < i_n$ and $n < \omega$;
- $\psi(\bar{c}_{i_1}, \dots \bar{c}_{i_n}) \leftrightarrow \psi(\bar{c}_{j_1}, \dots, \bar{c}_{j_n})$, whenever $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$, $n < \omega$ and $\psi \in L(T)$.

By the compactness theorem and the definition of X_n, T_1 has a model N_1. Call $\bar{a}_i = \bar{c}_i^{N_1}$ Notice also that the construction ensures that $\{\bar{a}_i : i < \omega\}$ is a D-set. Hence we have the ω-order property witnessed by indiscernibles.

We will use these to show that D is not stable. Let μ be a given cardinality. Define $\kappa = \min\{\kappa : 2^\kappa > \mu\}$. By compactness, using the indiscernibility of $\{\bar{a}_i : i < \omega\}$, we can get a D-set $\{\bar{a}_\eta : \eta \in {}^{\kappa \geq}2\}$ such that $\models \phi[\bar{a}_\eta, \bar{a}_\nu]$ if and only if $\eta \prec \nu$. Let $A = \bigcup_{\eta \in {}^{\kappa >}2} \bar{a}_\eta$. Then $|A| \leq \mu$, by choice of κ, and for $\eta \neq \nu \in {}^{\kappa}2$, we have that $\mathrm{tp}(\bar{a}_\eta/A) \neq \mathrm{tp}(\bar{a}_\nu/A)$. Indeed, there is a first $i < \kappa$ such that $\eta[i] \neq \nu[i]$, say $\eta[i] = 0$. But then $\psi(\bar{a}_{\eta\hat{}0}, \bar{x}) \in \mathrm{tp}(\bar{a}_\eta/A)$ and $\neg\psi(\bar{a}_{\eta\hat{}0}, \bar{x}) \in \mathrm{tp}(\bar{a}_\nu/A)$. Thus $|S_D(A)| \geq 2^\kappa > \mu$ and so D is not stable in μ. □

The next corollary tells us that if D is stable, we can find $\lambda < \beth_{(2^{|T|})^+}$ demonstrating this. Notice that if $D = D(T)$ we are in the first order case and the bound on the first stability cardinal is actually $2^{|T|}$.

COROLLARY I.2.11. *If D is stable, then there exists $\lambda < \beth_{(2^{|T|})^+}$ such that D is stable in λ.*

PROOF. Suppose that D is not stable in any $\lambda < \beth_{(2^{|T|})^+}$. Then, since $\beth_{(2^{|T|})^+}$ is a strong limit, for each $\lambda < \beth_{(2^{|T|})^+}$, we have $2^{2^\lambda} < \beth_{(2^{|T|})^+}$ and so D is not stable in 2^{2^λ}. Hence by Theorem .2.9, D has the λ^+-order property for all $\lambda < \beth_{(2^{|T|})^+}$ and so by Theorem .2.10 D is not stable. □

The next corollary is the order/stability dichotomy.

COROLLARY I.2.12. *D is stable if and only if D does not have the order property.*

PROOF. If D is not stable, then it is not stable in 2^{2^λ} for any $\lambda \geq |L(T)|$ so by Theorem .2.9, D has the λ-order property for every cardinal λ. For the converse, we use Theorem .2.10. □

I.3. The stability spectrum

In the first part of this section, combinatorial properties related to splitting are introduced for finite diagrams. They can be used to give another characterization of stability (see Corollary .3.7). In the second part, the focus is on a more delicate tool; *strong splitting*. It is a substitute for the notion of forking. The appropriate cardinal invariant and combinatorial property related to strong splitting are introduced. They are used to derive the Stability Spectrum Theorem (Theorem .3.17).

DEFINITION I.3.1.

(1) D satisfies $(*\lambda)$ if there exists an increasing continuous chain of D-sets $\{A_i : i \leq \lambda\}$ and $p \in S^n_D(A_\lambda)$ such that

$$p \restriction A_{i+1} \text{ splits over } A_i, \quad \text{for all } i < \lambda.$$

(2) D satisfies $(B*\lambda)$ if there exists a tree of types $\{p_\eta \in S_D(B_\eta) \mid \eta \in {}^{\lambda>}2\}$, and formulas $\phi_\eta(\bar{x}, \bar{a}_\eta)$ such that $p_\eta \subseteq p_\nu$ if $\eta \prec \nu$ and

$$\phi_\eta(\bar{x}, \bar{a}_\eta) \in p_{\eta\hat{}0} \quad \text{and} \quad \neg\phi_\eta(\bar{x}, \bar{a}_\eta) \in p_{\eta\hat{}1}.$$

The next two remarks are routine induction using the definition. As an illustration we prove the first one.

REMARK I.3.2. If there exists a type $p \in S_D(A)$ that splits over every subset of A of cardinality less than λ, then D satisfies $(*\lambda)$.

PROOF. Let $p \in S_D(A)$ be such that p splits over every subset B of A of cardinality less than λ. Construct an increasing continuous chain of sets $\{A_i : i \leq \lambda\}$ of cardinality less than λ demonstrating $(*\lambda)$ as follows. Let $A_0 = \emptyset$ and $A_\delta = \bigcup_{i<\delta} A_i$, if δ is a limit ordinal. If A_i is constructed of cardinality less than λ, then by assumption p splits over A_i. Hence, we can find $\bar{b}, \bar{c} \in A$ and $\phi(\bar{x}, \bar{y})$ such that $\mathrm{tp}(\bar{b}/A_i) = \mathrm{tp}(\bar{c}/A_i)$ and $\phi(\bar{x}, \bar{b}) \in p$ and $\neg\phi(\bar{x}, \bar{c}) \in p$. Let $A_{i+1} = A_i \cup \bar{b} \cup \bar{c}$. □

REMARK I.3.3. In the definitions of $(*\lambda)$ and $(B*\lambda)$ we may assume that $|A_i| < |i|^+ + \aleph_0$ and similarly that $|B_\eta| < |\ell(\eta)|^+ + \aleph_0$.

LEMMA I.3.4. *If D satisfies $(*\lambda)$, then D satisfies $(B*\lambda)$.*

PROOF. We first show that if $p \in S^n_D(A)$ splits over $B \subseteq A$, then there is a partial elementary mapping f such that $f \restriction B = id_B$ and p and $f(p)$ are contradictory types:

If p splits over B, then there are $\bar{b}, \bar{c} \in A$ and $\phi(\bar{x}, \bar{y})$ such that $\mathrm{tp}(\bar{b}/B) = \mathrm{tp}(\bar{c}/B)$ and $\phi(\bar{x}, \bar{b}) \in p$ and $\neg\phi(\bar{x}, \bar{c}) \in p$. Hence there is an elementary mapping f such that $f \restriction B = id_B$ and $f(\bar{b}) = \bar{c}$. Then clearly p and $f(p)$ are contradictory types.

Now assume that D satisfies $(*\lambda)$. By definition, there exists an increasing continuous chain of sets $\{A_i \mid i \leq \lambda\}$ and $p \in S^n_D(A_\lambda)$ such that $p \restriction A_{i+1}$ splits over A_i for $i < \lambda$. By Remark .3.3, we may assume that $|A_i| < |i|^+ + \aleph_0$. By the first paragraph, for each $i < \lambda$ there exists an elementary mapping f_i such that $A_i \subseteq \mathrm{dom}(f_i) \subseteq A_{i+1}$ and $f_i(p \restriction A_{i+1})$ and $p \restriction A_{i+1}$ are contradictory types.

Define G_η, p_η, B_η and F_η by induction on $\eta \in {}^{\lambda\geq}2$ such that:

(1) $p_\eta \in S_D(B_\eta)$.
(2) G_η is an elementary mapping with $\mathrm{dom}(G_\eta) = A_{\ell(\eta)}$ and $\mathrm{ran}(G_\eta) = B_\eta$.
(3) If $\nu \prec \eta$ then $G_\nu \subseteq G_\eta$, $p_\nu \subseteq p_\eta$, $B_\nu \subseteq B_\eta$ and $F_\nu \subseteq F_\eta$, and if $\ell(\eta)$ is a limit ordinal, we set $G_\eta = \bigcup_{i<\ell(\eta)} G_{\eta\restriction i}$, $p_\eta = \bigcup_{i<\ell(\eta)} p_{\eta\restriction i}$, and $B_\eta = \bigcup_{i<\ell(\eta)} B_{\eta\restriction i}$.
(4) $p_\eta = G_\eta(p \restriction A_{\ell(\eta)})$, and the types $p_{\eta\hat{}0}$ and $p_{\eta\hat{}1}$ are explicitly contradictory.
(5) F_η is an elementary mapping extending $G_{\eta\hat{}0} \circ f_{\ell(\eta)} \circ G_{\eta\hat{}1}$ with $\mathrm{dom}(F_\eta) = B_{\eta\hat{}0}$, such that $F_\eta \restriction B_\eta = id_{B_\eta}$ and $F_\eta(p_{\eta\hat{}0}) = p_{\eta\hat{}1}$.

This is enough. The tree of types $\{p_\eta \mid \eta \in {}^{\lambda\geq}2\}$ shows that D satisfies $(B * \lambda)$.

The construction is by induction on $\ell(\eta)$: For $\eta = \langle\rangle$, let $B_{\langle\rangle} = A_0$, $G_{\langle\rangle} = id_{A_0}$ and $p_{\langle\rangle} = p \restriction A_0$. If $\ell(\eta)$ is a limit ordinal use (3). Now assume that G_η, p_η, B_η are constructed for $\ell(\eta) = i$. Let $G_{\eta\hat{}0}$ be an extension of G_η with domain A_{i+1}. Define $B_{\eta\hat{}0} = \mathrm{ran}(G_{\eta\hat{}0})$ and $p_{\eta\hat{}0} = G_{\eta\hat{}0}(p \restriction A_{i+1})$. Now $G_{\eta\hat{}0} \circ f_{\ell(\eta)} \circ G_{\eta\hat{}1}$ is an elementary mapping with domain $\subseteq B_{\eta\hat{}0}$ which is the identity on B_η. Let F_η be an elementary mapping extending it with domain $B_{\eta\hat{}0}$. Set $B_{\eta\hat{}1} = \mathrm{ran}\, F_\eta$ and $p_{\eta\hat{}0} = F_\eta(p_{\eta\hat{}1})$. $\square$

The following theorem shows that the combinatorial properties $(*\lambda)$ and $(B * \lambda)$ contradict stability in λ.

THEOREM I.3.5. *If D satisfies $(*\lambda)$ or $(B * \lambda)$ then for every $\mu < 2^\lambda$, D is not stable in μ.*

PROOF. By the previous lemma, it is enough to show that if D satisfies $(B * \lambda)$ then for every $\mu < 2^\lambda$, D is not stable in μ.

Let $\mu < 2^\lambda$. Let $\kappa = \min\{\kappa \mid 2^\kappa > \mu\}$. Then $\lambda \geq \kappa$ so D satisfies $(B * \kappa)$.

By definition, there exists $p_\eta \in S_D(B_\eta)$ and $\phi_\eta(\bar{x}, \bar{a}_\eta)$ for $\eta \in {}^{\kappa>}2$, such that $p_\eta \subseteq p_\nu$ if $\eta \prec \nu$ and $\phi_\eta(\bar{x}, \bar{a}_\eta) \in p_{\eta\hat{}0}$ and $\neg\phi_\eta(\bar{x}, \bar{a}_\eta) \in p_{\eta\hat{}1}$. By Remark .3.3, we may assume that $|B_\eta| < |\ell(\eta)|^+ + \aleph_0$.

Let $B = \bigcup_{\eta\in {}^{\kappa>}2} B_\eta$. Then $|B| \leq \sum_{\eta\in {}^{\kappa>}2} |B_\eta| \leq \kappa \cdot 2^{<\kappa} \leq \mu$, by choice of κ and assumption on $|A_i|$. Now for each $\eta \in {}^{\kappa}2$, let a_η realize p_η. Define $q_\eta = \mathrm{tp}(a_\eta/B)$. Then for $\nu \neq \eta \in {}^{\kappa}2$, there is a first $i < \kappa$ such that $\eta[i] \neq \nu[i]$, say $\eta[i] = 0$ and $\nu[i] = 1$. Hence $p_{\eta\hat{}0} \subseteq q_\eta$ and $p_{\eta\hat{}1} \subseteq q_\nu$, so q_η and

q_ν are contradictory types. Therefore $|S_D(B)| \geq |\{q_\eta \mid \eta \in {}^\kappa 2\}| = 2^\kappa > \mu$, so D is not stable in μ. □

The next theorem is a sort of converse.

THEOREM I.3.6. *If there is a D-set A such that*

$$|S_D(A)| > |A|^{<\lambda} + \sum_{\mu<\lambda} 2^{|D|^\mu}$$

then D satisfies $(*\lambda)$.

PROOF. Let $\mu_0 = |A|^{<\lambda} + \sum_{\mu<\lambda} 2^{|D|^\mu}$. By Remark .3.2 it is enough to find a type $p \in S_D(A)$ which splits over every subset $B \subseteq A$ of cardinality less than λ.

Such a type p always exists: Otherwise for every $p \in S_D(A)$, there exists $B_p \subseteq A$ of cardinality less than λ such that p does not split over B_p. Since $|S_D(A)| > \mu_0 \geq |A|^{<\lambda}$, by the pigeonhole principle, we can find $S \subseteq S_D(A)$ of cardinality μ_0^+ and B such that p does not split over B, for each $p \in S$. But, by Corollary .2.8,

$$|\{p \in S_D(A) :\ p \text{ does not split over } B\}| \leq 2^{|D|^{|B|}} \leq \sum_{\mu<\lambda} 2^{|D|^\mu} \leq \mu_0,$$

a contradiction. □

This gives another characterization of instability. This characterization will be used in the Homogeneity Spectrum Theorem (Theorem .4.9). Notice that $(B * \lambda)$ can be used in lieu of $(*\lambda)$ in the following corollary.

COROLLARY I.3.7. *D is not stable if and only if D satisfies $(*\lambda)$, for every cardinal λ.*

PROOF. If D satisfies $(*\lambda)$ for every λ, then Theorem .3.5 implies that D is not stable in λ for every λ. Hence D is not stable.

For the converse, suppose that D is not stable and let λ be given. Then D is not stable in 2^{2^λ}. Hence, there exists a D-set A of cardinality 2^{2^λ} such that $|S_D(A)| > 2^{2^\lambda} = |A|^{<\lambda} + \sum_{\mu<\lambda} 2^{|D|^\mu}$. Therefore D satisfies $(*\lambda)$ by the previous theorem. □

For the second part, we will focus on strong splitting.

DEFINITION I.3.8. A type $p \in S^n(A)$ *splits strongly* over $B \subseteq A$ if there exists $\{\bar{a}_i : i < \omega\}$ an indiscernible sequence over B and $\phi(\bar{x}, \bar{y})$ such that $\phi(\bar{x}, \bar{a}_0)$ and $\neg\phi(\bar{x}, \bar{a}_1) \in p$.

A combinatorial property similar to $(*\lambda)$ is now defined in terms of strong splitting.

DEFINITION I.3.9. D satisfies $(C * \lambda)$ if there exists an increasing continuous chain of sets $\{A_i \mid i \leq \lambda\}$ and $p \in S^n_D(A_\lambda)$ such that

$$p \restriction A_{i+1} \text{ splits strongly over } A_i, \quad \text{for each } i < \lambda.$$

Clearly if D satisfies $(C * \lambda)$, then it satisfies $(*\lambda)$ and similarly to Remark .3.3, we may assume that $|A_i| < |i|^+ + \aleph_0$ in the definition of $(C * \lambda)$.

The next cardinal invariant plays the role of $\kappa(T)$ for the notion of strong splitting. It appears in the Stability Spectrum theorem.

DEFINITION I.3.10. Let

$$\kappa(D) = \min\{\kappa : \text{For all } p \in S_D(A) \text{ there is } B \subseteq A, |B| < \kappa \text{ such that } p \text{ does not split strongly over } B\ \}.$$

If it is undefined, we let $\kappa(D) = \infty$.

THEOREM I.3.11. *Let D be stable in λ. Then $\kappa(D)$ is well-defined and $\kappa(D) \leq \lambda$.*

PROOF. Suppose that $\kappa(D) > \lambda$. Then, by definition of $\kappa(D)$, there exists a D-set A and a type $p \in S_D(A)$ such that p splits strongly over every subset B of A of cardinality at most λ. Similarly to Remark .3.2 this implies that D satisfies $(C * \lambda)$. Hence, D satisfies $(*\lambda)$. By Theorem .3.5 D is not stable in λ, a contradiction. □

To deal with strong splitting, some understanding of indiscernibles is needed. Theorem .3.13 is one of the main results to produce indiscernible sequences in the presence of stability. Recall Lemma I.2.5 of [**Sh b**].

FACT I.3.12. *Let B and let $\{\bar{a}_i \mid i < \alpha\}$ be given. Consider $q_i = \operatorname{tp}(\bar{a}_i / B \cup \{\bar{a}_j \mid j < i\}) \in S_D(B \cup \{\bar{a}_j \mid j < i\})$ and suppose that*

(1) *If $i < j < \alpha$ then $q_i \subseteq q_j$;*
(2) *For each $i < \alpha$ the type q_i does not split over B.*

Then $\{\bar{a}_i \mid i < \alpha\}$ is an indiscernible sequence over B.

THEOREM I.3.13. *Let D be stable in λ. Let I be a set of finite sequences and let A be a set such that $I \cup A$ is a D-set. If $|A| \leq \lambda < |I|$ then there exists a subset of I of cardinality λ^+ which is an indiscernible set over A.*

PROOF. By the pigeonhole principle, there exists a subset J of I of cardinality λ^+ and $n < \omega$ such that $\bar{a} \in J$ implies $\ell(a) = n$. Write $J = \{\bar{a}_i : i < \lambda^+\}$.

CLAIM. There are D-sets B and C, $A \subseteq B \subseteq C$, such that every type in $S_D(B)$ is realized in C, and there exists a type $p \in S^n_D(C)$ such that for every D-set C_1 containing C of cardinality λ, there exists an extension $p_1 \in S^n_D(C_1)$ of p such that p_1 does not split over B and is realized in $J \setminus C$.

PROOF OF THE CLAIM. Assume that B, C and p as in the claim cannot be found. For each $i < \lambda$ construct D-sets A_i of cardinality at most λ such that every $p \in S_D(A_{i+1})$ which is realized in $J \setminus A_{i+1}$ splits over A_i.

This is possible: Let $A_0 = \emptyset$ and $A_\delta = \bigcup_{i<\delta} A_i$ for δ a limit. Now assume A_i of cardinality at most λ is already constructed. Then $|S_D(A_i)| \leq \lambda$ by stability in λ. Hence, there exists a D-set A^i of cardinality λ, containing A_i, realizing all the types over A_i. Now for any $p \in S^n_D(A^i)$, A_i, A^i and p do not satisfy the assumptions of the claim. Therefore, there exists C_p, a D-set, $C_p \supseteq A^i$ of cardinality λ such that every extension of p in $S^n_D(C_p)$ that is realized in $J \setminus C_p$ splits over A_i. Let $A_{i+1} = \bigcup_{p\in S^n_D(A^i)} C_p$. Then A_{i+1} is a D-set of cardinality at most λ with the desired property.

Let $A_\lambda = \bigcup_{i<\lambda} A_i$. Since J has cardinality λ^+, there is $\bar{a} \in J \setminus A^n_\lambda$. Let $p = \mathrm{tp}(\bar{a}/A_\lambda)$. By construction $p \restriction A_{i+1}$ splits over A_i so D satisfies $(*\lambda)$. Hence, D is not stable in λ by Theorem .3.5, a contradiction. □

Let B, C and $p \in S^n_D(C)$ be as in the claim. Construct $\{\bar{b}_i : i < \lambda^+\} \subseteq J$ by induction on $i < \lambda^+$ as follows. If $\bar{b}_j$ is defined for $j < i$ let $C_i = C \cup \{\bar{b}_j \mid j < i\}$ and $p_i \in S^n_D(C_i)$ be an extension of p which does not split over B and is realized in $J \setminus C^n_i$. Let $\bar{b}_i$ be in $J \setminus C^n_i$ realizing p_i. Then $\{\bar{b}_i \mid i < \lambda^+\}$ is an indiscernible sequence by Fact .3.12. Since D is stable, then it does not have the order property by Corollary .2.12 and hence $\{\bar{b}_i \mid i < \lambda^+\}$ is an indiscernible set, by Remark .2.4. □

The next two theorems prepare for the Stability Spectrum Theorem.

THEOREM I.3.14. *Let D be stable in λ. Let $\mu \geq \lambda$ be such that $\mu^{<\kappa(D)} = \mu$. Then D is stable in μ.*

PROOF. Suppose that D is not stable in μ. Let A be a D-set of cardinality μ such that $|S_D(A)| > |A|$. By assumption, $|S_D(A)| > |A|^{<\kappa(D)}$. Hence $|S_D(A)| \geq \lambda^{++}$. Since D is stable in λ, then that $\kappa(D) \leq \lambda$ by Theorem .3.11. Hence, for each $p \in S_D(A)$ there exists a subset $B_p \subseteq A$ of cardinality less than $\kappa(D)$ such that p does not split strongly over B_p. Since there are $|A|^{<\kappa(D)} = |A|$ such B_p's, by the pigeonhole principle, there exists a set $S \subseteq S_D(A)$ of cardinality λ^{++} and a D-set $B \subseteq A$ of cardinality less than $\kappa(D)$ such that p does not split strongly over B, for each $p \in S$.

Construct $\{\phi_i(x, \bar{a}_i) \mid i < \lambda^+\}$ and $p_i \in S$, for $i < \lambda^+$ such that

(*) $$\{\phi_j(x, \bar{a}_j) : j < i\} \cup \{\neg\phi_i(x, \bar{a}_i)\} \subseteq p_i.$$

To do this, define $S_i \subseteq S$ and $A_i \subseteq A$ for $i < \lambda^+$ such that

(1) $A_0 = \emptyset$, $A_\delta = \bigcup_{i<\delta} A_i$ for δ limit, and $A_i \subseteq A_{i+1}$;
(2) $|A_i| \leq \lambda$, for each $i < \lambda$;

(3) $S_i = \{p \in S \mid p$ is the unique extension of $p \restriction A_i\}$;
(4) A_{i+1} is a subset of A such that if $p \in S_D(A_i)$ has at least two contradictory extensions in S, then it has at least two extensions $q, r \in S$ such that $q \restriction A_{i+1} \neq r \restriction A_{i+1}$.

For $i = 0$ or i a limit ordinal, do (1). For the successor stage: If A_i is constructed and $q \in S_D(A_i)$ has two extensions q_1, $q_2 \in S$, then there is $\phi_q(x, \bar{y})$ and $\bar{a}_q \in A$ such that $\phi_q(x, \bar{a}_q) \in q_1$ and $\neg\phi_q(x, \bar{a}_q) \in q_2$. Since $|S_D(A_i)| \leq \lambda$, A_{i+1} of cardinality λ as in (4) can be found.

Notice that since $|S| = \lambda^{++}$ and $|\bigcup_{i<\lambda^+} S_i| \leq \sum_{i<\lambda^+} |S_D(A_i)| \leq \lambda^+ \cdot \lambda = \lambda^+$, there exists $p \in S \setminus \bigcup_{i<\lambda^+} S_i$. For each $i < \lambda^+$ consider $p \restriction A_i$. Since $p \notin S_i$, by definition of S_i the type $p \restriction A_i$ has at least two contradictory $q, r \in S$. By (4), we may assume that $q \restriction A_{i+1} \neq r \restriction A_{i+1}$. Hence, either $p \restriction A_{i+1} \neq q \restriction A_{i+1}$, or $p \restriction A_{i+1} \neq r \restriction A_{i+1}$. Thus, in either case, there is $p_i \in S$ such that $p \restriction A_{i+1} \neq p_i \restriction A_{i+1}$. Hence, there exist $\bar{a}_i \in A_{i+1}$ and $\phi_i(x, \bar{y}) \in L(T)$ such that $\phi_i(x, \bar{a}_i) \in p$ and $\neg\phi_i(x, \bar{a}_i) \in p_i$. This establishes (*)

Now for each $i < \lambda^+$, let b_i realize p_i. The set $\{b_i\hat{\ }\bar{a}_i : i < \lambda^+\}$ has cardinality λ^+ and B has cardinality less than $\kappa(D) \leq \lambda$, so by Theorem .3.13 there is a subset of $\{b_i\hat{\ }\bar{a}_i \mid i < \lambda^+\}$ of cardinality λ^+ which is indiscernible over B. Without loss of generality, we may assume that $\{b_i\hat{\ }\bar{a}_i \mid i < \lambda^+\}$ is indiscernible over B. By stability in λ we have $|S_D(\bigcup_{k<\lambda} \bar{a}_k)| \leq \lambda$. Hence, by the pigeonhole principle, there exist i and j with $\lambda < j < i < \lambda^+$ such that $p_i \restriction \bigcup_{k<\lambda} \bar{a}_k = p_j \restriction \bigcup_{k<\lambda} \bar{a}_k$. By choice of j, we have $\phi_j(x, \bar{a}_j) \in p_i$ and $\neg\phi_j(x, \bar{a}_j) \in p_j$. Now if $\phi_j(x, \bar{a}_0) \in p_i$ then since $\neg\phi_j(x, \bar{a}_j) \in p_j$, p_j splits strongly over B, since $\{\bar{a}_0, \bar{a}_j, \bar{a}_{j+1}, ...\}$ is indiscernible over B. And if $\phi_j(x, \bar{a}_0) \notin p_i$ then $\neg\phi_j(x, \bar{a}_0) \in p_i$, and since $\phi_j(x, \bar{a}_j) \in p_i$ then p_i splits strongly over B, since $\{\bar{a}_j, \bar{a}_0, \bar{a}_1, ...\}$ is indiscernible over B. This contradicts the choice of S and B. □

THEOREM I.3.15. *Let D be stable in λ. Let $\mu \geq \lambda$ be such that $\mu^{<\kappa(D)} > \mu$. Then D is not stable in μ.*

To prove this theorem, a proposition is needed.

PROPOSITION I.3.16. *Let D be stable in λ. Let $\chi \leq \lambda$ be a cardinal such that $\lambda^\chi > \lambda$. Let I be an indiscernible sequence. Then, for every $\bar{c} \in \mathfrak{C}$ and $\phi(\bar{x}, \bar{y}) \in L(T)$ either*

$$|\{\bar{a} \in I \ : \models \phi[\bar{a}, \bar{c}]\}| < \chi \quad \text{or} \quad |\{\bar{a} \in I \ : \models \neg\phi[\bar{a}, \bar{c}]\}| < \chi.$$

PROOF. Let I and $\phi(x, \bar{c})$ contradict the conclusion of the proposition. Then, without loss of generality $|I| = \chi$. Write $I = \{\bar{a}_i \mid i < \chi\}$. Since I is indiscernible, there exists $J = \{\bar{a}_i \mid i < \lambda\}$ containing I, indiscernible of cardinality λ. By the pigeonhole principle, either $\{i < \lambda \ : \models \phi[\bar{a}_i, \bar{c}]\}$ or $\{i < \lambda \ : \models \neg\phi[\bar{a}_i, \bar{c}]\}$ has cardinality λ. Without loss of generality, assume that it is the second. Hence, by a re-enumeration (recall that J is necessarily an indiscernible set), define $J_1 = \{\bar{a}_i : i < \chi + \lambda\}$ such that $\models \phi[\bar{a}_i, \bar{c}]$ if and only if $i < \chi$. Let $q = \mathrm{tp}(\bar{c}/J_1)$. Then for any $E \subseteq J_1$ of cardinality χ with complement of cardinality λ we can find a function $f_E \colon J_1 \to J_1$ with $f(\bar{a}_i) \in E$ if and only if $i < \chi$. Then, for two such sets $E_1 \neq E_2$, we have $f_{E_1}(q) \neq f_{E_2}(q)$. Hence $|S_D(J_1)| \geq \lambda^\chi > \lambda$, contradicting the stability in λ. $\square$

PROOF OF THE THEOREM. By assumption, there exists $\kappa < \kappa(D)$ such that $\kappa = \min\{\kappa \mid \mu^\kappa > \mu\}$. Let $\chi \leq \lambda$ such that $\chi = \min\{\chi \mid \lambda^\chi > \lambda\}$. Observe that $\mu^\kappa > \chi^\kappa$: Otherwise, $\lambda \leq \mu < \mu^\kappa \leq \chi^\kappa \leq \lambda^\kappa$, and so $\chi \leq \kappa$ by minimality of χ. Hence $\lambda < \mu^\kappa \leq \chi^\kappa = 2^\kappa$. But $(C * \kappa)$ holds and $\lambda \leq 2^\kappa$, so D is not stable in λ by Theorem .3.5, a contradiction.

Now, by definition of $(C * \kappa)$, there exists an increasing, continuous chain of D-sets $\{A_i \mid i \leq \kappa\}$ and a type $p \in S_D(A_\kappa)$ such that $|A_i| \leq |i| + \aleph_0$ and

$$p \upharpoonright A_{i+1} \text{ splits strongly over } A_i, \quad \text{for each } i < \kappa.$$

By definition of strong splitting, for each $i < \kappa$, there exist $\{\bar{a}^i_\alpha \mid \alpha < \omega\}$ indiscernible over A_i and $\phi_i(x, \bar{y}) \in L(T)$ such that both $\phi_i(x, \bar{a}^i_0)$, and $\neg\phi_i(x, \bar{a}^i_1)$ belong to $p \upharpoonright A_{i+1}$.

For each $\eta \in {}^{\kappa>}\mu$, construct a type p_η, a D-set B_η and an elementary mapping G_η, by induction on $\ell(\eta)$ such that:

(1) $p_\eta \in S_D(B_\eta)$ and if $\eta \prec \nu$ then $p_\eta \subseteq p_\nu$ and $B_\eta \subseteq B_\nu$;
(2) G_η is an elementary mapping from $A_{\ell(\eta)}$ onto B_η;
(3) $|B_\eta| \leq \kappa$;
(4) For each $c \in \mathfrak{C}$ the set $\{\alpha < \mu \mid c \text{ realizes } p_{\eta\hat{}\alpha}\}$ has cardinality less than χ.

Let $B_{\langle\rangle} = A_0$, $G_{\langle\rangle} = id_{A_0}$ and $p_{\langle\rangle} = p \upharpoonright A_0$. For η such that $\ell(\eta)$ is a limit ordinal, define everything by continuity. For the successor case, suppose that p_η, B_η and G_η have been constructed for η, with $\ell(\eta) = i$. Let F be an elementary mapping extending G_η with domain A_κ. Let $\bar{b}^i_\alpha = F(\bar{a}^i_\alpha)$, for $\alpha < \omega$. Then $\{\bar{b}^i_\alpha \mid \alpha < \omega\}$ is indiscernible over B_η. Hence, we can extend this set to $\{\bar{b}^i_\alpha \mid \alpha < \mu\}$ such that $\{\bar{b}^i_\alpha \mid \alpha < \mu\}$ is also indiscernible over B_η. For $\alpha < \mu$, let $G_{\eta\hat{}\alpha}$ be an elementary mapping extending G_η, with domain A_{i+1} such that $G_{\eta\hat{}\alpha}(\bar{a}^i_0) = \bar{b}^i_\alpha$ and $G_{\eta\hat{}\alpha}(\bar{a}^i_1) = \bar{b}^i_{\alpha+1}$. This is possible by indiscernibility. Let $p_{\eta\hat{}\alpha} = G_{\eta\hat{}\alpha}(p \upharpoonright A_{i+1})$ and $B_{\eta\hat{}\alpha} = \operatorname{ran} G_{\eta\hat{}\alpha}$. Hence (1)–(3) are satisfied. To see (4), observe that for each $\alpha < \mu$, both $\phi_i(x, \bar{b}^i_\alpha)$ and $\neg\phi_i(x, \bar{b}^i_{\alpha+1})$ belong to $p_{\eta\hat{}\alpha}$. Since $\{b^i_\alpha \mid \alpha < \mu\}$ is indiscernible and $\chi \leq \lambda < \lambda^\chi$, (4) follows from the previous proposition.

The construction implies the conclusion. Let $B = \bigcup_{\eta \in {}^{\kappa>}\mu} B_\eta$. Then $|B| \leq \mu^{<\kappa} \cdot \kappa = \mu$, by choice of κ. For each $\eta \in {}^{\kappa}\mu$, let $p_\eta = \bigcup_{i<\kappa} p_{\eta \upharpoonright i}$. By continuity, each p_η is a D-type and let a_η realize p_η. Then $\mathrm{tp}(a_\eta/B) \in S_D(B)$.

By (4), for each $c \in \mathfrak{C}$, the set $\{\eta \in {}^\kappa\mu \mid a_\eta = c\}$ has cardinality at most χ^κ and we observed that $\chi^\kappa < \mu^\kappa$. Hence, $|S_D(B)| > \mu$, so D is not stable in μ. □

We finish this section with the Stability Spectrum Theorem.

THEOREM I.3.17 (The Stability Spectrum). *Let D be a finite diagram. Either D is not stable, or D is stable and there exist cardinals $\kappa \leq \lambda < \beth_{(2^{|T|})^+}$ such that for every cardinal μ, D is stable in μ if and only if $\mu \geq \lambda$ and $\mu^{<\kappa} = \mu$.*

PROOF. If D is not stable, there is nothing to prove. If D is stable, let $\lambda(D)$ be the first cardinal λ for which D is stable λ. Then $\lambda(D) < \beth_{(2^{|T|})^+}$ by Corollary .2.11. Moreover, $\kappa(D)$ is defined and $\kappa(D) \leq \lambda(D)$ by Theorem .3.11.

Let μ be given. If $\mu < \lambda(D)$, then D is not stable in μ by choice of $\lambda(D)$. Suppose that $\mu \geq \lambda(D)$. If $\mu^{<\kappa(D)} = \mu$, then D is stable in μ by Theorem .3.14. If $\mu^{<\kappa(D)} > \mu$, then D is not stable in μ by Theorem .3.15. □

I.4. The homogeneity spectrum

The section is devoted to the proof of the Homogeneity Spectrum Theorem (Theorem .4.9). The proof will proceed by cases, and is broken into several theorems. There are two types of results. On the one hand there are theorems showing the existence of a (D, λ)-homogeneous model of cardinality λ from assumptions like stability in λ and $\lambda^{<\lambda}$. On the other hand, there are results showing that such models do not exist from the failure of these conditions. The combinatorial properties defined in the previous section and parts of the Stability Spectrum Theorem will play a crucial role.

THEOREM I.4.1. *Let $\lambda \geq |D|$ be such that $\lambda^{<\lambda} = \lambda$. Then there is a (D, λ)-homogeneous model of cardinality λ.*

PROOF. First, by Zermelo-König, λ is regular. By the downward Löwenheim-Skolem theorem, define an increasing continuous chain $\langle M_i \mid i < \lambda \rangle$ of D-models of cardinality λ, such that M_{i+1} realizes every D-type over every $A \subseteq M$ of cardinality less than λ. This is possible since we have only $\lambda^{<\lambda} = \lambda$ subsets of A of cardinality less than λ and only $|D|^{|A|} \leq \lambda^{<\lambda} = \lambda$ D-types over A. Let $M = \bigcup_{i<\lambda} M_i$. Then M has cardinality λ, and since λ is regular, M is (D, λ)-homogeneous. □

THEOREM I.4.2. *Let $\lambda \geq |D|$ be such that $\lambda^{<\lambda} > \lambda$. If D satisfies $(B * \lambda)$ then there is no (D, λ)-homogeneous model of cardinality λ.*

PROOF. Suppose $\lambda^{<\lambda} > \lambda \geq |D|$. Assume, by way of contradiction, that there is a (D, λ)-homogeneous model M of cardinality λ. Since D satisfies $(B * \lambda)$ there exist D-types $p_\eta \in S_D(B_\eta)$ and $\phi_\eta(\bar{x}, \bar{a}_\eta)$ for $\eta \in {}^{\lambda >}2$ such that $\phi_\eta(\bar{x}, \bar{a}_\eta) \in p_{\eta\hat{}0}$

and $\neg\phi_\eta(\bar{x}, \bar{a}_\eta) \in p_{\eta\hat{}1}$. In addition $p_\eta \subseteq p_\nu$ when $\eta \prec \nu$. By Remark .3.3, we may assume that $|B_\eta| < |\ell(\eta)|^+ + \aleph_0$. Hence, by (D,λ)-homogeneity of M, we may assume that $B_\eta \subseteq M$ for each $\eta \in {}^{\lambda>}2$.

For each $\mu < \lambda$ and $\eta \in {}^{\mu}2$, there are 2^μ types in $S_D(B_\eta)$. Each such type is realized in M, since M is (D,λ)-homogeneous and so $2^\mu \leq \lambda$, since M has cardinality λ. Hence, λ is singular, since otherwise $\lambda^{<\lambda} = \lambda$. Furthermore, λ is a strong limit (if there is $\mu < \lambda$ such that $2^\mu = \lambda$, then $\lambda^{\mathrm{cf}(\lambda)} = 2^{\mu \cdot \mathrm{cf}(\lambda)} \leq \lambda$, contradicting Zermelo-König).

Let $\kappa = \mathrm{cf}(\lambda)$ and let $\lambda_i < \lambda$ for $i < \kappa$ be increasing and continuous such that $\lambda = \sum_{i<\kappa} \lambda_i$. Let $A_i \subseteq M$ of cardinality λ_i for $i < \kappa$ such that $M = \bigcup_{i<\kappa} A_i$.

For each $i < \kappa$, define a sequence $\eta_i \in {}^{\lambda>}2$ and a finite set C_{i+1} such that

(1) If $i < j$ then $\eta_i \prec \eta_j$;
(2) C_{i+1} is a finite subset of $B_{\eta_{i+1}}$;
(3) The type $p_{\eta_{i+1}} \restriction C_i$ is not realized in A_i.

This is enough: Let $p = \bigcup_{i<\kappa} p_{\eta_i}$. Then $p \restriction \bigcup_{i<\kappa} C_i$ is a D-type (by continuity) over a set of cardinality κ, which is not realized in M. This contradicts the (D,λ)-homogeneity of M since $\kappa < \lambda$.

This construction is possible. Define $\eta_0 = \langle\rangle$, and for $\delta < \kappa$ a limit ordinal let $\eta_\delta = \bigcup_{i<\delta} \eta_i$. For the successor case, assume that $\eta_i \in {}^{\lambda>}2$ is constructed. Define $\tau_\alpha = \eta_i\hat{}0_\alpha$, where 0_α is a sequence of zeroes of order type α, for $\alpha < 2^{\lambda_i}$. Then $\tau_\alpha \in {}^{\lambda>}2$, since $\lambda_i < \lambda$ and λ is a strong limit.

We claim that there are $\alpha < \beta < (2^{\lambda_i})^+$ such that $\models \phi_{\tau_\alpha}[c, \bar{a}_{\tau_\alpha}] \leftrightarrow \phi_{\tau_\beta}[c, \bar{a}_{\tau_\beta}]$, for every $c \in A_i$.

Suppose that this is not the case. Let $A_i = \{c_\gamma \mid \gamma < \lambda_i\}$. Then, for every $\alpha < \beta < (2^{\lambda_i})^+$ there exists $\gamma < \lambda_i$ such that $\models \neg(\phi_{\tau_\alpha}[c_\gamma, \bar{a}_{\tau_\alpha}] \leftrightarrow \phi_{\tau_\beta}[c, \bar{a}_{\tau_\beta}])$. By the Erdős-Rado theorem, there is $\gamma < \lambda_i$ and an infinite set $S \subseteq (2^{\lambda_i})^+$ such that for every $\alpha < \beta$ in S we have $\models \neg(\phi_{\tau_\alpha}[c_\gamma, \bar{a}_{\tau_\alpha}] \leftrightarrow \phi_{\tau_\beta}[c, \bar{a}_{\tau_\beta}])$. This is an immediate contradiction.

Hence, let $\alpha < \beta$ be as in (*). Let $C_{i+1} = \bar{a}_{\tau_\alpha} \cup \bar{a}_{\tau_\beta}$ and let $\eta_{i+1} = \tau_\alpha\hat{}1$. Since $\phi_{\tau_\alpha}(x, \bar{a}_{\tau_\alpha})$ and $\neg\phi_{\tau_\beta}(x, \bar{a}_{\tau_\beta})$ are in $p_{\eta_{i+1}} \restriction C_i$, the type $p_{\eta_{i+1}}$ is omitted in A_i. This finishes the construction and proves the theorem. ☐

The next theorem is, in particular, an improvement of Proposition .3.16. It allows us to define *averages* (Definition .4.4). Averages are used in Theorem .4.6.

THEOREM I.4.3. *Let D be stable. Let I be an infinite indiscernible set over A of cardinality at least $\kappa(D)$. Let $\bar{b} \in \mathfrak{C}$. Then there is $J \subseteq I$ with $|J| < \kappa(D)$ such that $I \setminus J$ is indiscernible over $A \cup J \cup \bar{b}$.*

PROOF. Let $I = \{\bar{c}_i \mid i < \alpha\}$. Since D is stable, $\kappa(D)$ is defined by Theorem .3.11. Hence, there exists $B \subseteq A \cup I$ of cardinality less than $\kappa(D)$ such that the type $\mathrm{tp}(\bar{b}/A \cup I)$ does not split strongly over B. Let $J = B \setminus A$. Then $J \subseteq I$ has cardinality less than $\kappa(D)$. We will show that $I \setminus J$ is indiscernible over $A \cup J \cup \bar{b}$. Clearly, $I \setminus J$ is indiscernible over $A \cup J$. If $I \setminus J$ is not indiscernible over $A \cup J \cup \bar{b}$, then, there exist an integer $n < \omega$ and indices $i_0 \cdots < i_n$ such that $\mathrm{tp}(\bar{c}_0, \dots, \bar{c}_n/A \cup J \cup \bar{b}) \neq \mathrm{tp}(\bar{c}_{i_0}, \dots, \bar{c}_{i_n}/A \cup J \cup \bar{b})$. Then $\models \phi[\bar{c}_0, \dots, \bar{c}_n, \bar{a}, \bar{b}, \bar{c}]$ and $\models \neg\phi[\bar{c}_{i_0}, \dots, \bar{c}_{i_n}, \bar{a}, \bar{b}, \bar{c}]$, for some formula $\phi \in L(T)$, parameters $\bar{a} \in A$ and $\bar{c} \in J$. Let $\bar{d}_0 = \bar{c}_0\hat{\ }\dots\hat{\ }\bar{c}_n$ and $\bar{d}_1 = \bar{c}_{i_0}\hat{\ }\dots\hat{\ }\bar{c}_{i_n}$. By taking sequences from $I \setminus J$, it is easy to find $\{\bar{d}_i \mid i < \omega\}$ indiscernible over $A \cup J$. Thus $\{\bar{d}_i\hat{\ }\bar{a}\hat{\ }\bar{c} \mid i < \omega\}$ is indiscernible over $A \cup J$. Hence, the type $\mathrm{tp}(\bar{b}/A \cup I)$ splits strongly over $A \cup J$, a contradiction to the choice of B. $\square$

DEFINITION I.4.4. Let I be an indiscernible sequence of cardinality at least $\kappa(D)$. Let A be such that $A \cup I$ is a D-set. Define the *average of I over A*, by

$$\mathrm{Av}(I, A) = \{\phi(\bar{x}, \bar{a}) \mid \phi(\bar{x}, \bar{y}) \in L(T), \bar{a} \in A, \text{ and } \models \phi[\bar{b}, \bar{a}], \text{ for at least } \kappa(D) \text{ elements } \bar{b} \in I\}.$$

THEOREM I.4.5. *Let D be stable. Let I be an indiscernible sequence of cardinality at least $\kappa(D)$ and A be such that $A \cup I$ is a D-set. Then $\mathrm{Av}(I, A) \in S^n_D(A)$, where $n = \ell(\bar{a})$ for $\bar{a} \in I$. In addition, if $|I| > |A|$, then $\mathrm{Av}(I, A)$ is realized in I.*

PROOF. Averages are complete: Assume $\phi(\bar{x}, \bar{c}) \notin \mathrm{Av}(I, A)$, with $\bar{c} \in A$. Then by definition, the set $J \subseteq I$ of elements realizing $\phi(\bar{x}, \bar{c})$ has cardinality less than $\kappa(D)$. Thus, since $I \setminus J$ has cardinality at least $\kappa(D)$, and all elements in $I \setminus J$ realize $\neg\phi(\bar{x}, \bar{c})$, necessarily $\neg\phi(\bar{x}, \bar{c}) \in \mathrm{Av}(I, A)$. Averages are consistent: Let $\phi_1(x, \bar{c}_1), \dots, \phi_n(x, \bar{c}_n) \in \mathrm{Av}(I, A)$. Then, if $\bar{c} = \bar{c}_1\hat{\ }\dots\hat{\ }\bar{c}_n$, by Theorem .4.3, there is $J_{\bar{c}}$, $J_{\bar{c}} \subseteq I$ of cardinality less than $\kappa(D)$ such that $I \setminus J_{\bar{c}}$ is indiscernible over $\bar{c}$. Hence, since each $\phi_i(x, \bar{c}_i)$ was realized by at least $\kappa(D)$ elements of I, we can find one in $I \setminus J_{\bar{c}}$. But then, all elements in $I \setminus J_{\bar{c}}$ realize $\phi_i(x, \bar{c}_i)$ by indiscernibility ($1 \leq i \leq n$), so $\{\phi_1(x, \bar{c}_1), \dots, \phi_n(x, \bar{c}_n)\}$ is consistent. The last sentence follows similarly: For any $\bar{c} \in A$, every element of $I \setminus J_{\bar{c}}$ realizes $\mathrm{Av}(I, A) \upharpoonright \bar{c}$, since they realize every formula in it, and so if $|I| > |A|$, we can find $\bar{b} \in I \setminus \bigcup_{\bar{c} \in A} J_{\bar{c}}$ realizing $\mathrm{Av}(I, A)$. It remains to show that $\mathrm{Av}(I, A)$ is a D-type: Notice that if we stretch I to J, $I \subseteq J$ indiscernibles of cardinality greater than $|A|$, we have $\mathrm{Av}(I, A) = \mathrm{Av}(J, A)$. Then $\mathrm{Av}(I, A)$ is realized in J, thus in $\mathfrak{C}$, since J is a D-set, and so $\mathrm{Av}(I, A)$ is a D-type. $\square$

THEOREM I.4.6. *Let $\lambda \geq |D|$. If D is stable in λ, then there exists a (D, λ)-homogeneous model of cardinality λ.*

PROOF. Suppose first that λ is regular. Define an increasing continuous chain $\langle M_i \mid i < \lambda \rangle$ of models of cardinality λ, such that M_0 realizes all the types in D, and M_{i+1} realizes all the types over M_i. Such a construction is possible since D is stable in λ and $\lambda \geq |D|$. Let $M = \bigcup_{i<\lambda} M_i$. Then, M has cardinality λ and M is (D, λ)-homogeneous by regularity of λ.

Now suppose that λ is singular. Construct an increasing continuous chain of models $\langle M_i \mid i < \lambda \cdot \lambda \rangle$ as above of length $\lambda \cdot \lambda$. Let $M = \bigcup_{i<\lambda\cdot\lambda} M_i$. Notice that M has cardinality λ. We now show that it is (D, λ)-homogeneous. Let $A \subseteq M$ of cardinality less than λ and $p_0 \in S_D(A)$. We will find I indiscernibles of cardinality greater than $|A|$ with $p_0 = \mathrm{Av}(I, A)$. Let $p \in S_D(M)$ extending p_0 and choose $C \subseteq M$ of cardinality less than $\kappa(D)$ such that p does not split strongly over C. Since D is stable in λ, then $\lambda^{<\kappa(D)} = \lambda$ by Theorem .3.15. Hence, $\mathrm{cf}(\lambda) \geq \kappa(D)$. Thus, considering the sequence $\langle M_{\lambda\cdot i} \mid i < \lambda \rangle$ we can find $i < \lambda$ such that $C \subseteq M_{\lambda\cdot i}$.

We claim that p does not split over $M_{\lambda\cdot i+\lambda}$. Otherwise, there are $\bar{b}$ and $\bar{c}$ in M and $\phi(\bar{x}, \bar{y})$ such that $\phi(\bar{x}, \bar{b}) \in p$, $\neg\phi(\bar{x}, \bar{c}) \in p$ and

$$\mathrm{tp}(\bar{b}/M_{\lambda\cdot i+\lambda}) = \mathrm{tp}(\bar{c}/M_{\lambda\cdot i+\lambda}).$$

Let $q := \mathrm{tp}(\bar{b}/M_{\lambda\cdot i+\lambda})$. Now, since λ is singular, we have $\omega < \lambda$. Consider the following set

$$\{j < \lambda : q \restriction M_{\lambda\cdot i+\omega\cdot(j+1)} \text{ splits over } M_{\lambda\cdot i+\omega\cdot j}\}.$$

Since D is stable in λ, in particular $(*\lambda)$ fails so we can find γ with

$$\lambda \cdot i < \gamma < \gamma + \omega < \lambda \cdot \lambda$$

such that $q \restriction M_{\gamma+\omega}$ does not split over M_γ. For each $n < \omega$, we can choose $\bar{b}_n \in M_{\gamma+n+1}$ realizing $\mathrm{tp}(\bar{b}/M_{\gamma+n})$. Now, $\mathrm{tp}(\bar{b}_n/M_{\gamma+n})$ does not split over M_γ ($\forall n < \omega$) by monotonicity. Hence $\{\bar{b}_n \mid n < \omega\}$ are indiscernible over M_γ, by Fact .3.12. Similarly, both $\{\bar{b}_0, \bar{b}_1, \dots, \bar{b}\}$ and $\{\bar{b}_0, \bar{b}_1, \dots, \bar{c}\}$ are indiscernible over M_γ. In fact, since D is stable, D does not have the order property by Corollary .2.12, and thus they are indiscernible sets by Remark .2.4. Now suppose that for some $n < \omega$, the formula $\phi(\bar{x}, \bar{b}_n) \in p$. Then p splits strongly over C since

$$\{\bar{b}_n, \bar{c}, \bar{b}_{n+1}, \dots\} \text{ is indiscernible over } C.$$

Otherwise $\neg\phi(\bar{x}, \bar{b}_0) \in p$. Then p splits strongly over C because

$$\{\bar{b}, \bar{b}_0, \bar{b}_1, \dots\} \text{ is indiscernible over } C.$$

We have a contradiction in both cases, which proves the claim.

We now use the claim to prove the conclusion of the theorem. First, we may assume that $\lambda \cdot i = 0$, so p does not split over M_0. Now for each $\alpha < \lambda \cdot \lambda$, choose $a_\alpha \in M_{\alpha+1}$ realizing $p \restriction M_\alpha$. Since p does not split over M_0 the sequence

$I := \{a_\alpha \mid \alpha < \lambda \cdot \lambda\}$ is indiscernible. Let $\phi(x, \bar{a}) \in p_0$. There is $\alpha_0 < \lambda^2$ such that $\phi(x, \bar{a}) \in p_0 \restriction M_{\alpha_0}$, so we have that $\models \phi[a_\alpha, \bar{a}]$ for every $\alpha \geq \alpha_0$. Hence there are $\lambda \geq \kappa(D)$ many elements of I realizing $\phi(x, \bar{a})$, showing that $\phi(x, \bar{a}) \in \mathrm{Av}(I, A)$. So $\mathrm{Av}(I, A) \supseteq p_0$ and since both types are complete, we have $p_0 = \mathrm{Av}(I, A)$. Thus since $|I| > |A|$, there are elements of I realizing p_0. This shows that p_0 is realized in M. Hence M is (D, λ)-homogeneous. $\square$

The next lemma is an improvement of Corollary .2.8. It is needed in the proof of Theorem .4.8.

LEMMA I.4.7. *Let D be stable. Let $A \subseteq B$ be D-sets such that every D-type over A is realized in B. Fix $n < \omega$ and define*

$$\Gamma := \{p \in S^n_D(B) \mid p \text{ does not split over } A\}.$$

Then, for each $p \in \Gamma$, there is a sequence $\langle \bar{a}^p_i \mid i \leq \omega \rangle$ indiscernibles over A such that

(*) $\quad p \neq q \in \Gamma \quad$ *implies* $\quad \mathrm{tp}(\langle \bar{a}^p_i : i < \omega \rangle / A) \neq \mathrm{tp}(\langle \bar{a}^q_i : i < \omega \rangle / A).$

Moreover,

$$|\Gamma| \leq |\bigcup_{m<\omega} S^m_D(A)|^{\aleph_0} \leq |D|^{|A| + \aleph_0}.$$

PROOF. It is enough to establish (*), since the last statement follows from (*) by a computation.

For each $p \in \Gamma$, define

$$I_p := \langle \bar{a}^p_i : i < \kappa(D) \rangle,$$

by induction on $i < \kappa(D)$ such that $\mathrm{tp}(\bar{a}^p_i / B \cup \{\bar{a}^p_j : j < i\})$ extends p and does not split over A. This is possible by Lemma .2.7. By Fact .3.12 the sequence I_p is indiscernible over A. Hence, it is enough to show that

$$\mathrm{tp}(\langle \bar{a}^p_i : i < \kappa(D) \rangle / A) \neq \mathrm{tp}(\langle \bar{a}^q_i : i < \kappa(D) \rangle / A), \qquad \text{for } p \neq q \in \Gamma.$$

We will use the following claim.

CLAIM. If $\bar{b} \in B$ and $\bar{b}_1 \in \mathfrak{C}$ such that $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{b}_1/A)$, then

$$|\{i < \kappa(D) : \mathrm{tp}(\bar{b}\hat{\ }\bar{a}^p_0/A) \neq \mathrm{tp}(\bar{b}_1\hat{\ }\bar{a}^p_i/A)\}| < \kappa(D)$$

PROOF OF THE CLAIM. To show this, define $\{\bar{a}^p_i : \kappa(D) \leq i < \kappa(D)^+\}$, by induction on i ($\kappa(D) \leq i < \kappa(D)^+$) such that $\mathrm{tp}(\bar{a}^p_i / B \cup \{\bar{a}^p_j : j < i\} \cup \bar{b}_1)$ extends p and does not split over A. Hence, by Fact .3.12, $I' = \{\bar{a}^p_i : i < \kappa(D)^+\}$ is indiscernible. By construction

$$\mathrm{tp}(\bar{b}_1\hat{\ }\bar{a}^p_i/A) = \mathrm{tp}(\bar{b}\hat{\ }\bar{a}^p_i/A) = \mathrm{tp}(\bar{b}\hat{\ }\bar{a}^p_i/A), \quad \text{for } i \geq \kappa(D),$$

since $\bar{b} \in B$ and I_p is indiscernible over B. Thus

$$|\{i \in I' : \mathrm{tp}(\bar{b}\hat{\ }\bar{a}^p_0/A) = \mathrm{tp}(\bar{b}_1\hat{\ }\bar{a}^p_i/A)\}| > \kappa(D),$$

but then, all $\bar{a}_1 \in I'$ but a subset of cardinality less than $\kappa(D)$ are indiscernibles over $\bar{b} \cup \bar{b}_1$ and so

$$|\{i \in I' : \operatorname{tp}(\bar{b}\hat{\ }\bar{a}_0^p/A) \neq \operatorname{tp}(\bar{b}_1\hat{\ }\bar{a}_i^p/A)\}| < \kappa(D).$$

The claim follows since $I_p \subset I'$. □

Suppose by way of contradiction that there are $p \neq q \in \Gamma$ with

$$\operatorname{tp}(\langle \bar{a}_i^p : i < \kappa(D)\rangle/A) = \operatorname{tp}(\langle \bar{a}_i^q : i < \kappa(D)\rangle/A).$$

Since $p \neq q$, there is $\bar{b} \in B$ and $\phi(\bar{x}, \bar{y})$ such that $\phi(\bar{x}, \bar{b}) \in p$ and $\neg\phi(\bar{x}, \bar{b}) \in q$. By construction, $\models \phi[\bar{a}_i^p, \bar{b}]$ and $\models \neg\phi[\bar{a}_i^q, \bar{b}]$, for all $i < \kappa(D)$. Let f be an elementary mapping such that $f \restriction A = id_A$ and $f(\bar{a}_i^p) = \bar{a}_i^q$ for $i < \kappa(D)$. Clearly, f exists by assumption on p and q. Call $\bar{b}_1 = f^{-1}(\bar{b})$. By applying the claim, we know that $|\{i < \kappa(D) : \operatorname{tp}(\bar{b}\hat{\ }\bar{a}_0^p/A) \neq \operatorname{tp}(\bar{b}_1\hat{\ }\bar{a}_i^p/A)\}| < \kappa(D)$, hence let $\bar{a}_i^p$, ($i < \kappa(D)$) such that $\operatorname{tp}(\bar{b}\hat{\ }\bar{a}_0^p/A) = \operatorname{tp}(\bar{b}_1\hat{\ }\bar{a}_i^p/A)$. But, by definition of f, we know that $\operatorname{tp}(\bar{b}_1\hat{\ }\bar{a}_i^p/A) = \operatorname{tp}(\bar{b}\hat{\ }\bar{a}_i^q/A)$. Hence $\operatorname{tp}(\bar{b}\hat{\ }\bar{a}_0^p/A) = \operatorname{tp}(\bar{b}\hat{\ }\bar{a}_i^q/A)$. Since $\phi(\bar{x}, \bar{b}) \in \operatorname{tp}(\bar{b}\hat{\ }\bar{a}_0^p/A)$, we then must have $\models \phi[\bar{a}_i^q, \bar{b}]$, the desired contradiction. □

We now prove the last significant ingredient of the Homogeneity Spectrum Theorem.

THEOREM I.4.8. *Let $\lambda \geq |D|$ be such that $\lambda^{<\lambda} > \lambda$. Suppose that D is stable but not in λ If D does not satisfy $(*\lambda)$ then there is no (D, λ)-homogeneous model of cardinality λ.*

PROOF. By way of contradiction, assume that M is a (D, λ)-homogeneous model of cardinality λ. Let $\{A_\alpha \mid \alpha < \operatorname{cf}(\lambda)\}$ be an increasing continuous chain of sets such that $|A_\alpha| < \lambda$ and $M = \bigcup_{\alpha < \operatorname{cf}(\lambda)} A_\alpha$.

Since D is not stable in λ, there is a D-set B of cardinality λ such that $|S_D(A)| > \lambda$. Then, by Lemma .1.3 we may assume that $B \subseteq M$ since M is (D, λ)-homogeneous. Hence $|S_D(M)| > \lambda$.

We first claim that for each $p \in S_D(M)$, there is $\alpha < \operatorname{cf}(\lambda)$ such that p does not split over A_α.

Suppose not. Let $p \in S_D(M)$ such that p splits over every A_α. If λ is regular, then $\lambda = \operatorname{cf}(\lambda)$ and this implies that D satisfies $(*\lambda)$, a contradiction. Suppose that λ is singular. For each $\alpha < \operatorname{cf}(\lambda)$, choose $\bar{b}_\alpha$, $\bar{c}_\alpha$ in M and $\phi_\alpha(x, \bar{y})$ such that $\operatorname{tp}(\bar{b}_\alpha/A_\alpha) = \operatorname{tp}(\bar{c}_\alpha/A_\alpha)$ and $\phi_\alpha(x, \bar{b}_\alpha) \in p$ and $\neg\phi_\alpha(x, \bar{c}_\alpha) \in p$. Then $p \restriction \{\bar{b}_\alpha, \bar{c}_\alpha\}$ is not realized in A_α. Set $A := \bigcup_{\alpha < \operatorname{cf}(\lambda)}\{\bar{b}_\alpha, \bar{c}_\alpha\}$. Then $p \restriction A$ is not realized in $\bigcup_{\alpha < \operatorname{cf}(\lambda)} A_\alpha = M$. This contradicts the (D, λ)-homogeneity of M since $|A| \leq \operatorname{cf}(\lambda) < \lambda$. This proves the claim.

Now since $|S_D(M)| > \lambda$, by the pigeonhole principle, there exists $\Gamma \subseteq S_D(M)$ of cardinality λ^+ and $\alpha < \operatorname{cf}(\lambda)$, such that if $p \in \Gamma$, then p does not split over A_α. Since $A_\alpha \subseteq M$ of cardinality less than λ and M is (D, λ)-homogeneous, we are in the situation of the previous lemma. Thus for each $p \in \Gamma$ there is $\{\bar{a}_i^p : i \leq \omega\}$ an indiscernible set over A_α such that

$$p \neq q \quad \text{if and only if} \quad \operatorname{tp}(\langle \bar{a}_i^p : i < \omega \rangle / A_\alpha) \neq \operatorname{tp}(\langle \bar{a}_i^q : i < \omega \rangle / A_\alpha).$$

Using the (D, λ)-homogeneity of M and the fact that $|A_\alpha| < \lambda$, construct $\{\bar{b}_i^p : i \leq \omega\} \subseteq M$ for each $p \in \Gamma$ with the following two properties:

(1) $\operatorname{tp}(\langle \bar{b}_j^p : j \leq i \rangle / A_\alpha) = \operatorname{tp}(\langle \bar{a}_j^p : j \leq i \rangle / A_\alpha)$
(2) If $\operatorname{tp}(\langle \bar{a}_j^p : j \leq i \rangle / A_\alpha) = \operatorname{tp}(\langle \bar{a}_j^q : j \leq i \rangle / A_\alpha)$, then $\bar{b}_j^p = \bar{b}_j^q$ for every $j \leq i$.

We now show that

(*) $$\bar{b}_\omega^p \neq \bar{b}_\omega^q, \qquad \text{if } p \neq q \in \Gamma.$$

Let $p, q \in \Gamma$ such that $p \neq q$. By construction, we have that

$$\operatorname{tp}(\langle \bar{a}_j^p : j < \omega \rangle / A_\alpha) \neq \operatorname{tp}(\langle \bar{a}_j^q : j < \omega \rangle / A_\alpha).$$

Hence, there is a minimal $i < \omega$ such that

$$\operatorname{tp}(\bar{a}_0^p, \ldots, \bar{a}_i^p \bar{a}_{i+1}^p / A_\alpha) \neq \operatorname{tp}(\bar{a}_0^q, \ldots, \bar{a}_i^q \bar{a}_{i+1}^q / A_\alpha).$$

By minimality of i and (1), we have

(**) $$\operatorname{tp}(\bar{b}_0^p, \ldots, \bar{b}_i^p / A_\alpha) = \operatorname{tp}(\bar{b}_0^q, \ldots, \bar{b}_i^q / A_\alpha).$$

Now, we have the following equations

$$\begin{aligned}
\operatorname{tp}(\bar{b}_0^p, \ldots, \bar{b}_i^p \bar{b}_\omega^p / A_\alpha) &= \operatorname{tp}(\bar{a}_0^p, \ldots, \bar{a}_i^p \bar{a}_\omega^p / A_\alpha) && \text{(by definition (2))} \\
&= \operatorname{tp}(\bar{a}_0^p, \ldots, \bar{a}_i^p \bar{a}_{i+1}^p / A_\alpha) && \text{(by indiscernibility)} \\
&\neq \operatorname{tp}(\bar{a}_0^q, \ldots, \bar{a}_i^q \bar{a}_{i+1}^q / A_\alpha) && \text{(by choice of } i) \\
&= \operatorname{tp}(\bar{a}_0^q, \ldots, \bar{a}_i^q \bar{a}_\omega^q / A_\alpha) && \text{(by indiscernibility)} \\
&= \operatorname{tp}(\bar{b}_0^q, \ldots, \bar{b}_i^q \bar{b}_\omega^q / A_\alpha) && \text{(by definition (2))}
\end{aligned}$$

Hence (*) follows from the previous equations and (**).

Therefore (*) implies that we have $|\Gamma|$ many different elements $\bar{b}_\omega^p \in M$. This is a contradiction, since

$$|\Gamma| = \lambda^+ > \lambda = \|M\|.$$

This finishes the proof. □

We can now present the Homogeneity Spectrum Theorem.

THEOREM I.4.9 (The Homogeneity Spectrum). *Let λ be a cardinal. There is a (D, λ)-homogeneous model of cardinality λ if and only if $\lambda \geq |D|$ and either D is stable in λ or $\lambda^{<\lambda} = \lambda$.*

PROOF. The proof is divided into 5 cases.

Case 1: $\lambda < |D|$. Then, there can be no (D, λ)-homogeneous model M of cardinality λ, since we require that $D(M) = D$, and there are not enough elements in M to realize all the types in D.

Case 2: $\lambda \geq |D|$ and $\lambda^{<\lambda} = \lambda$. Then, there exists a (D, λ)-homogeneous model M of cardinality λ by Theorem .4.1.

Case 3: $\lambda \geq |D|$ and D is stable in λ. Then, there is a (D, λ)-homogeneous model M of cardinality λ by Theorem .4.6.

Case 4: $\lambda \geq |D|$, $\lambda^{<\lambda} > \lambda$ and D is not stable. Then, by Corollary .3.7, D satisfies $(*\lambda)$. Hence D satisfies $(B * \lambda)$ by Lemma .3.4. Therefore, there is no (D, λ)-homogeneous model M of cardinality λ by Theorem .4.2.

Case 5: $\lambda \geq |D|$, $\lambda^{<\lambda} > \lambda$ and D is stable but not in λ. This case is divided into two sub-cases according to whether D satisfies $(*\lambda)$. If D does satisfy $(*\lambda)$, then D also satisfies $(B * \lambda)$ by Lemma .3.4. Therefore the result follows from Theorem .4.2. If D does not satisfy $(*\lambda)$, then by Theorem .4.8 we have no (D, λ)-homogeneous model of cardinality λ.

The proof is complete. $\square$

CHAPTER II

The local order property in nonelementary classes

In the first order case, Victor Harnik and Leo Harrington in [**HaHa**], while presenting an alternative approach of forking to that of Saharon Shelah [**Sh b**], started a localized generalization of stability theory extending Saharon Shelah's Unstable Formula Theorem (Theorem II 2.2 [**Sh b**]). This work was later continued and extended by Anand Pillay in [**Pi**]. About ten years later Zoe Chatzidakis and Ehud Hrushovski in their deep study of the model theory of fields with an automorphism [**ChHr**] as well as Ehud Hrushovski and Anand Pillay [**HrPi1**] discovered natural examples of this phenomenon in algebra and obtained results in local stability for first order simple theories.

In parallel, Rami Grossberg and Saharon Shelah continued their study of stability and the order property in contexts where the compactness theorem fails; inside a model and for nonelementary classes (see for example [**Gr1**], [**Gr2**], [**GrSh1**], [**GrSh3**], [**Sh16**], and [**Sh300**]).

The goal of this chapter is to continue the study of local stability both in the first order case and in cases where the compactness theorem fails. When possible, we have tried to merge first order local stability with nonelementary stability theory and obtain results improving existing theorems in two directions. Four frameworks, listed in decreasing order of generality, are examined: (1) Inside a fixed structure; (2) For a general nonelementary class of structures; (3) For the class of models of a finite diagram; (4) For the first order case. Hence, the results of (1) hold for (2), those of (2) hold in (3), and the ones of (3) hold in (4). We study local versions of stability and the order property in (1) and (2). In (3) we look at the localized versions of the saturation spectrum. In (4), we also study local versions of the independence property and the strict order property. By *local*, we mean inside the set of realizations of a fixed type.

In (1), (2), and (3), since the compactness theorem fails, we cannot use the forking machinery or definability of types, as [**HaHa**], [**Pi**] and [**Sh b**] do. Hence, the methods used have a combinatorial and set-theoretic flavor. Note that by (2) we mean the study of models of an infinitary logic, or of the class $\mathrm{PC}(T_1, T, \Gamma)$ (see the beginning of Section 3 for a definition). Hence, in addition to the failure of the compactness theorem, we have to do without the existence of saturated or even homogeneous models, as such models do not exist in general in (1) and (2). Thus, frameworks (1) and (2) are more general than (3).

The basic *structure* assumption will be the impossibility of coding, via a formula in a given logic, a linear order of a certain length inside the set of realizations of a fixed type p. Note that there are two standard definitions refered to as the order property. (For example both are given in [**Sh394**].) In the first order nonlocal case, they are equivalent when the complexity of the formula used to code the order is of no importance. We chose this version for two reasons: as a structure assumption it is weaker than the other, and as a nonstructure assumption, the existence of long orders implies the existence of many nonisomorphic models (see Theorem VIII 3.2 in [**Sh b**]), even in nonelementary cases (see for example [**Sh16**] and [**GrSh1**]).

This chapter is organized as follows:

In Section 1, we study stability and order for the realizations of a type p inside a fixed model M. In particular, the model M may omit many types. Denote by $p(M)$ the set of realizations of p in M. We prove that the impossibility of coding a linear order of a certain length inside $p(M)$ implies local stability (Theorem .1.5). By local stability, we mean the usual definitions in terms of the number of types extending the fixed type p. This is used to prove the existence of indiscernibles (Theorem .1.9), as well as averages (Theorem .1.12).

In Section 2, we study these local notions for classes of models that fail to satisfy the compactness theorem. We obtain a characterization of local stability for such a class of models in terms of the failure of the local order property, and a partial version of the stability spectrum (Theorem .2.4).

In Section 3, we study local stability for the class of models of a finite diagram. We obtain all localized versions of the results of the first chapter: the local stability spectrum (Theorem .3.12) and the local homogeneity spectrum (Theorem .3.13).

Finally, in Section 4, we particularize our discussion to the first order case. We introduce local versions of the independence property and the strict order property. We prove the local version of Shelah's Trichotomy Theorem: the local order property is equivalent to the disjunction of the local independence property and the local strict order property (Corollary .4.4). We characterize the local independence property in terms of averages (Theorem .4.6) and give, as an application, a characterization of stable types in terms of averages when the ambient first order theory is simple (Corollary .4.9).

Credits have been given throughout the text when particular cases of these results were known, either in the local first order case, or the nonlocal nonelementary case.

II.1. Local notions inside a fixed model

In this section, we work inside a fixed structure M. Denote by $L(M)$ the set of first order formulas in the language of M [1]. We will say formulas for $L(M)$-formulas.

Let p be a fixed set of formulas (maybe with parameters in M) such that p is realized in M. Denote by $p(M)$ the set of elements of M realizing p.

Recall the notion of complete type inside a model. Let $A \subseteq M$, Δ be a set of $L(M)$-formulas and $\bar{c} \in M$. We let

$$\mathrm{tp}_\Delta(\bar{c}/A, M) = \{\phi(\bar{x}, \bar{a}) \mid \bar{a} \in A, \phi(\bar{x}, \bar{y}) \in \Delta \text{ or } \neg\phi(\bar{x}, \bar{y}) \in \Delta, M \models \phi[\bar{c}, \bar{a}]\}.$$

We omit Δ when $\Delta = L(M)$.

For $A \subseteq M$ and Δ a set of formulas, we let

$$S_{\Delta,p}(A, M) = \{\mathrm{tp}_\Delta(\bar{c}/A, M) \mid \bar{c} \in M \text{ and } \bar{c} \text{ realizes } p\ \}.$$

We omit Δ when $\Delta = L(M)$.

For a type q and a set A, we denote by $q \upharpoonright A$ the set of formulas in q with parameters in A. For a set of formulas Δ, we denote by $q \upharpoonright \Delta$ the set of instances in q of formulas of Δ.

The next two definitions are the main concept of this chapter.

DEFINITION II.1.1. For an infinite cardinal $\lambda \geq |L(M)|$, the model M is said to be (λ, p)-*stable* if $|S_p(A, M)| \leq \lambda$ for each $A \subseteq p(M)$ of cardinality at most λ.

Note that in the above definition we make demands only on subsets of $p(M)$. In fact, throughout the rest of this chapter, we will only deal with types $q \in S_p(A, M)$ such that $A \subseteq p(M)$.

DEFINITION II.1.2. M has the (λ, p)-*order property* if there exists a formula $\phi(\bar{x}, \bar{y}) \in L(M)$ and a set $\{\bar{a}_i \mid i < \lambda\} \subseteq p(M)$, such that

$$M \models \phi[\bar{a}_i, \bar{a}_j] \quad \text{if and only if} \quad i < j < \lambda.$$

The first theorem (Theorem .1.5) is a local version inside a model of Shelah's Theorem that the failure of the order property implies stability for complete, first order theories. A generalization of Shelah's theorem for nonelementary classes and in the local case will appear in the next section (Theorem .1.5). Theorem .1.5 will also be used in a key way to prove existence of indiscernibles (Theorem .1.9) and averages (Theorem .1.12) in this section. The technical tool needed to prove it is *splitting*. Recall the definition.

[1] This is arbitrary, we may consider for $L(M)$ a fragment of a larger logic, or even a subset with some weak closure properties.

DEFINITION II.1.3. Let $q \in S_p(B, M)$, with $B \subseteq p(M)$. Let $\Delta_1, \Delta_2 \subseteq L(M)$. The type q is said to (Δ_1, Δ_2)*-split over* A, if there exist elements $\bar{b}, \bar{c} \in B$ and a formula $\phi(\bar{x}, \bar{y}) \in \Delta_2$ such that $\mathrm{tp}_{\Delta_1}(\bar{b}/A, M) = \mathrm{tp}_{\Delta_1}(\bar{c}/A, M)$ and both $\phi(\bar{x}, \bar{b})$ and $\neg\phi(\bar{x}, \bar{c})$ belong to q. We simply say *splits* for $(L(M), L(M))$-splits.

The next fact is a variation on Exercise I.2.3 from [**Sh b**].

PROPOSITION II.1.4. *Let* $B \subseteq C \subseteq p(M)$ *and let* $A \subseteq M$. *Suppose that* B *realizes all the types in* $S_{\Delta_2,p}(A)$ *that are realized in* C. *Let* $q, r \in S_{\Delta_1,p}(C)$ *such that* q, r *do not* (Δ_1, Δ_2)*-split over* A. *If* $q \restriction B = r \restriction B$, *then* $q = r$.

PROOF. Suppose $q \neq r$. Then there exists $\phi(\bar{x}, \bar{y}) \in \Delta_1$ and $\bar{c} \in C$ such that $\phi(\bar{x}, \bar{c}) \in q$ and $\neg\phi(\bar{x}, \bar{c}) \in r$. Consider $\mathrm{tp}_{\Delta_2}(\bar{c}/A, M)$. By assumption on B, there exists $\bar{b} \in B$ such that $\mathrm{tp}_{\Delta_2}(\bar{b}/A, M) = \mathrm{tp}_{\Delta_2}(\bar{c}/A, M)$. Since neither q, nor r (Δ_1, Δ_2)-split over A, we have $\phi(\bar{x}, \bar{b}) \in q$ and $\neg\phi(\bar{x}, \bar{b}) \in r$. This contradicts the assumption that $q \restriction B = r \restriction B$. $\square$

The following theorem localizes results from [**Sh16**] and [**Gr1**]. The proof appearing in [**Sh16**] uses generalizations of a theorem of Paul Erdős and Michael Makkai appearing in [**ErMa**]. The proof given here is simpler and closer to [**Gr1**]. See Theorem 1..2.9.

THEOREM II.1.5. *Let* μ *and* λ *be cardinals such that* $\mu \geq |L(M)|$, $\lambda^\mu = \lambda$, *and* $\lambda \geq 2^{2^\mu}$. *If* M *does not have the* (μ^+, p)*-order property, then* M *is* (λ, p)*-stable.*

PROOF. Suppose that M is not (λ, p)-stable. Then, there exists $A \subseteq p(M)$ of cardinality λ such that $|S_p(A, M)| > \lambda$.

For each $q \in S_p(A, M)$, we have $(q \restriction \phi) \in S_{\phi,p}(A, M)$. Define

$$f \colon S_p(A, M) \to \Pi_{\phi \in L} S_{\phi,p}(A, M), \quad \text{by} \quad f(q) = (q \restriction \phi)_{\phi \in L(M)}.$$

Then, f is a well-defined injection. Observe that

$$|\Pi_{\phi \in L(M)} S_{\phi,p}(A, M)| \leq \lambda^{|L(M)|} \leq \lambda^\mu < \lambda^+ \leq |S_p(A, M)|.$$

By the pigeonhole principle, we can find $\phi \in L(M)$ such that $|S_{\phi,p}(A, M)| > \lambda$.

Fix $\phi(\bar{x}, \bar{y})$ as above and choose $\{\bar{a}_i \mid i < \lambda^+\} \subseteq p(M)$ such that $i \neq j$ implies $\mathrm{tp}_\phi(\bar{a}_i/A, M) \neq \mathrm{tp}_\phi(\bar{a}_j/A, M)$.

Write $\chi(\bar{y}, \bar{x}) := \phi(\bar{x}, \bar{y})$. Define $\langle A_i \mid i < \lambda \rangle$ an increasing continuous sequence of subsets of $p(M)$ containing A, each of cardinality at most λ, such that

(*) $\quad A_{i+1}$ realizes every type in $S_p(B, M)$, for each $B \subseteq A_i$ with $|B| \leq \mu$.

This is possible: Having constructed A_i of cardinality at most λ, there are at most $\lambda^\mu = \lambda$ subsets B of A_i of cardinality μ. Further, for each such B, we have

$|S_p(B,M)| \leq 2^\mu \leq \lambda$, so we can add the needed realizations in A_{i+1} from $p(M)$ while keeping $|A_{i+1}| \leq \lambda$.

We now claim that (*) allows us to choose, for every $i < \lambda^+$, an index j, with $i < j < \lambda^+$, such that for each $l < \mu^+$ the type $\mathrm{tp}_\phi(\bar{a}_j/A_{l+1}, M)$ (χ,ϕ)-splits over each $B \subseteq A_l$ of cardinality at most μ.

Otherwise, there is $i < \lambda^+$ such that for every index j, with $i < j < \lambda^+$, there exists $l < \mu^+$ and $B^j \subseteq A_l$ of cardinality μ such that $\mathrm{tp}_\phi(\bar{a}_j/A_{l+1}, M)$ does not (χ,ϕ)-split over B^j. By the pigeonhole principle (since $\lambda^+ \geq \mu$) we can find $S \subseteq \lambda^+$ of cardinality λ^+, an ordinal $l < \mu^+$, and $B \subseteq A_{l+1}$ of cardinality μ such that $\mathrm{tp}_\phi(\bar{a}_j/A_{l+1}, M)$ does not (χ,ϕ)-splits over B, for every $j \in S$. By (*) we can choose $C \subseteq A_{l+1}$ of cardinality at most 2^μ such that C realizes every type in $S_{\chi,p}(B,M)$. Then, since $|S_{\phi,p}(C,M)| \leq 2^{2^\mu} < \lambda^+$, by the pigeonhole principle, we may assume that $\mathrm{tp}_\phi(\bar{a}_j/C, M)$ is constant for $j \in S$. By Proposition .1.4, we must have $\mathrm{tp}_\phi(\bar{a}_j/A_{l+1}, M) = \mathrm{tp}_\phi(\bar{a}_i/A_{l+1}, M)$, for $i, j \in S$. This contradicts the choice of $\bar{a}_i$s and the fact that $A \subseteq A_{l+1}$.

Define $\{\bar{c}_l, \bar{d}_l, \bar{b}_l \mid l < \mu^+\} \subseteq A_{2l+2}$ and $B_l = \bigcup\{\bar{c}_k, \bar{d}_k, \bar{b}_k \mid k < l\}$ such that:

(1) $B_l \subseteq A_{2l}$ and $|B_l| \leq \mu$;
(2) $\mathrm{tp}_\chi(\bar{c}_l/B_l, M) = \mathrm{tp}_\chi(\bar{d}_l/B_l, M)$;
(3) Both $\phi(\bar{x}, \bar{c}_l)$ and $\neg\phi(\bar{x}, \bar{d}_l)$ belong to $\mathrm{tp}_\phi(\bar{a}_j/A_{2l}, M)$;
(4) $\bar{b}_l \in A_{2l+1}$ realizes both $\phi(\bar{x}, \bar{c}_l)$ and $\neg\phi(\bar{x}, \bar{d}_l)$.

This is possible: Let $B_0 = \emptyset$ and $B_l = \bigcup_{k<l} B_k$ when l is a limit ordinal. Having constructed $B_l \subseteq A_{2l}$ of cardinality at most μ, the type $\mathrm{tp}_\phi(\bar{a}_j/A_{2l})$ (χ,ϕ)-splits over B_l and hence there are $\bar{c}_l, \bar{d}_l \in A_{2l}$ with $\mathrm{tp}_\chi(\bar{c}_l/B_l, M) = \mathrm{tp}_\chi(\bar{d}_l/B_l, M)$ and $\phi(\bar{x}, \bar{c}_l)$ and $\neg\phi(\bar{x}, \bar{d}_l) \in \mathrm{tp}_\phi(\bar{a}_j/A_{2l}, M)$. Then, by construction we can find $\bar{b}_l \in A_{2l+1}$ realizing $\mathrm{tp}_\phi(\bar{a}_j/\bar{c}_l\bar{d}_l, M)$ so (4) is automatically satisfied.

Now, the set $\{\bar{b}_l \hat{}\, \bar{c}_l \hat{}\, \bar{d}_l \mid l < \mu^+\} \subseteq p(M)$ and the formula

$$\psi(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{y}_0, \bar{y}_1, \bar{y}_2) := \phi(\bar{x}_0, \bar{y}_1) \leftrightarrow \phi(\bar{x}_0, \bar{y}_2)$$

demonstrate that M has the (μ^+, p)-order property. □

The following definition generalizes the notion of relative saturation.

DEFINITION II.1.6. We say that a set $C \subseteq M$ is *relatively* (λ, p)-*saturated* if C realizes every $q \in S_p(B, M)$ for every $B \subseteq C$ such that $|B| < \lambda$.

The following lemma is a version of $\kappa(T) \leq |T|^+$ for the notion of splitting.

LEMMA II.1.7. *Let μ be a cardinal such that $\mu \geq |L(M)|$. Suppose that M does not have the (μ^+, p)-order property. Suppose that $B \subseteq p(M)$ is relatively (μ^+, p)-saturated. Then for each $q \in S_p(B, M)$ there is $A \subseteq B$ of cardinality at most μ such that q does not split over A.*

PROOF. Suppose, for a contradiction, that there exist a relatively (μ^+, p)-saturated set B and a type $q \in S_p(B, M)$, such that q splits over every $A \subseteq B$ of cardinality at most μ.

We will show that M has the (μ^+, p)-order property. Construct a sequence of sets $\langle A_i \mid i < \mu^+ \rangle$ such that:

(1) $A_0 = \emptyset$;
(2) $A_i = \bigcup_{j<i} A_j$, when i is a limit ordinal;
(3) $A_i \subseteq B$, for each $i < \mu^+$;
(4) $|A_i| \leq \mu$, for each $i < \mu^+$;
(5) There are $\phi_i \in L(M)$ and $\bar{a}_i$, $\bar{b}_i \in A_{i+1}$, such that $\mathrm{tp}(\bar{a}_i/A_i, M) = \mathrm{tp}(\bar{b}_i/A_i, M)$ and $\phi(\bar{x}, \bar{a}_i)$ and $\neg\phi(\bar{x}, \bar{b}_i)$ are in q;
(6) A_{i+1} contains $\bar{c}_i$ realizing $q \restriction (A_i \cup \bar{a}_i\bar{b}_i)$.

This is possible: For $i = 0$ or a limit ordinal, it is obvious. Suppose that A_i has been constructed. Since $|A_i| \leq \mu$ and $A_i \subseteq B$, q splits over A_i. Hence, there exist a formula $\phi_i \in L(M)$, and $\bar{a}_i$, $\bar{b}_i \in B$ demonstrating this. Since B is relatively (μ^+, p)-saturated, and $q \restriction (A_i \cup \bar{a}_i\bar{b}_i) \in S_p(A_i \cup \bar{a}_i\bar{b}_i, M)$, there exists $\bar{c}_i \in B$ realizing $q \restriction (A_i \cup \bar{a}_i\bar{b}_i)$. Let $A_{i+1} = A_i \cup \{\bar{a}_i, \bar{b}_i, \bar{c}_i\}$. All the conditions are satisfied.

This is enough: By the pigeonhole principle, since $\mu \geq |L(M)|$, we may assume that there exists $\phi \in L(M)$ such that $\phi_i = \phi$, for each $i < \mu^+$. Now consider $\{\bar{c}_i\hat{\ }\bar{a}_i\hat{\ }\bar{b}_i \mid i < \mu^+\}$ and the formula

$$\psi(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{y}_0, \bar{y}_1, \bar{y}_2) := \phi(\bar{x}_0, \bar{y}_1) \leftrightarrow \phi(\bar{x}_0, \bar{y}_2).$$

It is easy to see that they demonstrate that M has the (μ^+, p)-order property. □

The following fact is Lemma I.2.5 of [**Sh b**].

FACT II.1.8. *Let $B \subseteq p(M)$ and let $\{\bar{a}_i \mid i < \alpha\} \subseteq p(M)$ be given. Consider the type $q_i = \mathrm{tp}(\bar{a}_i/B \cup \{\bar{a}_j \mid j < i\}, M) \in S_p(B \cup \{\bar{a}_j \mid j < i\}, M)$ and suppose that*

(1) *If $i < j < \alpha$ then $q_i \subseteq q_j$;*
(2) *For each $i < \alpha$ the type q_i does not split over B.*

Then $\{\bar{a}_i \mid i < \alpha\}$ is indiscernible over B.

The next theorem is a generalization of two theorems. (1) When p is stable for every model of a first order theory, a version of this theorem appears in [**Pi**]. (2) When $p := \{\bar{x} = \bar{x}\}$, it appears in [**Gr1**].

THEOREM II.1.9. *Let μ and λ be cardinals such that $\mu \geq |L(M)|$, $\lambda^\mu = \lambda$, and $\lambda \geq 2^{2^\mu}$. If M does not have the (μ^+, p)-order property, then for every $I \subseteq p(M)$ and every $A \subseteq p(M)$ such that $|I| > \lambda \geq |A|$, there exists $J \subseteq I$ of cardinality λ^+ indiscernible over A.*

PROOF. Let $I = \{\bar{a}_i \mid i < \lambda^+\}$. By the pigeonhole principle, we may assume that $\ell(\bar{a}_i) = \ell(\bar{a}_j)$, for $i, j < \lambda^+$.

Define $\langle A_i \mid i < \lambda^+ \rangle \subseteq p(M)$ such that:

(1) $A_0 = A$;
(2) $A_i = \bigcup_{j<i} A_j$, when j is a limit ordinal;
(3) $A_i \subseteq p(M)$;
(4) $|A_i| \leq \lambda$, for every $i < \lambda^+$;
(5) A_{i+1} contains $\bar{a}_i$;
(6) A_{i+1} realizes every type in $S_p(B, M)$, for each $B \subseteq A_i$ of cardinality at most μ.

This is possible: For $i = 0$ it is clear. If i is a limit ordinal it is easy. Let us concentrate on the successor stage. Assume that A_i of cardinality λ has been constructed. By cardinal assumption, there are $\lambda = \lambda^\mu$ subsets B of A_i of cardinality μ, and for each such B we have $|S_p(B, M)| \leq 2^\mu \leq \lambda$. Hence, A_{i+1} satisfying (3)–(6) can be found.

Consider the following stationary subset of λ^+

$$S = \{i < \lambda^+ \mid cf(i) \geq \mu^+\}.$$

Let $r_i := \mathrm{tp}(\bar{a}_i/A_i, M)$. Then clearly $r_i \in S_p(A_i, M)$. Now, for each $i \in S$, since $\mathrm{cf}(i) \geq \mu^+$, the set A_i is relatively (μ^+, p)-saturated. Hence, by Lemma .1.7, there exists $B_i \subseteq A_i$ of cardinality at most μ such that r_i does not split over B_i. Furthermore, since $\mathrm{cf}(i) = \mu^+$, there exists $j < i$ such that $B_i \subseteq A_j$.

This shows that the function $f\colon S \to \lambda^+$ defined by

$$f(i) = \min\{j \mid B_i \subseteq A_j\},$$

is regressive. Hence, by Fodor's lemma (see Theorem 22 of [**Je**]), there is $S' \subseteq S$ of cardinality λ^+ and $i_0 < \lambda^+$ such that for each $i \in S'$ we have $B_i \subseteq A_{i_0}$. Since there are only $\lambda^\mu = \lambda$ subsets of A_{i_0} of size μ, we may assume, by the pigeonhole principle, that there exists a set $B \subseteq A_{i_0}$ such that $B_i = B$ for each $i \in S'$. Now, M does not have the (μ^+, p)-order property, and $\lambda^\mu = \lambda$, so Theorem .1.5 implies that M is (λ, p)-stable. Hence, $|S_p(A_{i_0}, M)| \leq \lambda$, and thus by the pigeonhole principle, we may further assume that $\mathrm{tp}(\bar{a}_i/A_{i_0}, M) = \mathrm{tp}(\bar{a}_j/A_{i_0}, M)$, for every $i, j \in S'$.

By re-enumerating if necessary, we may assume that $S' \setminus (i_0 + 1) = \lambda^+$. Now let

$$q_i := \mathrm{tp}(\bar{a}_i / A_{i_0} \cup \{\bar{a}_j \mid j < i\}) \in S_p(Ai_0 \cup \{\bar{a}_j \mid j < i\}).$$

By Proposition .1.4 we have that $q_i \subseteq q_j$ if $i < j$. Thus, all the assumptions of Fact .1.8 are satisfied, so $J = \{\bar{a}_i \mid i < \lambda^+\}$ is indiscernible over A, since $A \subseteq A_{i_0}$. This finishes the proof. □

In the previous theorem, we demanded that A be a subset of $p(M)$. The next remark summarizes what we can do when $A \subseteq M$ is not necessarily contained in $p(M)$. It follows from the previous theorem by considering an expansion of $L(M)$ with constants for elements in A.

REMARK II.1.10. Let $\mu \geq |L(T)|$ be a cardinal. Let $A \subseteq M$ be given and suppose that M does not have the (μ^+, p)-order property even allowing parameters from A. Let $\lambda^\mu = \lambda$ and $\lambda \geq 2^{2^\mu}$. Then, for every $I \subseteq p(M)$ of cardinality λ^+, there exists $J \subseteq I$ of cardinality λ^+ indiscernible over A.

The next definition defines averages without using $\kappa(D)$. In stable diagrams, both definitions are easily seen to be equivalent.

DEFINITION II.1.11. Let I be an infinite set of finite sequences. Let $A \subseteq M$. We define the *average of I over A in M* as follows

$$\mathrm{Av}(I, A, M) := \{\phi(\bar{x}, \bar{a}) \mid \bar{a} \in A, \phi(\bar{x}, \bar{y}) \in L(M),$$
$$\text{and } M \models \phi[\bar{c}, \bar{a}] \text{ for } |I| \text{ elements } \bar{c} \in I\}.$$

We will be interested in conditions guaranteeing that averages are well-defined. It is a known fact (see Lemma III 1.7 (1) of [**Sh b**]) that if M is a model of a complete, first order, stable theory T, then for every infinite set of indiscernibles I and $A \subseteq M$, the average $\mathrm{Av}(I, A, M)$ is a complete type over A. Also, if $|I| > |A| + \kappa(T)$, then the average is realized by an element of I (this is essentially Lemma III 3.9 of [**Sh b**]). A corresponding local result (Theorem .4.6) in the presence of the compactness theorem will be given in Section 4. Inside a fixed model, the situation is more delicate. The next theorem is a localization of Conclusion 1.11 in [**Sh300**]. Notice the similarity with the assumptions of Theorem .1.9.

THEOREM II.1.12. *Let μ and λ be cardinals such that $\mu \geq |L(M)|$, $\lambda^\mu = \lambda$, and $\lambda \geq 2^{2^\mu}$. If M does not have the (μ^+, p)-order property, then for every $I \subseteq p(M)$ of cardinality λ^+, there exists $J \subseteq I$ of cardinality λ^+ such that for each $A \subseteq p(M)$ the average $\mathrm{Av}(J, A, M)$ is a complete type over A. Moreover, if $|J| > |A|$, then $\mathrm{Av}(J, A, M) \in S_p(A, M)$.*

PROOF. Let $I = \{\bar{a}_\alpha \mid \alpha < \lambda^+\}$. We may assume by the pigeonhole principle that there exists $n < \omega$ such that $\ell(\bar{a}_\alpha) = n$, for each $\alpha < \lambda^+$.

We first essentially repeat the proof of Theorem .1.9 and construct a sequence $\langle A_\alpha \mid \alpha \leq \lambda^+\rangle$ such that:

(1) $A_0 = \emptyset$, $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$ when δ is limit, and $A_\alpha \subseteq A_{\alpha+1}$.
(2) $A_\alpha \subseteq p(M)$.
(3) $|A_\alpha| \leq \lambda$, for every $\alpha < \lambda^+$.
(4) $A_{\alpha+1}$ contains $\bar{a}_\alpha$.
(5) $A_{\alpha+1}$ realizes all types in $S_p(A_\alpha, M)$.

This is possible: Since M does not have the (μ^+, p)-order property, then M is (λ, p)-stable by Theorem .1.5. Hence, $|S_p(A_\alpha, M)| \leq \lambda$ inductively, for each $\alpha < \lambda^+$.

Now (5) implies that

(6) If $\mathrm{cf}(\delta) \geq \mu^+$ then A_δ is relatively (μ^+, p)-saturated.

As in the proof of Theorem .1.9, we can find a set $S \subseteq \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) \geq \mu^+\}$ of cardinality λ^+ and an ordinal $\alpha(*) = \min S$ such that

(7) For each $\alpha \in S$, the type $\mathrm{tp}(\bar{a}_\alpha/A_\alpha, M)$ does not split over $A_{\alpha(*)}$.
(8) If $\alpha, \beta \in S$ and $\alpha < \beta$ then $\mathrm{tp}(\bar{a}_\alpha/A_\alpha, M) \subseteq \mathrm{tp}(\bar{a}_\beta/A_\beta, M)$.

We claim that the set $J = \{\bar{a}_\alpha \mid \alpha \in S\}$ is as desired. To show this, we will show that

(*) For every $\bar{c} \in p(M)$ and $\phi(\bar{x}, \bar{y}) \in L(M)$, either

$$|\{\alpha \in S \ : \ M \models \phi[\bar{a}_\alpha, \bar{c}]\}| \leq \mu \quad \text{or} \quad |\{\alpha \in S \ : \ M \models \neg\phi[\bar{a}_\alpha, \bar{c}]\}| \leq \mu.$$

This implies the conclusion of the theorem: For $A \subseteq p(M)$, condition (*) implies that $\mathrm{Av}(J, A, M)$ is a consistent set of formulas over A, as each finite subset is realized by all but μ many elements of J. Since $\mathrm{Av}(J, A, M)$ is always complete, we have that $\mathrm{Av}(J, A, M)$ is a complete type over A. For the last sentence, notice that all but $|A| + |L(M)| + \mu$ elements of J realize $\mathrm{Av}(J, A, M)$. Hence, if $\lambda^+ > |A|$, then there exists $\bar{a}_\alpha \in J \subseteq p(M)$ realizing $\mathrm{Av}(J, A, M)$ (as $\lambda \geq \mu + |L(M)|$). This shows that $\mathrm{Av}(J, A, M) \in S_p(A, M)$.

Let $\bar{c} \in p(M)$ and $\phi(\bar{x}, \bar{y}) \in L(M)$ be given. Then $\mathrm{tp}(\bar{c}/A_\alpha, M) \in S_p(A_\alpha, M)$, since $\bar{c} \in p(M)$. Hence, by (5), we can find $\{\bar{c}_\alpha \mid \alpha \in S\} \subseteq p(M)$ satisfying

(9) $\bar{c}_\alpha \in A_{\alpha+2}$.
(10) $\bar{c}_\alpha$ realizes $\mathrm{tp}(\bar{c}/A_{\alpha+1}, M)$.

We will prove (*) by finding a set of ordinals E of cardinality μ such that either $\{\alpha \in S : M \models \phi[\bar{a}_\alpha, \bar{c}]\} \subseteq E$ or $\{\alpha \in S : M \models \neg\phi[\bar{a}_\alpha, \bar{c}]\} \subseteq E$.

We construct the set E, as well as a set $C \subseteq A_{\lambda^+}$ with the following properties:

(11) $|E| \leq \mu$ and $|C| \leq \mu$.
(12) $\lambda^+ \in E$.
(13) If $\alpha+1 \in E$ then $\alpha \in E$ and if $\delta \in E$ and $\mathrm{cf}(\delta) \leq \mu$ then $\sup(E \cap \delta) = \delta$.
(14) If $\delta \in E$ and $\mathrm{cf}(\delta) \geq \mu^+$, then $\mathrm{tp}(\bar{c}/A_\delta, M)$ does not split over $C \cap A_\delta$. Moreover, $C \cap A_\delta \subseteq A_{\sup(E\cap\delta)}$.

This is possible: Construct E_n and C_n of cardinality at most μ by induction on $n < \omega$. Let $E_0 = \{\lambda^+\}$ and $C_0 = \emptyset$. Then, by (6) and Lemma .1.7 we can find C_{n+1} of cardinality μ such that $\mathrm{tp}(\bar{c}/A_\delta, M)$ does not split over $C_{n+1} \cap A_\delta$ for each $\delta \in E_n$ with $\mathrm{cf}(\delta) \geq \mu^+$. Furthermore, we can add at most μ many ordinals to E_{n+1} to ensure that $C_{n+1} \subseteq A_{\sup(E_{n+1}\cap\delta)}$. Thus, $E = \bigcup_{n<\omega} E_n$ and $C = \bigcup_{n<\omega} C_n$ are as desired.

This is enough to prove (*). In fact, to show that $\{\alpha \in S : M \models \phi[\bar{a}_\alpha, \bar{c}]\} \subseteq E$ or $\{\alpha \in S : M \models \neg\phi[\bar{a}_\alpha, \bar{c}]\} \subseteq E$, it clearly suffices to show

(**) $$M \models \phi[\bar{a}_\alpha, \bar{c}] \leftrightarrow \phi[\bar{a}_\beta, \bar{c}], \quad \text{for every } \alpha, \beta \in S \setminus E.$$

Notice that by construction (11)–(14) the set $S \setminus E$ is partitioned into at most μ *intervals* of the form $\{\alpha \in S \mid \sup(E \cap \delta) \leq \alpha < \delta\}$, where $\delta \in E$ with $\mathrm{cf}(\delta) \geq \mu^+$. If such an interval is nonempty, then it must have size at least μ^+. We will make use of this and prove (**) in two stages. In the first part, we will show that (**) holds, provided α and β belong to the same interval, and then in the second part, that (**) holds also when α and β belong to different intervals.

Let $\delta \in E$ be such that $\mathrm{cf}(\delta) \geq \mu^+$. Denote by $\delta_0 = \sup(E \cap \delta)$. Now let $\alpha, \beta \in S$ such that $\delta_0 \leq \alpha < \beta < \delta$. Without loss of generality, assume that $M \models \phi[\bar{a}_\alpha, \bar{c}]$. Then $\phi(\bar{a}_\alpha, \bar{y}) \in \mathrm{tp}(\bar{c}/A_\delta, M)$. By (14) the type $\mathrm{tp}(\bar{c}/A_\delta, M)$ does not split over $C \cap A_\delta \subseteq A_{\delta_0}$. But, by (8), we have $\mathrm{tp}(\bar{a}_\alpha/A_{\delta_0}, M) = \mathrm{tp}(\bar{a}_\beta/A_{\delta_0}, M)$. Hence, by nonsplitting $\phi(\bar{a}_\beta, \bar{y}) \in \mathrm{tp}(\bar{c}/A_\delta, M)$ and so $M \models \phi[\bar{a}_\beta, \bar{c}]$.

To prove the second part, we first claim that

(†) $$M \models \phi[\bar{a}_{\alpha_1}, \bar{c}_{\beta_1}] \leftrightarrow \phi[\bar{a}_{\alpha_2}, \bar{c}_{\beta_2}], \quad \text{for every } \alpha_1 > \beta_1 \text{ and } \alpha_2 > \beta_2 \text{ in } S.$$

To see this, let $\gamma = \max(\alpha_1, \alpha_2)$. Then by (8) and (9) (recall that ordinals in S are limit), we have $M \models \phi[\bar{a}_{\alpha_1}, \bar{c}_{\beta_1}] \leftrightarrow \phi[\bar{a}_\gamma, \bar{c}_{\beta_1}]$ and also $M \models \phi[\bar{a}_{\alpha_2}, \bar{c}_{\beta_2}] \leftrightarrow \phi[\bar{a}_\gamma, \bar{c}_{\beta_2}]$. Now by (10) we have that $\mathrm{tp}(\bar{c}_{\beta_1}/A_{\alpha(*)}, M) = \mathrm{tp}(\bar{c}_\gamma/A_{\alpha(*)}, M)$, and by (9), both $\bar{c}_{\beta_1}, \bar{c}_{\beta_2} \in A_\gamma$. But by (7) the type $\mathrm{tp}(\bar{a}_\gamma/A_\gamma, M)$ does not split over $A_{\alpha(*)}$. Hence, $\phi(\bar{x}, \bar{c}_{\beta_1}) \in \mathrm{tp}(\bar{a}_\gamma/A_\gamma, M)$ if and only if $\phi(\bar{x}, \bar{c}_{\beta_2}) \in \mathrm{tp}(\bar{a}_\gamma/A_\gamma, M)$. Thus, $M \models \phi[\bar{a}_\gamma, \bar{c}_{\beta_1}] \leftrightarrow \phi[\bar{a}_\gamma, \bar{c}_{\beta_2}]$. This proves (†).

Now for the second part, let $\delta, \xi \in E$ with $\mathrm{cf}(\delta) \geq \mu^+$ and $\mathrm{cf}(\xi) \geq \mu^+$. Denote by $\delta_0 = \sup(E \cap \delta)$ and $\xi_0 = \sup(E \cap \xi)$. Assume that $\delta_0 < \xi_0$ and let $i \in S$ with $\delta_0 \leq i < \delta$ and $j \in S$ with $\xi_0 \leq j < \xi$. To show: $M \models \phi[\bar{a}_i, \bar{c}] \leftrightarrow$

$\phi[\bar{a}_j, \bar{c}]$. Suppose $M \models \neg(\phi[\bar{a}_i, \bar{c}] \leftrightarrow \phi[\bar{a}_j, \bar{c}])$. We will derive a contradiction by showing that M has the (μ^+, p)-order property.

Assume, without loss of generality, that $M \models \phi[\bar{a}_i, \bar{c}]$ and $M \models \neg\phi[\bar{a}_j, \bar{c}]$. We distinguish two cases.

Case 1: Suppose $M \models \phi[\bar{a}_j, \bar{c}_i]$ (recall $j > i$). Then, by (†), we have that $M \models \phi[\bar{a}_\alpha, \bar{c}_\beta]$, for every $\alpha, \beta \in S$ with $\alpha > \beta$. On the other hand since $M \models \neg\phi[\bar{a}_j, \bar{c}]$, the first part of this argument shows that $M \models \neg\phi[\bar{a}_\alpha, \bar{c}]$, for each $\alpha \in S$ with $\xi_0 \leq \alpha < \xi$. Hence, by (10), for each $\beta \in S$ with $\alpha \leq \beta$ we have that $M \models \neg\phi[\bar{a}_\alpha, \bar{c}_\beta]$. Thus, for $\alpha, \beta \in S \cap [\xi_0, \xi)$, we have

$$M \models \neg\phi[\bar{a}_\alpha, \bar{c}_\beta] \quad \text{if and only if} \quad \alpha \leq \beta.$$

This implies easily that M has the (μ^+, p)-order property.

Case 2: Suppose $M \models \neg\phi[\bar{a}_j, \bar{c}_i]$. Similarly to Case 1, we obtain the (μ^+, p)-order property by using the interval $S \cap [\delta_0, \delta)$ and the fact that $M \models \phi[\bar{a}_i, \bar{c}]$.

□

II.2. Local order and stability for nonelementary classes

In this short section, we will examine the stability of p with respect to all the models of a given class of models $\mathcal{K}$. Let us fix the concepts. We will work inside the class $\mathcal{K} = \mathrm{PC}(T_1, T, \Gamma)$. Recall that for $T \subseteq T_1$ and Γ a set of T_1-types over the empty set, we let

$$\mathrm{PC}(T_1, T, \Gamma) = \{M \restriction L(T) \ : \ M \models T_1 \text{ and } M \text{ omits every type in } \Gamma\}$$

We will denote by $\mu(\mathcal{K}) = \mu(|T_1|, |\Gamma|)$, the Hanf-Morley number for $\mathcal{K}$. The properties of Hanf-Morley numbers work in this more general context. Recall that $\mu(\lambda, \kappa)$ is the least cardinal μ with the property that for every $\mathrm{PC}(T_1, T, \Gamma)$ with $|T_1| \leq \lambda$ and $|\Gamma| \leq \kappa$, if $\mathrm{PC}(T_1, T, \Gamma)$ contains a model of cardinality μ, then it contains models of arbitrarily large cardinality. It is known for example that when $\kappa = |\Gamma| = 0$, then $\mu(\mathcal{K}) = \aleph_0$. For $|\Gamma| \geq 1$, then $\mu(\mathcal{K}) = \beth_{\delta(|T_1|, |\Gamma|)}$. Recall that $\delta(\lambda, \kappa)$ is the least ordinal δ with the property that for every $\mathrm{PC}(T_1, T, \Gamma)$ with $|T_1| \leq \lambda$ and $|\Gamma| \leq \kappa$, if $\mathrm{PC}(T_1, T, \Gamma)$ contains a model with a predicate whose order type is δ, then it contains a model where this predicate is not wellordered. Much is known about such numbers. Here are some of the known facts. First $\delta(\lambda, 0) = \omega$ and $\delta(\lambda, \kappa)$ is always a limit ordinal. We have monotonicity properties: if $\lambda_1 \leq \lambda_2$ and $\kappa_1 \leq \kappa_2$, then $\delta(\lambda_1, \kappa_1) \leq \delta(\lambda_2, \kappa_2)$. Also, if $1 \leq \kappa \leq \lambda$ then $\delta(\lambda, \kappa) = \delta(\lambda, 1)$. In general $\delta(\lambda, \kappa) \leq (2^\lambda)^+$. Finally, suppose $\kappa \leq \lambda$ and λ is a strong limit cardinal of cofinality $\aleph_0$, then $\delta(\lambda, \kappa) = \lambda^+$. See Lemma VII.5.1 and Theorem VII.5.5 of [**Sh b**] or [**Gr b**]

Choosing to carry out the theorems of this section in a PC-class is arbitrary. We could have chosen to study any sufficiently general class of models extending the first order case in which the compactness theorem fails. For example, the class of models of an infinitary sentence $\psi \in L_{\omega_1\omega}$ or $L_{\lambda^+\omega}$. All the results of this section hold for such classes and the proofs can usually be used verbatim.

As in the previous section, we will fix p a set of $L(T)$-formulas (with parameters).

We expand the definitions we made in the first section for the class $\mathcal{K}$.

DEFINITION II.2.1.

(1) Let λ be a cardinal. We say that p is *stable in* λ, if for every $M \in \mathcal{K}$, M is (λ, p)-stable.
(2) We say that p is *stable* if there exists a cardinal λ such that p is stable in λ.

DEFINITION II.2.2.

(1) We say that p has the λ-*order property* if there exists $M \in \mathcal{K}$ such that M has the (λ, p)-order property.
(2) We say that p has the *order property* if p has the λ-order property for every λ.

Using proof techniques similar to those used in Theorem .2.10 of Chapter I we observe:

FACT II.2.3. *The following conditions are equivalent.*

(1) p *has the order property;*
(2) p *has the* λ*-order property for every* $\lambda < \mu(\mathcal{K})$*;*
(3) p *has the* $\mu(\mathcal{K})$*-order property;*
(4) *There exists a model* $M \in \mathcal{K}$*, a formula* $\phi(\bar{x}, \bar{y})$*, and an indiscernible sequence* $\{\bar{a}_i \mid i < \mu(\mathcal{K})\} \subseteq p(M)$*, such that*

$$M \models \phi[\bar{a}_i, \bar{a}_j] \quad \textit{if and only if} \quad i < j < \mu(\mathcal{K}).$$

We now prove a version of the stability spectrum and the equivalence between local instability and local order. Nonlocal theorems of this vein appear in [**Sh16**].

THEOREM II.2.4. *The following conditions are equivalent.*

(1) p *is stable;*
(2) *There exists a cardinal* $\kappa(\mathcal{K}) < \mu(\mathcal{K}) + |L(T)|^+$ *such that* p *is stable in every* $\lambda \geq \mu(\mathcal{K})$ *satisfying* $\lambda^{\kappa(\mathcal{K})} = \lambda$.
(3) p *does not have the order property.*

PROOF. (2) $\Rightarrow$ (1) trivially.

(3) $\Rightarrow$ (2): Since p does not have the order property, by Fact .2.3 there exists a cardinal $\kappa < \mu(\mathcal{K})$ such that no model of $\mathcal{K}$ has the (κ^+, p)-order property. Let $\lambda \geq \mu(\mathcal{K})$. Then, automatically, since $\kappa < \mu(\mathcal{K})$ and $\mu(\mathcal{K})$ is either $\aleph_0$ or a strong limit, we have $\lambda \geq 2^{2^\kappa}$. Let $\kappa(\mathcal{K}) = \kappa + |L(T)|$. Hence, if $\lambda \geq \mu(\mathcal{K})$ satisfies $\lambda^{\kappa(\mathcal{K})} = \lambda$, and $M \in \mathcal{K}$, then Theorem .1.5 implies that M is (λ, p)-stable. Thus, p is stable in λ.

(1) $\Rightarrow$ (3): This is again a standard application of Hanf number techniques. We give just a sketch. Suppose p is stable in λ. Let T^* be an expansion of T_1 with Skolem functions, such that $|T^*| = |T_1|$. Let κ be smallest such that $2^\kappa > \lambda$. Using the order property and the methods of Morley, we can find $M^* \models T^*$ such that $M = M^* \restriction L(T) \in \mathcal{K}$, with $\phi(\bar{x}, \bar{y})$, and $\{\bar{a}_i \mid i < \omega\} \subseteq p(M)$ demonstrating the p-order property. Furthermore $\{\bar{a}_i \mid i < \omega\} \subseteq p(M)$ is T^*-indiscernible. Hence, by the compactness theorem, we can find a model $N^* \models T^*$ and a set $\{\bar{a}_\eta \mid \eta \in {}^{\kappa\geq}2\} \subseteq p(N^*)$ demonstrating the p-order property with respect to the lexicographic order. Furthermore, for every $n < \omega$

$$\mathrm{tp}(\bar{a}_{\nu_0}, \ldots, \bar{a}_{\nu_n}/\emptyset, N^*) = \mathrm{tp}(\bar{a}_0, \ldots, \bar{a}_n/\emptyset, M^*), \text{ for every } \nu_0 < \cdots < \nu_n.$$

We may assume that N^* is the Ehrenfeucht-Mostowski closure of $\{\bar{a}_\eta \mid \eta \in {}^{\kappa\geq}2\}$, since T^* has Skolem functions. Let $N = N^* \restriction L(T)$. Then $N \in \mathcal{K}$. Consider $A = \bigcup_{\eta \in {}^{\kappa>}2} \bar{a}_\eta \subseteq p(N)$. Then $|A| \leq 2^{<\kappa} \leq \lambda$ and $|S_p(A, N)| = 2^\kappa > \lambda$. Thus, N is not (λ, p)-stable, a contradiction. $\square$

REMARK II.2.5. In the first order case, $\mu(\mathcal{K}) = \aleph_0$ and so p is stable if and only if p is stable in every λ such that $\lambda^{|L(T)|} = \lambda$. In the first order case, most authors define *stable* types using (3) with $\mu(\mathcal{K}) = \aleph_0$.

II.3. Local stability and local homogeneity in finite diagrams

In this section, we examine the stability of p for the class of models of a finite diagram D. The framework of finite diagrams was discussed in the first section of Chapter I. The same notation is used.

We set the necessary definitions to localize all results of the first chapter. Note that some of them have already been established in a more general context, like the equivalence between local stability and the failure of the local order property.

Fix p a set of $L(T)$-formulas (maybe with a D-set of parameters). All the results of Section 1 and 2 of this chapter hold, as finite diagrams is a particular case of a PC-class.

We adopt the following notation. Note that there is a slight clash of notation between the subscript p employed here and the subscript p as it is used in the first chapter, but this shouldn't cause any ambiguity.

DEFINITION II.3.1. For A a D-set, let

$$S_{D,p}(A) = \{\mathrm{tp}(c/A) \mid A \cup c \text{ is a } D \text{ set and } c \text{ realizes } p\}.$$

Although the definition makes sense for any $A \subseteq M$, it will only be used when $A \subseteq p(M)$. The next definition is only a restatement of what we meant by *relatively* (λ, p) *saturated* in the first section of this chapter, when $M = \mathfrak{C}$.

DEFINITION II.3.2. A model M is (D, λ, p)-*homogeneous*, if M realizes every type in $S_{D,p}(A)$, for each $A \subseteq p(M)$ of cardinality less that λ.

We can relax the monster model assumption to:

HYPOTHESIS II.3.3. There exists a $(D, \bar{\kappa}, p)$-*homogeneous* D-model $\mathfrak{C}$, for some $\bar{\kappa}$ larger than any cardinal needed in this chapter.

We will work inside $p(\mathfrak{C})$. The results of Chapter I, Section 1 hold relativized to realizations of p. Thus, $\mathfrak{C}$ can be assumed to contain every D-set $A \subseteq p(M)$, for any D-model M. And also $\mathfrak{C}$ is homogeneous with respect to subsets of $p(\mathfrak{C})$. Write $S_{D,p}(A)$ for $S_{D,p}(A, \mathfrak{C})$.

We rephrase the definitions of local stability and local order.

DEFINITION II.3.4.

(1) D is (λ, p)-*stable* if $|S_{D,p}(A)| \leq \lambda$ for every $A \subseteq p(\mathfrak{C})$ of cardinality λ.
(2) D is p-*stable* if D is (λ, p)-stable for some cardinal λ.

DEFINITION II.3.5.

(1) D has the (λ, p)-*order property* if there exist a formula $\phi(\bar{x}, \bar{y}) \in L(T)$ and a set $\{\bar{a}_i \mid i < \lambda\} \subseteq p(\mathfrak{C})$, such that
$$\models \phi[\bar{a}_i, \bar{a}_j] \quad \text{if and only if} \quad i < j < \lambda.$$
(2) D has the p-*order property* if D has the (λ, p)-order property for every cardinal λ.

Then, all the statements of Chapter I Section 2 are true provided all the sets mentioned are taken inside $p(\mathfrak{C})$ and the local notions $S_{D,p}(A)$, p-order property, p-stability are used instead. Most of the proofs can be used without modification. The only kind of changes which are occasionally required are the obvious ones, for example: In the proof of Remark .2.4 add the requirement $p(\bar{c}_i)$ for $i < \lambda$ in the list of conditions, as well as a requirement that $\{\bar{c}_i \mid i < \lambda\}$ be indiscernible over the parameters of p. In the proof of Theorem .2.9, choose $\{\bar{a}_i \mid i < \mu^+\} \subseteq p(\mathfrak{C})$ and so on. The main result of Section 2 is the local version of the stability/order dichotomy. We state it for completeness

THEOREM II.3.6. *D is p-stable if and only if D does not have the p-order property.*

We need to introduce a local version of strong splitting:

DEFINITION II.3.7. Let $A \subseteq p(\mathfrak{C})$ and $q \in S_{D,p}(A)$. The type q *splits strongly* over $B \subseteq A$ if there exist $\{\bar{c}_n \mid n < \omega\} \subseteq p(\mathfrak{C})$, an indiscernible sequence over B, and a formula $\phi(\bar{x}, \bar{y})$ such that $\phi(\bar{x}, \bar{c}_1) \in q$, $\neg\phi(\bar{x}, \bar{c}_2) \in q$.

Define the localized version of $(*\lambda)$ as follows:

DEFINITION II.3.8. D satisfies $(p * \lambda)$ if there exists an increasing and continuous chain $\{A_i \mid i \leq \lambda\}$, with $A_i \subseteq p(\mathfrak{C})$, and a type $q \in S_{D,p}(A)$ such that $q \restriction A_{i+1}$ splits over A_i.

The localized version of $(B * \lambda)$ is defined similarly using subsets of $p(\mathfrak{C})$, call it $(p, B * \lambda)$. For $(C * \lambda)$, use subsets of $p(\mathfrak{C})$ and the definition of strong splitting above for $(C * \lambda)$, call it $(p, C * \lambda)$

The same lemmas can be shown with very similar proofs using the homogeneity of $\mathfrak{C}$ inside $p(\mathfrak{C})$. We obtain:

THEOREM II.3.9. *D is not p-stable if and only if $(p * \lambda)$ holds for every cardinal λ if and only if $(p, B * \lambda)$ holds for every cardinal λ.*

DEFINITION II.3.10. Let

$$\kappa(p, D) = \min\{\kappa \mid \text{For all } q \in S_{D,p}(A), A \subseteq p(\mathfrak{C}), \text{ there is } B \subseteq A, |B| < \kappa \text{ such that } q \text{ does not split strongly over } B\ \}.$$

If it is undefined, we let $\kappa(p, D) = \infty$.

Then, by inspecting the proofs, the local version of the existence of $\kappa(D)$ exists, when the diagram D is p-stable.

THEOREM II.3.11. *If D is (λ, p)-stable, then $\kappa(p, D) \leq \lambda$.*

This allows us to obtain a local version of the stability spectrum. The cardinal κ in the statement below is $\kappa(p, D)$ and the cardinal λ the first cardinal such that D is (λ, p)-stable.

THEOREM II.3.12. *Either D is not p-stable or D is p-stable and there exists cardinals $\kappa \leq \lambda < \beth_{(2^{|T|})^+}$ such that for every cardinal μ, the diagram D is (μ, p)-stable if and only if $\mu \geq \lambda$ and $\mu^{<\kappa} = \mu$.*

Finally making the necessary adaptations, the local homogeneity spectrum follows:

THEOREM II.3.13. *There exists a (D, λ, p)-homogeneous model of cardinality λ if and only if $\lambda \geq |S_{D,p}(\emptyset)|$ and $\lambda^{<\lambda} = \lambda$ or D is (λ, p)-stable.*

II.4. Local order, independence, and strict order in the first order case

In this section, we will fix a complete, first order theory T and obtain results for the class of models of T. As usual, we work inside the *monster model* $\mathfrak{C}$, a model which is $\bar{\kappa}$-saturated, for a cardinal $\bar{\kappa}$ larger than any cardinality mentioned in this chapter. Hence, all sets will be assumed to be inside $\mathfrak{C}$ and satisfaction is defined with respect to $\mathfrak{C}$. We will write $S_p(A)$ for $S_p(A, \mathfrak{C})$ and $\mathrm{Av}(I, A)$ for $\mathrm{Av}(I, A, \mathfrak{C})$ as is customary. As before, we fix a (nonalgebraic) T-type p. Denote by $\mathrm{dom}(p)$ the set of parameters of p.

All the results we have obtained so far hold with $\mu(\mathcal{K}) = \aleph_0$.

We first give local versions of Saharon Shelah's first order notion of independence and strict order property [**Sh b**].

For a statement $\mathbf{t}$ and a formula ϕ, we use the following notation: $\phi^{\mathbf{t}} = \neg\phi$ if the statement $\mathbf{t}$ is false and $\phi^{\mathbf{t}} = \phi$, if the statement $\mathbf{t}$ is true. We will use the same notation when $\mathbf{t} \in \{0, 1\}$, where 0 stands for false and 1 stands for truth.

DEFINITION II.4.1.

(1) We say that $\phi(\bar{x}, \bar{y})$ has the *p-independence property* if for every $n < \omega$ there exists $\{\bar{a}_i \mid i < n\} \subseteq p(\mathfrak{C})$ such that

$$p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_i)^{i \in w} \mid i < n\} \text{ is consistent, } \quad \text{for every } w \subseteq n.$$

We say that p has the *independence property* if there exists a formula $\phi(\bar{x}, \bar{y})$ with the p-independence property;

(2) A formula $\phi(\bar{x}, \bar{y})$ is said to have the *p-strict order property* if for every $n < \omega$ there exists $\{\bar{a}_i \mid i < n\} \subseteq p(\mathfrak{C})$ such that

$$\models \exists \bar{x}(\neg\phi(\bar{x}, \bar{a}_i) \wedge \phi(\bar{x}, \bar{a}_j)) \quad \text{if and only if} \quad i < j < n.$$

We say that p has the *strict order property* if there exists a formula $\phi(\bar{x}, \bar{y})$ with the p-strict order property.

PROPOSITION II.4.2. *If p has the independence property or the strict order property, then p has the order property.*

PROOF. Suppose first that p has the independence property. Then, some $\phi(\bar{x}, \bar{y})$ has the p-independence property. Hence, by the compactness theorem there exist $I = \{\bar{a}_i \mid i < \omega\} \subseteq p(M)$ such that for every $n < \omega$ and $w \subseteq n$ there exists $\bar{c} \in p(\mathfrak{C})$ realizing the formula $\bigwedge_{i<n} \phi(\bar{x}, \bar{a}_i)^{i \in w}$. We show that ϕ has the p-order property. For each $k < n$, let $\bar{c}_k \in p(\mathfrak{C})$ realize $\{\phi(\bar{x}, \bar{a}_i) \mid i < k\} \cup \{\neg\phi(\bar{x}, \bar{a}_i) \mid i \geq k, i < n\}$. Then, we use $\{\bar{c}_i {}^\frown \bar{a}_i \mid i < n\}$ and the compactness theorem to show that the formula $\psi(\bar{x}_0, \bar{x}_1; \bar{y}_0, \bar{y}_1) := \phi(\bar{x}_0, \bar{y}_1)$ has the p-order property.

Suppose that p has the strict order property. Let $\phi(\bar{x}, \bar{y})$ have the p-strict order property. Then, the formula $\psi(\bar{y}_1, \bar{y}_2) := \exists \bar{x}(\neg\phi(\bar{x}, \bar{y}_1) \wedge \phi(\bar{x}, \bar{y}_2))$ has the p-order property. $\square$

The next two results depend explicitly on the parameters of p.

THEOREM II.4.3. *Let $\phi(\bar{x}, \bar{y})$ be a formula with the p-order property. Then, either $\phi(\bar{x}, \bar{y})$ has the p-independence property, or there exist $\chi(\bar{x})$, the conjunction of finitely many formulas of p, an integer $n < \omega$ and a sequence $\eta \in {}^n2$ such that the formula $\chi(\bar{x}) \wedge \bigwedge_{l<n} \phi(\bar{x}, \bar{y}_l)^{\eta[l]}$ has the p-strict order property (maybe with parameters from $\mathrm{dom}(p)$).*

PROOF. By Fact .2.3 (4) there exists an indiscernible sequence $\{\bar{a}_i \mid i < \omega\} \subseteq p(\mathfrak{C})$ such that

$$\models \phi[\bar{a}_i, \bar{a}_j] \quad \text{if and only if} \quad i < j < \omega.$$

Further, by a standard compactness argument using Ramsey's Theorem, we may assume that $\{\bar{a}_i \mid i < \omega\}$ is indiscernible over $\mathrm{dom}(p)$, the set of parameters of p.

If $\phi(\bar{x}, \bar{y})$ does not have the p-independence property, then there exists $n < \omega$ and $w \subseteq n$ such that

(*) $$p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w} \mid l < n\} \text{ is not consistent.}$$

Let $w^* = \{n-|w|, n-|w|+1, n-|w|+2, \dots, n-1\}$. Since ϕ has the p-order property, we have that $\models \phi[\bar{a}_{n-|w|-1}, \bar{a}_l]$ if and only if $n-|w| \leq l$. Therefore, by definition of w^*, the tuple $\bar{a}_{n-|w|-1}$ realizes $p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w^*} \mid l < n\}$, and so

(**) $$p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w^*} \mid l < n\} \text{ is consistent.}$$

Now, construct a sequence $\langle w_i \mid i \leq i^* \rangle$ of subsets of n of cardinality $|w|$ such that $w_0 = w$, $w_{i^*} = w^*$, and for each $i < i^*$, there exists $k \in w_i$ such that $w_{i+1} = w_i \cup \{k+1\} \setminus \{k\}$. Notice that because of (*) and (**) and the definition of $\langle w_i \mid i \leq i^* \rangle$, we can find $i < i^*$ such that $p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w_{i+1}} \mid l < n\}$ is consistent, while $p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w_i} \mid l < n\}$ is not.

Let $k \in w_i$ such that $w_{i+1} = w_i \setminus \{k\} \cup \{k+1\}$ (note that $k+1 \notin w_i$). We then have,

(†)
$$p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w_i}, \neg\phi(\bar{x}, \bar{a}_k), \phi(\bar{x}, \bar{a}_{k+1}) \mid l < n, l \neq k, k+1\} \text{ is consistent}$$

and

$$p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l \in w_i}, \neg\phi(\bar{x}, \bar{a}_{k+1}), \phi(\bar{x}, \bar{a}_k) \mid l < n, l \neq k, k+1\} \text{ is inconsistent.}$$

Hence, by the finite character of consistency, we can find $\chi(\bar{x})$, the conjunction of finitely many formulas of p, such that

(‡) $$\models \neg\exists \bar{x}[\chi(\bar{x}) \wedge (\bigwedge_{l<n, l \neq k, l \neq k+1} \phi(\bar{x}, \bar{a}_l)^{l \in w_i}) \wedge \neg\phi(\bar{x}, \bar{a}_{k+1}) \wedge \phi(\bar{x}, \bar{a}_k)].$$

Define the formula $\psi(\bar{x}, \bar{y}, \bar{z})$, where $\bar{z} = \bar{z}_0, \dots, \bar{z}_{k-1}, \bar{z}_{k+2}, \dots, \bar{z}_{n-1}$ by

$$\chi(\bar{x}) \wedge \Big(\bigwedge_{l<n, l\neq k, l\neq k+1} \phi(\bar{x}, \bar{z}_l)^{l\in w_i} \Big) \wedge \phi(\bar{x}, \bar{y}).$$

To conclude the proof we show that $\psi(\bar{x}, \bar{y}, \bar{z})$ has the p-strict order property:

Let $m < \omega$ be given. For every $j < m$ we let

$$\bar{c}^j = \bar{a}_{k+j}\hat{\ }\bar{a}_0\hat{\ }\bar{a}_1\hat{\ }\dots\hat{\ }\bar{a}_{k-1}\hat{\ }\bar{a}_{m+k+2}\hat{\ }\dots\hat{\ }\bar{a}_{m+n-1},$$

where $\bar{a}_{k+j}$ is to be substituted for the $\bar{y}$-variable, and $\bar{a}_0\hat{\ }\dots\hat{\ }\bar{a}_{m+n-1}$ is to be substituted for the variable $\bar{z}_0\hat{\ }\dots\hat{\ }\bar{z}_{k-1}\hat{\ }\bar{z}_{k+2}\hat{\ }\dots\hat{\ }\bar{z}_{n-1}$.

It is enough to check that

$$\models \exists\bar{x}(\neg\psi(\bar{x}, \bar{c}^{j_1}) \wedge \psi(\bar{x}, \bar{c}^{j_2})) \quad \text{if and only if} \quad j_1 < j_2.$$

For convenience, denote by $\bar{c}$ the following sequence $\bar{a}_0 \dots \bar{a}_{k-1}\bar{a}_{k+2} \dots \bar{a}_{n-1}$. By indiscernibility of $\{\bar{a}_i \mid i < \omega\}$, we have the following equalities

(***)
$$\begin{aligned} \operatorname{tp}(\bar{c}^{j_1}, \bar{c}^{j_2}/\operatorname{dom}(p)) &= \operatorname{tp}(\bar{a}_k\bar{c}, \bar{a}_{k+1}\bar{c}/\operatorname{dom}(p)), && \text{if } j_1 < j_2,\\ &= \operatorname{tp}(\bar{a}_k\bar{c}, \bar{a}_k\bar{c}/\operatorname{dom}(p)), && \text{if } j_1 = j_2,\\ &= \operatorname{tp}(\bar{a}_{k+1}\bar{c}, \bar{a}_k\bar{c}/\operatorname{dom}(p)), && \text{if } j_1 > j_2. \end{aligned}$$

We distinguish three cases.

If $j_1 < j_2$. By the first equality of (***), it suffices to check $\models \exists\bar{x}(\neg\psi(\bar{x}, \bar{a}_k, \bar{c}) \wedge \psi(\bar{x}, \bar{a}_{k+1}, \bar{c})))$. This is true since $p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_l)^{l\in w_i}, \neg\phi(\bar{x}, \bar{a}_k), \phi(\bar{x}, \bar{a}_{k+1})) \mid l < n, l \neq k, l \neq k+1\}$ is consistent, by (†).

If $j_1 = j_2$, then by the second equality of (***) $\models \exists\bar{x}(\neg\psi(\bar{x}, \bar{c}^{j_1}) \wedge \psi(\bar{x}, \bar{c}^{j_2}))$ if and only if $\models \exists\bar{x}(\neg\psi(\bar{x}, \bar{a}_k, \bar{c}) \wedge \psi(\bar{x}, \bar{a}_k, \bar{c}))$. Therefore, we have $\models \neg[\exists\bar{x}(\neg\psi(\bar{x}, \bar{c}^{j_1}) \wedge \psi(\bar{x}, \bar{c}^{j_2}))]$.

If $j_1 > j_2$, then use the third equality of (***), and (‡) to conclude that $\models \neg[\exists\bar{x}(\neg\psi(\bar{x}, \bar{c}^{j_1}) \wedge \psi(\bar{x}, \bar{c}^{j_2}))]$. □

The next corollary is the local version of Shelah's Trichotomy Theorem (see Theorem II 4.7 of [**Sh b**]).

COROLLARY II.4.4. *Assume that p has no parameters. The type p has the order property if and only if p has the independence property or p has the strict order property.*

PROOF. Suppose p has the order property. Then some formula ϕ has the p-order property. Thus, by Theorem .4.3 p has the independence property or the strict order property (without parameters, since $\operatorname{dom}(p) = \emptyset$).

The converse is Proposition .4.2. □

The following is an improvement of Theorem II.2.20 of [**Sh b**].

LEMMA II.4.5. *The following conditions are equivalent*

(1) p *does not have the independence property;*
(2) *For every infinite indiscernible sequence* $I \subseteq p(\mathfrak{C})$ *and for every* $\phi(\bar{x}, \bar{y}) \in L(T)$ *there exists an integer* $n_\phi < \omega$ *such that for every* $\bar{c} \in p(M)$ *either*

$$|\{\bar{a} \in I \ : \models \phi[\bar{a}, \bar{c}]\}| \leq n_\phi \quad \textit{or} \quad |\{\bar{a} \in I \ : \models \neg\phi[\bar{a}, \bar{c}]\}| \leq n_\phi.$$

PROOF. (1) $\Rightarrow$ (2) Let $\phi(\bar{x}, \bar{y})$ and I be given. Suppose (2) fails. Then, by the compactness theorem, we can find $\bar{c} \in p(\mathfrak{C})$ and a sequence $\{\bar{a}_i \mid i < \omega\} \subseteq p(\mathfrak{C})$ indiscernible over $\mathrm{dom}(p)$ such that

(*) $\quad |\{i < \omega \ : \models \phi[\bar{a}_i, \bar{c}]\}| = \aleph_0 \quad \text{and} \quad |\{i < \omega \ : \models \neg\phi[\bar{a}_i, \bar{c}]\}| = \aleph_0.$

We are going to show that $\phi(\bar{x}, \bar{y})$ has the p-independence property. Let $n < \omega$ and $w \subseteq n$. It is enough to show that

(**) $\quad p(\bar{y}) \cup \{\phi(\bar{a}_i, \bar{y})^{i \in w} \mid i < n\}$ is consistent.

To see this, construct a strictly increasing sequence of integers $\langle i_m \mid m < n\rangle$ such that $\mathfrak{C} \models \phi[\bar{a}_{i_m}, \bar{c}]$ if and only if $m \in w$. This is easily done by induction using (*). By indiscernibility of $\{\bar{a}_i \mid i < \omega\}$, (**) holds if and only the set of formulas $p(\bar{y}) \cup \{\phi(\bar{a}_{i_m}, \bar{y})^{m \in w} \mid m < n\}$ is consistent, which is the case, since it is realized by $\bar{c}$.

(2) $\Rightarrow$ (1) Suppose that $\phi(\bar{x}, \bar{y})$ has the p-independence property and $I = \{\bar{a}_i \mid i < \omega\} \subseteq p(\mathfrak{C})$ demonstrate this. Then, for each $n < \omega$, and for each $w \subseteq n$ we have

$$p(\bar{x}) \cup \{\phi(\bar{x}, \bar{a}_i)^{i \in w} \mid i < n\} \text{ is consistent.}$$

Hence, by the compactness theorem, we can find an indiscernible sequence $J = \{\bar{b}_i \mid i < \omega\} \subseteq p(\mathfrak{C})$ and $\bar{c} \in p(\mathfrak{C})$ such that both $\{i < \omega \ : \models \phi[\bar{c}, \bar{b}_i]\}$ and $\{i < \omega \ : \models \neg\phi[\bar{c}, \bar{b}_i]\}$ are infinite. Hence both $\phi(\bar{c}, \bar{y})$ and $\neg\phi(\bar{c}, \bar{y})$ belong to $\mathrm{Av}(J, \bar{c})$. Thus $\mathrm{Av}(J, \bar{c})$ is not consistent, which contradicts (2). $\square$

We can now answer the question of when averages are well-defined and characterize types without the independence property.

THEOREM II.4.6. *The following conditions are equivalent:*

(1) p *does not have the independence property;*
(2) *For every infinite indiscernible sequence* $I \subseteq p(\mathfrak{C})$ *and every subset* $A \subseteq p(\mathfrak{C})$ *the average* $\mathrm{Av}(I, A)$ *is a complete type. Furthermore,* $\mathrm{Av}(I, A) \in S_p(A)$.

PROOF. (1) $\Rightarrow$ (2): Let $I, A \subseteq p(\mathfrak{C})$ and I be an infinite indiscernible sequence. By Lemma .4.5 (1) $\Rightarrow$ (2), we have that $\mathrm{Av}(I, A) \in S(A)$. Furthermore, since $I \subseteq p(\mathfrak{C})$, we have $\mathrm{Av}(I, A) \in S_p(A)$.

(2) $\Rightarrow$ (1): We prove the contrapositive. Suppose that p has the independence property. Then, by Lemma .4.5 (2) $\Rightarrow$ (1), there exists an infinite indiscernible sequence $I \subseteq p(\mathfrak{C})$ and $\bar{a} \in p(\mathfrak{C})$ such that both $\phi(\bar{x}, \bar{a})$ and $\neg\phi(\bar{x}, \bar{a})$ belong to $\mathrm{Av}(I, \bar{a})$. This contradicts (2). $\square$

We now give an easy characterization of stable types in simple theories. The following fact is due to Shelah and appears in [**Sh93**].

FACT II.4.7. *If T is simple then T does not have the strict order property.*

We make use of the following observation.

PROPOSITION II.4.8. *If the formula $\phi(\bar{x}, \bar{y}, \bar{b})$ with parameter $\bar{b} \in \mathfrak{C}$ has the p-strict order property, then T has the strict order property.*

PROOF. We show that T has the strict order property, by showing that $\phi(\bar{x}, \bar{y}, \bar{z})$ has the strict order property. But, for each $n < \omega$, there exists $\{\bar{a}_i \mid i < n\} \subseteq p(\mathfrak{C})$ such that

$$\models \exists \bar{x}(\neg\phi(\bar{x}, \bar{a}_i, \bar{b}) \wedge \phi(\bar{x}, \bar{a}_j, \bar{b})) \quad \text{if and only if} \quad i < j < n.$$

Thus, for each $n < \omega$, the set $\{\bar{a}_i\hat{\ }\bar{b} \mid i < n\}$ shows that $\phi(\bar{x}, \bar{y}, \bar{z})$ has the strict order property. $\square$

COROLLARY II.4.9. *Let T be simple. The following conditions are equivalent:*

(1) *p is stable;*
(2) *For every infinite indiscernible sequence $I \subseteq p(\mathfrak{C})$ and for every $A \subseteq p(\mathfrak{C})$, we have $\mathrm{Av}(I, A) \in S_p(A)$.*

PROOF. (1) $\Rightarrow$ (2): Let p be stable, then p does not have the order property by Theorem .2.4. Hence p does not have the independence property by Proposition .4.2. Hence, (2) follows from Theorem .4.6.

(2) $\Rightarrow$ (1): Suppose p is not stable. Then p has the order property by Theorem .2.4. Thus, p has either the independence property or the strict order property (maybe with parameters) by Theorem .4.3. Since T is simple, by Fact .4.7, we have that T does not have the strict order property. But, if p has the strict order property with parameters, then T has the strict order property by Proposition .4.8. Therefore, p has the independence property, and so (2) fails by Lemma .4.6. $\square$

CHAPTER III

Ranks and pregeometries in finite diagrams

The problem of categoricity has been a driving force in model theory since its early development in the late 1950's. For the countable first order case, M. Morley in 1965 [**Mo**] introduced a rank which captures $\aleph_0$-stability, and used it to construct prime models and give a proof of Łoš conjecture. In 1971, J. Baldwin and A. Lachlan [**BaLa**] gave an alternative proof using the fact that algebraic closure induces a pregeometry on strongly minimal sets. Their proof generalizes ideas from Steinitz's famous 1910 theorem of categoricity for algebraically closed fields. Łoš conjecture for uncountable languages was solved in 1970 by S. Shelah [**Sh**] introducing a rank which corresponds to the superstable case. Later, Shelah discovered a dependence relation called forking and more general pregeometries, and since then, these ideas have been extended to more and more general first order contexts, each of them corresponding to a specific rank: $\aleph_0$-stable, superstable, stable, and simple.

The problem of categoricity for nonelementary classes is quite considerably more involved. In 1971, H. J. Keisler [**Ke**] proved a categoricity theorem for Scott sentences $\psi \in L_{\omega_1\omega}$, which in a sense generalizes Morley's Theorem. To achieve this, Keisler made the additional assumption that ψ admits $\aleph_1$-homogeneous models. Later, Shelah produced an example of a categorical $\psi \in L_{\omega_1\omega}$ that does not have any $\aleph_1$-homogeneous model, using an example of L. Marcus [**Mr**]. So this is not the most general case. Since then, many of Shelah's hardest papers in model theory have been dedicated the categoricity problem and to the development of general classification theory for nonelementary classes. Among the landmarks, one should mention [**Sh48**] about sentences in $L_{\omega_1\omega}(Q)$ which answers a question of Harvey Friedman's list [**Fr**]. In [**Sh87a**] and [**Sh87b**] a version of Morley's Theorem is proved for a special kind of formulas $\psi \in L_{\omega_1\omega}$ which are called excellent. It is noteworthy that to deal with these nonelementary classes, these papers introduced several crucial ideas, among them stable amalgamation, 2-goodness and others, which are now essential parts of the proof of the "Main Gap" for first order, countable theories. Later, R. Grossberg and B. Hart continued the classification of excellent classes and gave a proof of the Main Gap for those classes [**GrHa**]. H. Kierstead also continued the study of sentences in $L_{\omega_1\omega}(Q)$ [**Ki**]. He introduced a generalization of strongly minimal formulas by replacing "nonalgebraic" by "there exists uncountably many" and obtained results about countable models of these classes using [**Sh48**]. In [**Sh300**], Shelah began the classification theory for universal classes (see also ICM 1986/videotape) and is currently working on a book entirely dedicated to them. He also started the classification of classes in a context somewhat more general than $\mathrm{PC}(T_1, T, \Gamma)$, see [**Sh88**], [**Sh576**] and [**Sh 600**]. In a related work, Grossberg started studying the classification of the class of models of ψ, for $\psi \in L_{\lambda^+\omega}$, under the assumption that there exists a "Universal Model" for ψ and studied relatively saturated substructures (see [**Gr1**] and [**Gr2**]). This seems to be a natural hypothesis, as we discussed in Chapter I.

There are several striking differences between the problem of categoricity for first order and the nonelementary case. First, it appears that classification for nonelementary classes is sensitive to the axioms of set theory. Second, the methods used are heavily combinatorial: there is no "forking" (though splitting and strong splitting are sometimes well-behaved), and the presence of pregeometries to understand systematically models of a given class is scarce. (A nice example of pregeometries is hidden in the last section of [**Sh48**] and only [**Ki**] has used them to study countable models.) However, stability was not developed originally for first order. As we saw in Chapter I, in 1970, Shelah published [**Sh3**], where he introduced some of the most fundamental ideas of classification theory (stability, splitting of types, existence of indiscernibles, several notions of prime models and so on). Let us describe Shelah's original definitions in this context (as opposed to the ones we presented in Chapter I). He considered classes of models which omit all types in $D(T) \setminus D$, for a fixed *diagram* $D \subseteq D(T)$. This class is usually denoted $\mathrm{EC}(T, \Gamma)$, where Γ stands for $D(T) \setminus D$. He made assumptions of two kinds (explicitly in his definition of stability): (1) restriction on the cardinality of the space of types realizable by the models, and (2) existence of models realizing many types. In fact, the context studied by Keisler in his categoricity result for $L_{\omega_1\omega}$, turns out to be the $\aleph_0$-stable case in the above sense.

In retrospect, it seems that what prevented the emergence of a smooth theory for $\aleph_0$-stable diagrams is the absence of a rank like Morley's rank. Considering the success of the use of pregeometries to understand models in the first order $\aleph_0$-stable case, if one hopes to lift these ideas to more general contexts, it appears that $\aleph_0$-stable diagrams constitute a natural test case. This is the main goal of this chapter. We try to develop what Shelah calls the structure part of the theory for the class $\mathrm{EC}(T, \Gamma)$, under the assumption that it is $\aleph_0$-stable (in the sense of [**Sh3**]). In fact, as in [**Sh54**], we assume that $\mathrm{EC}(T, \Gamma)$ contains a large homogeneous model (which follows from Shelah's original definition of stability for $\mathrm{EC}(T, \Gamma)$, see Theorem 3.4. in [**Sh3**]), so that the stability assumptions only deal with the cardinality of the spaces of types. This hypothesis allows us to do all the work in ZFC, in contrast to [**Sh48**], [**Sh87a**], [**Sh87b**] or [**Ki**] for example.

The chapter is organized as follows.

In Section 1, we introduce a rank for this framework which captures $\aleph_0$-stability (it does not generalize Morley rank, but generalizes what Shelah calls $R[p, L, 2]$). This rank differs from previously studied ranks in two ways: (1) it allows us to deal with general diagrams (as opposed to the atomic case or the first order case) and (2) the definition is relativized to a given set (which allows us to construct prime models). By analogy with the first order case, we call D totally transcendental when the rank is bounded. For the rest of the chapter, we only consider totally transcendental D, and we make no assumption on the cardinality of T. We study basic properties of this rank and introduce the notion of stationarity.

In Section 2, we examine the natural dependence relation that it induces on the subsets of the models. We are then able to obtain many of the classical properties of forking, which we summarize in Theorem .2.3. We also obtain stationary types with respect to this dependence relation, and they turn out to behave well: they satisfy in addition the symmetry property, and can be represented by averages.

In Section 3, we focus on pregeometries. Regular types are defined in the usual manner (but with this dependence relation instead of forking, of course), and the dependence relation on the set of realizations of a regular type yields a pregeometry. We can show that stationary types of minimal rank are regular, and this is used to show that they exist very often. We also consider a more concrete kind of regular types, which are called minimal. They could be defined independently by replacing "nonalgebraic" by "realized outside any model which contains the set of parameters" in the usual definition of strongly minimal formulas. (This can be done for any suitable class of models, as in the last section of [**Sh48**].) We could show directly that the natural closure operator induces a pregeometry on the set of realizations in any $(D, \aleph_0)$-homogeneous model. We choose not to do this, and instead we consider minimal types only when the natural dependence relation coincides with the one given by the rank. This allows us to use the results we have already obtained and have a picture which is conceptually similar to the first order totally transcendental case (where strongly minimal types are stationary and regular, and the unique nonforking extension is also the unique nonalgebraic one). Another reason is that the proofs are identical to those which use the rank, and this presentation permits us to skip them.

In Section 4, we use the rank to prove the existence of prime models for the class $\mathcal{K}$ of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram (this improves parts of Theorems 5.3 and 5.10 of [**Sh3**]).

In Section 5, we first prove a version of Chang's Conjecture for the class $\mathcal{K}$ (Theorem .5.2). We then introduce unidimensionality for diagrams. We are able to adapt techniques of Baldwin-Lachlan [**BaLa**] to this context for the categoricity proof. In fact, we obtain a picture strikingly similar to the first order totally transcendental case. (1) If D is totally transcendental, then $\mathcal{K}$ is categorical in some $\lambda > |T| + |D|$ if and only if $\mathcal{K}$ is categorical in every $\lambda > |T| + |D|$ if and only if every model of $\mathcal{K}$ is prime and minimal over the set of realizations of a minimal type if and only if every model of $\mathcal{K}$ of cardinality $> |T| + |D|$ is D-homogeneous. (2) If D is totally transcendental and if there is a model of $\mathcal{K}$ of cardinality above

$|T| + |D|$ which is not D-homogeneous, then for any $|T| + |D| \leq \mu \leq \lambda$, there exists maximally (D, μ)-homogeneous models in $\mathcal{K}$ of cardinality λ (see the definition below). If T is countable this implies, in particular, that for each ordinal α the class $\mathcal{K}$ has at least $|\alpha|$ models of cardinality $\aleph_\alpha$. (3) When $|T| < 2^{\aleph_0}$, the categoricity assumption on $\mathcal{K}$ implies that D is totally transcendental, if D is the set of isolated types of T. As a byproduct, this gives an alternative proof to Keisler's theorem which works so long as $|T| < 2^{\aleph_0}$ (whereas Keisler's soft $L_{\omega_1 \omega}$ methods do not generalize to uncountable languages).

Using regular types and prime models, we will give in chapter IV a decomposition theorem for the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram D, which follows from a more general abstract decomposition theorem.

III.1. Rank and stationary types

The framework of this chapter is the class of models of a finite diagram. The notation is as in the first chapter. We first introduce a rank for the class of D-models which generalizes the rank from [**Sh87a**]. We then prove basic properties of it which show that it is well-behaved and is natural for this class.

DEFINITION III.1.1. For any set of formulas $p(\bar{x}, \bar{b})$ with parameters in $\bar{b}$, and A a subset of $\mathfrak{C}$ containing $\bar{b}$, we define the *rank* $R_A[p]$. The rank $R_A[p]$ will be an ordinal, -1, or ∞ and we have the usual ordering $-1 < \alpha < \infty$ for any ordinal α. We define the relation $R_A[p] \geq \alpha$ by induction on α.

(1) $R_A[p] \geq 0$ if $p(\bar{x}, \bar{b})$ is realized in $\mathfrak{C}$;
(2) $R_A[p] \geq \delta$, when δ is a limit ordinal, if $R_A[p] \geq \alpha$ for every $\alpha < \delta$;
(3) $R_A[p] \geq \alpha + 1$ if the following two conditions hold:
   (a) There is $\bar{a} \in A$ and a formula $\phi(\bar{x}, \bar{y})$ such that
   $$R_A[p \cup \phi(\bar{x}, \bar{a})] \geq \alpha \quad \text{and} \quad R_A[p \cup \neg\phi(\bar{x}, \bar{a})] \geq \alpha;$$
   (b) For every $\bar{a} \in A$ there is $q(\bar{x}, \bar{y}) \in D$ such that
   $$R_A[p \cup q(\bar{x}, \bar{a})] \geq \alpha.$$

We write:

$R_A[p] = -1$ if p is not realized in $\mathfrak{C}$;
$R_A[p] = \alpha$ if $R_A[p] \geq \alpha$ but it is not the case that $R_A[p] \geq \alpha + 1$;
$R_A[p] = \infty$ if $R_A[p] \geq \alpha$ for every ordinal α.

For any set of formulas $p(\bar{x})$ over $A \subseteq \mathfrak{C}$, we let
$$R_A[p] = \min\{\, R_A[q] \mid q \subseteq p \restriction B, B \subseteq \operatorname{dom}(p),\ B \text{ finite } \}.$$

We write $R[p]$ for $R_{\mathfrak{C}}[p]$.

We need several basic properties of this rank. Some of them are purely technical and are stated here for future reference. Most of them are analogs of the usual properties for ranks in the first order case, with the exception of (2) and (3). The proofs vary from the first order context because of the second clause at successor stage, but they are all routine inductions.

LEMMA III.1.2. *Let A be a subset of $\mathfrak{C}$.*

(1) $R_A[\{ \bar{x} = \bar{c} \}] = 0$.
(2) *If p is over a finite set or p is complete, then $R_A[p] \geq 0$ if and only if there is $B \subseteq A$ and $q \in S_D(B)$ such that $p \subseteq q$.*
(3) *Let A be $(D, \aleph_0)$-homogeneous and let $\bar{a}, \bar{b} \in A$. If $\operatorname{tp}(\bar{a}/\emptyset) = \operatorname{tp}(\bar{b}/\emptyset)$ then $R_A[p(\bar{x}, \bar{b})] = R_A[p(\bar{x}, \bar{a})]$.*
(4) (Monotonicity) *If $p \vdash q$ and p is over a finite set, then $R_A[p] \leq R_A[q]$.*
(5) *If p is over $B \subseteq A$ and $f \in \operatorname{Aut}(\mathfrak{C})$ then $R_A[p] = R_{f(A)}[f(p)]$.*
(6) (Monotonicity) *If $p \subseteq q$ then $R_A[p] \geq R_A[q]$.*
(7) (Finite Character) *There is a finite $B \subseteq \operatorname{dom}(p)$ such that*

$$R_A[p] = R_A[p \restriction B].$$

(8) *If $R_A[p] = \alpha$ and $\beta < \alpha$, then there is q over A such that $R_A[q] = \beta$.*
(9) *If $R_A[p] \geq (|A| + 2^{|T|})^+$, then $R_A[p] = \infty$.*
Moreover, when A is $(D, \aleph_0)$-homogeneous, the bound is $(2^{|T|})^+$.

PROOF. (1) Trivial

(2) Suppose $p \subseteq q \in S_D(B)$, and $B \subseteq A$. Then q is realized in $\mathfrak{C}$, since $\mathfrak{C}$ is (D, χ)-homogeneous, and $q \in S_D(B)$. Hence p is realized in $\mathfrak{C}$ and $R_A[p] \geq 0$.

For the converse, if p is over a finite set, and $R_A[p] \geq 0$, then there is $\bar{c} \in \mathfrak{C}$ realizing p. Thus $\operatorname{tp}(\bar{c}/\operatorname{dom}(p))$ extends p and $\operatorname{tp}(\bar{c}/\operatorname{dom}(p)) \in S_D(\operatorname{dom}(p))$.

If p is complete, then there is $B \subseteq A$ such that $p \in S(B)$. Now let $\bar{c}$ (not necessarily in $\mathfrak{C}$) realize p. For every $\bar{b} \in B$, $R_A[p \restriction \bar{b}] \geq 0$, and so there is $\bar{c}' \in \mathfrak{C}$ realizing $p \restriction \bar{b}$. But $\operatorname{tp}(\bar{c}/\bar{b}) = p \restriction \bar{b} = \operatorname{tp}(\bar{c}'/\bar{b})$ since p is complete. Thus $\operatorname{tp}(\bar{c}\bar{b}/\emptyset) \in D$, so $p \in S_D(B)$.

(3) By symmetry, it is enough to show that for every ordinal α,

$$R_A[p(\bar{x}, \bar{b})] \geq \alpha \qquad \text{implies} \qquad R_A[p(\bar{x}, \bar{a})] \geq \alpha.$$

We prove that this is true for all types by induction on α.

- When $\alpha = 0$, we know that there is $\bar{c} \in \mathfrak{C}$ realizing $p(\bar{x}, \bar{a})$. Then, since $\operatorname{tp}(\bar{a}/\emptyset) = \operatorname{tp}(\bar{b}/\emptyset)$ and A is $(D, \aleph_0)$-homogeneous, there is $\bar{d} \in A$ such that $\operatorname{tp}(\bar{c}\bar{a}/\emptyset) = \operatorname{tp}(\bar{d}\bar{b}/\emptyset)$. But then $p(\bar{x}, \bar{b}) \subseteq \operatorname{tp}(\bar{d}/\bar{b})$. Hence $p(\bar{x}, \bar{b})$ is realized in $\mathfrak{C}$, so $R_A[p(\bar{x}, \bar{b})] \geq 0$.
- When α is a limit ordinal, this is true by induction.

- Suppose $R_A[p(\bar{x}, \bar{a})] \geq \alpha + 1$. First, there is $\bar{c} \in A$ and $\phi(\bar{x}, \bar{y}) \in L(T)$ such that both
$$R_A[p(\bar{x}, \bar{a}) \cup \phi(\bar{x}, \bar{c})] \geq \alpha \quad \text{and} \quad R_A[p(\bar{x}, \bar{a}) \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha.$$
Since A is $(D, \aleph_0)$-homogeneous, there is $\bar{d} \in A$ such that $\text{tp}(\bar{c}\bar{a}/\emptyset) = \text{tp}(\bar{d}\bar{b}/\emptyset)$. Therefore by induction hypothesis, both
$$R_A[p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{d})] \geq \alpha \quad \text{and} \quad R_A[p(\bar{x}, \bar{b}) \cup \neg\phi(\bar{x}, \bar{d})] \geq \alpha.$$
Second, for every $\bar{d} \in A$, there is $\bar{c} \in A$ such that $\text{tp}(\bar{c}\bar{a}/\emptyset) = \text{tp}(\bar{d}\bar{b}/\emptyset)$. Since $R_A[p(\bar{x}, \bar{a})] \geq \alpha + 1$, there is $q(\bar{x}, \bar{y}) \in D$, such that $R_A[p(\bar{x}, \bar{a}) \cup q(\bar{x}, \bar{c})] \geq \alpha$. Therefore, by induction hypothesis, $R_A[p(\bar{x}, \bar{b}) \cup q(\bar{x}, \bar{d})] \geq \alpha$. This shows that $R_A[p(\bar{x}, \bar{b})] \geq \alpha + 1$.

(4) Suppose $p \vdash q$. By definition of the rank, we may choose $q_0 \subseteq q$ over a finite set, such that $R_A[q_0] = R_A[q]$. Hence, since $p \vdash q_0$, it is enough to show the lemma when q is over a finite set also. Write $p = p(\bar{x}, \bar{b}) \vdash q = q(\bar{x}, \bar{a})$. We show by induction on α that for every such pair of types over finite sets, we have
$$R_A[p(\bar{x}, \bar{b})] \geq \alpha \quad \text{implies} \quad R_A[q(\bar{x}, \bar{b})] \geq \alpha.$$

- For $\alpha = 0$, this is true by definition.
- For α a limit ordinal, this is true by induction.
- Suppose $R_A[p(\bar{x}, \bar{b})] \geq \alpha + 1$. On the one hand, there is $\bar{c} \in A$ and $\phi(\bar{x}, \bar{y}) \in L(T)$ such that both
$$R_A[p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{c})] \geq \alpha \quad \text{and} \quad R_A[p(\bar{x}, \bar{b}) \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha.$$
But
$$p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{c}) \vdash q(\bar{x}, \bar{a}) \cup \phi(\bar{x}, \bar{c})$$
and similarly
$$p(\bar{x}, \bar{b}) \cup \neg\phi(\bar{x}, \bar{c}) \vdash q(\bar{x}, \bar{a}) \cup \neg\phi(\bar{x}, \bar{c}),$$
so by induction hypothesis, both
$$R_A[q(\bar{x}, \bar{a}) \cup \phi(\bar{x}, \bar{c})] \geq \alpha \quad \text{and} \quad R_A[q(\bar{x}, \bar{a}) \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha.$$
On the other hand, given any $\bar{c} \in A$, there is $r(\bar{x}, \bar{y}) \in D$, such that $R_A[p(\bar{x}, \bar{b}) \cup r(\bar{x}, \bar{c})] \geq \alpha$. But
$$p(\bar{x}, \bar{b}) \cup r(\bar{x}, \bar{c}) \vdash q(\bar{x}, \bar{a}) \cup r(\bar{x}, \bar{c}),$$
so by induction hypothesis, $R_A[q(\bar{x}, \bar{a}) \cup r(\bar{x}, \bar{c})] \geq \alpha$. Hence $R_A[q(\bar{x}, \bar{a})] \geq \alpha + 1$.

(5) First, choose $q(\bar{x}, \bar{a}) \subseteq p$, such that $R_A[q] = R_A[p]$ (this is possible by definition of the rank). Similarly, since $f(q) \subseteq f(p)$, we could have chosen q so that in addition $R_{f(A)}[f(q)] = R_{f(A)}[f(p)]$. Now, by symmetry, it is enough to show that if $R_A[q] \geq \alpha$ then $R_{f(A)}[f(q)] \geq \alpha$.

- For $\alpha = 0$ or α a limit ordinal, it is obvious by definition.

- Suppose $\alpha = \beta + 1$. First, there exists $\phi(\bar{x}, \bar{b})$ such that

$$R_A[q \cup \phi(\bar{x}, \bar{b})] \geq \beta \quad \text{and} \quad R_A[q \cup \neg\phi(\bar{x}, \bar{b})] \geq \beta.$$

  Thus, by induction hypothesis, we have

$$R_{f(A)}[f(q) \cup \phi(\bar{x}, f(\bar{b}))] \geq \beta \quad \text{and} \quad R_{f(A)}[f(q) \cup \neg\phi(\bar{x}, f(\bar{b}))] \geq \beta.$$

  Second, notice that for every $\bar{b} \in f(A)$, there is $\bar{c} \in A$, such that $f(\bar{c}) = \bar{b}$. Since $R_A[q] \geq \beta+1$, there exists $r(\bar{x}, \bar{y}) \in D$, such that $R_A[q \cup r(\bar{x}, \bar{c})] \geq \beta$. Hence, by induction hypothesis, $R_{f(A)}[f(q) \cup r(\bar{x}, \bar{b})] \geq \beta$. This shows that $R_{f(A)}[f(q)] \geq \beta + 1$.

(6) This is immediate by definition of the rank.

(7) By definition of the rank, let $B \in \mathrm{dom}(p)$ and $q \subseteq p \restriction B$ be such that $R_A[q] = R_A[p]$. Now, clearly $q \subseteq p \restriction B \subseteq p$, so $R_A[q] \geq R_A[p \restriction B] \geq R_A[p]$ by Lemma 6. So $R_A[p \restriction B] = R[p]$.

(8) Suppose there is α_0 such that $R_A[p] \neq \alpha_0$ for every p. We prove by induction on $\alpha \geq \alpha_0$, that for no type p do we have $R_A[p] = \alpha$.

- For $\alpha = \alpha_0$, this is the definition of α_0.
- Now suppose that there is p such that $R_A[p] = \alpha+1$. By 7, we may assume that p is over a finite set. Then there is $\bar{c} \in A$ and $\phi(\bar{x}, \bar{y}) \in L(T)$ such that both

$$R_A[p \cup \phi(\bar{x}, \bar{c})] \geq \alpha \quad \text{and} \quad R_A[p \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha.$$

  But by induction hypothesis, neither can be equal to α, so we must have both

$$R_A[p \cup \phi(\bar{x}, \bar{c})] \geq \alpha + 1 \quad \text{and} \quad R_A[p \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha + 1.$$

  Similarly, given any $\bar{c} \in A$, there is $q(\bar{x}, \bar{y}) \in D$, such that $R_A[p \cup q(\bar{x}, \bar{c})] \geq \alpha$. But, by induction hypothesis, we cannot have $R_A[p \cup q(\bar{x}, \bar{c})] = \alpha$, so $R_A[p \cup q(\bar{x}, \bar{c})] \geq \alpha+1$. But this shows that $R_A[p] \geq \alpha+2$, a contradiction.
- Suppose $\alpha > \alpha_0$ is a limit ordinal. Then $\alpha \geq \alpha_0 + 1$, so as in the previous case, there is $\bar{c} \in A$ and $\phi(\bar{x}, \bar{y}) \in L(T)$ such that both

$$R_A[p \cup \phi(\bar{x}, \bar{c})] \geq \alpha_0 \quad \text{and} \quad R_A[p \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha_0.$$

  But by induction hypothesis, for no β such that $\alpha > \beta \geq \alpha_0$ can we have $R_A[p \cup \phi(\bar{x}, \bar{c})] = \beta$ or $R_A[p \cup \neg\phi(\bar{x}, \bar{c})] = \beta$, so necessarily since α is a limit ordinal, we have

$$R_A[p \cup \phi(\bar{x}, \bar{c})] \geq \alpha \quad \text{and} \quad R_A[p \cup \neg\phi(\bar{x}, \bar{c})] \geq \alpha.$$

  Similarly, for any $\bar{c} \in A$, there is $q(\bar{x}, \bar{y}) \in D$, such that $R_A[p \cup q(\bar{x}, \bar{c})] \geq \alpha_0$ and hence by induction hypothesis $R_A[p \cup q(\bar{x}, \bar{c})] > \beta$ for any $\alpha_0 \leq \beta < \alpha$ so since α is a limit ordinal, we have $R_A[p \cup q(\bar{x}, \bar{c})] \geq \alpha$. But this shows that $R_A[p] \geq \alpha + 1$, a contradiction.

(9) By the previous lemma, it is enough to find $\alpha_0 < (|A| + 2^{|T|})^+$, (respectively $< (2^{|T|})^+$ if A is a $(D, \aleph_0)$-homogeneous model) such that

(*) $$R_A[p] \neq \alpha_0 \qquad \text{for every type over } A.$$

We do this by counting the number of possible values for the rank. By 7 it is enough to count the values achieved by types over finite subsets of A. But there are at most $|A|^{<\aleph_0} \leq |A| + \aleph_0$ finite subsets of A, and given any finite subset, there are only $2^{|T|}$ distinct types over it. Hence there are at most $|A| + 2^{|T|}$ many different ranks, and so by the pigeonhole principle (*) holds for some $\alpha_0 < (|A| + 2^{|T|})^+$.

When A is a $(D, \aleph_0)$-homogeneous model, the bound can be further reduced by a use of 3, since only the type of each of those finite subset of A is relevant. □

The next lemma shows that the rank is especially well-behaved when the parameter A is the universe of a $(D, \aleph_0)$-homogeneous model. This is used in particular to study $(D, \aleph_0)$-homogeneous models in the last two sections. Recall that $R[p]$ is an abbreviation for $R_{\mathfrak{C}}[p]$.

LEMMA III.1.3. (1) *If p is over a subset of a $(D, \aleph_0)$-homogeneous model M, then $R_M[p] = R[p]$.*
(2) *If p is over $M_1 \cap M_2$, with M_l $(D, \aleph_0)$-homogeneous, for $l = 1, 2$, we have $R_{M_1}[p] = R_{M_2}[p]$.*
(3) *If $q(\bar{x}, \bar{a}_1)$ and $q(\bar{x}, \bar{a}_2)$ are sets of formulas, with $a_1 \in M_1$ and $\bar{a}_2 \in M_2$ satisfying $\mathrm{tp}(\bar{a}_1/\emptyset) = \mathrm{tp}(\bar{a}_2/\emptyset)$, then $R_{M_1}[q(\bar{x}, \bar{a}_1)] = R_{M_2}[q(\bar{x}, \bar{a}_2)]$.*

PROOF. (1) First, by Finite Character, we may assume that p is over a finite set. Now we show by induction on α that

$$R_M[p] \geq \alpha \qquad \text{implies} \qquad R[p] \geq \alpha.$$

When $\alpha = 0$ or α is a limit, it is clear. Suppose $R_M[p] \geq \alpha + 1$. Then there is $\bar{b} \in M$ and $\phi(\bar{x}, \bar{y})$ such that both

$$R_M[p \cup \phi(\bar{x}, \bar{b})] \geq \alpha \quad \text{and} \quad R_M[p \cup \neg\phi(\bar{x}, \bar{b})] \geq \alpha.$$

By induction hypothesis, we have

$$R[p \cup \phi(\bar{x}, \bar{b})] \geq \alpha \quad \text{and} \quad R[p \cup \neg\phi(\bar{x}, \bar{b})] \geq \alpha.$$

Further, if $\bar{b} \in \mathfrak{C}$, choose $\bar{b}' \in M$, such that $\mathrm{tp}(\bar{b}/\bar{a}) = \mathrm{tp}(\bar{b}'/\bar{a})$. Since $R_M[p] \geq \alpha + 1$, there is $q(\bar{x}, \bar{y}) \in D$ such that $R_M[p \cup q(\bar{x}, \bar{b}')] \geq \alpha$. Thus, since $\mathfrak{C}$ is $(D, \aleph_0)$-homogeneous, by induction hypothesis we have $R[p \cup q(\bar{x}, \bar{b}')] \geq \alpha$, and so by Lemma .1.2 3 $R[p \cup q(\bar{x}, \bar{b})] \geq \alpha$. Hence $R[p] \geq \alpha + 1$.

For the converse, similarly by induction on α we show that

$$R[p] \geq \alpha \qquad \text{implies} \qquad R_M[p] \geq \alpha.$$

Again, for $\alpha = 0$ or α a limit, it is easy. Suppose $R[p] \geq \alpha + 1$. Then there is $\bar{b} \in \mathfrak{C}$ and $\phi(\bar{x}, \bar{y})$ such that both

$$R[p \cup \phi(\bar{x}, \bar{b})] \geq \alpha \quad \text{and} \quad R[p \cup \neg\phi(\bar{x}, \bar{b})] \geq \alpha.$$

Since M is $(D, \aleph_0)$-homogeneous, there exists $\bar{b}' \in M$, such that $\mathrm{tp}(\bar{b}/\bar{a}) = \mathrm{tp}(\bar{b}'/\bar{a})$. By Lemma .1.2 3, we have

$$R[p \cup \phi(\bar{x}, \bar{b}')] \geq \alpha \quad \text{and} \quad R[p \cup \neg\phi(\bar{x}, \bar{b}')] \geq \alpha.$$

Hence, by induction hypothesis, we have (since $\bar{b}' \in M$)

$$R_M[p \cup \phi(\bar{x}, \bar{b}')] \geq \alpha \quad \text{and} \quad R_M[p \cup \neg\phi(\bar{x}, \bar{b}')] \geq \alpha.$$

Also, for any $\bar{b} \in M$, since $\bar{b} \in \mathfrak{C}$ there is $q(\bar{x}, \bar{y}) \in D$ such that $R[p \cup q(\bar{x}, \bar{b})] \geq \alpha$. By induction hypothesis, we have $R_M[p \cup q(\bar{x}, \bar{b})] \geq \alpha$, which finishes to show that $R_M[p] \geq \alpha + 1$ and completes the proof.

(2) By (1) applied twice, $R_{M_1}[p] = R[p] = R_{M_2}[p]$.

(3) Since $R_{M_1}[q(\bar{x}, \bar{a}_1)] = R[q(\bar{x}, \bar{a}_1)] = R[q(\bar{x}, \bar{a}_2)] = R_{M_2}[q(\bar{x}, \bar{a}_2)]$. □

We now show that the rank is bounded when D is $\aleph_0$-stable. When $D = D(T)$, D has bounded rank if and only if the theory T is totally transcendental. Therefore, the rank may be bounded for diagrams that are*not* $\aleph_0$-stable. See Theorem .1.13 for a precise converse.

THEOREM III.1.4. *If D is stable in λ for some $\aleph_0 \leq \lambda < 2^{\aleph_0}$ then $R_A[p] < \infty$, for every type p and every subset A of $\mathfrak{C}$.*

PROOF. We prove the contrapositive. Suppose there is a subset A of $\mathfrak{C}$ and a type p over A such that $R_A[p] = \infty$. We construct sets $A_\eta \subseteq A$ and types p_η, for $\eta \in {}^{<\omega}2$, such that:

(1) $p_\eta \in S_D(A_\eta)$;
(2) $p_\eta \subseteq p_\nu$ when $\eta < \nu$;
(3) A_η is finite;
(4) $p_{\eta\hat{}0}$ and $p_{\eta\hat{}1}$ are contradictory;
(5) $R_A[p_\eta] = \infty$;

This is possible: Let $\mu = (2^{|T|})^+$ if A is a $(D, \aleph_0)$-homogeneous model, and $\mu = (|A| + 2^{|T|})^+$ otherwise. The construction is by induction on $n = \ell(\eta)$.

- For $n = 0$, by Finite Character we choose first $\bar{b} \in A$, such that $R_A[p] = R_A[p \upharpoonright \bar{b}] = \infty$. Since $R_A[p \upharpoonright \bar{b}] = \infty$, in particular $R_A[p \upharpoonright \bar{b}] \geq \mu + 1$ so there exists $q(\bar{x}, \bar{y}) \in D$, such that $R_A[(p \upharpoonright \bar{b}) \cup q(\bar{x}, \bar{b})] \geq \mu$. But then $p \upharpoonright \bar{b} \subseteq q(\bar{x}, \bar{b})$, $q(\bar{x}, \bar{b}) \in S_D(\bar{b})$ and $R_A[q(\bar{x}, \bar{b})] \geq \mu$, so $R_A[q(\bar{x}, \bar{b})] = \infty$ by Lemma .1.2 9. Therefore, we let $A_{<>} = \bar{b}$ and $p_{<>} = q(\bar{x}, \bar{b})$ and the conditions are satisfied.

- Assume $n \geq 0$ and that we have constructed $p_\eta \in S_D(A_\eta)$ with $\ell(\eta) = n$. Since $R_A[p_\eta] = \infty$, in particular $R_A[p_\eta] \geq (\mu + 1) + 1$. Hence, there is $\bar{a}_\eta \in A$ and $\phi(\bar{x}, \bar{y})$ such that

(*) $$R_A[p_\eta \cup \phi(\bar{x}, \bar{a}_\eta)] \geq \mu + 1 \quad \text{and} \quad R_A[p_\eta \cup \neg\phi(\bar{x}, \bar{a}_\eta)] \geq \mu + 1.$$

Let $A_{\eta\hat{\ }0} = A_{\eta\hat{\ }1} = A_\eta \cup \bar{a}_\eta \subseteq A$. Both $A_{\eta\hat{\ }0}$ and $A_{\eta\hat{\ }1}$ are finite, so (*) and the definition of the rank imply that there are $q_l(\bar{x}, \bar{y}) \in D$ for $l = 0, 1$, such that

$$R_A[p_\eta \cup \phi(\bar{x}, \bar{a}_\eta) \cup q_0(\bar{x}, A_{\eta\hat{\ }0})] \geq \mu$$

and

$$R_A[p_\eta \cup \neg\phi(\bar{x}, \bar{a}_\eta) \cup q_1(\bar{x}, A_{\eta\hat{\ }1})] \geq \mu.$$

Define $p_{\eta\hat{\ }0} := p_\eta \cup \phi(\bar{x}, \bar{a}_\eta) \cup q_0(\bar{x}, A_{\eta\hat{\ }0})$ and $p_{\eta\hat{\ }1} := p_\eta \cup \neg\phi(\bar{x}, \bar{a}_\eta) \cup q_1(\bar{x}, A_{\eta\hat{\ }1})$. Then $p_{\eta\hat{\ }l} \in S_D(A_{\eta\hat{\ }l})$ since $q_l(\bar{x}, A_{\eta\hat{\ }l}) \in S_D(A_{\eta\hat{\ }l})$ and $A_{\eta\hat{\ }l}$ is finite for $l = 0, 1$. Moreover, $p_{\eta\hat{\ }0}$ and $p_{\eta\hat{\ }1}$ are contradictory by construction. Finally $R_A[p_{\eta\hat{\ }l}] = \infty$, since $R_A[p_{\eta\hat{\ }l}] \geq \mu$. Hence all the requirements are met.

This is enough: For each $\eta \in {}^\omega 2$, define $A_\eta := \bigcup_{n \in \omega} A_{\eta \restriction n}$ and $p_\eta := \bigcup_{n \in \omega} p_{\eta \restriction n}$. We claim that $p_\eta \in S_D(A_\eta)$. Certainly $p_\eta \in S(A_\eta)$, so we only need to show that if $\bar{c} \models p_\eta$, then $A_\eta \cup \bar{c}$ is a D-set ($\bar{c}$ is not assumed to be in $\mathfrak{C}$). It is enough to show that $\mathrm{tp}(\bar{c}\bar{d}/\emptyset) \in D$ for every finite $\bar{d} \in A_\eta$. But, if $\bar{d} \in A_\eta$, then there is $n \in \omega$ such that $\bar{d} \in A_{\eta \restriction n}$. Since $\bar{c} \models p_{\eta \restriction n}$ and $p_{\eta \restriction n} \in S_D(A_{\eta \restriction n})$, then $\bar{c} \cup A_{\eta \restriction n}$ is a D-set, and therefore $\mathrm{tp}(\bar{c}\bar{d}/\emptyset) \in D$, which is what we wanted. Now that we have established that $p_\eta \in S_D(A_\eta)$, since $\mathfrak{C}$ is (D, χ)-homogeneous, there is $\bar{c}_\eta \in \mathfrak{C}$ such that $\bar{c}_\eta \models p_\eta$. Now let $C = \bigcup_{\eta \in {}^{<\omega}2} A_\eta$. Then $|C| = \aleph_0$ and if $\eta \neq \nu \in {}^\omega 2$, then $\mathrm{tp}(\bar{c}_\eta/C) \neq \mathrm{tp}(\bar{c}_\nu/C)$, since p_η and p_ν are contradictory. Therefore $|S_D(C)| \geq 2^{\aleph_0}$, which shows that D is not stable in λ for any $\aleph_0 \leq \lambda < 2^{\aleph_0}$. □

REMARK III.1.5. Recall that in [**Sh3**], D is stable in λ if and only if there is a (D, λ^+)-homogeneous model and $|S_D(A)| \leq \lambda$ for all D sets A of cardinality at most λ (this is Definition 2.1 of [**Sh3**]). The proof of the previous theorem shows that if D is stable in λ for some $\aleph_0 \leq \lambda < 2^{\aleph_0}$ in the sense of [**Sh3**] then $R_A[p] < \infty$ for all D-set A and D-type p. In other words, we do not really need $\mathfrak{C}$ for this proof.

By analogy with the first order case (see [**Sh b**] definition 3.1), we introduce the following definition. It is not difficult to see that if $D = D(T)$, D is totally transcendental if and only if T is a totally transcendental first order theory. In general however, the underlying theory may be unstable (even if the diagram is categorical).

DEFINITION III.1.6. We say that D is *totally transcendental* if $R_A[p] < \infty$ for every subset A of $\mathfrak{C}$ and every type p over A.

For the rest of the chapter, we will make the following hypothesis. We will occasionally repeat that D is totally transcendental for emphasis.

HYPOTHESIS III.1.7. D is totally transcendental.

In what follows, we shall show that when D is totally transcendental, the rank affords a well-behaved dependence relation on the subsets of $\mathfrak{C}$. We first focus on a special kind of types.

DEFINITION III.1.8. A type p is called *stationary* if for every B containing $\mathrm{dom}(p)$ there is a unique type $p_B \in S_D(B)$, such that p_B extends p and $R[p] = R[p_B]$.

Note that since our rank is not an extension of Morley's rank, one does not necessarily get the usual stationary types when the class is first order. The argument in the next lemma is a generalization of Theorem 1.4.(1)(b) in [**Sh87a**]. Recall that $p \in S_D(A)$ *splits over* $B \subseteq A$ if there exists $\phi(\bar{x}, \bar{y})$ and $\bar{a}, \bar{c} \in A$ with $\mathrm{tp}(\bar{a}/B) = \mathrm{tp}(\bar{c}/B)$, such that $\phi(\bar{x}, \bar{a}) \in p$ and $\neg\phi(\bar{x}, \bar{c}) \in p$.

LEMMA III.1.9. *Let M be a $(D, \aleph_0)$-homogeneous model and let $\bar{d} \in \mathfrak{C}$ realizing $p(\bar{x}, \bar{b})$ such that*

(*) $$R[\mathrm{tp}(\bar{d}/M)] = R[p(\bar{x}, \bar{b})] = \alpha.$$

Then, for any $A \subseteq \mathfrak{C}$ containing $\bar{b}$ there is a unique $p_A \in S_D(A)$ extending $p(\bar{x}, \bar{b})$, such that

$$R[p_A] = R[p(\bar{x}, \bar{b})] = \alpha.$$

Moreover, p_A does not split over $\bar{b}$.

PROOF. We first prove uniqueness. Suppose two different types p_A and $q_A \in S_D(A)$ extend $p(\bar{x}, \bar{b})$ and

$$R[p_A] = R[p(\bar{x}, \bar{b})] = R[q_A] = \alpha.$$

Then there is $\phi(\bar{x}, \bar{c}) \in p_A$ such that $\neg\phi(\bar{x}, \bar{c}) \in q_A$. Thus, by Monotonicity,

$$R[p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{c})] \geq R_A[p] = \alpha \quad \text{and} \quad R[p(\bar{x}, \bar{b}) \cup \neg\phi(\bar{x}, \bar{c})] \geq R_A[p] = \alpha.$$

Further, for every $\bar{c} \in \mathfrak{C}$, there is $\bar{c}' \in M$ such that $\mathrm{tp}(\bar{c}/\bar{b}) = \mathrm{tp}(\bar{c}'/\bar{b})$ since M is $(D, \aleph_0)$-homogeneous. Now write $q(\bar{x}, \bar{c}') = \mathrm{tp}(\bar{d}/\bar{c}')$, and notice that

$$R[p(\bar{x}, \bar{b}) \cup q(\bar{x}, \bar{c}')] \geq R[\mathrm{tp}(\bar{d}/\bar{b} \cup \bar{c}')] \geq R[\mathrm{tp}(\bar{d}/M)] = \alpha.$$

But $q(\bar{x}, \bar{y}) \in D$ by definition and so by Lemma .1.2 (2) $R[p(\bar{x}, \bar{b}) \cup q(\bar{x}, \bar{c})] \geq \alpha$ since $\mathrm{tp}(\bar{c}\bar{b}/\emptyset) = \mathrm{tp}(\bar{c}'\bar{b}/\emptyset)$. But this shows that $R[p(\bar{x}, \bar{b})] \geq \alpha + 1$, which contradicts (*).

We now argue that p_A does not split over $\bar{b}$. Suppose it does, and choose a formula $\phi(\bar{x}, \bar{y}) \in L(T)$ and sequences $\bar{c}_0, \bar{c}_1 \in A$ with $\mathrm{tp}(\bar{c}_0/\bar{b}) = \mathrm{tp}(\bar{c}_1/\bar{b})$ such that $\phi(\bar{x}, \bar{c}_0)$ and $\neg\phi(\bar{x}, \bar{c}_1)$ both belong to p_A. Then by Monotonicity,

$$R[p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{c}_0)] \geq R_A[p] = \alpha \quad \text{and} \quad R[p(\bar{x}, \bar{b}) \cup \neg\phi(\bar{x}, \bar{c}_1)] \geq R_A[p] = \alpha.$$

But $\mathrm{tp}(\bar{c}_0/\bar{b}) = \mathrm{tp}(\bar{c}_1/\bar{b})$ so by Lemma .1.2(3) we have

$$R[p(\bar{x}, \bar{b}) \cup \phi(\bar{x}, \bar{c}_1)] \geq \alpha.$$

An argument similar to the uniqueness argument in the first paragraph finishes to show that $R[p(\bar{x}, \bar{b})] \geq \alpha + 1$, which is again a contradiction to (*).

For the existence, let p_A be the following set of formulas with parameters in A:

$$\{\, \phi(\bar{x}, \bar{c}) \mid \text{There exists } \bar{c}' \in M \text{ such that } \mathrm{tp}(\bar{c}/\bar{b}) = \mathrm{tp}(\bar{c}'/\bar{b}) \text{ and } \models \phi[\bar{d}, \bar{c}'] \,\}.$$

By the nonsplitting part, using the fact that M is $(D, \aleph_0)$-homogeneous, we have that $\mathrm{tp}(\bar{d}/M)$ does not split over $\bar{b}$. Hence $p_A \in S_D(A)$ and does not split over $\bar{b}$. We show that this implies that $R[p_A] = R[\mathrm{tp}(\bar{d}/M)] = \alpha$. Otherwise, since p_A extends $p(\bar{x}, \bar{b})$, by Monotonicity we must have $R[p_A] \leq \alpha$, and therefore $R[p_A] < \alpha$. Let us choose $\bar{b}' \in A$ such that $\bar{b} \subseteq \bar{b}'$ and $R[p_A] = R[p_A \restriction \bar{b}']$. For convenience, we write $q(\bar{x}, \bar{b}') := p_A \restriction \bar{b}'$, and so $R[q(\bar{x}, \bar{b}')] < \alpha$. Now since M is $(D, \aleph_0)$-homogeneous, we can choose $\bar{b}'' \in M$ such that $\mathrm{tp}(\bar{b}''/\bar{b}) = \mathrm{tp}(\bar{b}'/\bar{b})$. Hence

$$(**) \qquad R[q(\bar{x}, \bar{b}')] = R[q(\bar{x}, \bar{b}')] < \alpha.$$

But by definition of p_A, we must have $q(\bar{x}, \bar{b}') \subseteq \mathrm{tp}(\bar{d}/M)$, so by Monotonicity we have $R[q(\bar{x}, \bar{b}')] \geq R[\mathrm{tp}(\bar{d}/M)] = \alpha$, which contradicts (**). ☐

COROLLARY III.1.10. *The following conditions are equivalent:*

(1) $p \in S_D(A)$ *is stationary.*
(2) *There is a* $(D, \aleph_0)$*-homogeneous model* M *containing* A *and* $\bar{d} \in \mathfrak{C}$ *realizing* p *such that* $R[\mathrm{tp}(\bar{d}/M)] = R[p]$.

DEFINITION III.1.11. The type $p \in S_D(A)$ is *based on* B if p is stationary and $R[p] = R[p \restriction B]$.

REMARK III.1.12.

(1) If p is stationary, there is a finite $B \subseteq \mathrm{dom}(p)$ such that p is based on B.
(2) If p is based on B, then $p \restriction B$ is also stationary and p is the only extension of $p \restriction B$ such that $R[p] = R[p \restriction B]$.
(3) If p is stationary and $\mathrm{dom}(p) \subseteq A \subseteq B$, then $p_A = p_B \restriction A$.
(4) Suppose $\mathrm{tp}(\bar{a}/\emptyset) = \mathrm{tp}(\bar{a}'/\emptyset)$. Then $p(\bar{x}, \bar{a}')$ is stationary if and only if $p(\bar{x}, \bar{a})$ is stationary. (Use an automorphism of $\mathfrak{C}$ sending $\bar{a}$ to $\bar{a}'$.)

Stationary types allow us to prove a converse of Theorem .1.4.

THEOREM III.1.13. *If* D *is totally transcendental then* D *is stable in every* $\lambda \geq |D| + |T|$*. In particular* $\kappa(D) = \aleph_0$.

PROOF. Let $\lambda \geq |D| + |T|$, and let A be a subset of $\mathfrak{C}$ of cardinality at most λ. Since $\lambda \geq |D| + |T|$, by using a countable, increasing chain of models we can find a $(D, \aleph_0)$-homogeneous model M containing A of cardinality λ. Since $|S_D(A)| \leq |S_D(M)|$, it is enough to show that $|S_D(M)| \leq \lambda$. Suppose that $|S_D(M)| \geq \lambda^+$. Since M is $(D, \aleph_0)$-homogeneous, each $p \in S_D(M)$ is stationary. Hence, for each $p \in S_D(M)$, we can choose a finite $B_p \subseteq M$ such that p is based on B_p. Since there are only λ many finite subsets of M, by the pigeonhole principle there is a fixed finite subset B of M such that λ^+ many types $p \in S_D(M)$ are based on B. Since $\lambda^+ > |S_D(B)| = |D|$, another application of the pigeonhole principle shows that there a single stationary type $q \in S_D(B)$ with λ^+ many extensions in $S_D(M)$ of the same rank. This contradicts the stationarity of q. Hence D is stable in λ.

For the last sentence, let $\lambda = \beth_\omega(|D|+|T|)$. By Zermelo-König, $\lambda^{\aleph_0} > \lambda$, hence by Theorem 1..3.17 $\kappa(D) = \aleph_0$. □

The following results show that stationary types behave nicely. Not only do they have the uniqueness and the extension properties, but they can be represented by averages. Surprisingly, it turns out that every type is reasonably close to a stationary type (this is made precise in Lemma .4.9).

DEFINITION III.1.14. Let $p \in S_D(A)$ be stationary and let α be an infinite ordinal. The sequence $I = \{ c_i \mid i < \alpha \}$ is called a *Morley sequence based on* p if for each $i < \alpha$ we have c_i realizes p_{A_i}, where $A_i = A \cup \{c_j \mid j < i\}$.

LEMMA III.1.15. *Let $p \in S_D(A)$ be stationary. If I is a Morley sequence based on p, then I is indiscernible over A.*

PROOF. By stationarity $p_{A_i} \subseteq p_{A_j}$ when $i < j$, and by the previous lemma each p_{A_i} does not split over A. Hence, a standard result (see for example [**Sh b**] Lemma I.2.5) implies that I is an indiscernible sequence over A. □

The definition of averages is rephrased using the fact that $\kappa(D) = \aleph_0$ for totally transcendental diagrams.

DEFINITION III.1.16. ($\kappa(D) = \aleph_0$) For I an infinite set of indiscernibles and A a set (with $I \cup A \subseteq \mathfrak{C}$), recall that

$$\mathrm{Av}(I, A) = \{ \phi(\bar{x}, \bar{a}) \mid \bar{a} \in A, \phi(\bar{x}, \bar{y}) \in L(T) \text{ and } |\phi(I, \bar{a})| \geq \aleph_0 \}.$$

LEMMA III.1.17. *Suppose $p \in S_D(A)$ is stationary and I is a Morley sequence based on p. Then for any B containing A we have that $p_B = \mathrm{Av}(I, B)$.*

PROOF. Let $B \subseteq \mathfrak{C}$ and write $I = \{c_i \mid i < \alpha\}$. Choose $c_i \in \mathfrak{C}$ for $\alpha \leq i < \alpha + \omega$ realizing p_{B_i}, where $B_i = B \cup \bigcup\{a_j \mid j < i\}$. Since $\mathrm{Av}(I, B) \in S_D(B)$ extends p, it is enough to show that $R[\mathrm{Av}(I, B)] = R[p]$. Suppose $R[\mathrm{Av}(I, B)] \neq R[p]$. Then, by Monotonicity, we must have $R[\mathrm{Av}(I, B)] < R[p]$. We can find a finite $C \subseteq B$ such that p is based on C and by Finite Character, we may assume in addition that

(*) $$R[\mathrm{Av}(I, B)] = R[\mathrm{Av}(I, C)] < R[p].$$

But, since C is finite and $\kappa(D) = \aleph_0$, by Theorem 1. .4.5 there is $c_i \in I$ for $\alpha \leq i < \alpha + \omega$ realizing $\mathrm{Av}(I, C)$, and since $C \subseteq B$, we must have $\mathrm{tp}(c_i/C) = \mathrm{Av}(I, C) = p_C$ (since c_i realizes p_{B_i}). But then, by choice of C we have $R[\mathrm{Av}(I, C)] = R[p_C] = R[p]$ which contradicts (*). □

LEMMA III.1.18. *Let I be an infinite indiscernible set, A be finite and $p = \mathrm{Av}(I, A)$ be stationary. Then for any $C \supseteq A$ we have $p_C = \mathrm{Av}(I, C)$.*

PROOF. Write $I = \{c_i \mid i < \alpha\}$, for $\alpha \geq \omega$ and let C be given. Choose $c_i \in \mathfrak{C}$ for $\alpha \leq i < \alpha + \omega$ realizing p_{C_i}, where $C_i = C \cup \bigcup\{c_j \mid j < i\}$. Let $I' = \{c_i \mid i < \alpha + \omega\}$ and notice that necessarily $\mathrm{Av}(I, B) = \mathrm{Av}(I', B)$ for any B. Suppose $p_C \neq \mathrm{Av}(I, C)$, then since $\mathrm{Av}(I, A) \subseteq \mathrm{Av}(I, C)$, we must have $R[\mathrm{Av}(I, C)] < R[p]$, so $R[\mathrm{Av}(I', C)] < R[p_C]$. Choose C' finite, with $A \subseteq C' \subseteq C$, such that $R[\mathrm{Av}(I', C)] = R[\mathrm{Av}(I', C')]$. Now there is $J \subseteq I'$ finite such that $I' \setminus J$ is indiscernible over C'. Choose $c_i \in I' \setminus J$ with $i > \alpha$. Then c_i realizes $\mathrm{Av}(I', C')$, so $\mathrm{Av}(I', C') = \mathrm{tp}(c_i/C') \subseteq p_{C_i}$ by choice of c_i. But then

$$R[\mathrm{Av}(I', C')] \geq R[p_{C_i}] = R[p] > R[\mathrm{Av}(I, C)] = R[\mathrm{Av}(I', C')],$$

a contradiction. □

III.2. Dependence relation

By analogy with the first order case (see for example [**Ba a**] or [**Ma**]), it is natural at this point to introduce the anchor symbol, used for nonforking in the first order case. We do not claim that the two notions coincide even when both are defined. First, forking may be better behaved. When $D = D(T)$, the relation $A \underset{B}{\smile} C$ we will define is very close to nonsplitting and in fact, nonsplitting satisfies all the axioms of Theorem .2.3. At the same time, forking is defined, but it is not clear that they coincide for general sets (the main obstacles are that the notions of extension, stationarity and symmetry hold only over models that are, in this case, $\aleph_0$-saturated). Second, forking may not work at all. Typically a diagram may be totally transcendental while the underlying theory is unstable. Thus, in addition to the problem of failure of the compactness theorem (which is key to proving many of the properties of forking), one could not expect forking to be so well-behaved.

DEFINITION III.2.1. Suppose A, B, $C \subseteq \mathfrak{C}$, with $B \subseteq A$. We say that

$$A \underset{B}{\smile} C \quad \text{if} \quad R[\mathrm{tp}(\bar{a}/B)] = R[\mathrm{tp}(\bar{a}/B \cup C)], \qquad \text{for every } \bar{a} \in A.$$

As in many other contexts, the symmetry property can be obtained from the failure of the order property.

THEOREM III.2.2 (Symmetry). *If* $\mathrm{tp}(\bar{a}/B)$ *and* $\mathrm{tp}(\bar{c}/B)$ *are stationary, then*

$$\bar{a} \underset{B}{\downarrow} \bar{c} \qquad \text{if and only if} \qquad \bar{c} \underset{B}{\downarrow} \bar{a}.$$

PROOF. First, D is stable by Theorem .1.13, and therefore does not have the order property by Corollary .2.12. Suppose, for a contradiction, that

$$R[\mathrm{tp}(\bar{c}/B \cup \bar{a})] < R[\mathrm{tp}(\bar{c}/B)] \quad \text{and} \quad R[\mathrm{tp}(\bar{a}/B \cup \bar{c})] = R[\mathrm{tp}(\bar{a}/B)].$$

Let $\lambda = \beth_{(2^{|T|})^+}$ and let $\mu = (2^\lambda)^+$. We show that D has the ∞-order property, by constructing an order of length λ. Choose $p(\bar{x}, \bar{y}, \bar{b}) \in S_D(\bar{b})$ with $\bar{b} \in B$, such that

$$R[\mathrm{tp}(\bar{a}/B \cup \bar{c})] = R[p(\bar{x}, \bar{c}, \bar{b})] = R[\mathrm{tp}(\bar{a}/B)]$$

and

$$R[\mathrm{tp}(\bar{c}/B \cup \bar{a})] = R[p(\bar{c}, \bar{y}, \bar{b})] < R[\mathrm{tp}(\bar{c}/B)].$$

Let $\bar{a}_\alpha, \bar{c}_\alpha \in \mathfrak{C}$ for $\alpha < \mu$ and $B_\alpha = \bigcup\{\bar{a}_\beta, \bar{c}_\beta \mid \beta < \alpha\}$ be such that:

(1) $B_0 = B$;
(2) $\bar{a}_\alpha$ realizes $\mathrm{tp}(\bar{a}/B)$ and $R[\mathrm{tp}(\bar{a}_\alpha/B_\alpha)] = R[\mathrm{tp}(\bar{a}/B)]$;
(3) $\bar{c}_\alpha$ realizes $\mathrm{tp}(\bar{c}, B)$ and $R[\mathrm{tp}(\bar{c}_\alpha/B_\alpha \cup \bar{a}_\alpha)] = R[\mathrm{tp}(\bar{c}/B)]$.

This is achieved by induction on $\alpha < \mu$. Let $B_0 := B$, $\bar{a}_0 := \bar{a}$ and $\bar{c}_0 := \bar{c}$. At stage α, we let first $B_\alpha := \bigcup\{\bar{a}_\beta, \bar{c}_\beta \mid \beta < \alpha\}$ which is well-defined by induction hypothesis. We then satisfy in this order (2) by stationarity of $\mathrm{tp}(\bar{a}/B)$, and (3) by stationarity of $\mathrm{tp}(\bar{c}/B)$.

This is enough: First, notice that $\bar{c}_\alpha$ does not realize $p(\bar{a}, \bar{y}, \bar{b})$, otherwise

$$R[\mathrm{tp}(\bar{c}_\alpha/B_\alpha \cup \bar{a}_\alpha)] \leq R[p(\bar{a}, \bar{y}, \bar{b})] < R[\mathrm{tp}(\bar{c}/B)],$$

contrary to the choice of $\bar{c}_\alpha$. Similarly, since $\mathrm{tp}(\bar{a}_\alpha/B) = \mathrm{tp}(\bar{a}/B)$ and $\bar{b} \in B$, then

$$R[p(\bar{a}_\beta, \bar{y}, \bar{b})] < R[\mathrm{tp}(\bar{c}/B)],$$

so $\bar{c}_\alpha$ does not realize $p(\bar{a}_\beta, \bar{y}, \bar{b})$ when $\alpha \geq \beta$.

Now suppose $\alpha < \beta$. Then $\bar{a}_\beta$ realizes $p(\bar{x}, \bar{c}, \bar{b})$ since by stationarity, we must have $\mathrm{tp}(\bar{a}_\beta/A \cup \bar{c}) = \mathrm{tp}(\bar{a}/B \cup \bar{c})$. Further, since $\mathrm{tp}(\bar{a}_\alpha/B_\alpha)$ does not split over B and $\mathrm{tp}(\bar{c}_\alpha/B) = \mathrm{tp}(\bar{c}/B)$ we must have $p(\bar{x}, \bar{c}_\alpha, \bar{b}) \subseteq \mathrm{tp}(\bar{a}_\alpha/B_\alpha)$. So $\bar{a}_\beta$ realizes $p(\bar{x}, \bar{c}_\alpha, \bar{b})$.

Let $\bar{d}_\alpha = \bar{c}_\alpha \bar{a}_\alpha$ and let $q(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2, \bar{b}) := p(\bar{x}_1, \bar{y}_2, \bar{b})$ (we may assume that q is closed under finite conjunction). Then, above construction shows that

(*) $$\bar{d}_\alpha \bar{d}_\beta \models q(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2, \bar{b}) \quad \text{if and only if} \quad \alpha < \beta < \mu,$$

i.e. we we have an order of length μ witnessed by the type q.

We use (*) to obtain an order of length λ witnessed by a formula as follows. On the one hand, (*) implies that for any $\phi(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{c}) \in q$, the following holds:

(**) $$\models \phi[\bar{d}_\alpha, \bar{d}_\beta, \bar{b}] \qquad \text{whenever } \alpha < \beta.$$

On the other hand, if $\alpha \geq \beta$, by (*) again, there is $\phi_{\alpha,\beta}(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{b}) \in q$, such that $\models \neg\phi_{\alpha,\beta}[\bar{d}_\alpha, \bar{d}_\beta, \bar{b}]$. Hence, by the Erdös-Rado Theorem, since $|q| \leq |T|$, we can find $S \subseteq \mu$ of cardinality λ and $\phi(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{b}) \in q$, such that

(***) $$\models \neg\phi[\bar{d}_\alpha, \bar{d}_\beta, \bar{b}] \qquad \text{whenever } \alpha \geq \beta, \quad \alpha, \beta \in S.$$

Therefore, (**) and (***) together show that we can find an order of length λ, which is the desired contradiction. □

We close this section by gathering together the properties of the anchor symbol. They are stated with the names of the first order forking properties to which they correspond.

THEOREM III.2.3.

(1) (Definition) $A \underset{B}{\downarrow} C$ *if and only if* $A \underset{B}{\downarrow} B \cup C$.
(2) (Existence) $A \underset{B}{\downarrow} B$
(3) ($\kappa(D) = \aleph_0$) *For all* $\bar{a}$ *and* C, *there is a finite* $B \subseteq C$ *such that* $\bar{a} \underset{B}{\downarrow} C$.
(4) (Invariance under automorphisms) *Let* $f \in \mathrm{Aut}(\mathfrak{C})$.

$$A \underset{B}{\downarrow} C \qquad \textit{if and only if} \qquad f(A) \underset{f(B)}{\downarrow} f(C).$$

(5) (Finite Character)

$$A \underset{B}{\downarrow} C \qquad \textit{if and only if} \qquad A' \underset{B}{\downarrow} C',$$

for every finite $A' \subseteq A$, *and finite* $C' \subseteq C$.
(6) (Monotonicity) *Suppose* A' *and* C' *contain* A *and* C *respectively and that* B' *is a subset of* B. *Then*

$$A \underset{B}{\downarrow} C \qquad \textit{implies} \qquad A' \underset{B'}{\downarrow} C'.$$

(7) (Transitivity) *If* $B \subseteq C \subseteq D$, *then*

$$A \underset{B}{\downarrow} C \quad \textit{and} \quad A \underset{C}{\downarrow} D \qquad \textit{if and only if} \qquad A \underset{B}{\downarrow} D.$$

(8) (Symmetry) *Let* M *is a* $(D, \aleph_0)$*-homogeneous model.*

$$A \underset{M}{\downarrow} C \qquad \textit{if and only if} \qquad C \underset{M}{\downarrow} A.$$

(9) (Extension) *Let M be a $(D, \aleph_0)$-homogeneous model. For every A, C there exists A' such that*

$$\mathrm{tp}(A/M) = \mathrm{tp}(A'/M) \quad \textit{and} \quad A' \underset{M}{\downarrow} C.$$

(10) (Uniqueness) *Let M be a $(D, \aleph_0)$-homogeneous model. If A, A' satisfy*

$$\mathrm{tp}(A/M) = \mathrm{tp}(A'/M) \qquad \textit{and both} \qquad A \underset{M}{\downarrow} C \quad \textit{and} \quad A' \underset{M}{\downarrow} C$$

then $\mathrm{tp}(A/MC) = \mathrm{tp}(A'/MC)$.

(11) *Suppose that $A \underset{M}{\downarrow} BC$ and $C \underset{M}{\downarrow} B$. Then $C \underset{M}{\downarrow} BA$.*

PROOF. (1) This is just by Definition .2.1.

(2) Immediate from Definition .2.1.

(3) By Finite Character of the rank and Definition .2.1.

(4) Follows from Lemma .1.2 5.

(5) Immediate by finite definition and finite character of the rank.

(6) Assume $C \underset{M}{\not\downarrow} A$. By Finite Character, $R[\mathrm{tp}(\bar{c}/M)] < R[\mathrm{tp}(\bar{c}/M)]$, for some $\bar{c} \in C$. Also by Finite Character, there exists $\bar{a} \in A$ such that $R[\mathrm{tp}(\bar{c}/M \cup \bar{a})] = R[\mathrm{tp}(\bar{c}/M)]$. Hence $\bar{c} \underset{M}{\not\downarrow} \bar{a}$. But, by Corollary .1.10, both $\mathrm{tp}(\bar{a}/M)$ and $\mathrm{tp}(\bar{c}/M)$ are stationary, so by Theorem .2.2 we must have $\bar{a} \underset{M}{\not\downarrow} \bar{c}$. By Finite Character, this shows that $A \underset{M}{\not\downarrow} C$.

(7) Let $\bar{a} \in A$. Then, $\bar{a} \underset{B}{\downarrow} C$ and $\bar{a} \underset{C}{\downarrow} D$ by Finite Character. By Definition .2.1 $R[\mathrm{tp}(\bar{a}/C)] = R[\mathrm{tp}(\bar{a}/B)]$ and $R[\mathrm{tp}(\bar{a}/D)] = R[\mathrm{tp}(\bar{a}/C)]$. So $R[\mathrm{tp}(\bar{a}/B)] = R[\mathrm{tp}(\bar{a}/D)]$, and $\bar{a} \underset{B}{\downarrow} D$. Hence, by Finite Character, we must have $A \underset{B}{\downarrow} D$. The converse is just by Monotonicity.

(8) Immediate by Theorem .2.2 and Corollary .1.10.

(9) Follows from Corollary .1.10 and Definition .2.1.

(10) Follows from Corollary .1.10 and Definition .2.1.

(11) First, notice that by Monotonicity, we must have $A \underset{M}{\downarrow} B$. By definition, this shows that for every $\bar{a} \in A$, we have

$$R[\mathrm{tp}(\bar{a}/MB)] = R[\mathrm{tp}(\bar{a}/M)].$$

Since $\mathrm{tp}(\bar{a}/M)$ is stationary, this implies that $\mathrm{tp}(\bar{a}/MB)$ is stationary. Similarly, using the assumption that $C \underset{M}{\downarrow} B$, we must have that $\mathrm{tp}(\bar{c}/MB)$ is stationary for every $\bar{c} \in C$.

Now, by Monotonicity, we have that $A \underset{MB}{\downarrow} BC$, so that by definition, we have $A \underset{MB}{\downarrow} C$. By Symmetry for stationary types, using the first paragraph, we can derive that $C \underset{MB}{\downarrow} A$. By definition, this means that $C \underset{MB}{\downarrow} AB$.

Finally, we use transitivity to show that $C \underset{MB}{\downarrow} AB$. Hence $C \underset{M}{\downarrow} MB$ implies $C \underset{M}{\downarrow} BA$.

□

III.3. Regular and minimal types

In this section, we prove the existence of various pregeometries for totally transcendental diagrams. First, we make the following definition (a similar definition appears in [**Sh48**]).

DEFINITION III.3.1.

(1) Let $\bar{a}$ be in M and $q(\bar{x}, \bar{a})$ be a type. We say that $q(\bar{x}, \bar{a})$ is *big for* M if $q(\bar{x}, \bar{a})$ is realized outside M;
(2) We say that $q(\bar{x}, \bar{a})$ is *big* if $q(\bar{x}, \bar{a})$ is big for any M containing $\bar{a}$;
(3) A type $q \in S_D(A)$ is *big (for M)* if $q \upharpoonright \bar{a}$ is big (for M) for every $\bar{a} \in A$.

In presence of the compactness theorem, big types are the same as nonalgebraic types. Even in the general case, we have a nice characterization of bigness when the types are stationary.

LEMMA III.3.2. *Let $q \in S_D(A)$ be stationary. The following conditions are equivalent:*

(1) *q is big for some $(D, \aleph_0)$-homogeneous M containing A;*
(2) *$R[q] \geq 1$;*
(3) *q is big.*

PROOF. (1) ⇒ (2): Since M is $(D, \aleph_0)$-homogeneous, by Lemma .1.3, $R[q] = R_M[q]$, so it is enough to show $R_M[q] \geq 1$. Let $\bar{a} \in A$ be such that $R_M[q] = R_M[q \upharpoonright \bar{a}]$. Since $q \upharpoonright \bar{a}$ is big for M, there exists $\bar{c} \notin M$ realizing $q \upharpoonright \bar{a}$. Also, since M is $(D, \aleph_0)$-homogeneous, there is $\bar{c}' \in M$ realizing $q \upharpoonright \bar{a}$. Hence

$$R_M[(q \upharpoonright \bar{a}) \cup \{\bar{x} = \bar{c}'\}] \geq 0 \quad \text{and} \quad R_M[(q \upharpoonright \bar{a}) \cup \{\bar{x} \neq \bar{c}'\}] \geq 0.$$

Moreover, for every $\bar{b} \in M$, $(q \upharpoonright \bar{a}) \cup \mathrm{tp}(\bar{c}/\bar{b})$ is realized by $\bar{c}$, and so

$$R_M[(q \upharpoonright \bar{a}) \cup \mathrm{tp}(\bar{c}/\bar{b})] \geq 0,$$

and $\mathrm{tp}(\bar{c}/\bar{b}) \in S_D(\bar{b})$. This shows that $R_M[q \upharpoonright \bar{a}] \geq 1$.

(2) $\Rightarrow$ (3): Suppose q is stationary, $R[q] \geq 1$ and M containing $\bar{a}$ are given. By taking a larger M if necessary, we may assume that M is $(D, \aleph_0)$-homogeneous. Since q is stationary, there exists $q_M \in S_D(M)$, such that $R[q_M] = R[q] \geq 1$. Let $\bar{c}$ realize q_M. If $\bar{c} \in M$, then $\{ x = \bar{c} \} \in q_M$, so

$$0 = R[\bar{x} = \bar{c}] \geq R[q_M] \geq 1,$$

which is a contradiction. Hence $\bar{c} \notin M$, so q is big for M.

(3) $\Rightarrow$ (1): Clear by definition. □

DEFINITION III.3.3. Let $p \in S_D(A)$ be a big, stationary type.

(1) We say that p is *regular for* M if $A \subseteq M$ and for every $B \subseteq M$ we have

$$\bar{a} \underset{A}{\perp} B \text{ and } \bar{b} \underset{A}{\not\perp} B \quad \text{imply} \quad \bar{a} \underset{A}{\perp} B \cup \bar{b}, \qquad \text{for all } \bar{a}, \bar{b} \in p(M).$$

(2) We say that p is *regular* if p is regular for $\mathfrak{C}$.

LEMMA III.3.4. *Let $p \in S_D(A)$ be a big, stationary type based on $\bar{c} \in A$. If $p \upharpoonright \bar{c}$ is regular, then p is regular.*

PROOF. First notice that stationarity and bigness are preserved (bigness is the content of Lemma .3.2). Suppose p is not regular. We will show that $p \upharpoonright \bar{c}$ is not regular. Let $\bar{a}, \bar{b} \models p$ and B be such that

$$\bar{a} \underset{A}{\perp} B, \quad \bar{b} \underset{A}{\not\perp} B \quad \text{and yet} \quad \bar{a} \underset{A}{\not\perp} B \cup \bar{b}.$$

Therefore $\mathrm{tp}(\bar{a}/A \cup B) = p_{A \cup B}$ and so by choice of $\bar{c}$ we have $\mathrm{tp}(\bar{a}/A \cup B) = (p \upharpoonright c)_{A \cup B}$, i.e. $\bar{a} \underset{\bar{c}}{\perp} A \cup B$. Now since $R[p] = R[p \upharpoonright \bar{c}]$,

$$R[\mathrm{tp}(\bar{b}/A \cup B)] < R[\mathrm{tp}(\bar{b}/A)] \quad \text{implies} \quad R[\mathrm{tp}(\bar{b}A \cup B)] < R[p \upharpoonright \bar{c}],$$

i.e. $\bar{b} \underset{\bar{c}}{\not\perp} A \cup B$. We show similarly that $\bar{a} \underset{\bar{c}}{\not\perp} A \cup B \cup \bar{b}$, which shows that $p \upharpoonright \bar{c}$ is not regular. □

REMARK III.3.5. If $p(\bar{x}, \bar{a})$ is regular and $\bar{a}' \in M$ is such that $\mathrm{tp}(\bar{a}/\emptyset) = \mathrm{tp}(\bar{a}'/\emptyset)$, then $p(\bar{x}, \bar{a}')$ is regular.

DEFINITION III.3.6. Let $p \in S_D(B)$, $B \subseteq M$ and $W = p(M) \setminus B \neq \emptyset$. Define

$$a \in \mathrm{cl}(C) \quad \text{if} \quad a \underset{B}{\not\perp} C, \qquad \text{for } a \in W \text{ and } C \subseteq W.$$

THEOREM III.3.7. *Let M be $(D, \aleph_0)$-homogeneous and let $p \in S_D(B)$ such that $B \subseteq M$ and p is realized in M. If p is regular then (W, cl) is a pregeometry.*

PROOF. We need to show that the four axioms of pregeometry hold (notice $W \neq \emptyset$).

(1) We show that for every $C \subseteq W$, $C \subseteq \mathrm{cl}(C)$.

Let $c \in C$, then $\{x = c\} \in \mathrm{tp}(c/A \cup C)$, hence

$$R[\mathrm{tp}(c/B \cup C)] = 0 < R[p],$$

so $c \not\perp_B C$ and thus $c \in \mathrm{cl}(C)$.

(2) We show that if $c \in \mathrm{cl}(C)$, there is $C' \subseteq C$ finite, such that $c \in \mathrm{cl}(C')$.

Let $c \in \mathrm{cl}(C)$. By Definition .3.6 $c \not\perp_B C$ so by Theorem .2.3 5 there exists $C' \subseteq C$ finite, such that $c \not\perp_B C'$, hence $c \in \mathrm{cl}(C')$.

(3) We show that if $a \in \mathrm{cl}(C)$ and $C \subseteq \mathrm{cl}(E)$, then $a \in \mathrm{cl}(E)$.

Write $C = \{c_i \mid i < \alpha\}$. Then $a \not\perp_B \{c_i \mid i < \alpha\}$. Suppose $a \perp_B E$. We show by induction on $i < \alpha$ that $a \perp_B E \cup \{c_j \mid j < i\}$.

- For $i = 0$ this is the assumption and for i a limit ordinal, this is true by Theorem .2.3 5.
- For the successor case, suppose it is true for i. Then $a \perp_B E \cup \{c_l \mid l < i\}$. Since $C \subseteq \mathrm{cl}(E)$, we have $c_i \not\perp_B E$, so by Theorem .2.3 6 $c_i \not\perp_B E \cup \{c_l \mid l < i\}$. Hence, since p is regular, we must have $a \perp_B E \cup \{c_l \mid l < i\} \cup c_i$.

Thus $a \perp_B E \cup C$, and since $C \subseteq C \cup E$, we must have $a \perp_B C$. Hence $a \notin \mathrm{cl}(C)$, which contradicts our assumption.

(4) We show that if $c \in \mathrm{cl}(Ca) \setminus \mathrm{cl}(C)$, then $a \in \mathrm{cl}(Cc)$.

Since symmetry has been shown only for stationary types, this statement is not immediate from Theorem .2.2.

Suppose that $c \not\perp_B Ca$ and $c \perp_B C$. Then $c \not\perp_C a$, since

$$R[\mathrm{tp}(c/B \cup Ca)] < R[\mathrm{tp}(c/B)] = R[\mathrm{tp}(c/B \cup C)].$$

Therefore c realizes $p_{B \cup C}$, so $\mathrm{tp}(c/B \cup C)$ is stationary. If $a \not\perp_B C$, then by Theorem .2.3 6 we must have $a \not\perp_B Cc$, and we are done.

Otherwise, $a \perp_B C$. Hence a realizes $p_{B \cup C}$ and so $\mathrm{tp}(a/B \cup C)$ is stationary. Therefore by Theorem .2.2 we must have $a \not\perp_C c$, a contradiction. Hence by Theorem .2.3 6, we have $a \not\perp_B Cc$, i.e. $a \in \mathrm{cl}(Cc)$.

□

We now show the connection between independent sets in the pregeometries, averages and stationarity.

LEMMA III.3.8. *Let $p(\bar{x}, \bar{c})$ be regular. Let I be infinite and independent in $p(\mathfrak{C}, \bar{c})$. Then I is indiscernible and for every B containing $\bar{c}$ we have $p_B = \mathrm{Av}(I, B)$.*

PROOF. Write $I = \{\bar{a}_i \mid i < \alpha\}$. Then since I is independent, $\bar{a}_{i+1} \models p_{A_i}$, where $A_i = \bar{c} \cup \{\bar{a}_j \mid j < i\}$. Thus I is a Morley sequence based on p, so the result follows from Lemmas .1.15 and .1.17. □

Now we turn to existence. In order to do this, we need a lemma.

LEMMA III.3.9. *Let M be $(D, \aleph_0)$-homogeneous, and $p(\bar{x}, \bar{c})$ over M be big and stationary. Then $p(\bar{x}, \bar{c})$ is regular if and only if $p(\bar{x}, \bar{c})$ is regular for M.*

PROOF. If $p(\bar{x}, \bar{c})$ is regular, then $p(\bar{x}, \bar{c})$ is clearly regular for M. Suppose $p(\bar{x}, \bar{c})$ is not regular. Then there are $B \subseteq \mathfrak{C}$, and $\bar{a}$, $\bar{b}$ realizing $p(\bar{x}, \bar{c})$, such that

$$\bar{a} \underset{\bar{c}}{\perp} B, \quad \bar{b} \underset{\bar{c}}{\not\perp} B, \quad \text{and} \quad \bar{a} \underset{\bar{c}}{\not\perp} B\bar{b}.$$

First, we may assume that B is finite: choose $B' \subseteq B$ such that

$$R[\mathrm{tp}(\bar{a}/B' \cup \bar{c}\bar{b})] = R[\mathrm{tp}(\bar{a}/B \cup \bar{c}\bar{b})]$$

and then choose $B'' \subseteq B$ finite, such that $\bar{b} \not\models p_B \restriction B''$. Hence, for $B_0 = B' \cup B'' \subseteq B$, we have

$$\bar{a} \underset{\bar{c}}{\perp} B_0, \quad \bar{b} \underset{\bar{c}}{\not\perp} B_0, \quad \text{and} \quad \bar{a} \underset{\bar{c}}{\not\perp} B_0\bar{b}.$$

Now, since M is $(D, \aleph_0)$-homogeneous and $\bar{c} \in M$, we can find $B_1, \bar{a}_1$ and $\bar{b}_1$ inside M such that $\mathrm{tp}(B_0\bar{a}\bar{b}/\bar{c}) = \mathrm{tp}(B_1\bar{a}_1\bar{b}_1/\bar{c})$. Therefore, by invariance we have:

$$\bar{a} \underset{\bar{c}}{\perp} B_1, \quad \bar{b} \underset{\bar{c}}{\not\perp} B_1, \quad \text{and} \quad \bar{a} \underset{\bar{c}}{\not\perp} B_1\bar{b}.$$

This shows that p is not regular for M. □

The following argument for the existence of regular types is similar to Claim V.3.5. of [**Sh b**]. However, since our basic definitions are different, we provide a proof.

THEOREM III.3.10 (Existence of regular types). *Let M and N be such that $M \subseteq N$ and $M \neq N$. If M and N are $(D, \aleph_0)$-homogeneous, then there exists $p(x, \bar{a})$ regular, realized in $N \setminus M$. In fact, if $p(x, \bar{a})$ is big and stationary, and has minimal rank among all big, stationary types over M realized in $N \setminus M$, then $p(x, \bar{a})$ is regular.*

PROOF. The first statement follows from the second. To prove the second statement, we first choose $c' \in N \setminus M$, be such that $\mathrm{tp}(c'/M)$ has minimal rank among all types over M realized in $N \setminus M$, say $R[\mathrm{tp}(c'/M)] = \alpha$. We then choose

$\bar{a} \in M$ such that $R[\mathrm{tp}(c'/M)] = R[\mathrm{tp}(c'/\bar{a})] = \alpha$. Write $\mathrm{tp}(c'/\bar{a}) = p(x, \bar{a})$ and notice that p is stationary and big for M, hence big, by Lemma .3.2.

By the previous lemma, to show that $p(x, \bar{a})$ is regular, it is equivalent to show that $p(x, \bar{a})$ is regular for M. For this, let $a, b \in p(M)$ and $B \subseteq M$ such that

$$a \underset{\bar{a}}{\perp} B \quad \text{and} \quad b \underset{\bar{a}}{\not\perp} B.$$

We must show that $a \underset{\bar{a}}{\perp} Bb$. Suppose, by way of contradiction that this is not the case. Then, by definition, we have $R[\mathrm{tp}(a/B\bar{a}b)] < \alpha$. We now choose $\bar{c}, \bar{d} \in B$ such that

$$R[\mathrm{tp}(a/B\bar{a}b)] = R[\mathrm{tp}(a/\bar{c}\bar{a}b)] < \alpha \quad \text{and} \quad R[\mathrm{tp}(b/B\bar{a})] = R[\mathrm{tp}(b/\bar{d}\bar{a})] < \alpha.$$

Since N is $(D, \aleph_0)$-homogeneous and $c', a, b, \bar{a}, \bar{c}, \bar{d} \in N$, there is $b' \in N$ such that $\mathrm{tp}(ab/\bar{a}\bar{c}\bar{d}) = \mathrm{tp}(a'b'/\bar{a}\bar{c}\bar{d})$. Now, $\mathrm{tp}(b'/\bar{a}\bar{d}) = \mathrm{tp}(b'/\bar{a}\bar{d})$, so

$$R[\mathrm{tp}(b'/M)] \leq R[\mathrm{tp}(b'/\bar{a}\bar{d})] = R[\mathrm{tp}(b/\bar{a}\bar{d})] < \alpha.$$

By minimality of α, we must have $b' \in M$. Hence $R[\mathrm{tp}(a'/M)] \leq R[\mathrm{tp}(a'/\bar{c}\bar{a}b')]$, so $R[\mathrm{tp}(a'/\bar{c}\bar{a}b')] = \alpha$. Now there is $f \in \mathrm{Aut}(\mathfrak{C})$ such that $f(a') = a$, $f(b') = b$ and $f \restriction \bar{c}\bar{a} = id_{\bar{c}\bar{a}}$, by choice of b'. Hence, by property of the rank

$$\alpha = R[\mathrm{tp}(a'/\bar{c}\bar{a}b')] = R[f(\mathrm{tp}(a'/\bar{c}\bar{a}b'))] = R[\mathrm{tp}(a/\bar{c}\bar{a}b)] < \alpha,$$

which is a contradiction. Hence $a \underset{\bar{a}}{\perp} Bb$, so that $p(x, \bar{a})$ is regular. □

By observing what happens when $N = \mathfrak{C}$ in above theorem, one discovers more concrete regular types. For this, we make the following definition. A similar definition in the context of $L_{\omega_1\omega}(Q)$ appears in the last section of [**Sh48**]. An illustration of why this definition is natural can be found in the proof of Lemma .5.11. In presence of the compactness theorem, S-minimal is the same as strongly minimal.

DEFINITION III.3.11.

(1) A big, stationary type $q(\bar{x}, \bar{a})$ over M is said to be ***S-minimal for*** M if for any $\theta(\bar{x}, \bar{b})$ over M not both $q(\bar{x}, \bar{a}) \cup \theta(\bar{x}, \bar{b})$ and $q(\bar{x}, \bar{a}) \cup \neg\theta(\bar{x}, \bar{b})$ are big for M.
(2) A big, stationary type $q(\bar{x}, \bar{a})$ is said to be ***S-minimal*** if $q(\bar{x}, \bar{a})$ is S-minimal for for every M containing $\bar{a}$.
(3) If $q \in S_D(A)$ is big and stationary, we say that q is ***S-minimal*** if $q \restriction \bar{a}$ is S-minimal for some $\bar{a}$.

REMARK III.3.12.

(1) Let M be $(D, \aleph_0)$-homogeneous model. Let $q(\bar{x}, \bar{c})$ be S-minimal for M. Let $W = q(M, \bar{c})$ and for $a \in W$ and $B \subseteq W$ define

$$a \in \mathrm{cl}(B) \quad \text{if} \quad \mathrm{tp}(a/B \cup \bar{c}) \text{ is not big (for } M).$$

Then it can be shown directly from the assumption that D is totally transcendental, that (W, cl) is a pregeometry.

(2) Let M be $(D, \aleph_0)$-homogeneous. If $q(x, \bar{c})$ has minimal rank among all big, stationary $q(x, \bar{c})$ over M, then the previous theorem shows that q is regular. But q is also S-minimal for M. As a matter of fact, if $a \mathop{\downarrow}_{\bar{c}} B$, then $R[\mathrm{tp}(a/B \cup \bar{c})] = R[q(\bar{x}, \bar{c})] \geq 1$ and $\mathrm{tp}(a/B \cup \bar{c})$ is stationary, so $\mathrm{tp}(a/b \cup \bar{c})$ is big, so $a \notin \mathrm{cl}(B)$. Conversely, if $a \not\mathop{\downarrow}_{\bar{c}} B$, then $R[\mathrm{tp}(a/B\bar{c})] < R[q(x, \bar{c}]$. But if $\mathrm{tp}(a/B \cup \bar{c})$ was big, then we could find $a' \notin M$ such that $\mathrm{tp}(a'/B \cup \bar{c}) = \mathrm{tp}(a/B \cup \bar{c})$, so

$$R[\mathrm{tp}(a'/M)] \leq R[\mathrm{tp}(a'/B \cup \bar{c})] = R[\mathrm{tp}(a/B \cup \bar{c})] < R[q(x, \bar{c})],$$

contradicting the minimality of $R[q(x, \bar{c})]$. Hence $\mathrm{tp}(a/B \cup \bar{c})$ is not big, and so $a \in \mathrm{cl}(B)$. In other words, both pregeometries coincide.

(3) Using the results that we have proven so far, it is not difficult to show that if M, N are $(D, \aleph_0)$-homogeneous, and $q(x, \bar{c})$ has minimal rank among all big, stationary types over M and $\bar{c}' \in N$ such that $\mathrm{tp}(\bar{c}/\emptyset) = \mathrm{tp}(\bar{c}'/\emptyset)$, then $q(x, \bar{c}')$ has minimal rank among all big, stationary types over N, hence if $q(x, \bar{c}')$ is S-minimal for N.

In the light of these remarks, we will make the following definition.

DEFINITION III.3.13. Let M be $(D, \aleph_0)$-homogeneous. A stationary type $q(\bar{x}, \bar{c})$ with $\bar{c} \in M$ is called *minimal* if $q(\bar{x}, \bar{c})$ is big and has minimal rank among all big, stationary types over M.

We close this section by summarizing above remark in the following theorem.

THEOREM III.3.14. (1) *For any $(D, \aleph_0)$-homogeneous model, there exists a minimal $q(x, \bar{c})$ with $\bar{c} \in M$.*

(2) *Minimal types are regular and moreover for every A containing $\bar{c}$, every set B and $a \models q_A$ we have*

$$\mathrm{tp}(a/A \cup B) \quad \textit{is big if and only if} \quad a \mathop{\downarrow}_{A} B.$$

PROOF. The first item is clear by definition. The second follows by Theorem .3.10, and Remark .3.12 2 and 3. □

III.4. Prime models

In this section, we consider the question of prime models. The rank is especially useful to study the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental D.

We give definitions from [**Sh3**] in more modern terminology.

DEFINITION III.4.1.

(1) We say that $p \in S_D(A)$ is *D^s_λ-isolated over* $B \subseteq A$, $|B| < \lambda$, if for any $q \in S_D(A)$ extending $p \restriction B$, we have $q = p$.
(2) We say that $p \in S_D(A)$ is *D^s_λ-isolated* if there is $B \subseteq A$, $|B| < \lambda$, such that p is D^s_λ-isolated over B.

We next verify Axioms X.1 and XI.1 from Chapter IV of [**Sh b**].

THEOREM III.4.2 (X.1). *Let $A \subseteq \mathfrak{C}$ and $\mu \geq \aleph_0$. Every $\phi(\bar{x}, \bar{a})$ over A realized in $\mathfrak{C}$ can be extended to a D^s_μ-isolated type $p \in S_D(A)$.*

PROOF. It is enough to show the result for $\mu = \aleph_0$.

Since $\mathfrak{C} \models \exists \bar{x} \phi[\bar{x}, \bar{a}]$, there exists $\bar{c} \in \mathfrak{C}$ such that $\mathfrak{C} \models \phi[\bar{c}, \bar{a}]$. Thus there exists is $p \in S_D(A)$, namely $\mathrm{tp}(\bar{c}/A)$, containing $\phi(\bar{x}, \bar{a})$. Since D is totally transcendental and $A \subseteq \mathfrak{C}$ we must have $R_A[p] < \infty$. Among all those $p \in S_D(A)$ containing $\phi(\bar{x}, \bar{a})$ choose one with minimal rank. Say $R_A[p] = \alpha \geq 0$.

We claim that p is $D^s_{\aleph_0}$-isolated. First, there is $\bar{b} \in A$ such that $R_A[p] = R_A[p \restriction \bar{b}]$. We may assume that $p \restriction \bar{b}$ contains $\phi(\bar{x}, \bar{a})$ by Lemma .1.2 6. Suppose that there is $q \in S_D(A)$, $q \neq p$, such that q extends $p \restriction \bar{b}$. Then $R_A[q] \geq \alpha$ by choice of p (since q contains $\phi(\bar{x}, \bar{a})$). Now, choose $\psi(\bar{x}, \bar{c})$ with $\bar{c} \in A$ such that $\psi(\bar{x}, \bar{c}) \in p$ and $\neg\psi(\bar{x}, \bar{c}) \in q$. Then since $(p \restriction \bar{b}) \cup \psi(\bar{x}, \bar{c}) \subseteq p$, by Lemma .1.2 6 we have

$$R_A[(p \restriction \bar{b}) \cup \psi(\bar{x}, \bar{c})] \geq R_A[p] \geq \alpha.$$

Similarly

$$R_A[(p \restriction \bar{b}) \cup \neg\psi(\bar{x}, \bar{c})] \geq R_A[q] \geq \alpha.$$

Now, given any $\bar{d} \in A$, $R_A[p \restriction \bar{b} \cup \bar{d}] \geq \alpha$ (again by Lemma .1.2 6). Since $p \in S_D(A)$, necessarily if we write $p \restriction \bar{d} = p(\bar{x}, \bar{d})$, then we have $p(\bar{x}, \bar{y}) \in D$ (since $p(\bar{x}, \bar{d}) \in S_D(\bar{d})$). Hence since $p \restriction \bar{b} \cup \bar{d} \vdash p \restriction b \cup p(\bar{x}, \bar{d})$) we have

$$R_A[(p \restriction b) \cup p(\bar{x}, \bar{d})] \geq R_A[p \restriction \bar{b} \cup \bar{d}] \geq \alpha.$$

But this shows that $R_A[p \restriction \bar{b}] \geq \alpha + 1$, a contradiction.

Hence p is the only extension of $p \restriction b$, so p is $D^s_{\aleph_0}$-isolated. ☐

THEOREM III.4.3 (XI.1). *Let μ be infinite and B, A be D-sets such that $B \subseteq A$. Every D^s_μ-isolated $r \in S_D(B)$ can be extended to a D^s_μ-isolated type $p \in S_D(A)$.*

PROOF. Since $\mathfrak{C}$ is (D, χ)-homogeneous, there exists $\bar{c} \in \mathfrak{C}$ realizing r. Hence there is $p \in S_D(A)$ extending r, namely $\mathrm{tp}(\bar{c}/A)$. Since D is totally transcendental and $A \subseteq \mathfrak{C}$ we must have $R_A[p] < \infty$. Among all those $p \in S_D(A)$ extending r choose one with minimal rank. Say $R_A[p] = \alpha \geq 0$.

We claim that p is D^s_μ-isolated. First, there is $\bar{b} \in A$ such that $R_A[p] = R_A[p \restriction \bar{b}]$. Also, since r is D^s_μ-isolated, there is $C \subseteq B$, $|C| < \mu$ such that $r \restriction C$ isolates r. We may assume that $R_A[r] = R_A[r \restriction C]$, by Lemma .1.2 7. We claim that $(r \restriction C) \cup (p \restriction \bar{b})$ isolates p. By contradiction, suppose that there is $q \in S_D(A)$ extending $(r \restriction C) \cup (p \restriction \bar{b})$ such that $q \neq p$. Notice that $r \subseteq q$, since r was isolated by $r \restriction C$, and hence $R_A[q] \geq R_A[p] = \alpha$ by choice of p. Now, choose $\psi(\bar{x}, \bar{a})$ with $\bar{a} \in A$ such that $\psi(\bar{x}, \bar{a}) \in p$ and $\neg\psi(\bar{x}, \bar{a}) \in q$. By Lemma .1.2 6 (since $(p \restriction \bar{b}) \cup \psi(\bar{x}, \bar{c}) \subseteq p$), we must have

$$R_A[(p \restriction \bar{b}) \cup \psi(\bar{x}, \bar{c})] \geq R_A[p] = \alpha.$$

Similarly

$$R_A[(p \restriction \bar{b}) \cup \neg\psi(\bar{x}, \bar{c})] \geq R_A[q] \geq \alpha.$$

Now, given any $\bar{d} \in A$ we have that $R_A[p \restriction \bar{b} \cup \bar{d}] \geq \alpha$ (again by Lemma .1.2 6). Since $p \in S_D(A)$, necessarily if we write $p \restriction \bar{d} = p(\bar{x}, \bar{d})$, then we have $p(\bar{x}, \bar{y}) \in D$ (since $p(\bar{x}, \bar{d}) \in S_D(\bar{d})$). Hence

$$R_A[(p \restriction b) \cup p(\bar{x}, \bar{d})] \geq R_A[p \restriction \bar{b} \cup \bar{d}] \geq \alpha,$$

since $p \restriction \bar{b} \cup \bar{d} \vdash (p \restriction b) \cup p(\bar{x}, \bar{d})$. But this shows that $R_A[p \restriction \bar{b}] \geq \alpha + 1$, a contradiction.

Hence p is the only extension of $(r \restriction C) \cup (p \restriction b)$, so p is D^s_μ-isolated. $\square$

Following Chapter IV of [**Sh b**], we set:

DEFINITION III.4.4.

(1) We say that $\mathcal{C} = \{\langle a_i, A_i, B_i \rangle \mid i < \alpha\}$ is a (D, λ)-*construction of* C *over* A if
  (a) $C = A \cup \bigcup\{a_i \mid i < \alpha\}$;
  (b) $B_i \subseteq A_i$, $|B_i| < \lambda$, where $A_i = A \cup \bigcup\{a_j \mid j < i\}$;
  (c) $\mathrm{tp}(a_i/A_i) \in S_D(A_i)$ is D^s_λ-isolated over B_i.

(2) We say that M is D^s_λ-*constructible over* A if there is a (D, λ)-construction for M over A.

(3) We say that M is D^s_λ-*primary over* A, if M is D^s_λ-constructible over A and M is (D, λ)-homogeneous.

(4) We say that M is D^s_λ-*prime over* A if
  (a) M is (D, λ)-homogeneous and
  (b) if N is (D, λ)-homogeneous and $A \subseteq N$, then there is $f : N \to M$ elementary such that $f \restriction A = id_A$.

(5) We say that M is D^s_λ-*minimal over* A, if M is D^s_λ-prime over A and for every (D, λ)-homogeneous model N, if $A \subseteq N \subseteq M$, then $M = N$.

REMARK III.4.5. We use the same notation as in [**Sh b**], except that we replace **F** by D to make it explicit that we deal exclusively with D-types (or equivalently, types realized in $\mathfrak{C}$). In particular, for example if M is $D^s_{\aleph_0}$-primary over A, then M is $D^s_{\aleph_0}$-prime over A.

THEOREM III.4.6 (Existence of prime models). *Let μ be an infinite cardinal and A be a D-set. Then, there is a D^s_μ-primary model M over A of cardinality $|A| + |T| + |D| + \mu$. Moreover, M is D^s_μ-prime over A.*

PROOF. See page 175 of [**Sh b**] and notice that we just established $X.1$ and $XI.1$. Observe that in the construction, each new element realizes a D-type, so that the resulting model is indeed a D-model. The optimal bound on the cardinality follows from Theorem .1.13. The second sentence follows automatically. □

REMARK III.4.7. A similar theorem, with a stronger assumption (D is $\aleph_0$-stable) and without the bound on the cardinality appears in [**Sh3**]. Note that D^s_μ-primary, is called $(D, \mu, 1)$-prime there.

It is natural to make the following conjecture.

CONJECTURE III.4.8. Let D be totally transcendental. Then for any A the $D^s_{\aleph_0}$-prime model over A is unique up to isomorphism fixing A.

Notice that this allows us to show how any type can be decomposed into stationary and isolated types. A similar result appears in [**Sh87a**].

LEMMA III.4.9. *Let $p \in S_D(A)$ and suppose $\bar{a}$ realizes p. Then there is $\bar{b} \in \mathfrak{C}$ such that*

(1) $\operatorname{tp}(\bar{b}/A)$ *is $D^s_{\aleph_0}$-isolated;*
(2) $\operatorname{tp}(\bar{a}/A\bar{b})$ *is stationary;*
(3) $R[\operatorname{tp}(\bar{a}/A\bar{b})] = R[\operatorname{tp}(\bar{a}/\bar{b})]$.

Furthermore, p does not split over a finite set.

PROOF. Let $\bar{a} \models p$. Let M be $D^s_{\aleph_0}$-primary model over A. Then $\operatorname{tp}(\bar{a}/M)$ is stationary since M is $(D, \aleph_0)$-homogeneous, and there is $\bar{b} \in M$ finite, such that $R[\operatorname{tp}(\bar{a}/M)] = R[\operatorname{tp}(\bar{a}/\bar{b})]$. Hence $R[\operatorname{tp}(\bar{a}/A\bar{b})] = R[\operatorname{tp}(\bar{a}/\bar{b})]$ by Lemma .1.2 6, and so $\operatorname{tp}(\bar{a}/A\bar{b})$ is stationary. Also, $\operatorname{tp}(\bar{b}/A)$ is $D^s_{\aleph_0}$-isolated, since M is $D^s_{\aleph_0}$-primary over A.

Finally, to see that p does not split over a finite set, assume $\bar{a} \models p$, $\operatorname{tp}(\bar{b}/A)$ is $D^s_{\aleph_0}$-isolated, $\operatorname{tp}(\bar{a}/A\bar{b})$ is stationary, and $R[\operatorname{tp}(\bar{a}/A\bar{b})] = R[\operatorname{tp}(\bar{a}/\bar{b})]$. Then there is $C \subseteq A$ finite, such that $\operatorname{tp}(\bar{b}/A)$ is $D^s_{\aleph_0}$-isolated over C. Also, since $\operatorname{tp}(\bar{a}/A\bar{b})$ is stationary, it does not split over $\bar{b}$. Now it is easy to see that p does not split over C: otherwise there are $\bar{c}_l \in A$, and $\phi(\bar{x}, \bar{y})$ such that $\operatorname{tp}(\bar{c}_1/C) = \operatorname{tp}(\bar{c}_2/C)$, $\bar{c}_l \in A$ for $l = 1, 2$, and $\models \phi[\bar{a}, \bar{c}_1]$ and $\models \neg\phi[\bar{a}, \bar{c}_2]$. But $\operatorname{tp}(\bar{b}/A)$ does not split over C, and so $\operatorname{tp}(\bar{c}_1/\bar{b}) = \operatorname{tp}(\bar{c}_2/\bar{b})$. However, this contradicts the fact that $\operatorname{tp}(\bar{a}/A\bar{b})$ does not split over $\bar{b}$. All the conditions are satisfied. □

This gives us an alternative and short proof that averages are well-defined, and in fact, allows us to give short proofs of all the facts in Theorem 1..4.5.

LEMMA III.4.10. *Let I be infinite and $A \subseteq \mathfrak{C}$. Then* $\mathrm{Av}(I, A) \in S_D(A)$

PROOF. Completeness is clear. To see that $\mathrm{Av}(I, A)$ is consistent, suppose that both $\phi(x, \bar{a})$ and $\neg\phi(x, \bar{a})$ are realized by infinitely many elements of I. But $\mathrm{tp}(\bar{a}/I)$ does not split over a finite set $B \subseteq I$ by the previous lemma. Hence, by choice of $\phi(x, \bar{a})$, we can find $b, c \in I \setminus B$ such that $\models \phi[b, \bar{a}]$ and $\models \neg\phi[c, \bar{a}]$. This however, shows that $\mathrm{tp}(\bar{a}/I)$ splits over B, since $\mathrm{tp}(b/B) = \mathrm{tp}(c/B)$ by indiscernibility of I and both $\phi(b, \bar{y})$, $\neg\phi(c, \bar{y}) \in \mathrm{tp}(\bar{a}/I)$. Now $\mathrm{Av}(I, A) \in S_D(A)$ since we can extend I to a D-set of indiscernible J of cardinality $|A|^+$, and then some element of J realizes $\mathrm{Av}(I, A)$. □

The following is a particular case of a theorem of Shelah in [**Sh54**]. We include it here not just for completeness, but because the proof is different from Shelah's original proof in finite diagrams and very similar in the conceptual framework to the first order case.

THEOREM III.4.11. *Let D be totally transcendental. If $\langle M_i \mid i < \alpha\rangle$ is an increasing chain of (D, μ)-homogeneous models, then $\bigcup_{i<\alpha} M_i$ is (D, μ)-homogeneous (μ infinite).*

PROOF. Let $M = \bigcup_{i<\alpha} M_i$ and notice that M is $(D, \aleph_0)$-homogeneous. Let $p \in S_D(A)$, $A \subseteq M$, $|A| < \mu$ and choose $q \in S_D(M)$ extending p. Then, by Corollary .1.10, q is stationary and there is $B \subseteq M$, finite such that q is based on B. Let $i < \alpha$, be such that $B \subseteq M_i$. Since M_i is (D, μ)-homogeneous, there is $I = \{a_j \mid j < \mu\} \subseteq M_i$ a Morley sequence for q_B. Then, by Lemma .1.17, $q_{AB} = \mathrm{Av}(I, A \cup B)$. But $|I| > |A \cup B|$, so by Theorem 1. .4.5 there is $a_j \in I$ realizing $\mathrm{Av}(I, A \cup B)$. But $q_{AB} \supseteq p$, so p is realized in M. This shows that M is (D, μ)-homogeneous. □

III.5. Chang's conjecture and categoricity

We now focus on the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram D. As we said in the introduction, there are several key examples. When D is the set of isolated types over the empty set, the class of $(D, \aleph_0)$-homogeneous models coincides with the class of D-models, that is we are studying the class of atomic models of a given first order theory. Recall that this class is especially important from the point of view of classification for nonelementary classes. A result due to S. Shelah [**Sh48**] shows that, for example, given any $\aleph_1$-categorical Scott sentence $\psi \in L_{\omega_1\omega}$, there exists a first order theory T (possibly unstable) with the property that for every cardinal λ, there is a one-to-one correspondence between the models of ψ of cardinality λ and the atomic models of T of cardinality λ. Another example is when $D = D(T)$. As we mentioned, when $D = D(T)$, then $\mathcal{K}$ is the class of $\aleph_0$-saturated models of a totally transcendental theory. This case is important, since it demonstrates, in particular, that all our estimates on the number of models are sharp.

DEFINITION III.5.1. Define

$$\mathcal{K} = \{\, M \mid M \text{ is } (D, \aleph_0)\text{-homogeneous} \,\}.$$

We first prove Chang's Conjecture for $\mathcal{K}$, when D is a totally transcendental diagram.

Chang's Conjecture for the class of models of countable first order theories is the following statement. If T is a theory in a language containing a unary predicate $P(x)$ and if there exists a model $M \models T$ such that $\|M\| = \lambda^+$ and $|P(M)| = \lambda$, for some infinite λ, then there exists $N \prec M$ of cardinality $\aleph_1$ such that $|P(N)| = \aleph_0$. It is known that this statement is, in fact, a large cardinal axiom. However, Chang's Conjecture holds for the class of models of T, when T satisfies additional assumptions. The next theorem implies Chang's Conjecture for the class $\mathcal{K}$, when D is a countable totally transcendental diagram and T a countable first order theory. Note the similarity with Theorem .5.8.

THEOREM III.5.2. *Let T be a first order theory in a language containing a unary predicate $P(x)$. Let $D \subseteq D(T)$ be a totally transcendental diagram. Let λ, μ, χ be cardinals such that $|D| + |T| \leq \mu < \chi \leq \lambda^+$. If there exists $M \in \mathcal{K}$ of cardinality λ^+ with $P(M)$ of cardinality λ, then there exists $N \in \mathcal{K}$ with $N \prec M$ of cardinality χ such that $P(N)$ has cardinality μ.*

PROOF. Let $M \in \mathcal{K}$ of cardinality λ^+ with $P(M)$ of cardinality λ. Since D is totally transcendental, then D is stable in λ by Theorem .1.13. Hence, by Theorem 1..3.13, there exists $\{a_i \mid i < \lambda^+\} \subseteq M \setminus P(M)$ indiscernible over $P(M)$. Let $A \subseteq P(M)$ of cardinality μ. Now choose $N^* \prec M$ of cardinality μ, with $N^* \in \mathcal{K}$, containing $A \cup \{a_i \mid i < \aleph_0\}$. Let $B = P(N^*)$. Then B has cardinality μ and $B \subseteq P(M)$. By Theorem .4.6, there exists a $D^s_{\aleph_0}$-primary model N over $B \cup \{a_i \mid i < \chi\}$ of cardinality χ. By using an automorphism fixing $B \cup \{a_i \mid i < \chi\}$ if necessary, we may assume that $N \subseteq M$. Thus, $N \prec M$.

We claim that $P(N)$ has cardinality μ. It is enough to show that $P(N) = B$. Suppose this is not the case and let $c \in P(N) \setminus B$. Consider the types $\operatorname{tp}(c/B) \subseteq \operatorname{tp}(c/B \cup \{a_i \mid i < \chi\})$. Note that the formula $P(x)$ belongs to $\operatorname{tp}(c/B)$ and that $\operatorname{tp}(c/B)$ is not realized inside B. Now since N is $D^s_{\aleph_0}$-primary over $B \cup \{a_i \mid i < \chi\}$, there exists a type $q(x, \bar{b}, a_{i_1}, \ldots, a_{i_n}) \in S_D(\bar{b} \cup \{a_{i_1}, \ldots, a_{i_n}\})$ with $\bar{b} \in B$ and $i_1 < \ldots i_n < \chi$ satisfying

$$q(x, \bar{b}, a_{i_1}, \ldots, a_{i_n}) \vdash \operatorname{tp}(c, B),$$

since $\operatorname{tp}(c/B \cup \{a_i \mid i < \chi\}) \vdash \operatorname{tp}(c/B)$. By indiscernibility of $\{a_i \mid i < \chi\}$ over B, we may assume that $i_1 < \ldots i_n < \aleph_0$. Hence $\bar{b} \cup \{a_{i_1}, \ldots, a_{i_n}\} \subseteq N^*$. Since N^* is $(D, \aleph_0)$-homogeneous, there exists $c^* \in N^*$ realizing $q(x, \bar{b}, a_{i_1}, \ldots, a_{i_n})$.

Thus, c^* realizes $\mathrm{tp}(c/B)$. But $P(x) \in \mathrm{tp}(c/B)$ so $c^* \in P(N^*) = B$, a contradiction. □

We now turn to Categoricity.

REMARK III.5.3. We will say that $M \in \mathcal{K}$ is *prime over* A or *minimal over* A, when M is $D^s_{\aleph_0}$-prime over A or $D^s_{\aleph_0}$-minimal over A respectively.

By analogy with the first order case, we set the following definition.

DEFINITION III.5.4. Let D be totally transcendental. We say that D is *unidimensional* if for every pair of models $M \subseteq N$ in $\mathcal{K}$ and minimal type $q(x, \bar{a})$ minimal over M,

$$q(M, \bar{a}) = q(N, \bar{a}) \quad \text{implies} \quad M = N.$$

Unidimensionality for a totally transcendental diagram D turns out to be a meaningful dividing line. When it fails, we can construct non-isomorphic models, like in the next theorem (this justifies the name), and when it holds we get a strong structural theorem (see Theorem .5.10, which implies categoricity). In fact, the conclusion of our next theorem is similar to (but stronger than) the conclusion of Theorem 6.9 of [**Sh3**] (we prove it for every μ, not just regular μ, and can obtain these models of cardinality exactly λ, not arbitrarily large). The assumptions of Theorem 6.9 of [**Sh3**] are weaker and the proof considerably longer. Actually, Corollary .5.16 makes the connection with Theorem 6.9 of [**Sh3**] clearer.

We first prove two technical lemmas which are similar to Lemma 3.4 and fact 3.2.1 from [**GrHa**] respectively. The proofs are straightforward generalizations and are presented here for the sake of completeness.

LEMMA III.5.5. *Let $p, q \in S_D(M)$ and $M \subseteq N$ be in $\mathcal{K}$. If $a \underset{M}{\perp} b$ for every $a \models q$ and $b \models p$, then $a \underset{N}{\perp} b$ for every $a \models q_N$ and $b \models p_N$.*

PROOF. Suppose not. Then there are $a \models p_N$ and $b \models q_N$ such that $a \underset{N}{\not\perp} b$. Choose $E \subseteq N$ finite such that $a \underset{ME}{\not\perp} b$ and $\mathrm{tp}(ab/N)$ is based on E. This is possible by Theorem .2.3 5 and by the fact that $\mathrm{tp}(ab/N)$ is stationary. Similarly, we can find $C \subseteq M$ finite, such that p_M and q_M are based on C and $a \underset{CE}{\not\perp} b$. Since $C \subseteq M$ finite and $M \in \mathcal{K}$, there exists $a^*, b^*, E^* \subseteq M$, such that $\mathrm{tp}(abE/C) = \mathrm{tp}(a^*b^*E^*/C)$, and so $a^* \underset{CE^*}{\not\perp} b^*$. Since $\mathrm{tp}(ab/N)$ is based on E, then $\mathrm{tp}(ab/CE)$ is stationary based on E, so $\mathrm{tp}(a^*b^*/CE^*)$ is stationary based on E^*. Therefore, we can choose $a'b' \models \mathrm{tp}(a^*b^*/CE^*)_M$, and by choice of C, necessarily $a' \models p_M$ and $b' \models q_M$.

Hence, by assumption on p_M, q_M, we have $a' \underset{M}{\perp} b'$, so also $a' \underset{CE^*}{\perp} b'$. But this implies $a^* \underset{CE^*}{\perp} b^*$, by choice of $a'b'$, a contradiction. □

LEMMA III.5.6. *Let N be (D, μ)-homogeneous. If $a \underset{N}{\perp} b$ and $\mathrm{tp}(a/Nb)$ is D^s_μ-isolated, then $a \in N$.*

PROOF. Since $p = \mathrm{tp}(a/Nb)$ is D^s_μ-isolated, there is $C \subseteq N$, $|C| < \mu$ such that $\mathrm{tp}(a/Cb)$ isolates p. Since $\mathrm{tp}(b/N)$ is stationary, we may assume that $\mathrm{tp}(b/N)$ does not split over C. Since, by Theorem .2.3 8 also $b \underset{N}{\perp} a$, so we may assume that $\mathrm{tp}(b/Na)$ does not split over C.

Since N is (D, μ)-homogeneous, there is $a' \in N$, such that $\mathrm{tp}(a/C) = \mathrm{tp}(a'/C)$. But since $\mathrm{tp}(b/Na)$ does not split over C, then $\mathrm{tp}(ab/C) = \mathrm{tp}(a'b/C)$. Hence $\mathrm{tp}(a/N) = \mathrm{tp}(a'/N)$, so that $a \in N$. □

Recall a definition from [**Sh3**].

DEFINITION III.5.7. Let M be a D-model. M is said to be *maximally (D, μ)-homogeneous* if M is (D, μ)-homogeneous, but not (D, μ^+)-homogeneous.

THEOREM III.5.8. *Suppose D is not unidimensional. Then there is a maximally (D, μ)-homogeneous model M of cardinality λ, for every $\lambda \geq \mu \geq |T| + |D|$.*

PROOF. Suppose D is totally transcendental and not unidimensional. Then there exists M, N in $\mathcal{K}$ and a minimal type $q(x, \bar{a})$ over M with the property that

(*) $$q(M, \bar{a}) = q(N, \bar{a}) \quad \text{and} \quad M \subseteq N, \quad M \neq N.$$

Using the Downward Löwenheim Skolem Theorem and prime models, we may assume that $|q(M, \bar{a})| \leq |T| + |D|$. Let $\lambda \geq \mu \geq |T| + |D|$ be given. We first show that we can find M, $N \in \mathcal{K}$ satisfying (*) such that in addition $\|M\| = |q(M, \bar{a})| = \mu$.

Since $M \neq N \in \mathcal{K}$, there is $b \in N \setminus M$, so $p = \mathrm{tp}(b/M) \in S_D(M)$ is big and stationary. This implies that $a' \underset{M}{\perp} b'$ for any $a' \models q_M$ and $b' \models p$ (by an automorphism sending b' to b, it is enough to see $a' \underset{M}{\perp} b$, but this is obvious, otherwise $\mathrm{tp}(a'/Mb)$ is not big, thus cannot be big for N by Lemma .3.2, hence it has to be realized in $N \setminus M$, which implies that $a' \in N \setminus M$, contradicting $q(M, \bar{a}) = q(N, \bar{a})$).

Construct $\langle M_i \mid i \leq \mu \rangle$ increasing and $I = \{a_i \mid i < \mu\}$, $a_i \notin M_i$ realizing q_{M_i}, such that:

(1) $M_{i+1} \in \mathcal{K}$ is $D^s_{\aleph_0}$-primary over $M_i \cup a_i$;
(2) $M_0 = M$;
(3) $M_i = \bigcup_{j<i} M_j$ when j is a limit ordinal;
(4) If b' realizes p_{M_i}, and N^* is $D^s_{\aleph_0}$-primary over $M_i \cup b'$, then $q(M_i, \bar{a}) = q(N^*, \bar{a})$.

This is enough: Consider N $D^s_{\aleph_0}$-primary over $M_\mu \cup b'$, where $b' \models p_{M_\mu}$. Then $b' \in N \setminus M_\mu$ and yet $q(M_\mu, \bar{a}) = q(N, \bar{a})$, so (*) holds. Furthermore, $\|M_\mu\| = |q(M_\mu, \bar{a})| = \mu$.

This is possible:

- For $i = 0$, this follows from the definition of q (send b' to b by an automorphism, fixing M, to obtain a realization of q_M in $N \setminus M$).
- If i is a limit ordinal, and $b' \models p_{M_i}$, then this implies that $b' \models p_{M_j}$, for any $j < i$. Also, if N^* is prime over $M_i \cup b'$, and $c \in N^* \setminus M_i$ realizes $q(x, \bar{a})$, then $\mathrm{tp}(c/M_i b')$ is $D^s_{\aleph_0}$-isolated over some $\bar{m}b$, and $\bar{m}b \in M_j$ for some $j < i$, hence $c \in M_j$ by induction hypothesis, a contradiction.
- For $i = j + 1$. Let $b' \models p_{M_j}$ and N^* be prime over $M_j \cup b'$. Suppose that $c \in N^* \setminus M_j$ realizes $q(x, \bar{a})$. Then, since $c \notin M_j$, we must have $\mathrm{tp}(c/M_j)$ is big, so $c \models q_{M_j}$. Hence, by Lemma .5.5 we have $c \underset{M_j}{\perp} b'$. But $\mathrm{tp}(c/M_j b')$ is $D^s_{\aleph_0}$-isolated, so by Lemma .5.6, we must have $c \in M_j$, a contradiction. Hence $q(M_i) = q(N^*)$ and we are done.

Let $M^* = M_\mu$, and fix $b \models p_{M^*}$. We now show that we can find a (D, μ)-homogeneous model $N \in \mathcal{K}$ of cardinality λ such that M^* and N satisfy (*). This implies the conclusion of the theorem: N is (D, μ)-homogeneous of cardinality λ; N is *not* (D, μ^+)-homogeneous, since N omits $q_{M^*} \in S_D(M^*)$, and $\|M^*\| = \mu$.

We construct $\langle N_i \mid i \leq \lambda \rangle$ increasing, and $b_i \notin N_i$ realizing p_{N_i} such that:

(1) $b_0 = b$ and N_0 is D^s_μ-primary over $M^* \cup b$;
(2) N_{i+1} is D^s_μ-primary over $N_i \cup b_i$;
(3) $N_i = \bigcup_{j<i} N_i$, when i is a limit ordinal;
(4) $\|N_i\| \leq \lambda$;
(5) N_i is (D, μ)-homogeneous;
(6) $q(N_i, \bar{a}) = q(M^*, \bar{a})$.

This is clearly enough: N_λ is as required.

This is possible: We construct N_i by induction on $i \leq \lambda$.

- For $i = 0$, let $N^* \subseteq N_0$ be $D^s_{\aleph_0}$-primary over $M^* \cup b$. By construction of M^*, we have $q(N^*, \bar{a}) = q(M^*, \bar{a})$, so it is enough to show that $q(N^*, \bar{a}) = q(N_0, \bar{a})$. Suppose not and let $c \in N_0 \setminus N^*$ realize $q(x, \bar{a})$.

Then, c realizes q_{N^*} since $\mathrm{tp}(c/N^*)$ is big, and further there is $A \subseteq M^*$, $|A| < \mu$ such that $\mathrm{tp}(c/Ab)$ isolates $\mathrm{tp}(c/M^*b)$. By Lemma .1.17 since I is based on q, we have $\mathrm{Av}(I, N^*) = q_{N^*}$, where $I = \{a_i \mid i < \mu\} \subseteq M^*$ defined above. But since both $\mathrm{tp}(c/Ab)$ and $\mathrm{tp}(c/M^*)$ are big, we must have $\mathrm{tp}(c/Ab) = \mathrm{Av}(I, Ab)$ and $\mathrm{tp}(c/M^*) = \mathrm{Av}(I, M^*)$. Hence $\mathrm{Av}(I, Ab) \vdash \mathrm{Av}(I, M^*)$. Now, by Theorem 1..4.5, we can find $I' \subseteq I$, $|I'| < \mu$ such that $I \setminus I'$ is indiscernible over Ab. Since $|I| = \mu$, then $I \setminus I' \neq \emptyset$ and all elements of $I \setminus I'$ realize $\mathrm{Av}(I, Ab)$, hence also $\mathrm{Av}(I, M^*) = q_{M^*}$. But this is impossible since $I \subseteq M^*$. Therefore $q(N_0, \bar{a}) = q(N^*, \bar{a}) = q(M^*, \bar{a})$.

- For i a limit ordinal, the only condition to check is that N_i is (D, μ)-homogeneous, but this follows from Theorem .4.11.
- For $i = j + 1$, by induction hypothesis, we have $q(N_j, \bar{a}) = q(M^*, \bar{a})$, so it is enough to show that $q(N_{j+1}, \bar{a}) = q(N_j, \bar{a})$. Suppose $c \in N_{j+1}$ realizes q. Since N_{j+1} is D^s_μ-primary over $N_j \cup b_j$, we have $\mathrm{tp}(c/N_j \cup b_j)$ is D^s_μ-isolated. But $c \underset{N_j}{\downarrow} b_j$, by Lemma .5.5. Therefore, by Lemma .5.6, we have that $c \in N_j$. This shows that $q(N_{j+1}, \bar{a}) = q(M^*, \bar{a})$.

This completes the proof. □

COROLLARY III.5.9. *Let D be totally transcendental. If $\mathcal{K}$ is categorical in some $\lambda > |T| + |D|$ then D is unidimensional.*

PROOF. Otherwise, there is a D-homogeneous model of cardinality λ and a maximally $(D, |T| + |D|)$-homogeneous model of cardinality λ. Hence $\mathcal{K}$ is not categorical in λ, since these models cannot be isomorphic. □

We now obtain strong structural results when D is unidimensional.

THEOREM III.5.10. *Let D be unidimensional. Then every $M \in \mathcal{K}$ is prime and minimal over $q(M, \bar{a})$, for any minimal type $q(x, \bar{a})$ over M.*

PROOF. Let $M \in \mathcal{K}$ be given. Since D is totally transcendental, there exists a minimal type $q(x, \bar{a})$ over M. Consider $A = q(M, \bar{a})$. To check minimality, suppose there was $N \in \mathcal{K}$, such that $A \subseteq N \subseteq M$. Since $q(N, \bar{a}) = A = q(M, \bar{a})$, we must have $N = M$, by unidimensionality of D. We now show that M is prime over A. Since D is totally transcendental, there is $M^* \in \mathcal{K}$ prime over A. Hence, we may assume that $A \subseteq M^* \subseteq M$. Now the minimality of M implies that $M = M^*$, so M is prime over A. Clearly, any other minimal type would have the same property. □

We next establish two lemmas, which are key results to carry out the geometric argument for the categoricity theorem.

LEMMA III.5.11. *Let $M \in \mathcal{K}$ and suppose that $q(x, \bar{a})$ is minimal over M. If $W = q(M, \bar{a})$ has dimension λ infinite, then W realizes every extension $p \in S_D(A)$ of type q, provided A is a subset of W of cardinality less than the dimension λ.*

PROOF. Let $p \in S_D(A)$ be given extending q. Let $c \in \mathfrak{C}$ realize p. If p is not big for M, then p is not realized outside M so $c \in M$. Hence $c \in W$ since p extends q. If however p is big for M, then p is big and then by Lemma .3.8 and Theorem .3.14 we have that $p = \mathrm{Av}(I, A)$, where I is any basis of W of cardinality λ. But $|I| = \lambda \geq |A|^+ + \aleph_0$, so by Theorem 1..4.5 and definition of averages, $\mathrm{Av}(I, A)$ is realized by some element of $I \subseteq W$. Hence p is realized in W. $\square$

LEMMA III.5.12. *Let D be unidimensional and let M be in $\mathcal{K}$ of cardinality $\lambda > |T| + |D|$. Suppose $q(x, \bar{a})$ is minimal over M. Then $q(M, \bar{a})$ has dimension λ.*

PROOF. Let $M \in \mathcal{K}$ be given and $q(x, \bar{a})$ be minimal. Construct $\langle M_\alpha \mid \alpha < \lambda\rangle$ strictly increasing and continuous such that $\bar{a} \in M_0$, $M_\alpha \subseteq M$ and $\|M_\alpha\| = |\alpha| + |T| + |D|$.

This is possible by Theorem .4.6: For $\alpha = 0$, just choose $M_0 \subseteq M$ prime over $\bar{a}$. For α a limit ordinal, let $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. At successor stage, since $\|M_\alpha\| \leq |\alpha| + |T| + |D| < \lambda$, there exists $a_\alpha \in M \setminus M_\alpha$, so we can choose $M_{\alpha+1} \subseteq M$ prime over $M_\alpha \cup a_\alpha$.

This is enough: Since D is unidimensional, we can find $c_\alpha \in M_{\alpha+1} \setminus M_\alpha$ realizing q. By definition, $\mathrm{tp}(c_\alpha / \bigcup\{c_\beta \mid \beta < \alpha\})$ is big, since $c_\alpha \notin M_\alpha$. Hence $c_\alpha \notin \mathrm{cl}(\bigcup\{c_\beta \mid \beta < \alpha\})$. Therefore $\{c_\alpha \mid \alpha < \lambda\}$ is independent and so $q(M, \bar{a})$ has dimension at least λ. Hence since $\|M\| = \lambda$, then $q(M, \bar{a})$ has dimension λ. $\square$

THEOREM III.5.13. *Let D be unidimensional. Then $\mathcal{K}$ is categorical in every $\lambda > |T| + |D|$.*

PROOF. Let $M_l \in \mathcal{K}$ for $l = 1, 2$ be of cardinality $\lambda > |T| + |D|$. Since D is totally transcendental, we can choose, $q(x, \bar{a}_1)$ minimal, with $\bar{a}_1 \in M_1$. Now, since M_2 is $(D, \aleph_0)$-homogeneous, we can find $\bar{a}_2 \in M_2$ such that $\mathrm{tp}(\bar{a}_1/\emptyset) = \mathrm{tp}(\bar{a}_2/\emptyset)$. Then $q(x, \bar{a}_1)$ is minimal also. Let $W_l = q(M_l, \bar{a}_l)$ for $l = 1, 2$. Since D is unidimensional, by Lemma .5.12, we have $\dim(W_l) = \lambda > |T| + |D|$. Hence, by Lemma .5.11 every type extending $q(x, \bar{a}_l)$ over a subset of W_l of cardinality less than λ is realized in W_l, for $l = 1, 2$. This allows us to construct by induction an elementary mapping g from W_1 onto W_2 extending $\langle \bar{a}_1, \bar{a}_2\rangle$. By Theorem .5.10, M_l is prime and minimal over W_l, for $l = 1, 2$. Hence, in particular M_1 is prime over W_1, so there is $f : M_1 \to M_2$ elementary extending g. But now $\mathrm{ran}(f)$ is a

$(D, \aleph_0)$-homogeneous model containing W_2, so by minimality of M_2 over W_2 we have $\mathrm{ran}(f) = M_2$. Hence f is also onto, and so M_1 and M_2 are isomorphic. $\square$

We can now summarize our results.

COROLLARY III.5.14. *Let D be totally transcendental. The following conditions are equivalent:*

(1) *$\mathcal{K}$ is categorical in every $\lambda > |T| + |D|$;*
(2) *$\mathcal{K}$ is categorical in some $\lambda > |T| + |D|$;*
(3) *D is unidimensional;*
(4) *Every $M \in \mathcal{K}$ is prime and minimal over $q(M, \bar{a})$, where $q(x, \bar{a})$ is any minimal type over M;*
(5) *Every model $M \in \mathcal{K}$ of cardinality $\lambda > |T| + |D|$ is D-homogeneous.*

PROOF. (1) implies (2) is trivial.
(2) implies (3) is Theorem .5.9.
(3) implies (1) is Theorem .5.13.
(3) implies (4) is Theorem .5.10.
(4) implies (3) is clear since prime models exist by Theorem .4.6.
(5) implies (1) is by back and forth construction, similarly to the corresponding proof with saturated models.
(1) implies (5) since for each $\lambda > |D| + |T|$ there exist a (D, λ)-homogeneous model of cardinality λ (e.g. by Theorem .4.6).

$\square$

COROLLARY III.5.15. *Let D be totally transcendental. If $\mathcal{K}$ is not categorical in some $\lambda_1 > |T| + |D|$, then*

(1) *If T is countable, then there are at least $|\alpha|$ models of cardinality $\aleph_\alpha$ in $\mathcal{K}$;*
(2) *For every $\lambda \geq \mu \geq |T| + |D|$ there is a maximally (D, μ)-homogeneous of cardinality λ.*

PROOF. (1) follows from (2). For (2), notice that D is not unidimensional by above Corollary, so the result follows from Theorem .5.8. $\square$

COROLLARY III.5.16. *Let D be totally transcendental. Suppose there is a maximally (D, μ)-homogeneous model of cardinality $\lambda > |T| + |D|$ for some $\lambda > \mu \geq \aleph_0$. Then for every $\lambda \geq \mu \geq |T| + |D|$ there is a maximally (D, μ)-homogeneous of cardinality λ.*

PROOF. Notice that $M \in \mathcal{K}$, and so $\mathcal{K}$ is not categorical in λ. Hence, by the previous corollary, D is not unidimensional, so the result follows from Theorem .5.8. $\square$

As a last Corollary, we obtain a generalization of Keisler's Theorem (notice that $\mathcal{K}$ is the class of atomic models in this case). We *do not* assume that D is totally transcendental.

COROLLARY III.5.17. *Let $|T| < 2^{\aleph_0}$, and suppose D is the set of isolated types of T. The following conditions are equivalent.*

(1) *$\mathcal{K}$ is categorical in every $\lambda > |T|$;*
(2) *$\mathcal{K}$ is categorical in some $\lambda > |T|$;*
(3) *D is totally transcendental and unidimensional;*
(4) *D is totally transcendental and every model of $\mathcal{K}$ is prime and minimal over $q(M, \bar{a})$, where $q(x, \bar{a})$ is any minimal type over M;*
(5) *Every model $M \in \mathcal{K}$ of cardinality $\lambda > |T| + |D|$ is D-homogeneous.*

PROOF. (5) implies (1) and (2) by back and forth construction. The rest of the proof follows from .5.14, since conditions (1), (2), (3) and (4) imply that D is totally transcendental. More precisely (1) and (2) imply that D is stable in $|T| < 2^{\aleph_0}$ and hence totally transcendental: this is a standard fact using Ehrenfeucht-Mostowski models. For (3) and (4) it is a hypothesis. $\square$

CHAPTER IV

Main gap and an abstract decomposition theorem

In [**Sh 131**], one of his most celebrated papers, Saharon Shelah proved what he called the *main gap* for the class of $\aleph_\epsilon$-saturated models of a complete first order theory T.

The result consists of showing that if there are less than the maximum number of nonisomorphic models of cardinality λ, for some cardinal λ greater than the cardinality of T, then the theory is superstable and satisfies NDOP. Using superstability and NDOP, Shelah shows that every model has a decomposition in terms of an independent tree of small models. Furthermore, under the same assumption as above, the tree must always be well-founded. This implies that the number of nonisomorphic models in each cardinal is bounded by a slow growing function. This exponential/slow growing dichotomy in the number of nonisomophic models is what is refered to as the main gap.

About six years later, Rami Grossberg and Bradd Hart [**GrHa**] realized that the main gap phenomenon is not limited to the elementary case. They proved the main gap for the class of models of an excellent Scott sentence (see [**Sh87b**] for the definition of excellence). The crucial property allowing a decomposition is also NDOP.

In this chapter, we introduce an axiomatic framework to prove decomposition theorems for a general class of models under NDOP. This framework is general enough to include in the same proof [**Sh 131**] and [**GrHa**], and includes the case of (D, μ)-homogeneous models of a totally transcendental diagram D introduced in Chapter IV. For nonstructure results using DOP (the failure of NDOP), the axiomatization needs several levels of saturation (or homogeneity, or fullness). We give a proof of the nonstructure parts of the theorem in the context of Chapter IV. This gives the main gap for the class $\mathcal{K}$ of (D, μ)-homogeneous models of a totally transcendental diagram D (for any infinite μ). Note that, since finite diagrams generalize the first order case, it is easy to see that the failure of a finite diagram to be totally transcendental does not imply the existence of many models. All the basic tools in place, we can also show, using the methods of [**Sh b**] or [**Ha**] that $\lambda \mapsto I(\lambda, \mathcal{K})$ is weakly monotonic (Morley's Conjecture) for sufficiently large λ.

This result was presented as an abstract in the European Meeting of the ASL in Leeds in June 1997. In January 1998, Shelah informed us that in a joint paper with Hyttinen [**HySh2**] includes a similar result to the one we present here for *superstable* diagrams. The paper is not yet available, however, we suspect that their decomposition falls within the axiomatic framework presented in Section 1.

This chapter is organized as follows.

In Section 1, we present the axiomatic framework. The aim of the framework is to capture the essential features of the various contexts where decomposition theorems using NDOP are known. A proof of a decomposition theorem is given under the parallel of NDOP (Theorem I.1.29).

In Section 2, we present the necessary orthogonality calculus to show that the class of $(D,\aleph_0)$-homogeneous models of a totally transcendental diagram D satisfies the axioms of Section 1. This implies that under NDOP, every $(D,\aleph_0)$-homogeneous model is prime and minimal over an independent tree of small models. We also prove several additional lemmas that will allow us to complete the main gap for this class.

In Section 3, we introduce DOP (the negation of NDOP) for the class of $(D,\aleph_0)$-homogeneous models of a totally transcendental diagram D. We show that DOP implies the existence of many nonisomorphic models (Theorem .3.1).

In Section 4, we work under the assumption that the class has NDOP. We introduce depth for the class of $(D,\aleph_0)$-homogeneous models of a totally transcendental diagram D. We prove that if a class is deep then it has many nonisomorphic models (Theorem .4.23). Finally, we derive the main gap (Theorem .4.25). Using the same methods, we can also derive the main gap for the class of (D,μ)-homogeneous models of a totally transcendental diagram D.

IV.1. The axiomatic framework and decomposition theorem

Let $\mathcal{K}$ be a class of models in a fixed similarity type L. Let $N^*, N^{**} \in \mathcal{K}$ be such that $N^* \subseteq N^{**}$. This section will study conditions on the subsets of N^* which guarantee that it can be decomposed.

All models will be in $\mathcal{K}$. All models, sets, and elements will be subsets of N^*. For a set $A \subseteq N^*$, denote by $S(A)$ the set of complete types over A realized in N^{**}.

At the end of this section, we included a short subsection describing some of the known cases that this axiomatic framework covers. The reader may want to consult it, in order to have a more concrete framework in mind.

We begin with a list of axioms.

We postulate the existence of a *dependence relation* on subsets of N^*, i.e. a relation on triples of sets, written $A \underset{B}{\downarrow} C$, satisfying the following axioms. Note that (5) is not used in the decomposition.

AXIOM IV.1.1 (Independence). *Let A, B, C and D be sets. Let M be a model.*

(1) *(Definition)* $A \underset{B}{\downarrow} C$ *if and only if* $A \underset{B}{\downarrow} B \cup C$;
(2) *(Triviality)* $A \underset{B}{\not\downarrow} A$;
(3) *(Finite Character)* $A \underset{B}{\downarrow} C$ *if and only if* $A' \underset{B}{\downarrow} C'$, *for all finite* $A' \subseteq A$, $C' \subseteq C$;
(4) *(Monotonicity) If* $A \underset{B}{\downarrow} C$ *and* $B \subseteq B_1 \subseteq M_1$ *and* $C' \subseteq C$, *then* $A \underset{C_1}{\downarrow} B'$;
(5) *($\aleph_0$-Local Character) For every* $\bar{a}$ *and* M, *there exists a finite* $B \subseteq M$ *such that* $\bar{a} \underset{B}{\downarrow} M$;
(6) *(Transitivity) If* $B \subseteq C \subseteq D$, *then* $A \underset{C}{\downarrow} D$ *and* $A \underset{B}{\downarrow} C$ *if and only if* $A \underset{B}{\downarrow} D$;
(7) *(Symmetry over models)* $A \underset{M}{\downarrow} C$ *if and only if* $C \underset{M}{\downarrow} A$.
(8) *(Invariance) Let* f *be a partial elementary mapping of* N^* *with* $A \cup B \cup C \subseteq \mathrm{dom}(f)$. *Then* $A \underset{B}{\downarrow} C$ *if and only if* $f(A) \underset{f(B)}{\downarrow} f(C)$.
(9) *(Concatenation) If* $A \underset{M}{\downarrow} BC$ *and* $C \underset{M}{\downarrow} B$ *then* $C \underset{M}{\downarrow} BA$.

We first examine independent families.

DEFINITION IV.1.2. We say that $\{B_i \mid i < \alpha\}$ is *independent over* M if

$$B_i \underset{M}{\downarrow} \bigcup \{B_j \mid j \neq i, j < \alpha\},$$

for every $i < \alpha$.

LEMMA IV.1.3. *Let* $\{B_i \mid i < \alpha\}$ *be a family of sets and assume that*

(*) $$B_{i+1} \underset{M}{\downarrow} \bigcup \{B_j \mid j < i\}, \quad \textit{for every } i < \alpha.$$

Then $\langle B_i \mid i < \alpha \rangle$ *is independent over* M.

PROOF. By finite character of independence, it is enough to prove this statement for α finite. We do this by induction on α, an integer.

For $\alpha = 1$, (*) implies that $B_1 \underset{M}{\downarrow} B_0$, so by symmetry over models we have $B_0 \underset{M}{\downarrow} B_1$, which shows that $\{B_0, B_1\}$ is independent over M.

Assume by induction that the statement is true for $\alpha < \omega$. Let $i \leq \alpha + 1$ be given. We must show that

(**) $$B_i \underset{M}{\downarrow} \bigcup \{B_j \mid j \leq \alpha + 1, j \neq i\}.$$

If $i = \alpha + 1$, then this is (*), so we may assume that $i \neq \alpha + 1$ and therefore (**) can be rewritten as

$$B_i \underset{M}{\perp} \bigcup \{B_j \mid j \leq \alpha, j \neq i\} \cup B_{\alpha+1}.$$

Notice that by induction hypothesis

$$B_i \underset{M}{\perp} \bigcup \{B_j \mid j \leq \alpha, j \neq i\}$$

Further, by (*), we have

$$B_{\alpha+1} \underset{M}{\perp} \bigcup \{B_j \mid j \leq \alpha, j \neq i\} \cup B_i.$$

Therefore (**) follows from the previous two lines by concatenation of independence. ☐

We will say that a set of sequences I is a tree if it is closed under initial segment. We will use the notation $\eta \prec \nu$ to mean that η is an initial segment of ν.

DEFINITION IV.1.4. Let I be a tree, we say that $\langle M_\eta \mid \eta \in I \rangle \subseteq M_1$ is a *system* if $M_\eta \in \mathcal{K}$ for each $\eta \in I$ and $M_\eta \subseteq M_\nu$ when $\eta \prec \nu \in I$.

The concept in the next definition is called system in stable amalgamation by Shelah.

DEFINITION IV.1.5. We say that $\langle M_\eta \mid \eta \in I \rangle \subseteq M_1$ is an *independent system* if it is a system satisfying in addition:

$$M_\eta \underset{M_{\eta^-}}{\perp} \bigcup_{\eta \not\prec \nu} M_\nu, \quad \text{for every } \eta \in I.$$

Under our axioms, independent systems are quite independent. For J a subtree of I, denote by $M_J := \bigcup \{M_\eta \mid \eta \in J\}$. The following is an abstract version of Shelah's generalized Symmetry Lemma. It appears in a similar way in [**Ma**].

LEMMA IV.1.6. *Let $\langle M_\eta \mid \eta \in I \rangle$ be an independent system. Then, for any I_1, I_2 subtrees of I, we have:*

(*)
$$M_{I_1} \underset{M_{I_1 \cap I_2}}{\perp} M_{I_2}$$

PROOF. By the finite character of independence, it is enough to prove (*) for finite trees I. We prove this by induction on $|I_1 \cup I_2|$.

First, if $I_2 \subseteq I_1$, then it is obvious. Thus, assume that there is $\eta \in I_2 \setminus I_1$, and choose η of maximal length. Let $J_2 := I_2 \setminus \{\eta\}$. Notice that by choice of η, we have $M_{I_1 \cap J_2} = M_{I_1 \cap I_2}$. By induction hypothesis, we have that

(*)
$$M_{I_1} \underset{M_{I_1 \cap I_2}}{\perp} M_{J_2}.$$

Since $M_{\eta^-} \subseteq M_{J_2}$, by monotonicity (*) implies that

(**) $$M_{I_1} \underset{M_{\eta^-}}{\downarrow} M_{J_2}.$$

By definition of independent system and monotonicity we have

(***) $$M_\eta \underset{M_{\eta^-}}{\downarrow} M_{I_1} \cup M_{J_2}.$$

Therefore, by concatenation applied to (**) and (***), we can conclude that

(†) $$M_{I_1} \underset{M_{\eta^-}}{\downarrow} M_{I_2}.$$

Now, using (*) and monotonicity we have

(‡) $$M_{I_1} \underset{M_{I_1 \cap I_2}}{\downarrow} M_{\eta^-}.$$

Thus, the transitivity property applied to (†) and (‡), implies that

$$M_{I_1} \underset{M_{I_1 \cap I_2}}{\downarrow} M_{I_2}.$$

This finishes the proof. □

We now introduce the dependence relation on types; the invariance of the dependence relation makes it natural. We set a few standard definitions in this context.

DEFINITION IV.1.7. (1) We say that $p \in S(A)$ is *free* over $B \subseteq A$ if for every $M \supseteq A$ and $\bar{a} \in M$ realizing p, we have $\bar{a} \underset{B}{\downarrow} A$;

(2) We say that $p \in S(A)$ is *stationary* if for every M containing A, there is a unique extension $p_M \in S(M)$ of p such that p_M is free over A.

(3) We say that the stationary type $p \in S(A)$ is *based on* B if p is free over B.

AXIOM IV.1.8 (Existence of Stationary types). *Let $M \in \mathcal{K}$. Then any $p \in S(M)$ is stationary.*

Note that by the local character of the dependence relation, any stationary type is based on a finite set.

We now introduce a strong independence between stationary types: orthogonality.

DEFINITION IV.1.9. Let $p \in S(B)$ and $q \in S(A)$ be stationary. We say that p is *orthogonal* to q, written $p \perp q$, if for every $M \in \mathcal{K}$ containing $A \cup B$ and for every $a \models p_M$ and $b \models q_M$, we have $a \underset{M}{\downarrow} b$.

By symmetry of independence, $p \perp q$ if and only if $q \perp p$. Also, if $p \in S(A)$ and $q \in S(B)$ are stationary with $A \subseteq B$, then by definition $p \perp q$ if and only if $p_B \perp q$.

We now expand this definition to orthogonality against models.

DEFINITION IV.1.10.

(1) Let $p \in S(A)$ be stationary. We say that p is *orthogonal to* M, written $p \perp M$, if p is orthogonal to each $q \in S(M)$.
(2) If $M_0 \subseteq M_1 \cap M_2$, we write that $M_1/M_0 \perp M_2$ if and only if $p \perp M_2$, for every $p \in S(M_0)$ realized in M_1.

We now concentrate on a special kind of types: regular types. It follows from the definition of that over the set of elements realizing a regular type, the dependence relation satisfies the axioms of a pregeometry.

DEFINITION IV.1.11. A stationary type $p \in S(A)$ is called *regular* if there exist a finite B and a set of formulas $p(x, \bar{b}) \subseteq p \upharpoonright B$, such that p is based on B and for every M containing A and every $q \in S(M)$ extending $p(x, \bar{b})$ either $q = p_M$ or $q \perp p$.

AXIOM IV.1.12 (Parallelism). *Let $p, q \in S(M)$, be regular types. Let N contain M. Then $p \perp q$ if and only if $p_N \perp q_N$.*

LEMMA IV.1.13. *Let $M \subseteq M_1$. If $p \in S(M)$ is regular, then $p_{M_1} \in S(M_1)$ is regular.*

PROOF. Let $p \in S(M)$ be regular. Let $B \subseteq M$ and $p(x, \bar{b})$ be as in the definition. Then, p_{M_1} is stationary based in B. Let $q \in S(N)$ extend $p(x, \bar{b})$. Then either $q = p_N = (p_{M_1})_N$ or $q \perp p$. Hence by definition of $\perp$ we have $q \perp p_{M_1}$. This shows that q is regular. □

AXIOM IV.1.14 (Existence of Regular types). *If $M \subseteq N$ and $M \neq N$, then there exists a regular type $p \in S(M)$ realized in $N \setminus M$.*

The next three axioms guarantees that it is enough to focus on regular types.

AXIOM IV.1.15 (Perp I). *Let $M, N \in \mathcal{K}$ such that $M \subseteq N$. Let $p \in S(N)$ be regular. Then $p \perp M$ if and only if $p \perp q$, for every regular type $q \in S(M)$.*

DEFINITION IV.1.16. We say that a model $M(A)$ is *prime* over $A \subseteq M_1$, if for every N containing A, there exists an elementary embedding $f \colon M(A) \to N$, which is the identity on A.

The next few axioms assert that prime models exist under some circumstances.

AXIOM IV.1.17 (Prime models).

(1) *Let $M_1 \in \mathcal{K}$. There exists a prime model $M_\emptyset \subseteq M_1$ over $\emptyset$;*
(2) *If $\bar{a} \in M_1 \setminus M$ (where $\bar{a}$ is finite) then there is a prime model $M(\bar{a}) \subseteq M_1$ over $M \cup \bar{a}$;*
(3) *If $\langle M_\eta \mid \eta \in I\rangle \subseteq M_1$ is an independent system, then there exists a prime model $M(A_I) \subseteq M_1$ over $\bigcup_{\eta\in I} M_\eta$.*

This is to establish connections with the dependence relation and orthogonality.

AXIOM IV.1.18 (Equivalence). *Let $M \in \mathcal{K}$ and let $p, q \in S(M)$ be regular and let $\bar{b} \notin M$ realize p. Then q is realized in $M(\bar{b}) \setminus M$ if and only if $p \not\perp q$.*

Note that by Equivalence, the relation $\not\perp$ among regular types (over the same base set) is an equivalence relation.

LEMMA IV.1.19. *Let $M_0 \subseteq M \subseteq M' \subseteq N$. Let $p \in S(M')$ be regular realized in $N \setminus M'$ and $q \in S(M)$ such that $p \not\perp q$. Let $r \in S(M_0)$ be regular. If $p \perp r$ then $q \perp r$.*

PROOF. By Lemma I.1.13, the types $r_{M'}$ and $q_{M'}$ are regular. By definition, $p \not\perp q_{M'}$. If $q \not\perp r$, then $q_{M'} \not\perp r_{M'}$. By the axiom of parallelism, $p \perp r$ if and only if $p \perp r_{M'}$ and $q \perp r$ if and only if $q_{M'} \perp r_{M'}$. The conclusion follows from the equivalence axiom. $\square$

AXIOM IV.1.20 (Prime base). *If M' is a prime model over $\bigcup_{\eta\in I} M_\eta$, where $\langle M_\eta \mid \eta \in I\rangle$ is an independent system and let $p \in S(M')$ is regular. Then there exists a finite subtree $J \subseteq I$ and a model M^* prime over $\bigcup_{\eta\in J} M_\eta$ such that p is based on M^*.*

AXIOM IV.1.21 (Dominance).

(1) *Let a be such that $\mathrm{tp}(a/M)$ is regular. For each C, if $a \underset{M}{\perp} C$ then $M(a) \underset{M}{\perp} C$;*
(2) *Let $\langle M_\eta \mid \eta \in I\rangle$ be an independent system and let $M(A_I)$ be prime over it. Then, for each C, if $\bigcup\{M_\eta \mid \eta \in I\} \underset{M}{\perp} C$ then $M(A_I) \underset{M}{\perp} C$.*

LEMMA IV.1.22. *Let $p = \mathrm{tp}(a/M)$ be regular and suppose that $p \perp M_1$, with $M_1 \subseteq M$. Then $M(a)/M \perp M_1$.*

PROOF. By axiom (Perp I) it is enough to show that any regular type $q \in S(M)$ realized in $M(a) \setminus M$ is orthogonal to any regular type r over M_1. But, if q is regular realized in $M(a) \setminus M$, then by Equivalence we must have $q \not\perp p$. Since $p \perp M_1$ by assumption, then $p \perp r$. Then, by definition, $q \perp r$ if and only if $q \perp r_M$. Hence, we conclude by Equivalence. $\square$

LEMMA IV.1.23. *Let $M_0 \subseteq M$ and let $\bar{a}_1, \bar{a}_2$ such that $\bar{a}_1 \underset{M}{\downarrow} \bar{a}_2$. Suppose that $\mathrm{tp}(\bar{a}_i/M) \perp M_0$, for $i = 1, 2$. Let B be such that $B \underset{M}{\downarrow} M_0$, then $\bar{a}_1\bar{a}_2 \underset{M}{\downarrow} B$.*

PROOF. By finite character of independence, it is enough to prove this for finite B. Let $\bar{b}$ be finite such that

(*) $$\bar{b} \underset{M}{\downarrow} M_0,$$

First, since $\mathrm{tp}(\bar{a}_2/M) \perp M_0$, (*) implies that

(**) $$\bar{a}_2 \underset{M}{\downarrow} \bar{b}.$$

Thus, by symmetry, we must have $\bar{b} \underset{M}{\downarrow} \bar{a}_2$, which shows that $\bar{b} \models \mathrm{tp}(\bar{b}/M) \restriction M\bar{a}_2$. By assumption, we have that

(***) $$\bar{a}_1 \underset{M}{\downarrow} \bar{a}_2,$$

and thus $\bar{a}_1 \models \mathrm{tp}(\bar{a}_1/M) \restriction M\bar{a}_2$. But, $\mathrm{tp}(\bar{a}_1/M) \perp \mathrm{tp}(\bar{b}/M)$, so by definition, we must have $\bar{a}_1 \underset{M\bar{a}_2}{\downarrow} \bar{b}$. By the first axiom of the dependence relation, we have $\bar{a}_1 \underset{M\bar{a}_2}{\downarrow} \bar{b}\bar{a}_2$. By transitivity using (***), we obtain $\bar{a}_1 \underset{M}{\downarrow} \bar{b}\bar{a}_2$. Hence, by the concatenation property of independence and (†) again, we can derive

$$\bar{a}_1\bar{a}_2 \underset{M_0}{\downarrow} \bar{b},$$

which is what we wanted. □

COROLLARY IV.1.24. *Let $M \subseteq N$. Let $\langle A_i \mid i < \alpha \rangle$ be independent over N, such that $A_i/N \perp M$, for each $i < \alpha$. Let B be such that $B \underset{M}{\downarrow} N$. Then $\bigcup\{A_i \mid i < \alpha\} \underset{M}{\downarrow} B$.*

PROOF. By finite character of independence and monotonicity, we may assume that $\alpha < \omega$. We prove the statement by induction on α and use the previous lemma at the successor step. □

COROLLARY IV.1.25. *Let $\langle M_\eta \mid \eta \in I \rangle$ be a system satisfying:*

(1) *$\langle M_\eta \mid \eta^- = \nu, \eta \in I \rangle$ is independent over M_ν, for every $\nu \in I$;*
(2) *The type $\mathrm{tp}(M_\eta/M_{\eta^-}) \perp M_{\eta^{--}}$, for every $\eta \in I$.*

Then $\langle M_\eta \mid \eta \in I \rangle$ is an independent system.

PROOF. By the finite character of independence, we may assume that I is finite. We prove this statement by induction on $|I|$. First, notice that if there is no $\eta \in I$ such that η^{--} exists, then the result follows from (1). We must show that

$$M_\eta \underset{M_{\eta^-}}{\downarrow} \bigcup\{M_\nu \mid \eta \not\prec \nu, \nu \in I\}.$$

Choose $\nu \in I$ of maximal length such that $\eta \not\prec \nu$. Let

$$I_1 := \{\rho \mid \eta \not\prec \rho, \nu^- \prec \rho \text{ and } \rho \neq \nu\}.$$

Then, by (1), the system

(*) $$\langle M_\rho, M_\nu \mid \rho \in I_1\rangle, \text{ is independent over } M_{\nu^-}.$$

Let

$$I_2 := \{\rho \mid \eta \not\prec \rho, \nu^- \not\prec \rho\}.$$

By induction hypothesis

(**) $$M_{\nu^-} \underset{M_{\nu^{--}}}{\downarrow} M_{I_2} M_\eta.$$

Hence, by the previous corollary, using (*), symmetry on (**) and the fact that $M_\rho / M_{\nu^-} \perp M_{\nu^{--}}$, for $\rho \in I_1$ or $\rho = \nu$, we conclude that

(***) $$M_\nu M_{I_1} \underset{M_{\nu^-}}{\downarrow} M_{I_2} M_\eta.$$

Now, by induction hypothesis, we must have $M_{I_2} \underset{M_{\nu^-}}{\downarrow} M_\eta$, so by concatenation, we must have

(†) $$M_\eta \underset{M_{\nu^-}}{\downarrow} M_{I_1} M_{I_2} M_\nu.$$

Now, $M_\eta \underset{M_{\eta^-}}{\downarrow} M_{\nu^-}$ by monotonicity and induction hypothesis. Therefore, using (†), transitivity and the definition of I_1 and I_2, we conclude that

$$M_\eta \underset{M_{\eta^-}}{\downarrow} \bigcup\{M_\nu \mid \eta \not\prec \nu, \nu \in I\}.$$

□

DEFINITION IV.1.26. N^* has NDOP if for every $M_0, M_1, M_2 \subseteq N^*$ such that $M_1 \underset{M_0}{\downarrow} M_2$, for every $M' \subseteq N^*$ prime over $M_1 \cup M_2$ and for every regular type $p \in S(M')$. If p is realized in $N^* \setminus M'$, then either $p \not\perp M_1$ or $p \not\perp M_2$.

THEOREM IV.1.27. *Suppose N^* has NDOP. Let $M, M_\eta \subseteq N^*$, for $\eta \in I$ be such that $\langle M_\eta \mid \eta \in I\rangle$ is an independent system and M is prime over it. Let $a \in N^* \setminus M$ be such that $\mathrm{tp}(a/M)$ is regular. Then there is η such that $\mathrm{tp}(a/M) \not\perp M_\eta$.*

PROOF. Let $p = \mathrm{tp}(a/M)$. Suppose that $p \perp M_\eta$ for every $\eta \in I$. By the prime base axiom and parallelism we may assume that I is finite. We will obtain a contradiction to NDOP by induction on $|I|$.

If $I = \{\eta \restriction k : k < n\}$, it is obvious because $\bigcup_{\nu \in I} M_\nu = M_\eta$, so by definition of prime, we have $M' = M_\eta$. But $p \not\perp p$ by triviality of independence. Therefore, $p \not\perp M_\eta$ by definition.

Otherwise, there exists $\nu \in I$ such that both subtrees $I_1 := \{\eta : \eta \in I\ \nu \prec \eta\}$ and $I_2 := \{\eta : \eta \in I\ \nu \not\prec \eta\}$ are nonempty. By the third axiom on prime models, we can choose M_k prime over $\bigcup_{\eta \in I_k} M_\eta$ for $k = 1, 2$. By induction hypothesis, we have

$$p \perp M_1 \qquad \text{and} \qquad p \perp M_2.$$

Furthermore, since $\langle M_\eta \mid \eta \in I \rangle$ is an independent system, we have

$$\bigcup_{\eta \in I_1} M_\eta \underset{M_\nu}{\perp} \bigcup_{\eta \in I_2} M_\eta.$$

Therefore, by the symmetry of independence and dominance, we must have

$$M_1 \underset{M_\nu}{\perp} M_2.$$

But, M' is prime over $M_1 \cup M_2$. This contradicts the fact that N^* has NDOP. $\square$

An ω-tree is simply a tree of height at most ω.

DEFINITION IV.1.28. We say that $\langle M_\eta, a_\eta \mid \eta \in I \rangle$ is a *decomposition of* N^* *over* M if it satisfies the following conditions:

(1) I is an ω-tree;
(2) $\langle M_\eta \mid \eta \in I \rangle$ is a system with $M_\eta \subseteq N^*$ for each $\eta \in I$;
(3) If η^{--} exists for $\eta \in I$, then $M_\eta / M_{\eta^-} \perp M_{\eta^{--}}$;
(4) For every $\nu \in I$ the system $\langle M_\eta \mid \eta^- = \nu, \eta \in I \rangle$ is independent over M_ν.
(5) $M_{\langle\rangle} = M$ and M_η is prime over $M_{\eta^-} \cup a_\eta$;
(6) For every $\eta \in I$, the type $\mathrm{tp}(a_\eta / M_{\eta^-})$ is regular.

We say that $\langle M_\eta, a_\eta \mid \eta \in I \rangle$ is a *decomposition of* N^* if it is a decomposition of N^* over $M_{\langle\rangle}$ the prime model over the empty set.

We can introduce an ordering between decompositions of N^* over M as follows: We say that

$$\langle M_\eta, a_\eta \mid \eta \in I \rangle \prec \langle N_\eta, b_\eta \mid \eta \in J \rangle$$

if $I \subseteq J$ and for every $\eta \in I$ we have

$$M_\eta = N_\eta, \quad \text{and} \quad a_\eta = b_\eta.$$

It is now easy to show that the set of decompositions of N^* is inductive: Let $\langle \mathcal{S}_i \mid i < \alpha \rangle$ be a chain of decompositions $\mathcal{S}_i = \langle M^i_\eta, a^i_\eta \mid \eta \in I^i \rangle$. First, let $I := \bigcup_{i<\alpha} I^i$. Then I is an ω-tree. Hence, we can define the system $\mathcal{S} := \langle M_\eta, a_\eta \mid \eta \in I \rangle$, by $M_\eta := M^i_\eta$ if $\eta \in I^i$ and $a_\eta := a^i_\eta$, if $\eta \in I^i$. This is well-defined since $\langle \mathcal{S}_i \mid i < \alpha \rangle$ is chain. We need to check that $\mathcal{S}$ is a decomposition of N^*. The only nontrivial fact is to check that for every $\nu \in I$ the system

$$\langle M_\eta \mid \eta^- = \nu, \eta \in I \rangle$$

is independent over M_ν. If it failed, then by finite character, there would be a finite set $F \subseteq I$ such that

$$\langle M_\eta \mid \eta^- = \nu, \eta \in F \rangle$$

is not independent. By then, there exists $i < \alpha$ such that $F \subseteq I^i$, contradicting the fact that $\mathcal{S}_i$ is a decomposition of N^*.

Recall that we say that a model N is *minimal* over A if prime models exist over A and if $M(A) \subseteq N$ is prime over A, then $N = M(A)$. Note that a decomposition as in the next theorem is called *complete*.

THEOREM IV.1.29. *Suppose N^* has NDOP. Then for every $M \subseteq N^*$, there exists $\langle M_\eta, a_\eta \mid \eta \in I \rangle$ a decomposition of N^* over M such that N^* is prime and minimal over $\bigcup_{\eta \in I} M_\eta$.*

PROOF. First, notice that the set of decompositions of N^* over M is not empty. Therefore, by Zorn's Lemma, since the set of decompositions of N^* over M is inductive, there exists a maximal decomposition

(*) $$\langle M_\eta, a_\eta \mid \eta \in I \rangle.$$

By Lemma I.1.25, we know that $\langle M_\eta \mid \eta \in I \rangle$ is an independent system. Therefore, by the third axiom for prime models, there exists $M' \subseteq N^*$ prime over $\bigcup_{\eta \in I} M_\eta$. We will show that $M' = N^*$. This will show that N^* is prime and minimal over $\bigcup_{\eta \in I} M_\eta$.

Suppose that $M' \neq N^*$. Then, by the axiom of existence of regular types, there exists a regular type $p \in S(M')$ realized in $N^* \setminus M'$. We are going to contradict the maximality of $\langle M_\eta, a_\eta \mid \eta \in I \rangle$. Since N^* has NDOP, by Theorem I.1.27, there exists $\eta \in I$ such that $p \not\perp M_\eta$. Choose η of smallest length such that $p \not\perp M_\eta$. By axiom (Perp I), there exists a regular type $q \in S(M_\eta)$ such that $p \not\perp q$. Since q is stationary, we can choose $q_{M'}$ the unique free extension of q to the prime model M'. Then, by Lemma I.1.13, the type $q_{M'}$ is regular. Since $p \not\perp q$ and $p \in S(M')$, by definition $p \not\perp q_{M'}$. By Equivalence, since p is realized in $N \setminus M'$, there exists $a \in N \setminus M'$ realizing $q \restriction M'$. Hence $\mathrm{tp}(a/M') = q_{M'}$ and by choice of $q_{M'}$, this implies that

(**) $$a \underset{M_\eta}{\perp} M'.$$

Since $\mathrm{tp}(a/M_\eta)$ is regular and $a \in N \setminus M_\eta$, by the second axiom on prime models, there exists a prime model $M(a) \subseteq N^*$ over $M_\eta \cup a$. By dominance and (**) we must have

$$M(a) \underset{M_\eta}{\perp} M'.$$

Thus, by monotonicity of independence and choice of M', we conclude that

(***) $$M(a) \underset{M_\eta}{\perp} \bigcup \{M_\nu \mid \nu^- = \eta,\ \nu \in I\}.$$

But $\{M_\nu \mid \nu^- = \eta\}$ is independent by definition of decomposition. Thus, (***) and Lemma I.1.3 implies that

$$\{M_\nu, M(a) \mid \nu^- = \eta,\ \nu \in I\}$$

is independent over M_η. Suppose now that η^- exists. By choice of η we must have $p \perp M_{\eta^-}$. Since $p \not\perp \mathrm{tp}(a/M_\eta)$, we must have by Lemma I.1.19 and axiom (Perp I) that $\mathrm{tp}(a/M_\eta) \perp M_{\eta^-}$. Hence, by Lemma I.1.22, we must have $M(a)/M_\eta \perp M_{\eta^-}$. This shows that we can add $\mathrm{tp}(a/M_\eta)$ and $M(a)$ to (*) and still have a decomposition of N^*. This contradicts the maximality of (*). Thus N^* is prime and minimal over $\bigcup_{\eta \in I} M_\eta$. □

COROLLARY IV.1.30. *If N^* has NDOP, there exists a complete decomposition of N^*.*

PROOF. By the previous theorem since by axiom on prime models there exists a prime model over the empty set. □

The same proof shows:

COROLLARY IV.1.31. *If N^* has NDOP and N^* is prime over a decomposition $\langle M_\eta \mid \eta \in I\rangle$ of N^* over M, then $\langle M_\eta \mid \eta \in I\rangle$ is a complete decomposition of N^* over M.*

IV.1.0.1. *Examples.* The abstract decomposition given in the section above generalizes the known NDOP cases.

There are several classical first order cases. The first one is for $\aleph_0$-saturated models of a totally transcendental theory T. A second one is for $\aleph_\epsilon$-saturated models of a superstable theory T. And finally, for the class of models of a totally transcendental theory T. In each case, N^{**} can be taken to be the monster model for T. The independence relation is forking. Regular types in the first two cases are just the regular types in the sense of first order. In the last case, they correspond to strongly regular types. The prime models are the $F^s_{\aleph_0}$-primary models, the $\aleph_\epsilon$-primary models for the second case, and the $F^t_{\aleph_0}$-primary models in the third case. All the results needed to apply the theorem can be found in [**Sh b**].

In the nonelementary case, there is one published example: the models of an excellent Scott sentence in $L_{\omega_1,\omega}$ [**GrHa**]. The model N^{**} can be taken to be any sufficiently large full model over N^*. The dependence relation is that afforded by the rank. Regular types are the SR types. The existence of prime models follows from excellence (see [**Sh87a**], [**Sh87b**], and [**GrHa**]).

The aim of the next section is to prove that the axiomatic framework developed in this section holds for the class $\mathcal{K}$ of $(D, \aleph_0)$-homogeneous models of a totally transcendental D. Let N^* be a D-model and $N^{**} = \mathfrak{C}$.

The *dependence relation* is given by the rank; the axioms for independence were verified in Theorem .2.3. The *stationary types* correspond to the ones in Chapter III, and the axiom postulating their existence follows from Corollary .1.10. As for *regular types*, they are defined slightly differently in the previous section, but by inspecting the proof, one sees easily that their existence follows from Theorem .3.10. Finally, the prime models are the $D^s_{\aleph_0}$-models of Chapter III. Then their existence follows from Theorem .4.6. By definition of isolation, Axiom (Prime base) also holds immediately, since stationary types are based on a finite set.

This leaves us with the proof of Parallelism, Equivalence, and Dominance. These results are part of what is called Orthogonality Calculus.

Note also that in each of the known cases, the failure of NDOP implies the existence of many nonisomorphic models. This will be the object of section 3 for totally transcendental diagrams.

IV.2. Orthogonality calculus in finite diagrams

In this section, the context is that of totally transcendental diagrams. We already establised in Chapter III that many of the axioms of the previous section hold for totally transcendental diagrams. We will now develop what is refered to as *orthogonality calculus* for this context and show that the remaining axioms used to obtain an abstract decomposition theorem also hold for the class of $(D, \aleph_0)$-homogenous models of a totally transcendental diagram D.

Note that some results falling under orthogonality calculus were already proved in the last section of Chapter III.

The next few lemmas show Dominance.

First, for D-sets A and B, we say that $A \subseteq_{TV} B$, if every D-type over finitely many parameters in A realized in B is realized in A. The subscript TV stands for Tarski-Vaught.

LEMMA IV.2.1. *Let M be $(D, \aleph_0)$-homogeneous. Suppose $\bar{a} \underset{M}{\downarrow} \bar{b}$. Then, for every $\bar{m} \in M$ the type $\mathrm{tp}(\bar{b}/\bar{m}\bar{a})$ is realized in M.*

PROOF. By symmetry, $\bar{b} \underset{M}{\downarrow} \bar{a}$. Hence, by taking a larger $\bar{m}$ if necessary, we may assume that $\text{tp}(\bar{b}/M\bar{a})$ does not split over $\bar{m}$. By $(D, \aleph_0)$-homogeneity of M, we can find $\bar{b}' \in M$, such that $\text{tp}(\bar{b}/\bar{m}) = \text{tp}(\bar{b}'/\bar{m})$. We claim that $\text{tp}(\bar{b}/\bar{m}\bar{a}) = \text{tp}(\bar{b}'/\bar{m}\bar{a})$. If not, there exists a formula $\phi(\bar{x}, \bar{m}, \bar{a})$ such that $\models \phi[\bar{b}, \bar{m}, \bar{a}]$ and $\models \neg\phi[\bar{b}', \bar{m}, \bar{a}]$. But, $\text{tp}(\bar{b}/\bar{m}) = \text{tp}(\bar{b}'/\bar{m})$, so $\text{tp}(\bar{a}/M\bar{b})$ splits over $\bar{m}$, a contradiction. □

The next lemma is standard.

LEMMA IV.2.2. *Let A, B be D-sets such that $A \subseteq_{TV} B$. If $\text{tp}(\bar{c}/A)$ is $D^s_{\aleph_0}$-isolated, then $\text{tp}(\bar{c}/A) \vdash \text{tp}(\bar{c}/B)$.*

PROOF. Let $q(\bar{x}, \bar{a}) \vdash \text{tp}(\bar{c}/A)$, with $\bar{a} \in A$. Suppose that $\text{tp}(\bar{c}/A) \nvdash \text{tp}(\bar{c}/B)$. Then, there exists $\bar{b} \in B$ and a formula $\phi(\bar{x}, \bar{y})$ such that $q(\bar{x}, \bar{a}) \cup \phi(\bar{x}, \bar{b})$ and $q(\bar{x}, \bar{a}) \cup \neg\phi(\bar{x}, \bar{b})$ are both realized in $\mathfrak{C}$. By assumption, there exists $\bar{b}' \in A$ realizing be such that $\text{tp}(\bar{b}/\bar{a}) = \text{tp}(\bar{c}/\bar{c})$. Hence, by an automorphism fixing $\bar{a}$ and sending $\bar{b}$ to $\bar{b}'$, both $q(\bar{x}, \bar{a}) \cup \phi(\bar{x}, \bar{b}')$ and $q(\bar{x}, \bar{a}) \cup \neg\phi(\bar{x}, \bar{b}')$ are realized in $\mathfrak{C}$. This contradicts the choice of $q(\bar{x}, \bar{a})$. □

Recall that we denote by $M(A)$ the $D^s_{\aleph_0}$-primary model over $M \cup A$.

THEOREM IV.2.3 (Dominance). *Let M be $(D, \aleph_0)$-homogeneous and A be a D-set. For each B, if $A \underset{M}{\downarrow} B$, then $M(A) \underset{M}{\downarrow} B$.*

PROOF. By finite character of independence, it is enough to show that if $\bar{a} \underset{M}{\downarrow} \bar{b}$, then $\bar{c} \underset{M}{\downarrow} \bar{b}$, for each finite $\bar{c} \in M(\bar{a})$. Let $\bar{c} \in M(\bar{a})$ be given. Then $\text{tp}(\bar{c}/M\bar{a})$ is $D^s_{\aleph_0}$-isolated. Hence, by assumption and Lemma .2.1, $\text{tp}(\bar{c}/M\bar{a}) \vdash \text{tp}(\bar{c}/M\bar{a}\bar{b})$. Therefore, $\bar{c} \underset{M}{\downarrow} \bar{b}$. □

Recall the definitions of orthogonality.

DEFINITION IV.2.4. Let $p \in S_D(B)$ and $q \in S_D(A)$ be stationary. We say that p is *orthogonal* to q, written $p \perp q$, if for every D-model M containing $A \cup B$ and for every $a \models p_M$ and $b \models q_M$, we have $a \underset{M}{\downarrow} b$;

Then, by Lemma .5.5 of Chapter III, we can immediately simplify the definition: for $p, q \in S_D(M)$, we have $p \perp q$ if and only if $\bar{a} \underset{M}{\downarrow} \bar{b}$ for every $\bar{a} \models p$ and $\bar{b} \models q$.

The following lemma is a particular case of Lemma .5.6 of Chapter III.

LEMMA IV.2.5. *Let M be $(D, \aleph_0)$-homogeneous. If $\bar{a} \underset{M}{\perp} \bar{b}$ and $\mathrm{tp}(\bar{a}/M\bar{b})$ is $D^s_{\aleph_0}$-isolated, then $\bar{a} \in M$.*

LEMMA IV.2.6. *Let $\mathrm{tp}(\bar{a}/M\bar{b})$ be isolated, and $\mathrm{tp}(\bar{b}/M)$ be regular. Suppose that $\bar{a} \underset{M}{\not\perp} \bar{b}$. Then, for any $\bar{c}$ if $\bar{a} \underset{M}{\perp} \bar{c}$, then $\bar{b} \underset{M}{\perp} \bar{c}$.*

PROOF. Suppose that $\bar{b} \underset{M}{\not\perp} \bar{c}$. By symmetry, we have that $\bar{c} \underset{M}{\not\perp} \bar{b}$. Let $q(\bar{x}, \bar{m}, \bar{b}) \subseteq \mathrm{tp}(\bar{c}/M\bar{b})$ be such that

$$R[q(\bar{z}, \bar{m}, \bar{b})] = R[\mathrm{tp}(\bar{c}/M\bar{b})] < R[\mathrm{tp}(\bar{c}/M)].$$

Without loss of generality, since $\bar{a} \underset{M}{\not\perp} \bar{b}$, we can choose $p(\bar{x}, \bar{m}, \bar{b}) \subseteq \mathrm{tp}(\bar{a}/M\bar{b})$ be such that

$$R[p(\bar{y}, \bar{m}, \bar{a})] = R[\mathrm{tp}(\bar{b}/M\bar{a})] < R[\mathrm{tp}(\bar{b}/M)]$$

and also

$$R[p(\bar{b}, \bar{m}, \bar{x})] = R[\mathrm{tp}(\bar{a}/M\bar{b})] < R[\mathrm{tp}(\bar{a}/M)].$$

Choose $\bar{c}' \in M$ such that $tp(\bar{c}/\bar{m}) = \mathrm{tp}(\bar{c}'/\bar{m})$. Since $\bar{a} \underset{M}{\perp} \bar{c}$, we have in particular that $\mathrm{tp}(\bar{a}/M\bar{c})$ does not split over $\bar{m}$ so that $\mathrm{tp}(\bar{c}/\bar{m}\bar{a}) = \mathrm{tp}(\bar{c}'/\bar{m}\bar{a})$. Thus, $\bar{b}$ realizes the following type

(*) $$p(\bar{y}, \bar{m}, \bar{a}) \cup q(\bar{a}, \bar{m}, \bar{y}) \cup \mathrm{tp}(\bar{b}/\bar{m}).$$

Since $\mathrm{tp}(\bar{a}/M\bar{b})$ is isolated, we may assume that $M(\bar{a}) \subseteq M(\bar{b})$. Now choose $\bar{b}' \in M(\bar{a})$ realizing (*). If $\bar{b}' \in M$, then $R[\mathrm{tp}(\bar{a}/M)] \leq R[\mathrm{tp}(\bar{a}/\bar{m}\bar{b}')] = R[p(\bar{b}', \bar{m}, \bar{x})]$, a contradiction. Hence $\bar{b}' \notin M$ and so $\bar{b}' \underset{M}{\not\perp} \bar{b}$, by the previous lemma. Thus $\mathrm{tp}(\bar{b}'/M)$ extends $\mathrm{tp}(\bar{b}/\bar{m})$ and is not orthogonal to it, thus since $\mathrm{tp}(\bar{b}/M)$ is regular based on $\bar{m}$, we must have $\mathrm{tp}(\bar{b}'/M) = \mathrm{tp}(\bar{b}/M)$. This is a contradiction, since then, $\bar{b}'$ realizes $q(\bar{c}', \bar{m}, \bar{y})$. □

The next corollary is Equivalence.

COROLLARY IV.2.7 (Equivalence). *Let $M \in \mathcal{K}$, let $p, q \in S_D(M)$ be regular, and let $\bar{b} \notin M$ realize p. Then q is realized in $M(\bar{b}) \setminus M$ if and only if $p \not\perp q$.*

PROOF. Let $\bar{b} \in M$ realize p. Let $M(\bar{b})$ be $D^s_{\aleph_0}$-primary over $M \cup \bar{b}$.

Let $\bar{a} \in M(\bar{b}) \setminus M$. Then $\mathrm{tp}(\bar{a}/M\bar{b})$ is $D^s_{\aleph_0}$-isolated. If $p \perp q$, then $\bar{b} \underset{M}{\perp} \bar{a}$. Hence, by symmetry $\bar{a} \underset{M}{\perp} \bar{b}$, and so $\bar{a} \in M$, by Lemma .2.5, a contradiction.

For the converse, suppose that $p \not\perp q$. This implies that there is $\bar{a} \models q$ such that

$$\bar{a} \underset{M}{\not\downarrow} \bar{b}.$$

Let $q(\bar{x}, \bar{m}, \bar{b}) \subseteq \mathrm{tp}(\bar{a}/M\bar{b})$ be such that

$$R[q(\bar{x}, \bar{m}, \bar{b})] = R[\mathrm{tp}(\bar{a}/M\bar{b})] < R[q].$$

Since q is regular, we may further assume that q is based on $\bar{m}$. Thus, the element $\bar{a}$ realizes the type

(*) $$q(\bar{x}, \bar{m}, \bar{b}) \cup q \restriction \bar{m}.$$

Since $M(\bar{b})$ is in particular $(D, \aleph_0)$-homogeneous, there is $\bar{a}' \in M(\bar{b})$ realizing the type (*). Since $M(\bar{b})$ is $(D, \aleph_0)$-primary, we must have that $\mathrm{tp}(\bar{a}'/M\bar{b})$ is isolated. Thus, since $\bar{b} \underset{M}{\not\downarrow} \bar{a}$, we must have by the Lemma .2.6 that $\bar{a}' \underset{M}{\not\downarrow} \bar{a}$. This implies that $\mathrm{tp}(\bar{a}'/M)$ is an extension of the regular type $q \restriction \bar{m}$ which is not orthogonal to q. Hence, since q is regular, we must have $q = \mathrm{tp}(\bar{a}'/M)$. This shows that q is realized (by $\bar{a}'$) in $M(\bar{b})$. □

We prove the axiom of Parallelism.

THEOREM IV.2.8 (Parallelism). *Let $p, q \in S(M)$ be regular types. Let N contain M. Then $p \perp q$ if and only if $p_N \perp q_N$.*

PROOF. Certainly, if $p \perp q$, then $p_N \perp q_N$. Now suppose that $p_N \perp q_N$. Let $\bar{b} \models p_N$. Then $\bar{b} \models p$ and by Equivalence, $p \perp r$ if and only if r is realized in $M(\bar{b})$, the prime model over $M \cup \bar{b}$. Suppose that there is $\bar{a} \in M(\bar{b}) \setminus M$ realizing r. Let $N(\bar{b})$ be the prime model over $N \cup \bar{b}$. Then $\bar{a} \in N(\bar{b})$. But, notice that $\mathrm{tp}(\bar{a}/M\bar{b})$ is $D^s_{\aleph_0}$-isolated, and $M\bar{b} \subseteq_{TV} N\bar{b}$. Hence, $tp(\bar{a}/M\bar{b}) \vdash \mathrm{tp}(\bar{a}/N\bar{b})$. This implies $\bar{a} \underset{N}{\downarrow} \bar{b}$ and $\bar{a} \notin N$. But, by stationarity, $\bar{a} \models q_N$. Hence, q_N is realized in $N(\bar{b}) \setminus N$, so $q_N \not\perp p_N$, a contradiction. □

We encountered Morley sequences when we talked about stationary types in the previous chapter. The definition can be made for any type.

DEFINITION IV.2.9. Let $p \in S_D(A)$. We say that $\langle \bar{a}_i \mid i < \omega \rangle$ is a *Morley sequence* for p if

(1) The sequence $\langle \bar{a}_i \mid i < \omega \rangle$ is indiscernible over A;
(2) For every $i < \omega$ we have $\bar{a}_i \underset{A}{\downarrow} A \cup \{\bar{a}_j \mid j < i\}$.

The next fact was established in the previous chapter.

FACT IV.2.10. *If $p \in S_D(A)$ is stationary, then there is a Morley sequence for p.*

The next theorem is Axiom (Perp I).

THEOREM IV.2.11 (Perp I). *Let $p \in S_D(N)$ be regular, $M \subseteq N$. Then $p \perp M$ if and only if $p \perp q$, for every regular $q \in S(M)$.*

PROOF. One direction is obvious. Suppose that $p \not\perp M$. We will find a regular type $q \in S_D(M)$ such that $p \not\perp q$.

Since p is regular, there exists a finite set $\bar{f} \subseteq N$ such that p is regular over $\bar{f}$. Write $p(\bar{x}, \bar{f})$ for the stationary type $p_{\bar{f}}$. Also, there exists a finite set $\bar{e} \subseteq M$ such that $\mathrm{tp}(\bar{f}/M)$ is based on $\bar{e}$. Since $p \not\perp M$, there exists a stationary type $r \in S(M)$ such that $p \not\perp r$. By monotonicity, we can find $\bar{a} \models p$, $\bar{b} \models r_N$ such that $\bar{a} \underset{\bar{e}}{\not\downarrow} \bar{f}\bar{b}$.

Since M is $(D, \aleph_0)$-homogeneous, there exists $\langle \bar{f}_i \mid i < \omega \rangle \subseteq M$, a Morley sequence for $\mathrm{tp}(\bar{f}/\bar{e})$. Let $p_i := p(\bar{x}, \bar{f}_i)_M$. This is well-defined since $p(\bar{x}, \bar{f})$ is stationary and $\mathrm{tp}(\bar{f}/\bar{e}) = \mathrm{tp}(\bar{f}_i/\bar{e})$, so $p(\bar{x}, \bar{f}_i)$ is stationary.

For each $i < \omega$, we can choose $M_i \subseteq N$ such that there is an automorphism g_i with $g_i(\bar{f}) = \bar{f}_i$, $g_i(\bar{c}) = \bar{c}$ and $g_i(M) = M_i$. Since p_{M_i} is regular and $p_i = g^{-1}(p_{M_i})$, then

(*) $$p_i \text{ is regular, for each } i < \omega.$$

A similar reasoning using an automorphisms sending $\bar{f}\bar{f}_0$ to $\bar{f}_i\bar{f}_j$ shows that

(**) $$p \perp p_0 \text{ implies } p_i \perp p_j, \text{ for every } i \neq j < \omega.$$

Finally, using the fact that $p \not\perp r$, we can derive

(***) $$p_i \not\perp r, \text{ for every } i < \omega.$$

If we show that $p \not\perp p_0$, then (*) implies the conclusion of the lemma. Suppose, for a contradiction, that $p \perp p_0$. By (***) we can find $\bar{b}' \models r$ and $\bar{a}_i \models p_i$, such that $\bar{b}' \underset{M}{\not\downarrow} \bar{a}_i$ and $\bar{a}_i \notin M$, for each $i < \omega$. Now (**) implies that $\bar{a}_{j+1} \underset{M}{\downarrow} \{\bar{a}_i \mid i \leq j\}$, for every $j < \omega$. Hence, by (*) and Lemma .2.5, we have $\bar{a}_{i+1} \notin M_i$, where M_i is $D^s_{\aleph_0}$-primary over $M \cup \{\bar{a}_j \mid j < i\}$. Let N be $D^s_{\aleph_0}$-primary over $M \cup \{\bar{a}_j \mid j < \omega\}$. Since $\kappa(D) = \aleph_0$, there exists $n < \omega$ such that $\bar{b}' \underset{M_n}{\downarrow} N$. Hence, by monotonicity, $\bar{b}' \underset{M_n}{\downarrow} \bar{a}_n$. By symmetry over models, $\bar{a}_n \underset{M_n}{\downarrow} \bar{b}'$. But $\bar{a}_n \underset{M}{\downarrow} \{\bar{a}_i \mid i < n\}$, and so $\bar{a}_n \underset{M}{\downarrow} M_n$, by dominance and symmetry. Hence, by transitivity of the independence relation, we have $\bar{a}_n \underset{M}{\downarrow} \bar{b}'$, so $\bar{b}' \underset{M}{\downarrow} \bar{a}_n$, a contradiction. ∎

We now prove two additional lemmas that will be used in the next section.

LEMMA IV.2.12. *If $p \in S(M_1)$ is regular, $p \perp M_0$, and $M_1 \underset{M_0}{\downarrow} M_2$, then $p \perp M_2$.*

PROOF. Suppose that $p \not\perp M_2$. Then, by definition, there exists $q \in S(M_2)$ such that $p \not\perp q$. By definition, there is $N \supseteq M_1 \cup M_2$ such that

(*) $$p_N \not\perp q_N.$$

We are going to find a type $q' \in S(M_0)$ such that $p \not\perp q'$.

Since p and q are stationary, there exist finite sets $\bar{c} \subseteq M_1$, $\bar{d} \subseteq M_2$, and $\bar{e} \subseteq M_0$ such that p is based on $\bar{c}$, q is based on $\bar{d}$, and both $\mathrm{tp}(\bar{c}/M_0)$ and $\mathrm{tp}(\bar{d}/M_0)$ are based on $\bar{e}$.

By (*) and finite character, there exist a set $F \subseteq N$, and $\bar{a}$, $\bar{b}$ such that

(**) $$\bar{a} \models p_{M_1 M_2 F}, \quad \bar{b} \models q_{M_1 M_2 F}, \quad \text{but} \quad \bar{a} \underset{M_1 M_2 F}{\not\downarrow} \bar{b}.$$

By monotonicity, we may assume that $\bar{c}\bar{d}\bar{e} \subseteq F$. Since $\mathrm{tp}(\bar{a}\bar{b}/N)$ is stationary, we may also assume that $\mathrm{tp}(\bar{a}\bar{b}/M_1M_2F)$ is stationary based on F. Finally, we may further assume that $R[\mathrm{tp}(\bar{a}/\bar{c})] < R[\mathrm{tp}(\bar{a}/\bar{c}F)]$.

Since M_0 is $(D, \aleph_0)$-homogeneous, we can choose $\bar{d}' \in M_0$ such that $\mathrm{tp}(\bar{d}'/\bar{e}) = \mathrm{tp}(\bar{d}/\bar{e})$. By stationarity, we have $\mathrm{tp}(\bar{c}\bar{d}\bar{e}/\emptyset) = \mathrm{tp}(\bar{c}\bar{d}\bar{e}/\emptyset)$. Now choose $F' \subseteq M_1$ such that $\mathrm{tp}(\bar{c}\bar{d}\bar{e}F/\emptyset) = \mathrm{tp}(\bar{c}\bar{d}'\bar{e}F/\emptyset)$. Finally, let $\bar{a}'\bar{b}' \in \mathfrak{C}$ such that $\mathrm{tp}(\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}F/\emptyset) = \mathrm{tp}(\bar{a}'\bar{b}'\bar{c}\bar{d}'\bar{e}F'/\emptyset)$.

By invariance under automorphism, we obtain: $R[\mathrm{tp}(\bar{a}'/\bar{c})] = R[\mathrm{tp}(\bar{a}'/F')]$ and $R[\mathrm{tp}(\bar{b}'/\bar{d}')] = R[\mathrm{tp}(\bar{b}'/F')]$, since these statements are true without the $'$.

Now let $q' := \mathrm{tp}(\bar{b}'/\bar{d}')_{M_0} \in S(M_0)$. Such a type exists since $\mathrm{tp}(\bar{b}'/\bar{d}')$ is stationary. We claim that $p \not\perp q'$. Otherwise, by the previous remark, we have $p \perp q'_{M_1}$. Now, let $\bar{a}''\bar{b}'' \models \mathrm{tp}(\bar{a}'\bar{b}'/F')$. We have $\bar{a}'' \models p$, $\bar{b}'' \models q'_{M_1}$ and so $\bar{a}'' \models p_{M_1\bar{b}''}$. But then $R[\mathrm{tp}(\bar{a}''/\bar{c})] = R[\mathrm{tp}(\bar{a}''/\bar{b}''F')]$ This contradicts the fact that $\mathrm{tp}(\bar{a}'\bar{b}'/F') = \mathrm{tp}(\bar{a}''\bar{b}''/F')$. ☐

LEMMA IV.2.13. *Let $p, q \in S_D(M)$ be regular. Let $\bar{a} \notin M$ realize p. If $p \not\perp q$, then there exists $\bar{b} \in M(\bar{a}) \setminus M$ realizing q such that $M(\bar{a}) = M(\bar{b})$.*

PROOF. By equivalence, there exists $\bar{b} \in M(\bar{a}) \setminus M$ realizing q. By definition of prime, it is enough to show that $\mathrm{tp}(\bar{a}/M\bar{b})$ is $D^s_{\aleph_0}$-isolated.

Let $\bar{c} \in M$ be finite such that p is regular over $\bar{c}$, and write $p(\bar{x}, \bar{c}) = p \restriction \bar{c}$. Now, since $\mathrm{tp}(\bar{b}/M\bar{a})$ is $D^s_{\aleph_0}$-isolated, there exists $r_1(\bar{y}, \bar{a})$ over M isolating $\mathrm{tp}(\bar{b}/M\bar{a})$. By a previous lemma, we know that $\bar{a} \underset{M}{\not\downarrow} \bar{b}$, so let $r_2(\bar{x}, \bar{b})$ witness this.

We claim that the following type isolates $\mathrm{tp}(\bar{a}/M\bar{b})$:

(*) $$p(\bar{x},\bar{c}) \cup r_1(\bar{b},\bar{x}) \cup r_2(\bar{x},\bar{b}).$$

Let $\bar{a}' \in M(\bar{a})$ realize (*). Then, $\bar{a}' \notin M$ by choice of r_1. Hence, $\bar{a} \underset{M}{\not\perp} \bar{a}'$ so by choice of $p(\bar{x},\bar{c})$, we have $\mathrm{tp}(\bar{a}'/M) = \mathrm{tp}(\bar{a}/M)$. Thus, $\mathrm{tp}(\bar{a}/M\bar{b}) = \mathrm{tp}(\bar{a}'/M\bar{b})$ using $r_2(\bar{a}',\bar{y})$. □

We can now show using the language of Section 1.

THEOREM IV.2.14. *Let $\mathcal{K}$ be the class of $(D,\aleph_0)$-homogeneous models of a totally transcendental diagram D. Let $N \in \mathcal{K}$ have NDOP. Then N has a complete decomposition.*

PROOF. All the axioms of Section 1 have been checked for $\mathcal{K}$. □

REMARK IV.2.15. Similary to the methods developed in this section for the class of $(D,\aleph_0)$-homogeneous models of a totally transcendental diagram D, we can check all the axioms for the class of (D,μ)-homogeneous models of a totally transcendental diagram D, for any infinite μ. This implies that if $\mathcal{K}$ is the class of (D,μ)-homogeneous models of a totally transcendental diagram D and if $N \in \mathcal{K}$ has NDOP, then N has a complete decomposition (in terms of models of $\mathcal{K}$).

IV.3. DOP in finite diagrams

Let $\mathcal{K}$ be the class of $(D,\aleph_0)$-homogeneous models of a totally transcendental diagram. In the language of the axiomatic framework, we take $N^{**} = \mathfrak{C}$. We say that $\mathcal{K}$ satisfies DOP if there exists $N^* \in \mathcal{K}$ which does not have NDOP. Recall that $\lambda(D) = |D| + |T|$.

CLAIM. Suppose that $\mathcal{K}$ has DOP. Then there exists $M, M_i, M' \in \mathcal{K}$ for $i = 1, 2$ such that

(1) $M_1 \underset{M}{\perp} M_2$;
(2) M' is prime over $M_1 \cup M_2$;
(3) $\|M'\| = \lambda(D)$;
(4) $M_i = M(\bar{a}_i)$, for $i = 1, 2$;
(5) There exists a regular type $p \in S(M')$ such that $p \perp M_i$, for $i = 1, 2$;
(6) The type p is based on $\bar{b}$ and $\mathrm{tp}(\bar{b}/M_1 \cup M_2)$ is isolated over $\bar{a}_1\bar{a}_2$.

PROOF. By assumption, there exists $N^* \in \mathcal{K}$ which fails to have NDOP. Then, there exist $M_i \in \mathcal{K}$ inside N^*, for $i \leq 2$ with $M_1 \underset{M_0}{\perp} M_2$, there exists $M'' \subseteq N^*$ which is $D^s_{\aleph_0}$-primary over $M_1 \cup M_2$ and there exists a regular type $p \in S(M'')$ such that $p \perp M_i$, for $i = 1, 2$.

Let $\bar{b} \in M''$ be a finite set such that p is based on $\bar{b}$. Let $\bar{a}_i \in M_i$, for $i = 1, 2$ be such that $\mathrm{tp}(\bar{b}/M_1 \cup M_2)$ is $D^s_{\aleph_0}$-isolated over $\bar{a}_1\bar{a}_2$. Let $M \in \mathcal{K}$, $M \subseteq M_0$ of cardinality $\lambda(D)$ be such that $\bar{a}_1 \underset{M}{\downarrow} M_0$. Such a model exists using local character and prime models. Let $M(\bar{a}_i)$ be prime over $M \cup \bar{a}_i$, for $i = 1, 2$. Then, by Dominance, Transitivity, and Monotonicity, we have $M(\bar{a}_1) \underset{M}{\downarrow} M(\bar{a}_2)$. By axiom on prime there exists $M' \subseteq M''$ prime over $M(\bar{a}_1) \cup M(\bar{a}_2)$. We may assume that $B \subseteq M'$. Let $p' = p \restriction M'$. Then $p' \in S(M')$ is regular based on $\bar{b}$ and $p'_{M''} = p$. It remains to show that $p' \perp M(\bar{a}_i)$, for $i = 1, 2$. Let $r \in S(M(\bar{a}_i)$ be regular. Then r_{M_i} is regular by our axiom. Furthermore, by definition, $p' \perp r$ if and only if $p' \perp r_{M'}$. By Parallelism, since $M' \subseteq M''$, it is equivalent to show that $p \perp r_{M''}$. But, $r_{M_i} \in S(M_i)$ is regular, $p \perp M_i$, and $r_{M''} = (r_{M_i})_{M''}$. Therefore, by choice of p we have $p \perp r_{M''}$, which finishes the proof. □

Let $\mu > \lambda(D)$ be a cardinal (for the following construction, we may have $\mu \geq \lambda(D)$, but the strict inequality is used in the last claim). Let $\langle M_i \mid i < \mu \rangle$ be independent over a model $M \subseteq M_i$. Suppose that $\|M_i\| = \lambda(D)$. Let $R \subseteq [\mu]^2$ and suppose that $M_s = M(M_i \cup M_j)$, for $s = (i, j)$. Such a model exist for each $s \in [\mu]^2$ by the axioms on prime. Then, by Dominance and the axiom on primes, the following system is independent:

(*) $$\langle M_i \mid i < \mu \rangle \cup \{M\} \cup \langle M_s \mid s \in R \rangle.$$

Hence, there exists a model M_R prime over $\bigcup_{i<\mu} M_i \cup \bigcup_{s \in R} M_s$.

Let $s = (i, j)$ and suppose that there exists a regular type $p_s \in S(M_s)$ such that $p_s \perp M_i$, $p_s \perp M_j$. Let I_s be a Morley sequence for p_s of length μ. (Such a sequence exists since $\mathfrak{C}$ is (D, μ^+)-homogeneous. Then, by Dominance, definition of a Morley sequence, and axiom on prime, there exists $N_s = M_s(I_s)$.

The next claim will allows us to choose prime models over complicated independent systems with some additional properties.

CLAIM. The system $\mathcal{S}_R = \langle M_i \mid i < \mu \rangle \cup \{M\} \cup \langle N_s \mid s \in R \rangle$ is an independent system.

PROOF. By definition, it is enough to show that $N_s \underset{M_i}{\downarrow} D$, when $D = \bigcup_{t \in R, t \neq s} N_t$. By finite character, it is enough to show this for R finite. We prove this by induction on the cardinality of R. When R is empty or has at most one element, there is nothing to do. Suppose that $R = \{s_i \mid i \leq n\} \cup \{s\}$. We show that we can replace M_{s_i} by N_{s_i} and M_s by N_s and still have an independent system. By (*), it is enough to show that if $M_s \underset{M_{s^-}}{\downarrow} D$, then $N_s \underset{M_{s^-}}{\downarrow} D$, for $D = \bigcup_{i \leq n} N_{s_i}$. Using the axioms of the dependence relation, it is enough to show that $N_s \underset{M_s}{\downarrow} D$. By induction hypothesis, we have

$$(**)\qquad N_s \underset{M_s}{\downarrow} \bigcup_{i<n} N_{s_i} \quad \text{and} \quad N_{s_n} \underset{M_{s_n}}{\downarrow} \bigcup_{i<n} N_{s_i}.$$

Now, either $s \cap s_n$ is empty so $M_s \underset{M}{\downarrow} M_{s_n}$ by (*) or they extend j and so $M_s \underset{M_j}{\downarrow} M_{s_n}$, by (*) again. Since $M \subseteq M_j$, in either case, $p_s \perp M_j$, by choice of p_j. Hence $p_s \perp M_{s_n}$ using Lemma .2.12. By induction hypothesis, there exists N' a prime model over $\bigcup_{i<n} N_{s_i}$. Hence, by (**) and Dominance $N_s \underset{M_s}{\downarrow} N'$ and $N_{s_n} \underset{M_{s_n}}{\downarrow} N'$. Hence, using again by Lemma .2.12, we have $p_s \perp N'$. Thus, $I_s \underset{M_{s_n}}{\downarrow} N'$ and $I_s \underset{N'}{\downarrow} N_{s_n}$. Therefore $I_s \underset{M_s}{\downarrow} N' \cup N_{s_n}$. By Dominance $N_s \underset{M_s}{\downarrow} N' \cup N_{s_n}$. We are done by monotonicity. □

We will now use DOP to construct systems as in the claim.

Let the situation be as in the first claim. Write $p(\bar{x}, \bar{b}) = p \restriction \bar{b}$. Let $\langle \bar{a}_1^\alpha \bar{a}_2^\alpha \alpha < \mu \rangle$ be a Morley sequence for $\mathrm{tp}(\bar{a}_1 \bar{a}_2 / M)$. Such a Morley sequence exists by assumption on $\mathfrak{C}$ and stationarity over models. Let M_i^α be prime over $M \cup \bar{a}_i^\alpha$, for $i = 1, 2$. Such a prime model exists by the axioms. Then $M_1^\alpha \underset{M}{\downarrow} M_2^\beta$ for every $\alpha < \beta$, by Dominance. By axiom on prime there exists $M^{\alpha\beta}$ prime over $M_1^\alpha \cup M_2^\beta$. Let $\bar{b}^{\alpha\beta}$ be the image of $\bar{b}$ in $M^{\alpha\beta}$. Let $p^{\alpha\beta} = p(\bar{x}, \bar{b}^{\alpha\beta})_{M^{\alpha\beta}} \in S(M^{\alpha\beta})$, which exists and is regular since p is based on $\bar{b}$. Thus, $p^{\alpha\beta} \perp M_1^\alpha$ and $p^{\alpha\beta} \perp M_2^\beta$. Let $I^{\alpha\beta}$ be a Morley sequence of length μ for $p^{\alpha\beta}$. Let $N^{\alpha\beta}$ be prime over $M^{\alpha\beta} \cup I^{\alpha\beta}$. Then, for the claim, for each $R \subseteq [\mu]^2$, the system

$\mathcal{S}_R = \{M\} \cup \langle M_i^\alpha : \alpha < \mu, i = 1, 2 \rangle \cup \langle N^{\alpha\beta} : \langle \alpha, \beta \rangle \in R \rangle$ is an independent system.

Hence, there exists M_R prime over it.

The final claim explains the name of Dimensional Order Property: It is possible to code the relation R (in particular an order in the following theorem) by looking at dimensions of indiscernibles in a model M_R. Note that the converse holds also, namely that the following property characterizes DOP (we do not prove this fact as it is not necessary to obtain the main gap). Recall $\mu > \lambda(D)$.

CLAIM. The pair $\langle \alpha, \beta \rangle \in R$ if and only if there exists $\bar{c} \in M_R$ with the property that $\mathrm{tp}(\bar{a}_1 \bar{a}_2 \bar{b} / \emptyset) = \mathrm{tp}(\bar{a}_1^\alpha \bar{a}_2^\beta \bar{c} / \emptyset)$ and for every prime $M^* \subseteq M_R$ over $M \cup \bar{a}_1^\alpha \bar{a}_2^\beta$ containing $\bar{c}$ there exists a Morley sequence for $p(\bar{x}, \bar{c})_{M^*}$ of length μ.

PROOF. If the pair $\langle \alpha, \beta \rangle \in R$, then $p^{\alpha\beta}$ is based on $\bar{b}^{\alpha\beta}$. Furthermore, $I^{\alpha\beta}$ is a Morley sequence of length μ for $p^{\alpha\beta}$ in M_R. Let M' be prime over $M \cup \bar{a}_1^\alpha \bar{a}_2^\beta$ containing $\bar{b}^{\alpha\beta}$, then $p(\bar{x}, \bar{b}^{\alpha\beta})_{M'}$ is realized by every element of $I^{\alpha\beta}$ except possibly $\lambda(D)$ many. Hence, there exists a Morley sequence of length μ, since $\mu > \lambda(D)$.

For the converse, let $\alpha < \beta < \mu$ be given such that $(\alpha, \beta) \notin R$. Let $t = (\alpha, \beta)$. Let $\bar{c} \subseteq M_R$ finite as in the claim. By using an automorphism, we have that $\mathrm{tp}(\bar{c}/\bar{a}_1^\alpha \bar{a}_2^\beta)$ isolates $\mathrm{tp}(\bar{c}/M_1^\alpha M_2^\beta)$ and hence there exists $M_t \subseteq M_R$ prime over $M_1^\alpha M_2^\beta$ containing $\bar{c}$. By assumption on $\bar{c}$, there exists $I \subseteq M_R$ a Morley sequence for $p(\bar{x}, \bar{c})_{M_t}$ of length μ. Let N_t be prime over $M_t(I)$, which exists by assumption on prime. By the previous claim, the following system is independent

$$\{M\} \cup \langle M_i^\alpha \mid \alpha < \mu, i = 1, 2\rangle \cup \langle N^{\alpha\beta} \mid \langle \alpha, \beta\rangle \in R\rangle \cup \{N_t\}.$$

Thus, in particular $N_t \underset{M_t}{\downarrow} \bigcup_{i<\mu} M_i \cup \bigcup_{s\in R} N_s$. Hence, $\bar{a} \underset{M_t}{\downarrow} \bigcup_{i<\mu} M_i \cup \bigcup_{s\in R} N_s$, for each $\bar{a} \in I$. By Dominance $\bar{a} \underset{M_t}{\downarrow} M_R$ and so $\bar{a} \in M_t$. This is a contradiction.

□

All the technology is now in place to apply the methods of [**Sh b**] or [**GrHa**] with the previous claim and to derive:

THEOREM IV.3.1. *Suppose that $\mathcal{K}$ has DOP. Then, $\mathcal{K}$ contains 2^λ nonisomorphic models of cardinality λ, for each $\lambda > |D| + |T|$.*

THEOREM IV.3.2. *Suppose that the class of (D, μ)-homogeneous models of a totally transcendental diagram D has DOP. Then, for each $\lambda > |D| + |T| + \mu$ there are 2^λ nonisomorphic (D, μ)-homogeneous models of cardinality λ.*

IV.4. Depth and the main gap

We have now showed that if every model $(D, \aleph_0)$-homogeneous model of a totally transcendental diagram D has NDOP, then every such model admits a decomposition. We will introduce an equivalence between decompositions, as well as the notion of depth, in order to compute the spectrum function for $\mathcal{K}$. Most of the treatment will be done under the assumption that $\mathcal{K}$ has NDOP.

DEFINITION IV.4.1. We say that $\mathcal{K}$ has NDOP if every $N \in \mathcal{K}$ has NDOP.

We introduce the *depth* of a regular type.

DEFINITION IV.4.2. Let $p \in S_D(M)$ be regular. We define the *depth* of p, written $\mathrm{Dep}(p)$. The depth $\mathrm{Dep}(p)$ will be an ordinal, -1, or ∞ and we have the usual ordering $-1 < \alpha < \infty$ for any ordinal α. We define the relation $\mathrm{Dep}(p) \geq \alpha$ by induction on α.

(1) $\mathrm{Dep}(p) \geq 0$ if p is regular;
(2) $\mathrm{Dep}(p) \geq \delta$, when δ is a limit ordinal, if $\mathrm{Dep}(p) > \alpha$ for every $\alpha < \delta$;

(3) $\operatorname{Dep}(p) \geq \alpha + 1$ if there exists $\bar{a}$ realizing p and a regular type $r \in S_D(M(\bar{a}))$ such that $r \perp M$ and $\operatorname{Dep}(r) \geq \alpha$.

We write:

$\operatorname{Dep}(p) = -1$ if p is not regular;
$\operatorname{Dep}(p) = \alpha$ if $\operatorname{Dep}(p) \geq \alpha$ but it is not the case that $\operatorname{Dep}(p) \geq \alpha + 1$;
$\operatorname{Dep}(p) = \infty$ if $\operatorname{Dep}(p) \geq \alpha$ for every ordinal α.

We let $\operatorname{Dep}(\mathcal{K}) = \sup\{\operatorname{Dep}(p)+1 \mid M \in \mathcal{K}, p \in S_D(M)\}$. This is called the *depth* of $\mathcal{K}$.

LEMMA IV.4.3. *Let $p \in S_D(M)$ be regular with $\operatorname{Dep}(p) < \infty$. Let $\bar{a} \models p$ with $r \in S_D(M(\bar{a}))$ regular with $r \perp M$. Then $\operatorname{Dep}(r) < \operatorname{Dep}(p)$.*

PROOF. This is obvious, by definition of depth, if $\operatorname{Dep}(r) = \operatorname{Dep}(p)$ is as above, then $\operatorname{Dep}(p) \geq \operatorname{Dep}(p) + 1$, contradicting $\operatorname{Dep}(p) < \infty$. ☐

LEMMA IV.4.4. *Let $p \in S_D(M)$ be regular. If $\operatorname{Dep}(p) < \infty$ and $\alpha \leq \operatorname{Dep}(p)$, then there exists q regular such that $\operatorname{Dep}(q) = \alpha$.*

PROOF. By induction on $\operatorname{Dep}(p)$. For $\operatorname{Dep}(p) = 0$ it is clear. Assume that $\operatorname{Dep}(p) = \beta + 1$. Let $\bar{a} \models p$ and let $r \in S_D(M(\bar{a}))$ be such that $r \perp M$ and $\operatorname{Dep}(r) \geq \beta$. Then, by the previous lemma, $\operatorname{Dep}(r) = \beta$. Hence, we are done by induction. Assume that $\operatorname{Dep}(p) = \delta$, where δ is a limit ordinal. Let $\alpha < \delta$. Then, $\operatorname{Dep}(p) > \alpha$ by definition, so there exist $\bar{a} \models p$ and $r \in S_D(M(\bar{a}))$ regular such that $r \perp M$ and $\operatorname{Dep}(r) \geq \alpha$. By the previous lemma $\operatorname{Dep}(r) < \operatorname{Dep}(p)$, so we are done by induction. ☐

We first show that the depth respects the equivalence relation $\not\perp$.

LEMMA IV.4.5. *Let $p, q \in S_D(M)$ be regular such that $p \not\perp q$. Then $\operatorname{Dep}(p) = \operatorname{Dep}(q)$.*

PROOF. By symmetry, it is enough to show that $\operatorname{Dep}(p) \leq \operatorname{Dep}(q)$. We show by induction on α that $\operatorname{Dep}(p) \geq \alpha$ implies $\operatorname{Dep}(q) \geq \alpha$. For $\alpha = 0$ or α a limit ordinal, it is obvious. Suppose that $\operatorname{Dep}(p) \geq \alpha + 1$, and let $\bar{a}$ realize p and $r \in S_D(M(\bar{a}))$ be such that $\operatorname{Dep}(r) \geq \alpha$ and $r \perp M$. Since $p \not\perp q$, by Lemma .2.13, there exists $\bar{b}$ realizing q such that $M(\bar{a}) = M(\bar{b})$. This implies that $\operatorname{Dep}(q) \geq \alpha + 1$. ☐

LEMMA IV.4.6. *Suppose $\mathcal{K}$ has NDOP. Let $M \subseteq N$, with $M, N \in \mathcal{K}$. Let $p \in S_D(M)$ be regular. Then $\operatorname{Dep}(p) = \operatorname{Dep}(p_N)$.*

PROOF. We first show that $\operatorname{Dep}(p) \geq \operatorname{Dep}(p_N)$. By induction on α, we show that $\operatorname{Dep}(p) \geq \alpha$ implies $\operatorname{Dep}(p_N) \geq \alpha$.

For $\alpha = 0$ it follows from the fact that p_N is regular. For α a limit ordinal it follows by induction. Suppose $\operatorname{Dep}(p) \geq \alpha+1$. Let $\bar{a}$ realize p and $r \in S_D(M(\bar{a}))$ regular be such that $\operatorname{Dep}(r) \geq \alpha$ and $r \perp M$. Without loss of generality, we may assume that $\bar{a} \underset{M}{\downarrow} N$. Hence, by Dominance $M(\bar{a}) \underset{M}{\downarrow} N$. Since $r \perp M$, then Lemma .2.12 implies that $r \perp N$. By induction hypothesis $\operatorname{Dep}(r_{N(\bar{a})}) \geq \alpha$. Hence $\operatorname{Dep}(p_N) \geq \alpha + 1$.

The converse uses NDOP. We show by induction on α that $\operatorname{Dep}(p_N) \geq \alpha$ implies $\operatorname{Dep}(p) \geq \alpha$. For $\alpha = 0$ or α a limit ordinal, this is clear. Suppose $\operatorname{Dep}(p_N) \geq \alpha + 1$. Let $\bar{a}$ realize p_N. Then $\bar{a} \underset{M}{\downarrow} N$, so by Dominance $M(\bar{a}) \underset{M}{\downarrow} N$. Consider N' $D^s_{\aleph_0}$-primary over $M(\bar{a}) \cup N$. We may assume that $N' = N(\bar{a})$. Hence, there is $r \in S_D(N')$ regular such that $\operatorname{Dep}(r) \geq \alpha$ and $r \perp N$. Hence, by NDOP, we must have $r \not\perp M(\bar{a})$. Therefore, by (Perp I) there exists a regular type $q \in S_D(M(\bar{a})$ such that $r \not\perp q$. But, since $r \perp M$, also $q \perp M$. Moreover, by Parallelism, $r \not\perp q_{N'}$ and since $q_{N'}$ is regular, the previous lemma shows that $\operatorname{Dep}(q_{N'}) = \operatorname{Dep}(r) \geq \alpha$. Hence, by induction hypothesis, $\operatorname{Dep}(q) \geq \alpha$. This implies that $\operatorname{Dep}(p) \geq \alpha + 1$. ☐

Let $\lambda(D) = |D| + |T|$. As we saw in Chapter III, if D is totally transcendental, then D is stable in $\lambda(D)$.

LEMMA IV.4.7. *Let $\mathcal{K}$ have NDOP. If* $\operatorname{Dep}(\mathcal{K}) \geq \lambda(D)^+$ *then* $\operatorname{Dep}(\mathcal{K}) = \infty$.

PROOF. Let p be regular based on B. Let M be $D^s_{\aleph_0}$-primary over the empty set. Then $\|M\| \leq \lambda(D)$. By an automorphism, we may assume that $B \subseteq M$. Then, by Lemma .4.6, we have $\operatorname{Dep}(p) = \operatorname{Dep}(p \upharpoonright M)$. Thus, since $|S_D(M)| \leq \lambda(D)$, there are at most $\lambda(D)$ possible depths. By Lemma .4.4, they form an initial segment of the ordinals. This proves the lemma. ☐

DEFINITION IV.4.8. The class $\mathcal{K}$ is called *deep* if $\operatorname{Dep}(\mathcal{K}) = \infty$.

The next theorem is the main characterization of deep $\mathcal{K}$. A class $\mathcal{K}$ is deep if and only if a natural partial order on $\mathcal{K}$ is not well-founded. This will be used to construct nonisomorphic models in Theorem .4.23.

THEOREM IV.4.9. *$\mathcal{K}$ is deep if and only if there exists a sequence $\langle M_i, \bar{a}_i \mid i < \omega\rangle$ such that*

(1) M_0 *has cardinality* $\lambda(D)$*;*
(2) $\operatorname{tp}(\bar{a}_i/M_i)$ *is regular;*
(3) M_{i+1} *is prime over* $M_i \cup \bar{a}_i$*;*
(4) $M_{i+1}/M_i \perp M_{i-1}$*, if* $i > 0$*.*

PROOF. Suppose that $\mathcal{K}$ is deep. Prove by induction on $i < \omega$ that a sequence satisfying (1)–(4) exists and that in addition

(5) $\mathrm{Dep}(\mathrm{tp}(\bar{a}_i/M_i)) = \infty$.

This is possible. For $i = 0$, let $M \in \mathcal{K}$ and $p \in S_D(M)$ be regular such that $\mathrm{Dep}(p) \geq \lambda(D)^+ + 1$. Such a type exists since $\mathcal{K}$ is deep. Now, let B be finite such that p is regular over B. Let $M_0 \in \mathcal{K}$ contain B be of cardinality $\lambda(D)$. Then, since $p = (p \restriction M_0)_M$, we have $\mathrm{Dep}(p \restriction M_0)$ by Lemma .4.6. Let $\bar{a}_0$ realize $p \restriction M_0$. By the previous fact, $\mathrm{Dep}(\mathrm{tp}(\bar{a}_0/M_0)) = \infty$. Now assume that $\bar{a}_i, M_i$ have been constructed. Let M_{i+1} be prime over $M_i \cup \bar{a}_i$. By (5), we must have $\mathrm{Dep}(\mathrm{tp}(\bar{a}_i/M_i)) \geq \lambda(D)^+ + 1$, so there exists $\bar{a}_{i+1}$ realizing $\mathrm{tp}(\bar{a}_i/M_i)$ and a regular type $p_i \in S_D(M_{i+1})$ such that $\mathrm{Dep}(p_i) \geq \lambda(D)^+$ and $p_i \perp M_i$. Let $\bar{a}_{i+1}$ realize p_i, then (1)–(5) hold.

For the converse, suppose there exists $\langle M_i, \bar{a}_i \mid i < \omega \rangle$ satisfying (1)–(4). We show by induction on α that $\mathrm{Dep}(\mathrm{tp}(\bar{a}_i/M_i)) \geq \alpha$, for each $i < \omega$. This is clearly enough since then $\mathrm{Dep}(\mathrm{tp}(\bar{a}_0/M_0)) = \infty$. For $\alpha = 0$, this is given by (2), and for α a limit ordinal, this is by induction hypothesis. For the successor case, assume that $\mathrm{Dep}(\mathrm{tp}(\bar{a}_i/M_i)) \geq \alpha$, for each $i < \omega$. Fix i. Then by (4) $\mathrm{tp}(\bar{a}_{i+1}/M_{i+1}) \perp M_i$. By (2) $\mathrm{tp}(\bar{a}_{i+1}/M_{i+1})$ is regular and by (3) $M_{i+1} = M_i(\bar{a}_i)$. By induction hypothesis $\mathrm{Dep}(\mathrm{tp}(\bar{a}_{i+1}/M_{i+1})) \geq \alpha$, hence $\mathrm{Dep}(\mathrm{tp}(\bar{a}_i/M_i) \geq \alpha + 1$ by definition of depth. □

We will find it convenient to introduce dominance.

DEFINITION IV.4.10. We say that A *dominates* B *over* M if for every set C, if $A \underset{M}{\perp} C$ then $B \underset{M}{\perp} C$.

We rephrase some of the results we have obtained in the following remark.

REMARK IV.4.11. For any set A, A dominates $M(A)$ over M. Thus, if $M \subseteq N$, and $\bar{a} \in N \setminus M$ there always is a model M' such that $\bar{a} \in M' \subseteq N$ and M' is maximally dominated by $\bar{a}$ over M, i.e. M' is dominated by $\bar{a}$ over M and every model contained in N strictly containing M' is *not* dominated by $\bar{a}$ over M.

We introduce triviality. The name comes from the fact that the pregeometry on the set of realizations of a trivial type is trivial.

DEFINITION IV.4.12. A type $p \in S_D(M)$ is *trivial* if for every $M', N \in \mathcal{K}$ such that $M \subseteq M' \subseteq N$ and for every set $I \subseteq p_{M'}(N)$ of pairwise independent sequences over M', then I is a Morley sequence for $p_{M'}$.

If p is trivial, $\bar{a} \models p$ and $\bar{a}$ dominates by B over M, then we say that B/M is *trivial*.

REMARK IV.4.13. If $\mathrm{tp}(\bar{a}/M)$ is trivial, then $M(\bar{a})/M$ is trivial.

The next lemma says essentially that all the regular types of interest are trivial.

LEMMA IV.4.14. *If $\mathcal{K}$ has NDOP, then if $p \in S_D(M)$ is regular with $\mathrm{Dep}(p) > 0$, then p is trivial.*

PROOF. Suppose $p \in S_D(M)$ is not trivial. Without loss of generality $\bar{a}_i$ for $i \leq 2$ be pairwise independent over M such that $\{\bar{a}_i \mid i \leq 2\}$ is not. Since $\mathrm{Dep}(p) > 0$, by using an automorphism, we can find $r \in S_D(M(\bar{a}_0))$ regular such that $r \perp M$.

Let $N = M(\bar{a}_0, \bar{a}_1, \bar{a}_2)$. Let $M' \subseteq N$ be maximal such that $\bar{a}_1\bar{a}_2 \underset{M}{\downarrow} M'$. Thus, we may assume that $N = M'(\bar{a}_0, \bar{a}_1, \bar{a}_2)$. Since $\bar{a}_0$ realizes p, and $M'/M \perp p$, we have $\bar{a}_0 \underset{M}{\downarrow} M'$. Hence, by Lemma .2.12, we must have $r \perp M'$. By the previous remark, choose $M_i \subseteq N$ maximally dominated by $\bar{a}_i$ over M'. By choice of M_i we have $M_1 \underset{M}{\downarrow} M_2$. Thus, by definition of M_i and NDOP, necessarily N is $D^s_{\aleph_0}$-primary over $M_1 \cup M_2$.

Now, since $M'/M \perp p$, we have $\bar{a}_0\bar{a}_i \underset{M}{\downarrow} M'$. Hence $M(\bar{a}_0) \underset{M'}{\downarrow} M_i$, for $i = 1, 2$. By Lemma .2.12, we have $r_N \perp M_i$ for $i = 1, 2$, contradicting NDOP. □

The next lemmas are used to calculate the spectrum function.

LEMMA IV.4.15. *Assume $\mathcal{K}$ has NDOP. Let $\langle M_\eta \mid \eta \in J\rangle$ be a complete decomposition of N^* over M. Let I be a subtree of J. Then there exists $N_I \subseteq N^*$ and $N_\eta \subseteq N^*$ for each $\eta \in J \setminus I$ such that $\{N_I\} \cup \{N_\eta \mid \eta \in J \setminus I\}$ is a complete decomposition of N^* over N_I.*

PROOF. Define $N_I \subseteq N^*$ and $N_\eta \subseteq N^*$ for $\eta \in J \setminus I$ as follows

(1) N_I is $D^s_{\aleph_0}$-primary over $\bigcup\{M_\eta \mid \eta \in I\}$;
(2) $N_I \underset{M_I}{\downarrow} M_J$;
(3) $N_\eta = N_{\eta^-}(M_\eta)$ for $\eta \in J \setminus I$ and when $\eta^- \in I$ then $N_\eta = N_I(M_\eta)$;
(4) $N_\eta \underset{M_\eta}{\downarrow} \bigcup_{\eta \prec \nu} M_\nu$.

This is easily done and one checks immediately that it satisfies the conclusion of the lemma.

□

We now define an equivalence relation on decompositions.

DEFINITION IV.4.16. Let $\langle M_\eta \mid \eta \in I\rangle$ be a complete decomposition of N^*. Define an equivalence relation $\sim$ on $I \setminus \{\langle\rangle\}$ by

$$\eta \sim \nu \quad \text{if and only if} \quad M_\eta/M_{\eta^-} \not\perp M_\nu/M_{\nu^-}.$$

By Equivalence, this is indeed an equivalence relation. By the following lemma, any two sequences in the same $\sim$-equivalence class have a common predecessor.

LEMMA IV.4.17. *If $\langle M_\eta \mid \eta \in I\rangle$ is a decomposition of N^*, then for $\eta, \nu \in I \setminus \{\langle\rangle\}$ such that $\eta^- \neq \nu^-$ we have $M_\eta/M_{\eta^-} \perp M_\nu/M_{\nu^-}$.*

PROOF. Let $\eta, \nu \in I \setminus \{\langle\rangle\}$ such that $\eta^- \neq \nu^-$. Let u be the largest common sequence of η^- and ν^-. We have $M_{\eta^-} \underset{M_u}{\perp} M_{\nu^-}$, by independence of the decomposition. By definition $M_\eta/M_{\eta^-} \perp M_{\eta^{--}}$. Hence, by Lemma .2.12, we have $M_\eta/M_{\eta^-} \perp M_u$ and also $M_\eta/M_{\eta^-} \perp M_{\nu^-}$. Therefore $M_\eta/M_{\eta^-} \perp M_\nu/M_{\nu^-}$. $\square$

The next lemma will be used inductively.

LEMMA IV.4.18. *Let $\langle M_\eta \mid \eta \in I\rangle$ and $\langle N_\nu \mid \nu \in J\rangle$ be a complete decompositions of N^* over M. Let $I' = \{\eta \in I \mid \eta^- = \langle\rangle\}$ and $J' = \{\nu \in J \mid \nu^- = \langle\rangle\}$. Then there exists a bijection $f : I' \to J'$ such that*

(1) *f preserves $\sim$-classes;*
(2) *If $\eta \in I'$ and M_η/M is trivial then $M_\eta \underset{M}{\not\perp} N_{f(\eta)}$.*

PROOF. Choose a representative for each $\not\perp$-class among the regular types of $S_D(M)$. Build the bijection by pieces. For each regular $p \in S_D(M)$, the cardinalities of $\{\eta \in I \mid M_\eta/M \not\perp p\}$ and $\{\nu \in J \mid N_\nu/N \not\perp p\}$ are equal and both equal to the dimension of $p(N^*)$ by construction. If p is not trivial, then choose any bijection between the two sets. If p is trivial, for each $\eta \in I$ such that $M_\eta/M \not\perp p$ there exists exactly one $\nu \in J'$ such that $M_\eta \underset{M}{\not\perp} N_\nu$. Let f send each such η to their corresponding ν. Since there is no relation between p's belonging to different equivalence classes, this is enough. $\square$

The following quasi-isomorphism will be relevant for the isomorphism type of models.

DEFINITION IV.4.19. Two ω-trees I, J are said to be *quasi-isomorphic*, if there exists a partial function f from I to J such that

(1) f is order-preserving;
(2) For each $\eta \in I$ all but at most $\lambda(D)$ many successors of η are in $\mathrm{dom}(f)$;
(3) For each $\nu \in J$ all but at $\lambda(D)$ many successors of ν are in the $\mathrm{ran}(f)$.

A function f as above is called a *quasi-isomorphism*.

THEOREM IV.4.20. *Let $\langle M_\eta \mid \eta \in I\rangle$ and $\langle M'_\nu \mid \nu \in J\rangle$ be complete decompositions of N^*. Then there exists a $\sim$-class preserving quasi-isomorphism from I to J.*

PROOF. For each $\eta \in I$, let $I^+_\eta = \{\nu \in I \mid \nu^- = \eta\}$. We define a partial class preserving function f_η from I^+_η into J as follows. Then M_η has cardinality $\lambda(D)$, so we can find I_0 and J_0 of cardinality at most $\lambda(D)$ such that there exists $N \subseteq N^*$ containing M_η, such that M is $D^s_{\aleph_0}$-primary over both $\bigcup\{M_\nu \mid \nu \in I_0\}$ and $\bigcup\{M'_\nu \mid \nu \in J_0\}$. By Lemma .4.15 and Lemma .4.18, there exists a partial function f_η from $I^+_\eta \setminus I_0$ into J satisfying conditions (1) and (2) in Lemma .4.18.

Now let $f = \bigcup_{\eta\in I} f_\eta$ (we let $f_{\langle\rangle}$ map $\langle\rangle$ to $\langle\rangle$). Clearly f is well-defined, since the domains of all the f_η's are disjoint. Further, by construction, the condition involving $\lambda(D)$ is satisfied.

It remains to show that f is one-to-one and order preserving. We check order preserving and leave one-to-one to the reader. Let $\eta \prec \nu \in I$ be given. We may assume that $\eta \neq \langle\rangle$. Then, by Lemma .4.14, we have M_η/M_{η^-} is trivial. We are going to compute $f(\eta)$ and $f(\nu)$. Recall that $f(\eta) = f_{\eta^-}(\eta)$. In the notation of Lemma .4.15 and of the first paragraph, we have

$$\langle N_\zeta : \zeta \in I \setminus I_0\rangle \cup \{N\} \quad \text{and} \quad \langle N'_\zeta : \zeta \in J \setminus J_0\rangle \cup \{N\},$$

two complete decompositions of N^* over N. By Lemma .4.18, we have

$$N_{\eta^-} \underset{N}{\not\downarrow} N'_{f(\eta^-)}.$$

Then, necessarily $M_\nu/M_{\nu^-} \not\perp M'_{f(\nu)}/M'_{f(\nu^-)}$ and any sequence $\sim$-related to ν is $\prec$-above η. Consider the following independent tree

$$\langle N'_\zeta : \zeta \in I \setminus I_0, f^-_\eta(\eta) \not\prec \zeta\rangle \cup \langle N_\zeta : \eta \in I, \eta \prec \zeta\rangle \cup \{N\}.$$

By triviality of M_η/M_{η^-}, it is a decomposition of N^* over N. Hence, by Lemma .4.17 we have $M_\nu/M_{\nu^-} \perp N_\zeta/N_{\zeta^-}$, for each $\zeta \in I \setminus I_0, f^-_\eta(\eta) \not\prec \zeta$. This implies that the $\sim$-class of $f_{\eta^-}(\nu)$ is above $f_\nu(\eta^-)$. Thus, f is order preserving. □

In order to construct many nonisomorphic models, we will need a special kind of trees. For an ω-tree I and $\eta \in I$, denote by $I_\eta = \{\nu \in I \mid \eta \prec \nu\}$. We write $I_\eta \cong I_\nu$ if both trees are isomorphic as trees.

DEFINITION IV.4.21. An ω-tree I is called *ample* if for every $\eta \in I$, with $\eta^- \in I$, we have

$$|\{\nu \in I : \nu^- = \eta^- \text{ and } I_\nu \cong I_\eta\}| > \lambda(D).$$

We now state a fact about ample ω-trees. If I is a tree, by definition every $\eta \in I$ is well-founded in the order of I. The *rank* of η in I will be the natural rank associated with the well-foundedness relation on η in I.

FACT IV.4.22. *Let I, J be ample trees. Let f be a quasi-isomorphism from I to J. Then for each $\eta \in \mathrm{dom}(f)$, the rank of η in I is equal to the rank of $f(\eta)$ in J.*

In the next proof, write $\ell(\eta)$ for the level of η.

THEOREM IV.4.23. *If $\mathcal{K}$ is deep, for each $\mu > \lambda(D)$, there are 2^μ nonisomorphic models of cardinality μ.*

PROOF. Let $\mu > \lambda(D)$. Since $\mathcal{K}$ is deep by Theorem .4.9, there exists $\langle M_i, \bar{a}_i \mid i < \omega\rangle$ such that

(1) M_0 has cardinality $\lambda(D)$;
(2) $\mathrm{tp}(\bar{a}_i/M_i)$ is regular;
(3) M_{i+1} is prime over $M_i \cup \bar{a}_i$;
(4) $M_{i+1}/M_i \perp M_{i-1}$, if $i > 0$.

Let $p = \mathrm{tp}(\bar{a}_0/M_0)$. Then p is regular based on a finite set B. We will find 2^μ non-isomorphic models of size μ with B fixed. This implies the conclusion of the theorem since $\mu^{<\aleph_0} = \mu$.

For each $X \subseteq \mu$ of size μ, let I_X be an ample ω-tree with the property that the set of ranks of elements of the first level of I_X is exactly X. Such a tree clearly exists ($\mu > \lambda(D)$). Define the following system $\langle M^X_\eta \mid \eta \in I_X\rangle$:

(1) $M^X_{\langle\rangle} = M_0$;
(2) If $\eta_0 \prec \cdots \prec \eta_n \in I_X$, we have $\mathrm{tp}(M^X_{\eta_0} \dots M^X_{\eta_n}/\emptyset) = \mathrm{tp}(M_{\ell(\eta_0)} \dots M_{\ell(\eta_n)}/\emptyset)$.

This is easy to do and by choice of $\langle M_i, \bar{a}_i \mid i < \omega\rangle$ this is a decomposition. Let M_X be a $D^s_{\aleph_0}$-primary model over $\bigcup\{M^X_\eta \mid \eta \in I_X\}$. Then $M_X \in \mathcal{K}$ has cardinality μ. By NDOP, $\langle M_i, \bar{a}_i \mid i < \omega\rangle$ is a complete decomposition of M_X over M_0.

We claim that for $X \neq Y$ as above, $M_X \not\cong_B M_Y$. Let $X, Y \subseteq \mu$ of cardinality μ be such that $X \neq Y$. Suppose $M_X \cong_B M_Y$. Then, by Theorem .4.20, there exists a class-preserving quasi-isomorphism between I_X and I_Y. Since B is fixed, the first level of I_X is mapped to the first level of I_Y. By the previous fact, we conclude that $X = Y$, a contradiction. □

We have shown that deep diagrams have many models. The usual methods (see [**Sh b**] for example) can be used to compute the spectrum of $\mathcal{K}$ when $\mathcal{K}$ is *not* deep. Recall that when $\mathcal{K}$ has NDOP but is not deep then $\mathrm{Dep}(\mathcal{K}) < \lambda(D)^+$, by Lemma .4.7.

THEOREM IV.4.24. *If $\mathcal{K}$ has NDOP but is not deep, then for each ordinal α with $\aleph_\alpha \geq \lambda(D)$, we have $I(\aleph_\alpha, \mathcal{K}) \leq \beth_{\mathrm{Dep}(\mathcal{K})}(|\alpha|^{2^{|T|}}) < \beth_{\lambda(D)^+}(|\alpha|)$.*

This proves the *main gap* for the class $\mathcal{K}$ of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram D.

THEOREM IV.4.25 (Main Gap). *Let $\mathcal{K}$ be the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram D. Then, either $I(\aleph_\alpha, \mathcal{K}) = 2^{\aleph_\alpha}$, for each ordinal α such that $\aleph_\alpha > |T| + |D|$, or $I(\aleph_\alpha, \mathcal{K}) < \beth_{(|T|+|D|)^+}(|\alpha|)$, for each α such that $\aleph_\alpha > |T| + |D|$.*

PROOF. If $\mathcal{K}$ has DOP (Theorem .3.1) or has NDOP but is deep (Theorem .4.23), then $\mathcal{K}$ has the maximum number of models. Otherwise, $\mathcal{K}$ has NDOP and is not deep and the bound follows from Theorem .4.24. $\square$

Similar methods using the existence of D^s_μ-prime models for totally transcendental diagrams allow us to prove the main gap for (D, μ)-homogeneous models of a totally transcendental diagram D.

THEOREM IV.4.26. *Let $\mathcal{K}$ be the class of (D, μ)-homogeneous models of a totally transcendental diagram D. Then, either $I(\aleph_\alpha, \mathcal{K}) = 2^{\aleph_\alpha}$, for each ordinal α such that $\aleph_\alpha > |T| + |D| + \mu$, or $I(\aleph_\alpha, \mathcal{K}) < \beth_{(|T|+|D|)^+}(|\alpha|)$, for each α such that $\aleph_\alpha > |T| + |D| + \mu$.*

Finally, similarly to [**GrHa**] or [**Ha**], it is possible to show that for α large enough, the function $\alpha \mapsto I(\aleph_\alpha, \mathcal{K})$ is non-decreasing, for the class $\mathcal{K}$ of (D, μ)-homogeneous models of a totally transcendental diagram D.

CHAPTER V

Forking in pregeometries

At the center of classification theory for the first order case is the notion of forking. Forking is a dependence relation discovered by S. Shelah. It satisfies the following properties in the first order stable case, see [**Sh b**].

(1) (Finite character) The type p does not fork over B if and only if every finite subtype $q \subseteq p$ does not fork over B.
(2) (Extension) Let p be a type which does not fork over B. Let C be given containing the domain of p. Then there exists $q \in S(C)$ extending p such that q does not fork over B;
(3) (Invariance) Let $f \in \mathrm{Aut}(\mathfrak{C})$ and p be a type which does not fork over B. Then $f(p)$ does not fork over $f(B)$.
(4) (Existence) The type p does not fork over its domain;
(5) (Existence of $\kappa(T)$) For every type p, there exists a set $B \subseteq \mathrm{dom}(p)$ such that p does not fork B;
(6) (Symmetry) Let $p = \mathrm{tp}(\bar{a}/B\bar{c})$. Suppose that p does not fork over B. Then $\mathrm{tp}(\bar{c}/B\bar{a})$ does not fork over B;
(7) (Transitivity) Let $B \subseteq C \subseteq A$. Let $p \in S(A)$. Then p does not fork over B if and only if p does not fork over C and $p \restriction C$ does not fork over B.

Already in the introduction of Chapter III of [**Sh b**], S. Shelah states what is important about the forking relation is that it satisfies properties (1)–(7). S. Shelah stated another property named by S. Buechler [**Bu1**] the Pairs Lemma (see Proposition I.1.16 for the statement) as one of the basic properties of forking, which he proved in [**Sh b**] using the Finite Equivalence Relation Theorem. Later Baldwin in his book [**Ba a**] presented an axiomatic treatment of forking in stable theories. This allowed Baldwin to derive abstractly Shelah's Pairs Lemma from the other properties of forking. Following these ideas, it has now become common to characterize various stability conditions in terms of the axiomatic properties that forking satisfies.

A major problem in the classification theory for nonelementary classes is to find a dependence relation which is as well-behaved as forking for first order theories. See for example [**Sh48**], [**Sh87a**], [**Sh87b**], [**GrHa**], [**Ki**], or [**HySh1**]. See also Chapter III. The situation in nonelementary classes is very different from the first order case. In the first order case, the Extension property for forking comes for free; it holds for any theory and is a consequence of the compactness theorem.

This is in striking contrast with the nonelementary cases; the Extension property is usually among the most problematic and does not hold over sets in general for any of the dependence relations introduced thus far.

A general dependence relation satisfying all the formal properties of forking has thus not been found yet for nonelementary classes. There are, however, several cases where pregeometries appear; that is sets with a closure operation satisfying the properties of linear dependence in a vector space. In the first order case, the pregeometries are the sets of realizations of a *regular* type, and the dependence is the one induced by forking and thus satisfies automatically many additional properties. In nonelementary classes the situation is different.

Let us describe several nonelementary examples. The first three examples have in common that there exists a rank, giving rise to a reasonable dependence relation. However the *Extension* property and the *Symmetry* property fail in general (they hold over sufficiently "rich" sets). The rank introduced for these classes are generalizations of what S. Shelah calls $R[\cdot, L, 2]$. Intuitively, a formula has rank $\alpha+1$ if it can be partitioned in *two* pieces of rank α with some additional properties that are tailored to each context (see Chapter III, for example). It is noteworthy that extensions of Morley rank are inadequate, as partitioning a formula in countably many pieces makes sense only when the compactness theorem holds. In the last example, no rank is known, but pregeometries exist.

Categorical sentences in $L_{\omega_1\omega}(Q)$: Shelah started working on this context [**Sh48**] to answer a question of J.T.Baldwin: Can a sentence in $L(Q)$ have exactly one uncountable model? Shelah answers this question negatively using **V=L** (and later using different methods within ZFC) while developing very powerful concepts. One of the main tools is the introduction of a rank. This rank is bounded under the parallel to $\aleph_0$-stability. It gives rise to a dependence relation and pregeometries. Later, H. Kierstead [**Ki**] uses these pregeometries to obtain some results on the countable models of these sentences.

Excellent Scott sentences: In [**Sh87a**] and [**Sh87b**] S.Shelah introduces a simplification of the rank of [**Sh48**]. S. Shelah identifies the concept of *excellent Scott sentences* and proves (among many other things) the parallel to Morley's Theorem for them. Again, this rank induces a dependence relation on the subsets of the models. Later, R. Grossberg and B. Hart [**GrHa**] proved the existence of pregeometries (regular types) for this dependence relation and used it to prove the Main Gap for excellent Scott sentences.

Totally transcendental diagrams: In Chapter III, we introduced a rank for $\aleph_0$-stable diagrams. Diagrams for which the rank is bounded are called *totally transcendental*. Recall that the rank gives rise to a dependence relation on the subsets of the models and pregeometries exist often. This is used to give a proof of categoricity generalizing the Baldwin-Lachlan Theorem.

Superstable diagrams: In [**HySh1**], Hyttinen and Shelah study stable finite diagrams under the additional assumption that $\kappa(D) = \aleph_0$. Such diagrams are called *superstable*. They introduce a relation between sets A, B and an element a, written $a \downarrow_B A$. The main result is that the parallel of regular types exist. More precisely, for every pair of "sufficiently saturated" models $M \subseteq N$, $M \neq N$, there exists a type p realized in $N \setminus M$ such that the relation $a \downarrow_M C$ (standing for $a \notin \mathrm{cl}(C)$) induces a pregeometry among the realizations of p in N.

Thus, pregeometries seem to appear naturally in nonelementary classes, while general well-behaved dependence relations are hard to find. The goal of the first section of this chapter is to recover from *any* pregeometry a dependence relation over the subsets of the pregeometry that satisfies all the formal properties of forking. This is, of course, particularly useful when the pregeometry itself was *not* induced by forking.

A similar endeavor was attempted by John Baldwin in the early eighties. In [**Ba**], J.Baldwin examined some pregeometries and several dependence relations in the first order case. From a pregeometry, he defines the relation $a \underset{B}{\downarrow} C$, by $a \in \mathrm{cl}(B \cup C) - \mathrm{cl}(B)$. He did not however introduce $A \underset{B}{\downarrow} C$, where A is a *tuple* or a *set* as opposed to an element, which we do (see Definition I.1.7). This is a crucial step; it is built-in in the model theory of first order, since forking is naturally defined for types of any arity. To make this more precise, fix T a first order stable theory. Let us write

$$\bar{a} \overset{*}{\underset{B}{\downarrow}} C \quad \text{for} \quad \mathrm{tp}(\bar{a}/B \cup C) \text{ does not fork over } B.$$

Inside a regular type $p(x) \in S(B)$, the relation $a \in \mathrm{cl}(C)$ given by $a \overset{*}{\underset{B}{\not\downarrow}} C$ gives rise to a pregeometry. But, the relation $\bar{a} \overset{*}{\underset{B}{\downarrow}} C$ is defined in general whether or not $\bar{a}$ and C consist of elements realizing p. Inside the pregeometry, the relation $\bar{a} \overset{*}{\underset{B}{\downarrow}} C$ holds (defined with forking) if and only if the relation $\bar{a} \underset{B}{\downarrow} C$ holds (defined formally from our definition using the closure operator of the pregeometry). This is a consequence of the Pairs Lemma, which holds for first order simple theories. When we start from an abstract pregeometry (or an abstract dependence relation), we do *not* have the formalism of types or the Pairs Lemma. Therefore the relation $\bar{a} \underset{B}{\downarrow} C$ has to be introduced for tuples, using the relation $a \underset{B}{\downarrow} C$ for elements. As a consequence, suppose we are given the corresponding notion of a regular type $p \in S(B)$ in a nonelementary context. Suppose there is some ambient dependence relation, written $A \overset{*}{\underset{B}{\perp}} C$ such that over realizations of p the relation $a \in \mathrm{cl}(C)$, given by $a \overset{*}{\underset{B}{\not\perp}} C$, induces a pregeometry. Then, the truth value of the relation $\bar{a} \overset{*}{\underset{B}{\perp}} C$ (given from the ambient dependence relation) and $\bar{a} \underset{B}{\perp} C$ (defined from the closure operation in the pregeometry) may not coincide. They will coincide only if the Pairs Lemma holds for the dependence relation (and this fact is not known in general for nonelementary cases). Therefore, this abstract formalism allows us to introduce for nonelementary classes a (possibly) *better* dependence relation, inside the pregeometry.

In Section 2, we present S. Buechler's characterization of local modularity with parallel lines [**Bu1**] in this general context. This also has esthetic value as it allows one carry out this work in the general context of combinatorial geometry, without logic.

In Section 3, we supplement this work with some observations of a set-theoretic nature, as well as a discussion of stable systems.

In Section 4, an abstract framework is presented where, using the dependence relation defined is this chapter, a generalization of Zilber-Hrushovski group configuration theorem can be derived. A rather lengthy introduction was added.

V.1. Forking in pregeometries

Recall a few well-known facts about pregeometries.

DEFINITION V.1.1. A *pregeometry* is a pair (W, cl), where W is a set and cl is a function $\mathrm{cl}\colon \mathcal{P}(W) \to \mathcal{P}(W)$ satisfying the following four properties

(1) (Monotonicity) For every set $X \in \mathcal{P}(W)$ we have $X \subseteq \mathrm{cl}(X)$;
(2) (Finite Character) If $a \in \mathrm{cl}(X)$ then there is a finite set $Y \subseteq X$, such that $a \in \mathrm{cl}(Y)$;
(3) (Transitivity) Let $X, Y \in \mathcal{P}(W)$. If $a \in \mathrm{cl}(X)$ and $X \subseteq \mathrm{cl}(Y)$ then $a \in \mathrm{cl}(Y)$;
(4) (Exchange Property) For $X \in \mathcal{P}(W)$ and $a, b \in W$, if $a \in \mathrm{cl}(Xb)$ but $a \notin \mathrm{cl}(X)$, then $b \in \mathrm{cl}(Xa)$.

We always assume $\mathrm{cl}(\emptyset) \neq W$.

The next two basic properties are standard and easy.

FACT V.1.2. *If* (W, cl) *is a pregeometry and* $B \subseteq C \subseteq W$*, then* $\mathrm{cl}(B) \subseteq \mathrm{cl}(C)$.

FACT V.1.3. *If* (W, cl) *is a pregeometry and* $B \subseteq W$*, then* $\mathrm{cl}(\mathrm{cl}(B)) = \mathrm{cl}(B)$.

DEFINITION V.1.4. Let (W, cl) be a pregeometry.

(1) For $X \subseteq W$, we say that X is *closed* if $X = \mathrm{cl}(X)$;
(2) $I \subseteq W$ is *independent* if for every $a \in I$, we have $a \notin \mathrm{cl}(I \setminus \{a\})$;
(3) We say that $I \subseteq A$ *generates* A, if $\mathrm{cl}(I) = \mathrm{cl}(A)$;
(4) A *basis* for a set $A \subset W$ is an independent set I generating $\mathrm{cl}(A)$;
(5) For $X \subseteq W$, the *dimension of X*, written $\dim(X)$, is the cardinality of a basis for $\mathrm{cl}(X)$.

FACT V.1.5. *Using the axioms of pregeometry, one can show that for every set, bases exist and that the dimension is well-defined see for example Appendix in* [**Gr a**]

DEFINITION V.1.6. Let $G = (W, \mathrm{cl})$ be a pregeometry.

(1) A bijection $f\colon W \to W$ is an *automorphism of* G if for every $a \in W$ and $A \subseteq W$ we have
$$a \in \mathrm{cl}(A) \quad \text{if and only if} \quad f(a) \in \mathrm{cl}(f(A)).$$
We denote $Aut_A(G)$ the set of automorphisms of G fixing A pointwise.
(2) We say that G is *homogeneous* if for every $a, b \in W$ and $A \subseteq W$, such that $a \notin \mathrm{cl}(A)$ and $b \notin \mathrm{cl}(A)$ there is an automorphism of G, fixing A pointwise and taking a to b.

The next definition is the main concept of this chapter.

DEFINITION V.1.7. Let (W, cl) be a pregeometry. Let A, B and C be subsets of W. We say that A *depends on* C *over* B, if there exist $a \in A$ and a finite $A' \subseteq A$ (possibly empty) such that
$$a \in \mathrm{cl}(B \cup C \cup A') \setminus \mathrm{cl}(B \cup A').$$

If A depends on C over B, we write $A \underset{B}{\not\perp} C$;

If A does not depend on C over B, we write $A \underset{B}{\perp} C$.

REMARK V.1.8. An alternative definition with $A' = \emptyset$ does not permit a smooth extension to sets $A \underset{B}{\perp} C$ when A is not a singleton.

REMARK V.1.9. $A \underset{B}{\perp} C$ if and only if $A \cup B \underset{B}{\perp} C \cup B$. Hence, we will often assume that $B \subseteq A \cap C$.

We now prove that the properties of forking in simple theories hold with this formalism, directly from the axioms of a pregeometry.

PROPOSITION V.1.10 (Finite Character). *Let* (W, cl) *be a pregeometry. Let* A, B *and* C *be subsets of* W. *Then*
$$A \underset{B}{\perp} C \quad \textit{if and only if} \quad A' \underset{B}{\perp} C',$$

for every finite $A' \subseteq A$ *and finite* $C' \subseteq C$.

PROOF. If $A \not\downarrow_B C$, then there exist $a \in A$, and a finite $A' \subseteq A$ such that

$$a \in \mathrm{cl}(B \cup C \cup A') \setminus \mathrm{cl}(B \cup A').$$

By Finite Character, there exist a finite $C' \subseteq C$ such that $a \in \mathrm{cl}(B \cup C' \cup A')$. Hence $A' \not\downarrow_B C'$, by definition.

For the converse, if there exist a finite $A' \subseteq A$ and a finite $C' \subseteq C$ such that $A' \not\downarrow_B C'$, then we can find $a \in A'$ and $A'' \subseteq A'$ such that

$$a \in \mathrm{cl}(B \cup C' \cup A'') \setminus \mathrm{cl}(B \cup A'').$$

Since $C' \subseteq C$, we have $a \in \mathrm{cl}(B \cup C \cup A'')$, by Fact I.1.2. Hence, $A \not\downarrow_B C$, by definition. $\square$

PROPOSITION V.1.11 (Continuity). *Let* (W, cl) *be a pregeometry. Let* $\langle C_i \mid i < \alpha\rangle$ *be a continuous increasing sequence of sets in* W, *and* $A, B \subseteq W$.

(1) *If* $A \downarrow_B C_i$ *for every* $i < \alpha$, *then* $A \downarrow_B \bigcup_{i<\alpha} C_i$.
(2) *If* $C_i \downarrow_B A$ *for every* $i < \alpha$, *then* $\bigcup_{i<\alpha} C_i \downarrow_B A$.

PROOF. By Finite Character. $\square$

PROPOSITION V.1.12 (Invariance). *Let* $G = (W, \mathrm{cl})$ *be a pregeometry. Let* A, B *and* C *be subsets of* W *and let* $f \in Aut(G)$. *Then*

$$A \downarrow_B C \quad \textit{if and only if} \quad f(A) \downarrow_{f(B)} f(C).$$

PROOF. Note that since the inverse of an automorphism is an automorphism, it is enough to show one direction. Assume that $A \not\downarrow_B C$ and let $a \in A$ and $A' \subseteq A$ finite be such that

$$a \in \mathrm{cl}(B \cup C \cup A') \setminus \mathrm{cl}(B \cup A').$$

Then $f(a) \in \mathrm{cl}(f(B \cup C \cup A')) \setminus \mathrm{cl}(f(B \cup A'))$, by definition of automorphism. But since f is a bijection

$$f(a) \in \mathrm{cl}(f(B) \cup f(C) \cup f(A')) \setminus \mathrm{cl}(f(B) \cup f(A')).$$

Therefore, $f(A) \not\downarrow_{f(B)} f(C)$ by definition. $\square$

PROPOSITION V.1.13 (Monotonicity). *Let* (W, cl) *be a pregeometry. Let* A, B *and* C *be subsets of* W. *Suppose* $A \downarrow_B C$.

(1) *If* $A' \subseteq A$ *and* $C' \subseteq C$*, then* $A' \underset{B}{\perp} C'$*;*

(2) *If* $B' \subseteq C$*, then* $A \underset{B \cup B'}{\perp} C$*.*

PROOF. (1) Suppose that $A' \underset{B}{\not\perp} C'$. Let $a \in A'$ and $A^* \subseteq A'$ finite such that

$$a \in \mathrm{cl}(B \cup C' \cup A^*) \setminus \mathrm{cl}(B \cup A^*).$$

Then, by Fact I.1.2, we have $a \in \mathrm{cl}(B \cup C \cup A^*) \setminus \mathrm{cl}(B \cup A^*)$. But $a \in A$ and $A^* \subseteq A$, so $A \underset{B}{\not\perp} C$.

(2) Suppose $A \underset{B \cup B'}{\not\perp} C$. Let $a \in A$ and $A' \subseteq A$ finite such that

$$a \in \mathrm{cl}(B \cup B' \cup C \cup A') \setminus \mathrm{cl}(B \cup B' \cup A').$$

Since $B' \subseteq C$, we have $\mathrm{cl}(B \cup B' \cup C \cup A') = \mathrm{cl}(B \cup C \cup A')$. Also, $\mathrm{cl}(B \cup A') \subseteq \mathrm{cl}(B \cup B' \cup A')$. Hence $a \in \mathrm{cl}(B \cup C \cup A') \setminus \mathrm{cl}(B \cup A')$. Therefore $A \underset{B}{\not\perp} C$. □

PROPOSITION V.1.14 (Symmetry). *Let* (W, cl) *be a pregeometry. Let* A, B *and* C *be subsets of* W*. Then*

$$A \underset{B}{\perp} C \quad \text{if and only if} \quad C \underset{B}{\perp} A.$$

PROOF. Suppose that $A \underset{B}{\not\perp} C$. Choose $a \in A$ and a finite $A' \subseteq A$ such that

(*) $$a \in \mathrm{cl}(B \cup C \cup A') \setminus \mathrm{cl}(B \cup A').$$

By Finite Character and (*), there exist $c \in C$ and a finite (and possibly empty) $C' \subseteq C$ such that

(**) $$a \in \mathrm{cl}(B \cup C' \cup c \cup A') \quad \text{and} \quad a \notin \mathrm{cl}(B \cup C' \cup A').$$

Therefore, by the Exchange Property, we have

$$c \in \mathrm{cl}(B \cup C' \cup A' \cup a).$$

But $c \notin \mathrm{cl}(B \cup C' \cup A')$, (**). Hence,

$$c \in \mathrm{cl}(B \cup C' \cup A' \cup a) \setminus \mathrm{cl}(B \cup C' \cup A').$$

Therefore, $C \underset{B}{\not\perp} A'$, for some finite subset A' of A. Hence, $C \underset{B}{\not\perp} A$, by Finite Character. □

PROPOSITION V.1.15 (Transitivity). *Let* (W, cl) *be a pregeometry. Let* A, B, C *and* D *be subsets of* W *such that* $B \subseteq C \subseteq D$*. Then,*

$$A \underset{C}{\perp} D \quad \text{and} \quad A \underset{B}{\perp} C \quad \text{if and only if} \quad A \underset{B}{\perp} D.$$

PROOF. Suppose first that $A \underset{B}{\not\downarrow} D$. Choose $a \in A$ and a finite $A' \subseteq A$ such that

$$a \in \mathrm{cl}(D \cup A') \setminus \mathrm{cl}(B \cup A').$$

Either $a \in \mathrm{cl}(C \cup A')$, and so

$$a \in \mathrm{cl}(C \cup A') \setminus \mathrm{cl}(B \cup A'),$$

which implies that $A \underset{B}{\not\downarrow} C$. Or $a \notin \mathrm{cl}(C \cup A')$, and therefore

$$a \in \mathrm{cl}(D \cup A') \setminus \mathrm{cl}(C \cup A'),$$

which implies that $A \underset{C}{\not\downarrow} D$.

The converse follows by Monotonicity since $B \subseteq C \subseteq D$. $\square$

The following is proved in [**Sh b**] directly using the finite equivalence relation theorem. The proof that it follows from the other axioms of forking is due to J. Baldwin. We present it here for completeness.

PROPOSITION V.1.16 (Pairs Lemma). *Let $G = (W, \mathrm{cl})$ be a pregeometry. Let A, B, C and D be subsets of W such that $C \subseteq B \cap D$. Then*

$$A \cup B \underset{C}{\downarrow} D \quad \text{if and only if} \quad A \underset{C \cup B}{\downarrow} D \cup B \quad \text{and} \quad B \underset{C}{\downarrow} D.$$

PROOF. Notice first, that by definition

(*) $$A \underset{C \cup B}{\downarrow} D \cup B \quad \text{if and only if} \quad A \underset{C \cup B}{\downarrow} D.$$

Therefore, by Symmetry and (*), it is equivalent to show that

$$D \underset{C}{\downarrow} A \cup B \quad \text{if and only if} \quad D \underset{C \cup B}{\downarrow} A \quad \text{and} \quad D \underset{C}{\downarrow} B,$$

which is true by Transitivity. $\square$

REMARK V.1.17. Let (W, cl) is a pregeometry. Let A, B, C and D be subsets of W. Then

$$AD \underset{B}{\downarrow} C \quad \text{if and only if} \quad A \underset{B}{\downarrow} CD.$$

PROOF. Suppose $A \underset{B}{\downarrow} CD$. Then, by Monotonicity we have $A \underset{B}{\downarrow} D$. Therefore, by Symmetry, we have $D \underset{B}{\downarrow} D$. By Transitivity, we have $A \underset{BD}{\downarrow} CD$. Hence, $AD \underset{B}{\downarrow} C$ by Concatenation.

For the converse, suppose that $A \underset{B}{\perp} CD$. Then by Symmetry we must have $CD \underset{B}{\perp} A$. Hence, by the first paragraph, we know that $C \underset{B}{\perp} AD$, so by Symmetry, also $AD \underset{B}{\perp} C$. $\square$

This finishes the list of usual properties of forking. We now prove a few propositions relating closure and $\perp$.

PROPOSITION V.1.18 (Closed Set Theorem). *Let (W, cl) be a pregeometry. Let A, B and C be subsets of W. Then*

$$A \underset{B}{\perp} C \quad \textit{if and only if} \quad A' \underset{B'}{\perp} C',$$

provided that $\mathrm{cl}(A \cup B) = \mathrm{cl}(A' \cup B')$, $\mathrm{cl}(B) = \mathrm{cl}(B')$ *and* $\mathrm{cl}(C \cup B) = \mathrm{cl}(C' \cup B')$.

PROOF. It is clearly enough to prove one direction. Furthermore, by Symmetry, it is enough to show that $A \underset{B}{\perp} C$ implies $A \underset{B'}{\perp} C'$. Suppose that $A \underset{B'}{\not\perp} C'$. Let $a \in A$ and $A^* \subseteq A$ be such that

$$a \in \mathrm{cl}(B' \cup C' \cup A^*) \setminus \mathrm{cl}(B' \cup A^*).$$

But, it follows from the assumption that $\mathrm{cl}(B' \cup C' \cup A^*) = \mathrm{cl}(B \cup C \cup A^*)$ and $\mathrm{cl}(B' \cup A^*) = \mathrm{cl}(B \cup A^*)$. Therefore

$$a \in \mathrm{cl}(B \cup C \cup A^*) \setminus \mathrm{cl}(B \cup A^*),$$

which implies that $A \underset{B}{\not\perp} C$. $\square$

REMARK V.1.19. In view of the previous result, when $A \underset{B}{\perp} C$, we can first choose a basis B' of B, and choose $A' \subseteq A$ and $C' \subseteq C$, independent over B (or equivalently B'), such that $\mathrm{cl}(A \cup B) = \mathrm{cl}(A' \cup B)$ and $\mathrm{cl}(C \cup B) = \mathrm{cl}(C' \cup B)$, and thus $A' \underset{B}{\perp} C'$ and also $A' \underset{B'}{\perp} C'$.

PROPOSITION V.1.20. *Let (W, cl) be a pregeometry. Let A, B and C be subsets of W.*

$$A \underset{B}{\perp} C \quad \textit{implies} \quad \mathrm{cl}(A \cup B) \cap \mathrm{cl}(C \cup B) = \mathrm{cl}(B).$$

PROOF. Certainly $\mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B) \cap \mathrm{cl}(C \cup B)$. Suppose that the reverse inclusion does not hold, and let $a \in \mathrm{cl}(A \cup B) \cap \mathrm{cl}(C \cup B)$ such that $a \notin \mathrm{cl}(B)$. Then $a \in \mathrm{cl}(C \cup B) \setminus \mathrm{cl}(B)$, so $\mathrm{cl}(A \cup B) \underset{B}{\not\perp} C$. But the previous proposition implies that $A \underset{B}{\not\perp} C$, which is a contradiction. $\square$

REMARK V.1.21. In view of the definition and symmetry, when we look at $A \underset{B}{\perp} C$, we will generally assume that $B \subseteq A$ and $B \subseteq C$. Further, because of the closed set theorem, we may assume that A, B and C are closed, and finally, that $B = A \cap C$.

V.2. Buechler's theorem

We list a few more definitions.

DEFINITION V.2.1. Let (W, cl) be a pregeometry.

(1) (W, cl) is called *modular* if for every closed subsets S_1 and S_2 of W we have

$$\dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2);$$

(2) (W, cl) is called *locally modular* if for every closed subsets S_1 and S_2 of W we have

$$\dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2),$$

provided that $S_1 \cap S_2 \neq \emptyset$;

(3) (W, cl) is called *projective* if for every $a, b \in W$ and $C \subseteq W$ such that

$$a \in \mathrm{cl}(C \cup \{b\}),$$

there exists $c \in C$ such that $a \in \mathrm{cl}(\{c, b\})$.

REMARK V.2.2. It is not too difficult to see that a pregeometry is projective if and only if it is modular.

DEFINITION V.2.3. Let (W, cl) be a pregeometry.

(1) A closed set $L \subseteq W$ is a *line* if $\dim(L) = 2$;
(2) Two disjoint lines L_1 and L_2 are *parallel* if $\dim(L_1 \cup L_2) = 3$.

DEFINITION V.2.4. Let $G = (W, \mathrm{cl})$ be a pregeometry and $A \subseteq W$. Define the *localization of G at A*, written $G_A = (W_A, \mathrm{cl}_A)$, by

$$W_A = W \setminus A \quad \text{and} \quad \mathrm{cl}_A(X) = \mathrm{cl}(X \cup A) \setminus A, \text{ for } X \subseteq W_A.$$

REMARK V.2.5. It is easy to see that if G is a pregeometry, then G_A is a pregeometry. In G_A, we denote the dimension of X by $\dim(X/A)$.

REMARK V.2.6. If $G = (W, \mathrm{cl})$ is locally modular, then G_A is modular for any finite subset A of $W \setminus \mathrm{cl}(\emptyset)$.

PROPOSITION V.2.7. *Let (W, cl) be a pregeometry. Let S_1, S_2 be finite dimensional closed sets satisfying $S_0 = S_1 \cap S_2$. Then,*

$$S_1 \underset{S_0}{\perp} S_2 \quad \textit{if and only if} \quad \dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2).$$

PROOF. Suppose first that $S_1 \underset{S_0}{\downarrow} S_2$. Let I be a basis for S_0, and let $I_i \supseteq I$ be a basis for S_i for $i = 1, 2$. Clearly, $\mathrm{cl}(S_1 \cup S_2) = \mathrm{cl}(I_1 \cup I_2)$. We claim, in addition, that $I_1 \cup I_2$ is independent. Otherwise there is $a \in \mathrm{cl}(I_1 \cup I_2 \setminus \{a\})$. Without loss of generality, we may assume that $a \in I_1$. Now, since I_1 is independent, $a \notin \mathrm{cl}(I_1 \setminus \{a\})$, thus

$$a \in \mathrm{cl}(I_1 \cup I_2 \setminus \{a\}) \setminus \mathrm{cl}(I_i \setminus \{a\}), \quad \text{for } i = 1, 2.$$

We may also assume that $a \notin I$. To see this, assume that $a \in I$. Choose $I_i' \subseteq I_i \setminus I$, minimal with respect to inclusion, such that $a \in \mathrm{cl}(I_1' \cup I_2' \cup I \setminus \{a\})$, $I_i' \neq \emptyset$, for $i = 1, 2$. By the Exchange Property, there is $b \notin I$, such that

$$b \in \mathrm{cl}(I_1' \cup I_2' \cup I \cup \{b\}) \subseteq \mathrm{cl}(I_1 \cup I_2 \setminus \{b\}).$$

But, if $a \notin I$, then $\mathrm{cl}(I_1 \setminus \{a\}) = \mathrm{cl}(I \cup I_1 \setminus \{a\})$ so

$$a \in \mathrm{cl}(I_2 \cup (I_2 \setminus \{a\})) \setminus \mathrm{cl}(I \cup (I_2 \setminus \{a\})),$$

which means that $S_1 \underset{S_0}{\not\downarrow} S_2$, a contradiction. Hence $I_1 \cup I_2$ is independent. Therefore $\dim(S_1 \cup S_2) = |I_1 \cup I_2|$. But $|I_1 \cup I_2| + |I| = |I_1| + |I_2|$, so

$$\dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2).$$

For the converse, suppose $S_1 \underset{S_0}{\not\downarrow} S_2$. Let $a \in S_1$ and $A_1 \subseteq S_1$ such that

$$(*) \qquad a \in \mathrm{cl}(S_2 \cup A_1) \setminus \mathrm{cl}(S_0 \cup A_1).$$

Choose a such that A_1 has minimal cardinality. This implies that $A_1 \cup \{a\}$ is independent over S_0, and A_1 is independent over S_2. Thus, we can pick a basis I_0 for S_0, and extend $I_0 \cup A_1 \cup \{a\}$ to a basis I_1 of S_1. Now choose I_2' disjoint from I_0, such that $I_0 \cup I_2'$ is a basis of S_2. But, $I_0 \cup A_1 \cup \{a\} \cup I_2'$ is not independent by (*). Hence

$$\dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) < \dim(S_1) + \dim(S_2),$$

which finishes the proof. □

In the previous section, we showed that in any pregeometry, there is a relation that satisfies all the properties that forking satisfies in the context of simple theories. This allows us to show a theorem of Buechler [**Bu1**], originally proved for stable theories, when the pregeometry comes from forking.

THEOREM V.2.8 (Buechler). *Let $G = (W, \mathrm{cl})$ be a pregeometry. Then G is locally modular if and only if G_A has no parallel lines for every finite $A \subseteq W$, such that $A \not\subseteq \mathrm{cl}(\emptyset)$.*

PROOF. Suppose first that there is a finite $A \subseteq W$, such that $A \not\subseteq \mathrm{cl}(\emptyset)$ and G_A contain parallel lines. Thus, let L_1 and L_2 be disjoint lines in G_A such that

$\dim(L_1 \cup L_2/A) = 3$. Let $L_i' = \mathrm{cl}(L_i \cup A)$ for $i = 1, 2$. Then $A \subseteq L_1' \cap L_2'$, so $L_1' \cap L_2' \not\subseteq \mathrm{cl}(\emptyset)$, L_i' is closed for $i = 1, 2$, and

$$\dim(L_1' \cup L_2') + \dim(L_1' \cap L_2') \neq \dim(L_1') + \dim(L_2').$$

This shows that G is not locally modular.

For the converse, suppose that G is not locally modular. Then there are closed S_1 and S_2 subsets of W such that $S_1 \cap S_2 \not\subseteq \mathrm{cl}(\emptyset)$ and

$$\dim(S_1 \cup S_2) + \dim(S_1 \cap S_2) \neq \dim(S_1) + \dim(S_2).$$

We may assume that S_1 and S_2 are finite dimensional. Let $S_0 = S_1 \cap S_2$. By Proposition I.2.7, this implies that $S_1 \underset{S_0}{\not\downarrow} S_2$.

Let $\mathcal{D}$ be the set of pairs of integers $\langle d_1, d_2 \rangle$ such that there are closed sets S_1 and S_2 such that

- $S_0 = S_1 \cap S_2$ and $S_0 \not\subseteq \mathrm{cl}(\emptyset)$;
- $d_1 = \dim(S_1/S_0)$ and $d_2 = \dim(S_2/S_0)$;
- $S_1 \underset{S_0}{\not\downarrow} S_2$.

By assumption $\mathcal{D} \neq \emptyset$. Choose $\langle d_1, d_2 \rangle$ minimal with respect to the lexicographic order. We claim that $\langle d_1, d_2 \rangle = \langle 2, 2 \rangle$. Note that this is enough to prove the theorem since $\mathrm{cl}_{S_0}(S_1 \setminus S_0)$ and $\mathrm{cl}_{S_0}(S_2 \setminus S_0)$ are parallel lines in G_{S_0}.

Certainly, $d_1 > 1$. Otherwise, $\dim(S_1/S_0) = 1$ and since $S_1 \underset{S_0}{\not\downarrow} S_1$ there must exist $a \in S_1 \setminus S_0$ such that $a \in \mathrm{cl}(S_2) \setminus \mathrm{cl}(S_0)$. Since S_2 and S_0 are closed, we have $a \in S_1 \cap S_2 \setminus S_0$, a contradiction, since $S_1 \cap S_2 = S_0$.

We now show that $d_1 < 3$. Suppose $d_1 = \dim(S_1/S_0) \geq 3$. We will show that this contradicts the minimality of d_1. We first show that

(*) $$S_1 \cap \mathrm{cl}(S_2 a) = \mathrm{cl}(S_0 a), \qquad \text{for any } a \in S_1 \setminus S_0.$$

First, notice that $S_0 a \subseteq S_1$ and $S_0 a \subseteq \mathrm{cl}(S_2 a)$, so

$$S_1 \cap \mathrm{cl}(S_2 a) \supseteq \mathrm{cl}(S_0 a), \qquad \text{for any } a \in S_1 \setminus S_0.$$

Hence, if (*) does not hold, it is because for some $a \in S_1 \setminus S_0$, there exists

$$b \in (S_1 \cap \mathrm{cl}(S_2 a)) \setminus \mathrm{cl}(S_0 a).$$

By definition, this implies that $\{ a, b \} \underset{S_0}{\not\downarrow} S_2$.

Let $S_1' = \mathrm{cl}(S_0 ab)$. Then $S_1' \cap S_2 = S_0$ and $S_0 \not\subseteq \mathrm{cl}(\emptyset)$. Furthermore $S_1' \underset{S_0}{\not\downarrow} S_2$. But $\dim(S_2/S_0) = d_2$ and $\dim(S_1'/S_0) = 2 < 3 \leq d_1$, which contradicts the minimality of d_1. Therefore, (*) holds.

Now, since $S_1 \underset{S_0}{\not\downarrow} S_2$, there exist $a \in S_1$ and a finite $A \subseteq S_1$ such that

(**) $$a \in \mathrm{cl}(S_2 \cup A) \setminus \mathrm{cl}(S_0 \cup A).$$

But $A \not\subseteq S_0$. Otherwise, by (**) we have $a \in \mathrm{cl}(S_2) \setminus \mathrm{cl}(S_0)$. This shows that $a \in S_2 \setminus S_1$ since S_2 and S_0 are closed. But $a \in S_1$, so $a \in (S_1 \cap S_2) \setminus S_0 = \emptyset$, which is impossible. Hence, there is $b \in A \setminus S_0$. Then, since $Ab = A$, we have

$$a \in \mathrm{cl}(S_2 \cup A) \setminus \mathrm{cl}(S_0 b \cup A).$$

Hence $S_1 \underset{S_0 \cup b}{\not\downarrow} S_2$.

Now consider $S_2' := \mathrm{cl}(S_2 b)$. Then, $S_1 \underset{S_0 \cup b}{\not\downarrow} S_2$ implies that $S_1 \underset{S_0 \cup b}{\not\downarrow} S_2'$. By (*) we have $S_1 \cap S_2' = \mathrm{cl}(S_0 b)$. Finally, $\dim(S_1/(S_0 b)) < \dim(S_1/S_0) = d_1$ and $d_2 = \dim(S_2/S_0) = \dim(S_2'/S_0 b)$. This contradicts the minimality of d_1. We prove similarly that $d_2 = 2$, which finishes the proof. □

V.3. Some “set theory”

In this section, we gather several observations with a set-theoretic flavor. The next theorem is a generalization of a lemma from J. Baumgartner, M. Foreman and O. Spinas [**BFS**]. Although the proof is easy, it does not follow from the fact that two resolutions of the same model coincide on a club, as we do not have control over the cardinality of the closures. The value of this theorem is that it makes it possible to attach a club as an invariant of the pregeometry.

THEOREM V.3.1. *Let $G = (W, \mathrm{cl})$ be a pregeometry. Suppose $\dim(W) = \lambda$ is regular and uncountable. Let $I = \{ a_i \mid i < \lambda \}$ and $J = \{ b_i \mid i < \lambda \}$ be bases of W. Then*

$$C = \{\ i < \lambda\ :\ \mathrm{cl}(\{ a_j \mid j < i \}) = \mathrm{cl}(\{ b_j \mid j < i \})\ \}$$

is a closed and unbounded subset of λ.

PROOF. We first show that C is closed. Let $\delta = \sup(\delta \cap C)$. Then, for any $i < \delta$ there is $i_1 \in C$ such that $i < i_1 < \delta$. Hence, by definition of C

(*) $$\mathrm{cl}(\{ a_j \mid j < i_1 \}) = \mathrm{cl}(\{ b_j \mid j < i_1 \}).$$

Lemma 4 and (*) implies that $a_i \in \mathrm{cl}(\{ b_j \mid j < \delta \})$. Hence,

$$\{ a_j \mid j < \delta \} \subseteq \mathrm{cl}(\{ b_j \mid j < \delta \}),$$

and therefore

$$\mathrm{cl}(\{ a_j \mid j < \delta \}) \subseteq \mathrm{cl}(\{ b_j \mid j < \delta \}),$$

by Fact I.1.2 again. The other inclusion is similar and so

$$\mathrm{cl}(\{ a_j \mid j < \delta \}) \supseteq \mathrm{cl}(\{ b_j \mid j < \delta \}).$$

This shows that $\delta \in C$, by definition of C.

We now show that C is unbounded in λ. Let $i < \lambda$ be given. We construct $i_n < \lambda$ for $n \in \omega$ increasing with $i_0 = i$ such that

(1) $\mathrm{cl}(\{ a_j \mid j < i_n \}) \subseteq \mathrm{cl}(\{ b_j \mid j < i_{n+1} \})$ if n is even;
(2) $\mathrm{cl}(\{ b_j \mid j < i_n \}) \subseteq \mathrm{cl}(\{ a_j \mid j < i_{n+1} \})$ if n is odd.

This is enough: Let $i(*) = \sup\{ i_n \mid n \in \omega \}$. Then $i(*) < \lambda$ since λ is regular uncountable. Further $\mathrm{cl}(\{ a_j \mid j < i(*) \}) = \mathrm{cl}(\{ b_j \mid j < i(*) \})$, since if $i < i(*)$, then there is i_n with n even such that $i < i_n$, so

$$a_i \in \mathrm{cl}(\{ a_j \mid j < i_n \}) \subseteq \mathrm{cl}(\{ b_j \mid j < i_{n+1} \}) \subseteq \mathrm{cl}(\{ b_j \mid j < i(*) \}),$$

hence

$$\mathrm{cl}(\{ a_j \mid j < i(*) \}) \subseteq \mathrm{cl}(\{ b_j \mid j < i(*) \}).$$

The other inclusion is proved similarly. Thus $i < i(*) \in C$, which shows that C is unbounded.

This is possible: Given $i < \lambda$, we let $i_0 = i$. Assume that $i_n < \lambda$ has been constructed. Suppose n is even. For each $j < i_n$, we have that $a_j \in W = \mathrm{cl}(\{ b_j \mid j < \lambda \})$ since J is a basis. By Finite Character, there is a finite $S_j \subseteq \lambda$ such that $a_j \in \mathrm{cl}(\{ b_k \mid k \in S_j \})$. Let $k_j = \sup S_j < \lambda$, so $a_j \in \mathrm{cl}(\{ b_l \mid l \leq k_j \})$, and by increasing k_j if necessary, we may assume that $k_j \geq i_n$. Set $i_{n+1} = \sup\{ k_j + 1 \mid j < i_n \}$. Then $i_{n+1} < \lambda$ since λ is regular and satisfies our requirement. The case when n is odd is handled similarly. □

PROPOSITION V.3.2 (Downward Theorem). *Let $G = (W, \mathrm{cl})$ be a pregeometry. Let A, B and C be subsets of W. Suppose $A \underset{B}{\perp} C$ and A' is a subset of A, of cardinality at most λ, for λ an infinite cardinal. Then there is $B' \subseteq B$ of cardinality at most λ such that $A' \underset{B'}{\perp} C$.*

PROOF. Let $A' \subseteq A$ of cardinality λ be given. Let $\{ \langle a_i, A_i \rangle \mid i < \lambda \}$ be an enumeration of all the pairs such that $a_i \in A'$ and $A_i \subseteq A'$ is finite. Such an enumeration is possible since λ is infinite. Since $A \underset{C}{\perp} B$, necessarily

(*) $$a_i \notin \mathrm{cl}(B \cup C \cup A_i) \setminus \mathrm{cl}(B \cup A_i), \quad \text{for every } i < \lambda.$$

Hence, either $a_i \notin \mathrm{cl}(B \cup C \cup A_i)$, or $a_i \in \mathrm{cl}(B \cup A_i)$. If the latter holds, by Finite Character, we can find a finite $B_i \subseteq B$ such that $a_i \in \mathrm{cl}(B_i \cup A_i)$. We let $B_i = \emptyset$, if $a_i \notin \mathrm{cl}(B \cup A_i)$. Let $B' = \bigcup B_i$. Then $B' \subseteq B$, and $|B'| \leq \lambda$.

We claim that $A' \underset{B'}{\perp} C$. Otherwise, there exist $a \in A'$ and a finite $A^* \subseteq A'$, such that

(**) $$a \in \mathrm{cl}(B' \cup C \cup A^*) \setminus \mathrm{cl}(B' \cup A^*).$$

Choose $i < \lambda$ such that $a = a_i$ and $A^* = A_i$. Thus, $a_i \in \mathrm{cl}(B' \cup C \cup A_i)$, and so by Fact I.1.2 we have $a_i \in \mathrm{cl}(B \cup C \cup A_i)$. Therefore, by (*) we have that

$a_i \in \mathrm{cl}(B \cup A_i)$. Hence $a_i \in \mathrm{cl}(B_i \cup A_i)$ by construction. But $B_i \subseteq B'$, and so $a_i \in \mathrm{cl}(B' \cup A_i)$ by Fact I.1.2. This contradicts (**) since $A^* = A_i$. □

COROLLARY V.3.3. *Let $G = (W, \mathrm{cl})$ be a pregeometry. Let A, B and C be subsets of W. Suppose that A, B and C have cardinality at least λ for some λ infinite. If $A \underset{C}{\perp} B$, then we can find $A' \subseteq A$, $B' \subseteq B$ and $C' \subseteq C$ of cardinality λ, such that $A' \underset{C'}{\perp} B'$.*

PROOF. By the previous theorem using monotonicity. □

PROPOSITION V.3.4 (Ultraproducts of Pregeometries). *Let I be a set and $\mathfrak{D}$ an $\aleph_1$-complete ultrafilter on I. Suppose that (W_i, cl_i) is a pregeometry for each $i \in I$. Consider $W = \Pi_{i \in I} W_i$ and for $a \in W$ and $B \subseteq W$, define*

$$a \in \mathrm{cl}(B) \quad \textit{if} \quad \{\, i \in I \mid a(i) \in \mathrm{cl}_i(B(i)) \,\} \in \mathfrak{D}.$$

Then (W, cl) is a pregeometry.

PROOF. We only show Finite Character, since all the other axioms of a pregeometry are routine. Suppose $a \in \mathrm{cl}(B)$. Then $J = \{\, i \in I \mid a(i) \in \mathrm{cl}_i(B(i)) \,\} \in \mathfrak{D}$, and by Finite Character of cl_i, for each $i \in J$, there is a finite $B'(i) \subseteq B(i)$, such that $a(i) \in \mathrm{cl}_i(B'(i))$. Let $J_n = \{\, i \in J \mid B'(i)$ has n elements$\}$. Then

$$\{\, i \in J \mid a(i) \in \mathrm{cl}_i(B'(i)) \,\} = \bigcup_{n<\omega} J_n.$$

Hence, by $\aleph_1$-completeness, there exist $n < \omega$ such that $J_n \in \mathfrak{D}$. We now write $B'(i) = \{\, b^i_1, \dots, b^i_n \,\}$ for $i \in J_n$. Let $A = \{\, f_1, \dots, f_n \,\} \subseteq B$ be given by $f_k(i) = b^i_k$ when $i \in J_n$ and $f_k(i) \in B(i)$ arbitrary when $i \notin J_n$. Then

$$\{\, i \in I \mid a(i) \in \mathrm{cl}_i(A(i)) \,\} \supseteq J_n \in \mathfrak{D},$$

by construction. Hence $\{\, i \in I \mid a(i) \in \mathrm{cl}_i(A(i)) \,\} \in \mathfrak{D}$. Thus, $a \in \mathrm{cl}(A)$ and A is a finite subset of B, which is what we needed. □

We now introduce stable systems, a notion originally developed in model theory. They are used for example in [**Sh87a**], [**Sh87b**] and later in the proof of the main gap [**Sh b**]. See also [**Ma**].

DEFINITION V.3.5. Let $G = (W, \mathrm{cl})$ be a pregeometry.

(1) We call $S = \langle A_s \mid s \in I \rangle$ a *system*, if $A_s \subseteq W$, I is a subset of $\bigcup I$ closed under subsets and $s \subseteq t$ implies $A_s \subseteq A_t$. We denote by s^- the immediate predecessor of s in I if one exists;
(2) We call $S = \langle A_s \mid s \in I \rangle$ a *stable system*, if S is a system which satisfies in addition

$$A_s \underset{A_{s^-}}{\perp} \bigcup \{A_t \mid t \not\supseteq s, t \in I\}, \qquad \text{for every } s, t \in I.$$

PROPOSITION V.3.6 (Generalized Symmetry Lemma). *Let $G = (W, \mathrm{cl})$ be a pregeometry. Let $S = \langle A_s \mid s \in I\rangle$ be a system. Suppose there is an enumeration $I = \langle s(i) \mid i < \alpha\rangle$ such that*

(1) $s(i) \subseteq s(j)$ *implies* $i \leq j$, *for every* $i, j < \alpha$;
(2) $A_{s(i)} \underset{A_{s(i)^-}}{\perp} \bigcup\{A_{s(j)} \mid j < i\}$.

Then S is a stable system.

PROOF. By Finite Character, we may assume that I is finite. We prove this by induction on $|I|$. The base case is obvious. Suppose it is true for $|I| = n < \omega$. Suppose $I = \langle s(i) \mid i \leq n\rangle$ is an enumeration satisfying (1) and (2). Assume for a contradiction that S is not a stable system. By induction hypothesis, we have either

(*) $$A_{s(n)} \underset{A_{s(n)^-}}{\not\perp} \bigcup\{A_{s(j)} \mid s(j) \not\subseteq s(n)\},$$

or there exists $i < n$ with $s(i) \not\subseteq s(n)$ such that

(**) $$A_{s(i)} \underset{A_{s(i)^-}}{\not\perp} \bigcup\{A_{s(j)} \mid s(j) \not\subseteq s(i), s(j) \neq s(n)\} \cup A_{s(n)}.$$

By assumption, we know that

(†) $$A_{s(n)} \underset{A_{s(n)^-}}{\perp} \bigcup\{A_{s(j)} \mid j < n\}.$$

By (1), we have that

$$\bigcup\{A_{s(j)} \mid s(j) \not\subseteq s(n)\} \subseteq \bigcup\{A_{s(j)} \mid j < n\}.$$

Hence (*) is impossible, by Monotonicity and (†).

Now if $s(i) \subseteq s(n)$, then $s(i)^- \subseteq s(n)^-$. Hence, $A_{s(i)^-} \subseteq A_{s(n)^-}$ since S is a system. By Monotonicity used twice, (†) implies that

$$A_{s(n)} \underset{A_{s(i)^-}}{\perp} \bigcup\{A_{s(j)} \mid s(j) \not\subseteq s(i), s(j) \neq s(n)\} \cup A_{s(i)}.$$

But this and Remark I.1.17 contradicts (**). Hence $s(i) \not\subseteq s(n)$. □

V.4. Abstract group configuration

A central result in Geometric Stability Theory is the presence in very general circumstances of a definable group among the definable (maybe infinitely definable) sets of a model. This is referred to by W. Hodges [**Ho**] as the Zilber Group Configuration Theorem, and by others as the Hrushovski Group Configuration Theorem. We will call it the Hrushovski-Zilber Group Configuration Theorem. It has an ancient flavor; it is in a line of work which dates back to Veblen and Young around 1910. The general template is the emergence of algebraic structures from certain geometric configurations.

The Hrushovski-Zilber Group configuration Theorem for the first order, countable $\aleph_1$-categorical case is due to Boris Zilber [**Zi**]. It builds on the methods of Baldwin-Lachlan [**BaLa**]). It was extended to stable theories by Ehud Hrushovski [**Hr1**] (see also the exposition of Elizabeth Bouscaren [**Bo**]). This generalization was done using S. Shelah's notions of forking, regular types and p-simple technology.

These methods have since developed into a field of its own. See for example the recent books of Steve Buechler [**Bu a**] and Anand Pillay [**Pi b**]. They have been used to answer classical logical questions, for example B. Zilber's solution to the finitely axiomatization problem [**Zi**]; to general classification theory questions, for example E. Hrushovski's proof that unidimensional stable theories are superstable [**Hr2**], S. Buechler's work on Vaught's Conjecture [**Bu2**], and have found several applications outside of model theory [**ChHr**], [**HrPi1**], [**HrPi2**], [**EvHr1**], [**EvHr2**], [**Hr3**].

Our aim in this section is to separate the model-theoretic aspects from the combinatorial geometry in the Hrushovski-Zilber Group Configuration Theorem to enable us to transfer this tool to non first order contexts.

The setting of the Hrushovski-Zilber Group Configuration Theorem is the following. We have a pregeometry where the closure operation comes from forking. Technically speaking, the pregeometry is the set of realization of a stationary type p with the additional property that the closure operation given by

$$a \in \mathrm{cl}(B) \quad \text{if and only if} \quad \mathrm{tp}(a/B \cup \mathrm{dom}(p)) \text{ forks over } \mathrm{dom}(p).$$

Here are several of the key ingredients in the first order case that are used. (1) The notion of types (2) The fact that the pregeometry comes from forking guarantees that the ambient dependence relation is well-behaved. (3) (T stable) Every type is definable. (4) Work in T^{eq}, which allows one to use the Canonical Basis Theorem. All these results rely on the compactness theorem.

As described in the introduction of this chapter, there are several nonelementary contexts, where pregeometries arise and where one may hope to apply these ideas. In each of the contexts we described (categorical sentences in $L_{\omega_1\omega}(Q)$, Excellent Scott sentences, totally transcendental or superstable finite diagrams) some facts allow us to circumvent the difficulties posed by the absence of the compactness theorem. In each of them, we have (1) a good notion of types. (2) In spite of the fact that the dependence relation is not necessarily as well-behaved as forking, there exists pregeometries. By work started in Chapter IV, this implies that we can define another dependence relation which satisfies all the formal properties of forking for first order theories. (3) In many of them, there is a notion of stationary types and those are definable. (4) There are several ways (as yet unpublished, some due to myself, some to Saharon Shelah) of introducing substitutes to T^{eq} and get the Canonical Basis Theorem.

The aim of this section is to take into account the technology available (or being developed in nonelementary classes) to find some natural axioms (behind which the logical framework is hidden) under which group configurations may yield a group. Let us make this more precise. The Hrushovski-Zilber Group Configuration Theorem states in essence that if in a *definable* pregeometry we have a certain dependence configuration (called group configuration, see the figure next to Hypothesis I.4.16), then there exists a *definable* group.

There are two steps in the Hrushovski-Zilber Group Configuration Theorems.

Step 1: Starting from the group configuration, where the dependence relation is forking, to obtain a similar group configuration, where in addition, some points are *uniquely determined* by others. This is often called the unique definability condition.

Step 2: From this special configuration, one derives a definable group.

Both steps rely on the general properties of forking and the canonical basis theorem for stable theories.

Step 1 seems decidedly model-theoretic and there is little hope for general conditions for the existence of an abstract theorem generalizing it. However, Step 2 is amenable.

There are two aspects of definability: By *syntactic definability*, we mean some model theoretic notion; we live in an ambient model M satisfying some axioms (not necessarily first order) and have a notion of formula. A set A is said to be syntactically definable over B if there exists a set of formulas p over B such that $a \in A$ if and only if a realizes all the formulas in p. Now given an automorphism group Γ, there is also a notion of *semantic definability*. We say that a set A is semantically definable over B if for every $f \in \Gamma$ fixing B pointwise, f fixes A setwise.

Now, syntactic definability implies semantic definability in case the automorphism group is (a subgroup) of the automorphism group of the model M. The converse is more delicate.

In this section, we work inside a pregeometry (W, cl), given with an automorphism group Γ. We require that the pregeometry be homogeneous with respect to this automorphism group, which in our context means that the automorphism group is rich. We then have a notion of semantic definable sets (henceforth just called definable). We also consider a subcollection $\mathfrak{D}$ of definable sets (which in the applications are going to be the syntactically definable sets). We assume that $\mathfrak{D}$ satisfies an axiom parallel to (i) the definability of (stationary) types and another axiom parallel to (ii) the canonical basis theorem. If we assume in addition that in the unique definability condition of Step 2, the definable sets are in $\mathfrak{D}$, then this implies the existence of a group, which is equal to a (potentially infinite) intersection of sets in $\mathfrak{D}$.

If one is interested in applications to model theory for nonelementary classes and in particular issues of definability, we will be given a natural notion of syntactical definable sets and this theorem will give a definable group in this language (provided this notion satisfies the condition of $\mathfrak{D}$). All the first order notions for definability used so far belong to this set and the axioms hold in the well-known first order cases.

We can also look at this without a notion of syntactically definable sets. This allows us to ignore $\mathfrak{D}$, that is to assume that $\mathfrak{D}$ is the set of all semantically definable sets. Then, we do not need an axiom on definability of stationary types and just consider the canonical basis theorem for semantically definable sets. This gives a very smooth theorem in the context of combinatorial geometry.

The presentation owes much to [**Ho**], [**Bo**] and [**EvHr1**]. In fact, the setting of [**EvHr1**] is a particular case of our setting: Let $K \subseteq L$ be algebraically closed fields. The pregeometry (W, cl) is given by $W = L \setminus K$ and $a \in \mathrm{cl}(C)$ if and only if a is in the algebraic closure (in L) of the field generated by $K \cup C$. The automorphism group Γ is $\mathrm{aut}(L/K)$. All the axioms are satisfied. Using the fact that they work in algebraically closed fields, they managed to obtain additional information on the definable groups.

V.4.1. The context. Let (W, cl) be a pregeometry and Γ be a group of automorphisms of (W, cl).

We always assume $\mathrm{cl}(\emptyset) \neq W$, in fact we will make the following assumption:

HYPOTHESIS V.4.1. We assume that (W, cl) is infinite dimensional.

NOTATION V.4.2. (1) *We denote Γ_X the group of automorphisms of (W, cl) fixing X pointwise.*

(2) *Given a sequence A of elements of W. We denote by $\Gamma_X(A)$ the orbit of A under Γ_X, namely*

$$\Gamma_X(A) = \{f(A) \mid f \in \Gamma_X\}.$$

For a sequence $A = \langle a_i \mid i < \alpha \rangle$, we write $f(A)$ for $\langle f(a_i) \mid i < \alpha \rangle$.

In the previous chapter we introduced the following relation between subsets of a pregeometry. For convenience and readability, we use the usual notation $\perp$

DEFINITION V.4.3. Let (W, cl) be a pregeometry. Let A, B and C be subsets of W. We say that A *depends on C over B*, if there exist $a \in A$ and a finite

$A' \subseteq A$ (possibly empty) such that

$$a \in \mathrm{cl}(B \cup C \cup A') \setminus \mathrm{cl}(B \cup A').$$

If A depends on C over B, we write $A \underset{B}{\not\downarrow} C$;

If A does not depend on C over B, we write $A \underset{B}{\downarrow} C$.

We proved in the first section of this chapter that this dependence relation satisfies the familiar axioms of forking, as introduced by Shelah.

FACT V.4.4 (Forking Relations).

(1) (Definition) $A \underset{B}{\downarrow} C$ *if and only if* $A \underset{B}{\downarrow} B \cup C$*;*
(2) (Existence) $A \underset{C}{\downarrow} C$.
(3) (Finite Character) $A \underset{C}{\downarrow} B$ *if and only if* $A' \underset{C}{\downarrow} B'$ *for every finite* $A' \subseteq A$ *and finite* $B' \subseteq B$*;*
(4) (Invariance) *If* $f \in \Gamma$*, then* $A \underset{B}{\downarrow} C$ *if and only if* $f(A) \underset{f(B)}{\downarrow} f(C)$*;*
(5) (Monotonicity) *Let* $B \subseteq B_1 \subseteq C' \subseteq C$*. Then* $A \underset{B}{\downarrow} C$ *implies* $A \underset{B_1}{\downarrow} C'$*;*
(6) (Symmetry) $A \underset{B}{\downarrow} C$ *if and only if* $C \underset{B}{\downarrow} A$*;*
(7) (Transitivity) *If* $B \subseteq C \subseteq D$*, then* $A \underset{B}{\downarrow} D$ *if and only if* $A \underset{B}{\downarrow} C$ *and* $A \underset{C}{\downarrow} D$*;*
(8) ($\kappa(T) = \aleph_0$) *For every* $\bar{a}$ *and* C *there exists* $B \subseteq C$, $|B| < \aleph_0$*, i.e. finite, such that* $\bar{a} \underset{B}{\downarrow} C$*;*
(9) (Closed Set) $A \underset{B}{\downarrow} C$ *if and only if* $\mathrm{cl}(A) \underset{\mathrm{cl}(B)}{\downarrow} \mathrm{cl}(C)$.

REMARK V.4.5. Definition, Existence, Finite Character, Invariance, Monotonicity, $\kappa(T) = \aleph_0$ and Closed Set are obvious. The difficulty is to obtain (6) and (7).

The first axiom corresponds to the extension property of forking as well as some saturation.

AXIOM V.4.6 (Extension). *Let* $\bar{a}$ *be given and* X *be finite dimensional. Then, there exists* $\bar{a}' \in \Gamma(\bar{a})$ *such that* $\bar{a}' \downarrow X$.

The next axioms correspond the uniqueness of the nonforking extension. I call it Homogeneity because a pregeometry satisfying H1 is called homogeneous. The axioms H2 and H3 have a similar flavor and in first order model theoretic cases follow from the same facts: stationarity and saturation.

AXIOM V.4.7 (Homogeneity).

H1 *If* $a, b \notin \mathrm{cl}(X)$ *then there is* $f \in \Gamma_{\mathrm{cl}(X)}$ *such that* $f(a) = b$*;*
H2 *If* $\bar{a}_1 \in \Gamma_X(\bar{a}_2)$*,* $\bar{b}_1 \in \Gamma_X(\bar{b}_2)$ *and* $\bar{a}_i \underset{X}{\perp} \bar{b}_i$ *for* $i = 1, 2$*, then there is* $g \in \Gamma_X$ *such that* $g(\bar{a}_1) = \bar{a}_2$ *and* $g(\bar{b}_1) = \bar{b}_2$*;*
H3 *If* $\bar{a} \underset{X}{\perp} \bar{b}$*,* $\bar{a}' \underset{X}{\perp} \bar{b}$ *and* $\bar{a} \in \Gamma_X(\bar{a}')$*, then* $\bar{a} \in \Gamma_{\mathrm{cl}(X\bar{b})}(\bar{a}')$.

FACT V.4.8. *If* $\dim(X) < \dim(W)$ *and* $|\,\Gamma_X(a)| < \aleph_0$*, then* $a \in \mathrm{cl}(X)$.

PROOF. Since W is infinite dimensional, there exists an infinite set $\{a_n \mid n < \omega\} \subseteq W \setminus \mathrm{cl}(X)$. By Homogeneity, if $a \notin \mathrm{cl}(X)$, then $\{a_n \mid n < \omega\} \subseteq \Gamma_{\mathrm{cl}(X)}(a) \subseteq \Gamma_X(a)$, a contradiction. □

The next definition is a substitute for the logical notions of algebraic or definable closure.

DEFINITION V.4.9.

(1) We say that a is in the *definable closure* of X, if $|\Gamma_X(a)| = 1$, i.e. $\Gamma_X(a) = \{a\}$. We write $a \in \mathrm{dcl}(X)$, if a is in the definable closure of X;
(2) We say that a is in the *algebraic closure* of X, if $|\Gamma_X(a)| < \aleph_0$. We write $a \in \mathrm{acl}(X)$, if a is in the algebraic closure of X.

REMARK V.4.10. For small dimensional sets $X \subseteq W$ and elements $a \in W$, Fact I.4.8 implies that if $a \in \mathrm{acl}(X)$ or $a \in \mathrm{dcl}(X)$, then $a \in \mathrm{cl}(X)$.

Finally, we introduce the notions that can be used to bypass the general $\mathfrak{C}^{eq}$ technology, in particular Shelah's Canonical Basis Theorem.

DEFINITION V.4.11.

(1) We say that a set $A \subseteq W^n$ is *definable over* $X \subseteq W$, if every $f \in \Gamma_X$ fixes A setwise;
(2) We say that $X \subseteq W$ is the *support* of a set $A \subseteq W^n$ if for every $f \in \Gamma$, f fixes A setwise if and only if f fixes X pointwise.

FACT V.4.12.

(1) *Any automorphism* f *fixes* X *pointwise if and only if* f *fixes* $\mathrm{dcl}(X)$ *pointwise, so by definition of support, we have* $X = \mathrm{dcl}(X)$.
(2) *The support of* A *is unique if it exists. Let* X *and* Y *be supports of* A*, Let* $f \in \Gamma$ *fixing* X *pointwise. Then* f *fixes* A *setwise since* X *is a support and so* f *fixes* Y *pointwise since* Y *is a support also. Thus* $\Gamma_X(Y) = Y$ *so* $\mathrm{dcl}(Y) = Y \subseteq \mathrm{dcl}(X)$. *We are done by symmetry.*

REMARK V.4.13. By the previous fact, if A has support X, we define $\dim(A) := \dim(X)$ and $A \underset{B}{\perp} C$ if and only if $\operatorname{supp}(A) \underset{\operatorname{supp}(B)}{\perp} \operatorname{supp}(C)$. All these notions are well-defined and satisfy all the facts we have already proved. There will be no ambiguity since we will not deal with $A \subseteq W$.

We consider a collection $\mathfrak{D}$ of definable sets (without) parameters. We require that $\mathfrak{D}$ be closed under union and intersection, projections, product and permutation. We do *not* require closure under complementation. For clarity, we use the usual first order notation with formulas. For example, by $\phi(\bar{x}) \in \mathfrak{D}$ we mean a definable subset of $W^{\ell(\bar{x})}$. We write $\models \phi[a, \bar{b}]$ to say that $(a, \bar{b})$ is in the definable set $\phi(\bar{x})$.

We require that if $a \in \operatorname{dcl}(\bar{b})$, then there is $\phi(x, \bar{y}) \in \mathfrak{D}$ such that $\models \phi[a, \bar{b}]$ and for every a' such that $\models \phi[a', \bar{b}]$, we have $a = a'$. We also require that the sets of if $\mathfrak{D}$ are compatible with Γ, i.e. if $\models \phi[a, \bar{b}]$, then also $\models \phi[f(a), f(\bar{b})]$ for $f \in \Gamma$.

Now on to the last axioms.

AXIOM V.4.14 (Definability of types). *Let $\bar{a}$, $\bar{b} \in W$ and $R \in \mathfrak{D}$ be a relation on the orbits of $\bar{a}$ and $\bar{b}$. Then there is $d_R \in \mathfrak{D}$ such that for all $\bar{a}' \in \Gamma(\bar{a})$ we have $\bar{a}' \in d_R \in \mathfrak{D}$ if and only if for every $\bar{b}' \in \Gamma(\bar{b})$ if $\bar{a}' \perp \bar{b}'$, then $(\bar{a}', \bar{b}') \in R$.*

AXIOM V.4.15 (Canonical Basis). *If $E(\bar{x}, \bar{y}) \in \mathfrak{D}$ is an equivalence relation over orbits of W, then each equivalence class $\bar{b}/E$ has a support.*

V.4.2. The group configuration. We show that if a pregeometry, its automorphism group and the collection of definable sets satisfy our list of axioms, then the special group configuration gives rise to a definable group.

HYPOTHESIS V.4.16. There exist b_i, a_i for $i = 1, 2, 3$, sequences of dimension 1, such that

(1) All sequences are pairwise independent;
(2) $\dim(b_1 b_2 b_3) = 2$, $\dim(b_i a_j a_k) = 2$, for all $i \neq j \neq k$, and
$$\dim(b_1 b_2 b_3 a_1 a_2 a_3) = 3;$$
(3) $a_2 \in \operatorname{dcl}(b_1 a_3)$, $a_1 \in \operatorname{dcl}(b_2 a_3)$, and $a_3 \in \operatorname{dcl}(b_1 a_2) \cap \operatorname{dcl}(b_2 a_1)$.

Given sets of sequences A, B, we will denote $A + B$, the set
$$A + B = \{(\bar{a}, \bar{b}) \mid \bar{a} \in A, \bar{b} \in B, \text{ and } \bar{a} \perp \bar{b}\}.$$

Given $(b_1', b_2') \in \Gamma(b_1) + \Gamma(b_1)$ and $a \in \Gamma(a_2) \setminus \operatorname{cl}(b_1' b_2')$, we define $h_{(b_1', b_2')}(a)$ as follows. Choose $f \in \Gamma$ such that $f(b_1) = b_1'$, $f(b_2) = b_2'$ and

$f(a_2) = a$. To do this, choose first $\sigma \in \Gamma$ such that $\sigma(b_1) = b_1'$, $\sigma(b_2) = b_2'$. Clearly σ exists by Axiom H2, since $b_i' \in \Gamma(b_i)$ for $i = 1, 2$. Then, choose $\tau \in \Gamma$ such $\tau(\sigma(a_2)) = a$ and $\tau \restriction b_1 b_2 = id$. This is possible by Axiom H1 since by assumption on the configuration $a_2 \notin \mathrm{cl}(b_1 b_2)$, so $\sigma(a_2) \notin \mathrm{cl}(b_1' b_2')$ and $a \notin \mathrm{cl}(b_1' b_2')$ by choice of a.

We now make a few observations. First, $f(a_3)$ is uniquely determined since $a_3 \in \mathrm{dcl}(b_1 a_2)$. Indeed, suppose $g(b_1) = b_1'$ and $g(a_2) = a$. Then $g^{-1} f \in \Gamma_{b_1 a_2}$ so $g^{-1} f(a_3) = a_3$. Therefore $g(a_3) = f(a_3)$. Second, notice that $f(a_1)$ is uniquely determined, since $a_1 \in \mathrm{dcl}(b_2 a_3)$. Indeed, suppose $g(b_2) = b_2'$ and $g(a_3) = f(a_3)$. Then $g^{-1} f \in \Gamma_{b_2 a_3}$, so $g^{-1} f(a_1) = a_1$, and $g(a_1) = f(a_1)$.

We define $h_{(b_1', b_2')}(a) = f(a_1)$. In view of the previous considerations, this is well-defined and furthermore $f(a_1) \in \Gamma(a_2)$. Notice also that $\langle \bar{b}_1', \bar{b}_2', a, a' \rangle \in \mathfrak{D}$, for all $h_{(b_1', b_2')}(a) = a'$, using projections and intersection.

We wish to extend the action of $\Gamma(b_1) + \Gamma(b_1)$ on all elements of $\Gamma(a_2)$. To do this, we define the following relation on $\Gamma(b_1) + \Gamma(b_1)$:

$$(b_1', b_2') \sim (b_1'', b_2'') \quad \text{if} \quad h_{(b_1', b_2')}(a) = h_{(b_1'', b_2'')}(a), \quad \text{for all } a \in \Gamma(a_2) \setminus \mathrm{cl}(b_1' b_2' b_1'' b_2'').$$

CLAIM. $\sim$ is an equivalence relation on $\Gamma(b_1) + \Gamma(b_1)$.

PROOF. Reflexivity and Symmetry are obvious. To see that Transitivity holds, we first show that we can replace "for all" by "there exists" in the definition of a. Indeed, suppose that $a, a' \in \Gamma(a_2) \setminus \mathrm{cl}(b_1' b_2' b_1'' b_2'')$ and that $h_{(b_1', b_2')}(a) = h_{(b_1'', b_2'')}(a)$. By Axiom H1, there exists $\sigma \in \Gamma_{\mathrm{cl}(b_1' b_2' b_1'' b_2'')}$ such that $\sigma(a) = a'$. Notice that $h_{(b_1', b_2')}(\sigma(a)) = \sigma(h_{(b_1', b_2')}(a)$ and similarly, $h_{(b_1'', b_2'')}(\sigma(a)) = \sigma(h_{(b_1'', b_2'')}(a)$, and hence $h_{(b_1', b_2')}(a') = h_{(b_1'', b_2'')}(a')$. Transitivity now follows easily. □

We denote by $[b_1', b_2']$ the equivalence class of (b_1', b_2') under $\sim$. It now follows from Axiom I.4.14 that $\sim \in \mathfrak{D}$. Hence, each $[b_1', b_2'] \in \mathfrak{D}$ and by Axiom I.4.15 must have a support. Clearly, $\mathrm{supp}[b_1', b_2'] \, \mathrm{cl}(b_1', b_2') \subseteq W$.

Let $H = \{[b_1', b_2'] \mid (b_1', b_2') \in \Gamma(b_1) + \Gamma(b_1)\}$.

Notice that Γ acts transitively on the elements of H in the following sense: if $[b_1, b_2], [c_1, c_2]$ are elements of H, there is $f \in \Gamma$ such that $f([b_1, b_2]) = [c_1, c_2]$. To see this, recall that $c_i \in \Gamma(b_i)$ for $i = 1, 2$ and that by definition of H we have that each sequence is independent and $b_1 \downarrow b_2$ and $c_1 \downarrow c_2$. Hence, by Axiom H2 there exists $f \in \Gamma$ such that $f(b_i) = c_i$ for $i = 1, 2$. Then $f([b_1, b_2]) = [f(b_1), f(b_1)] = [c_1, c_2]$ as required.

Notice also that by Axiom I.4.15, every element $\alpha \in H$ has a support X_α, so that we can extend forking and dimensions on elements of H. Elements of H are called *germs* and they each act on $\Gamma(a_2)$. We will want to compose germs, but we will want to make sure that the composition is also an element of H. For this, some more work is needed. We can express H by an infinite intersection of elements of $\mathfrak{D}$ by Axiom I.4.14.

LEMMA V.4.17. $[b_1, b_2] \subseteq \mathrm{cl}(b_3)$ *and therefore* $[b_1, b_2] \perp b_i$ *for* $i = 1, 2$.

PROOF. First, observe that

$$[b_1, b_2](a_2) = a_1 \in \mathrm{cl}(a_2 b_3), \tag{*}$$

by definition and the configuration. We want to show that

$$X := \mathrm{supp}([b_1, b_2]) \subseteq \mathrm{cl}(b_3).$$

By definition of support, it is enough to show that for all $f \in \Gamma_{\mathrm{cl}(b_3)}$, we have

$$f([b_1, b_2]) = [b_1, b_2],$$

i.e. $[b_1, b_2]$ is fixed setwise by f.

For this, fix $f \in \Gamma_{\mathrm{cl}(b_3)}$ and let $a \in \Gamma(a_2) \setminus \mathrm{cl}(X f(X) b_1 b_2 b_3)$.

We claim that $[b_1, b_2](a) \in \mathrm{cl}(a b_3)$. To see this, it is enough to find an automorphism $\sigma \in \Gamma_{\mathrm{cl}(b_1 b_2 b_3)}$ such that $\sigma(a_2) = a$ and then applying σ to (*). But the existence of σ follows from H1 if we can show that $a_2 \notin \mathrm{cl}(b_1 b_2 b_3)$. This follows from the configuration. Suppose $a_2 \in \mathrm{cl}(b_1 b_2 b_3)$. Then $a_1 \in \mathrm{cl}(b_1 b_2 b_3)$, since $a_1 \in \mathrm{cl}(a_2 b_3)$ and also $a_3 \in \mathrm{cl}(b_1 b_2 b_3)$ since $a_3 \in \mathrm{cl}(a_1 b_2)$. This is a contradiction since

$$\dim(a_1 a_2 a_3 b_1 b_2 b_3) = 3 \neq 2 = \dim(b_1 b_2 b_3).$$

Now choose $g \in \Gamma_{\mathrm{cl}(b_3 a)}$ such that $g \restriction X = f \restriction X$.

Then, we have the following equalities:

$$\begin{aligned}
[b_1, b_2](a) &= g([b_1, b_2](a)) && \text{(since } [b_1, b_2](a) \in \mathrm{cl}(b_3 a)\text{)} \\
&= g([b_1, b_2])(g(a)) && (g \text{ is an automorphism}) \\
&= g([b_1, b_2])(a) && (g(a) = a) \\
&= f([b_1, b_2])(a) && (f \restriction X = g \restriction X)
\end{aligned}$$

Thus, by definition of the germs, $[b_1, b_2] = f([b_1, b_2])$. This finishes the proof. □

The elements of H act on $\Gamma(a_2)$. It makes sense to compose them. Let α, $\beta \in H$. We write $\alpha * \beta$ for an element $\gamma \in H$, such that for all $a \in \Gamma(a_2)$ such that $a \perp \alpha\beta\gamma$, we have $\gamma(a) = \alpha(\beta(a))$. Such γ's do not necessarily exist. We will show that, in fact, $\alpha * \beta$ exists if $\alpha \perp \beta$.

Suppose $[c_1, c_2]$ and $[c_2, c_3]$ are in H. Then in this case, it is easy to see that $[c_1, c_2] * [c_2, c_3] = [c_1, c_3]$. We will show that, in fact, this is the typical situation when $\alpha \perp \beta$. This is done by the following lemmas.

LEMMA V.4.18. *If* c_1, c_2 *and* $c_3 \in \Gamma(b_1)$ *are such that* $\dim(c_1c_2c_3) = 3$, *then* $[c_1c_2] \downarrow [c_2, c_3]$

PROOF. Let $X = \mathrm{supp}([c_1, c_2])$ and $Y = \mathrm{supp}([c_2, c_3])$. Suppose $[c_1c_2] \not\downarrow [c_2, c_3]$. Then, by definition, $X \not\downarrow Y$, and so $X \subseteq Y$ since they are both closed, and furthermore $X = Y$ since they have dimension 1. By the dimension, $c_1 \downarrow c_2c_3$, so since $Y \subseteq \mathrm{cl}(c_2c_3)$, we must have $c_1 \downarrow c_2Y$. But, since $X = Y$, we now have also $c_1 \downarrow c_2X$. Since $c_2 \downarrow X$, we thus have $\dim(c_1c_2X) = 3$. But $\mathrm{cl}(c_1c_2X) = \mathrm{cl}(c_1c_2)$, so that's impossible. □

LEMMA V.4.19. *Let* $\alpha, \beta \in H$. *If* $\alpha \downarrow \beta$, *then there exist* c_1, c_2 *and* c_3 *such that* $\alpha = [c_1, c_2]$, $\beta = [c_2, c_3]$ *and* $\dim(c_1, c_2, c_3) = 3$.

PROOF. Notice that Γ acts transitively over $H + H$, via the supports: let $\alpha_1 \downarrow \beta_1$ and $\alpha_2 \downarrow \beta_2$. Denote by X_{α_i}, (respectively X_{β_i}) the supports of α_i (respectively β_i). Then, by definition $X_{\alpha_i} \downarrow X_{\beta_i}$, for $i = 1, 2$ and further, $X_{\alpha_1} \in \Gamma(X_{\alpha_2})$, and $X_{\beta_1} \in \Gamma(X_{\beta_2})$ by a homogeneous axiom. The result follows by Stationarity. Now, by the previous lemma, if c_1, c_2 and $c_3 \in \Gamma(a)$ are such that $\dim(c_1c_2c_3) = 3$, then $[c_1c_2] \downarrow [c_2, c_3]$. Thus, by transitivity, we can find $f \in \Gamma$ such that $f([c_1, c_2]) = \alpha$ and $f([c_2, c_3]) = \beta$. Thus, $\alpha = [f(c_1), f(c_2)]$ and $\beta = [f(c_2), f(c_3)]$. Clearly, $\dim(f(c_1), f(c_2), f(c_3)) = 3$. We are done. □

LEMMA V.4.20. *If* $\alpha, \beta \in H$ *with* $\alpha \downarrow \beta$, *then* $\alpha * \beta$ *is a well-defined element of* H. *Moreover,* $\alpha * \beta \downarrow \alpha$ *and* $\alpha * \beta \downarrow \beta$.

PROOF. Choose c_1, c_2 and c_3 with $\dim(c_1, c_2, c_3) = 3$, such that $\alpha = [c_1, c_2]$ and $\beta = [c_2, c_3]$. Check that $\gamma = [c_1, c_3]$. The rest is now immediate. □

Define an equivalence relation on $H + H$,

$$(\alpha_1, \beta_1) \approx (\alpha_2, \beta_2) \quad \text{if} \quad \alpha_1 * \beta_1(e) = \alpha_2 * \beta_2(e),$$

for every $e \in \Gamma(j)$ such that $\alpha_1 * \beta_1(e)$ and $\alpha_2 * \beta_2(e)$ are both defined. Let G be the set of equivalence classes. Let us call $[\alpha, \beta]$ the equivalence class of (α, β) under $\approx$. We define

$$[\alpha_1, \alpha_2] * [\beta_1, \beta_2] = [\gamma, \delta],$$

where $\alpha_1 * \alpha_2 * \beta_1 * \beta_2 \approx \gamma * \delta$, and $(\gamma, \delta) \in H + H$. By considerations similar to H, G can be expressed by an infinite intersection of sets in $\mathfrak{D}$, and also its product by Axiom I.4.15.

The next claim shows that G is closed under composition.

CLAIM. $(G, *)$ is closed under composition.

PROOF. Let $[\alpha_1, \alpha_2]$, $[\beta_1, \beta_2] \in G$ be given. Then $\alpha_1 \perp \alpha_2$ and $\alpha_1 * \alpha_2 \perp \alpha_i$, for $i = 1, 2$ by a previous lemma. Similarly, $\beta_1 \perp \beta_2$ and $\beta_1 * \beta_2 \perp \beta_i$, for $i = 1, 2$.

We distinguish two cases. Let $\alpha := \alpha_1 * \alpha_2$. If $\alpha \perp \beta_1 * \beta_2$, then both $[\alpha, (\beta_1 * \beta_2)]$ and $[\beta_1, \beta_2] \in G$, and obviously $\alpha_1 * \alpha_2 * \beta_1 * \beta_2 \approx \alpha * (\beta_1 * \beta_2)$.

If $\alpha \not\perp \beta_1 * \beta_2$, then $\alpha \in \mathrm{cl}(\beta_1 * \beta_2)$ and so since $\beta_1 * \beta_2 \perp \beta_1$, also $\alpha \perp \beta_1$. Thus, $\alpha \perp \beta_1$ and $\beta_1 \perp \beta_2$.

First, choose $\delta \in H$ such that $\delta \perp \alpha\beta_1\beta_2$. In particular, $\beta_1 \perp \delta$. Now choose $\delta_1 \in H$ such $\delta_1 \perp \delta$. Then $\delta_1 * \delta$ is well-defined, and $\delta_1 * \delta \perp \delta$. Since Γ acts transitively on $H+H$, we can find $g \in \Gamma$ such that $g(\delta) = \delta$ and $g(\delta * \delta_1) = \beta_1$. Thus, $\beta_1 = g(\delta_1) * \delta$. Call $g(\delta_1) = \delta' \in H$. Then $\alpha * \beta_1 * \beta_2 = (\alpha * \delta) * (\delta' * \beta_2)$. We are done in we can show that $\alpha \perp \delta$ and $\delta' \perp \beta_2$. Certainly $\alpha \perp \delta$ by choice of δ. Now if $\delta' \not\perp \beta_2$, then $\beta_2 \in \mathrm{cl}(\delta')$. But $\beta_1 \in \mathrm{cl}(\delta, \delta')$ so $\delta' \in \mathrm{cl}(\beta_1 \delta)$. Hence, $\dim(\delta, \beta_1, \beta_2) = 2$, contradicting the choice of δ.

This finishes the proof. □

LEMMA V.4.21. $(G, *)$ *is a group.*

PROOF. G is nonempty. Since H is closed under inverse, it is easy to see that the inverse of $[\alpha, \beta]$ is $[\beta^{-1}, \alpha^{-1}]$, so G is closed under inverse. The previous claim shows that G is closed under composition. Finally, $(G, *)$ acts on $\Gamma(a_2)$ as described. □

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